SPOTLIGHT TILING

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Abstract. This article introduces spotlight tilings of rectilinear regions. Spotlight tiling differs from the customary methods of tiling, and is defined inductively as follows: a tile begins in a northwest corner of a region, and extends as far south or east as possible. Some distinguishing aspects of spotlight tiling include that the order in which tiles are placed in a region affects what tiles may be placed subsequently, and the number of tiles in a spotlight tiling of a particular region is not fixed. A thorough examination of spotlight tilings of rectangles is presented, including the distribution of such tilings using a fixed number of tiles, and how the directions of the tiles themselves are distributed. The spotlight tilings of several other regions are studied, and suggest that further analysis of spotlight tilings will continue to yield elegant results and enumerations.

1. Introduction

The study of tilings of a region (or, dually, of perfect matchings of a region) is a well studied topic in combinatorics and statistical mechanics. Customarily, there is a finite set S of distinct tiles which may be used repeatedly to tile a particular region or family of regions. The natural questions to ask are: what regions may be tiled by the elements of the set S? how many ways are there to tile a region R by elements of S? how do these answers change if more restrictions are imposed on S? For example, the number of domino tilings of an $m \times n$ rectangle, where m is the number of rows and n is the number of columns, was computed by Kasteleyn in [1]. This formula is quite complicated, and its generating function is shown below, where m is assumed to be even (since at least one dimension must be even):

$$Z_{m,n}(x,y) = \prod_{i=1}^{m/2} \prod_{j=1}^{n} \left[2 \left(x^2 \cos^2 \frac{i\pi}{m+1} + y^2 \cos^2 \frac{j\pi}{n+1} \right)^{1/2} \right].$$

The number of tilings of an $m \times n$ rectangle can become much simpler if a few restrictions are made on S and R. For example, suppose the region R is colored as a checkerboard having a black upper-left square, with alternating black and white squares in each column or row. Restrict the set S to contain vertical dominos of both colorings (one with a white top square and one with a black top square), and only the horizontal domino with a black left square. Then, with these definitions, the number of tilings of an $m \times n$ region R by elements of S is

$$\begin{cases} 0 & : \quad m \text{ and } n \text{ are both odd;} \\ 1 & : \quad m \text{ is even;} \\ \left(\frac{m+1}{2}\right)^{n/2} & : \quad m \text{ is odd and } n \text{ is even.} \end{cases}$$

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These numbers are sequence A133300 of [2].

Additionally, tiling results typically do not depend on the order in which the tiles are placed. Because the set S of allowable tiles does not change as each tile is placed in the region, tiles may be considered to be placed simultaneously.

This article introduces a new type of tiling, and provides a sample of results answering the most basic questions about this method. In this paper, tiles are placed in the region sequentially, and after each placement the set of allowable tiles may change. To be specific, first a particular corner direction is specified (*northwest* for the duration of this article). At each stage a tile is placed with one end point in a "corner," as defined by the chosen direction, and the tile must extend as far as possible from this corner either horizontally or vertically. This type of tiling is called a *spotlight* tiling, in reference to the fact that the method of tiling is like placing a spotlight in one of the specified corners and turning it to point horizontally or vertically so that it shines as far as possible until it reaches an obstruction.

Spotlight tilings of rectangles are examined thoroughly below, including a description of various statistics, such as the number of tiles (spotlights) needed and the average number of tiles used in a spotlight tilings of the rectangle. Additionally, the spotlight tilings of certain other regions which are similar to rectangles are studied. The nature of spotlight tiling means that many of the results obtained below are recursive in nature.

The most basic rectilinear region is an $m \times n$ rectangle. Therefore, in the analysis of this new type of tiling, attention is primarily focused on tilings of rectangles, in terms of their enumeration and their properties. This will be the substance of Section 3. For example, in addition to determining the number of spotlight tilings of an $m \times n$ rectangle, more detailed statistics about these tilings will be studied. Unlike other sorts of tilings, where the number of tiles required to cover a region is fixed, the number of spotlight tiles used in a particular spotlight tiling depends on the tiling itself. The distribution of the number of these tiles will be part of the discussion in Section 3. Following this discussion, in Section 4, attention will be turned to spotlight tilings of rectilinear regions which are formed from rectangles by removing squares at the corners. The recursive nature of these tilings leads naturally to recursive enumeration formulae. In some cases, these equations will be left in a recursive format, as it is simpler to read them in this manner. In other situations, when a closed form itself is quite elegant, both the recursive and the closed formulae will be given. Finally, in Section 5, the spotlight tilings of a certain family of frame-like regions is explored. The paper concludes with a brief discussion of how spotlight tilings may be studied further in Section 6.

2. Definitions

The basic definitions and notation of this article are outlined below.

Definition 2.1. A *region* is the dual of a finite connected induced subgraph of $\mathbb{Z}^2$.

As mentioned in the introduction, the spotlight tilings discussed in this paper rely on the choice of a particular direction and type of corner, in this case a northwest corner.

Definition 2.2. A *northwest corner* in a region is a square that is bound above and on the left by the boundary edge of the region.

For example, the four northwest corners of the region in Figure 1 have been shaded.

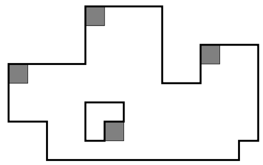


FIGURE 1. A region with four northwest corners, which are marked by shading.

As discussed in the introduction, the tilings in this paper are of a different nature from those usually studied. Instead of choosing form a finite set of tiles, the possible tiles themselves are defined by the region.

Definition 2.3. A *spotlight tile* with an endpoint in square s extends as far east horizontally or south vertically from s as possible, terminating at the boundary of the region, or when it encounters a tile that has already been placed.

Definition 2.4. Given a region R, a *spotlight tiling* of R is defined recursively as follows. Each connected component of a region is tiled individually, so suppose that R is connected. Choose any northwest corner $s \in R$. Place a spotlight tile with an endpoint in s, extending either horizontally (east) or vertically (south) as far as possible. If R' is the region remaining after placing this spotlight, then the spotlight tiling of R is completed by finding a spotlight tiling of R'.

A spotlight tiling of a 3×4 rectangle is depicted in Figure 2. The complete tiling is the last image in the figure, having been built successfully from the previous images.

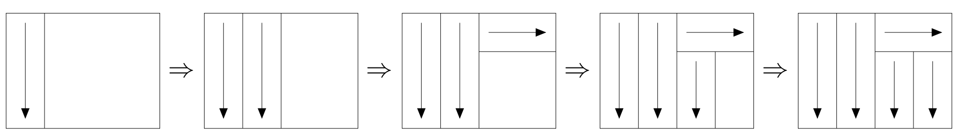


FIGURE 2. The recursive construction of a spotlight tiling of a 3×4 rectangle. The arrows are provided here only to highlight the direction (horizontal or vertical) of each spotlight tile.

Although spotlight tiles are placed sequentially in a region, two spotlight tilings are considered distinct only if they look different once all the tiles are in place. In other words, if there is more than one order in which the tiles can be placed in the region, this alone does not distinguish one tiling from another. Moreover, the direction (horizontal or vertical) of a spotlight tile is obvious except in certain cases of tiles of length one, where the direction of such a tile will not be specified as uniquely horizontal or vertical.

Definition 2.5. If the last tile placed in a spotlight tiling has length 1, it is a *HV-tile*, referring to the fact that the tile's direction could be considered to be either horizontal or vertical.

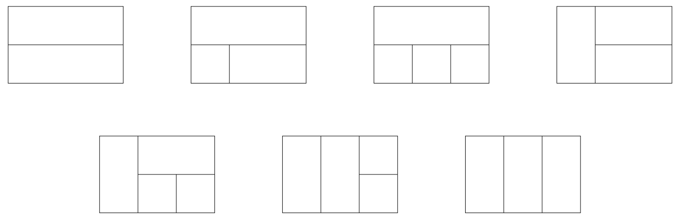


Figure 3. The seven distinct spotlight tilings of a 2×3 rectangle. In the third, fifth, and sixth of these, the last (southeast-most) tile is a HV-tile.

The seven different spotlight tilings of a 2×3 rectangle are depicted in Figure 3.

Definition 2.6. Let $R_{m,n}$ denote an $m \times n$ rectangle. The set of spotlight tilings of $R_{m,n}$ is denoted $\mathcal{T}_{m,n}$, and $T_{m,n} = |\mathcal{T}_{m,n}|$.

As depicted in Figure 3, $T_{2,3} = 7$.

The recursive definition of spotlight tiling means that

$$\begin{aligned} \mathcal{T}_{m,n} = &\{\text{one } (1 \times n)\text{-tile together with } t \mid t \in \mathcal{T}_{m-1,n}\} \\ &\cup \{\text{one } (m \times 1)\text{-tile together with } t \mid t \in \mathcal{T}_{m,n-1}\}. \end{aligned} \tag{1}$$

3. Spotlight tilings of rectangles

The first goal of this examination of spotlight tilings is a thorough understanding of spotlight tilings of rectangles. Since the definition of a spotlight tiling gives no preference to horizontal or vertical tiles, all results in this section should be symmetric with respect to m and n. In particular, it should be the case that $T_{m,n} = T_{n,m}$.

A precise formula for $T_{m,n}$ is straightforward to compute, based on the recursive nature of Definition 2.4.

Theorem 3.1. *For all $m, n \geq 1$,*

$$T_{m,n} = \binom{m+n}{m} - \binom{m+n-2}{m-1}. \tag{2}$$

Proof. Definition 2.4 gives the recursive formula

$$T_{m,n} = T_{m-1,n} + T_{m,n-1} \tag{3}$$

for all $m, n > 1$. Additionally, if $m > 1$, then

$$T_{m,1} = T_{m-1,1} + 1. \tag{4}$$

Since $T_{1,1} = 1$, equation (4) implies that $T_{m,1} = m$ for all $m \geq 1$. Therefore (2) is satisfied for $n = 1$ and any m. Supposing inductively that the result holds whenever the dimensions of the rectangle sum to less than k, consider an $m \times n$ rectangle where $m + n = k$. Then, using equation (3),

$$\begin{aligned} T_{m,n} &= T_{m-1,n} + T_{m,n-1} \\ &= \binom{m+n-1}{m-1} - \binom{m+n-3}{m-2} + \binom{m+n-1}{m} - \binom{m+n-3}{m-1} \\ &= \binom{m+n}{m} - \binom{m+n-2}{m-1}, \end{aligned}$$

Thus the result holds for all $m, n \geq 1$. □

Notice that equation (2) is symmetric in m and n, as required. The values of $T_{m,n}$ for small m and n are displayed in Table 1. Additionally, these are sequence A051597 of [2].

$T_{m,n}$	$n=1$	2	3	4	5	6	7
$m=1$	1	2	3	4	5	6	7
2	2	4	7	11	16	22	29
3	3	7	14	25	41	63	92
4	4	11	25	50	91	154	246
5	5	16	41	91	182	336	582
6	6	22	63	154	336	672	1254
7	7	29	92	246	582	1254	2508

TABLE 1. The number of spotlight tilings of $R_{m,n}$, for $m, n \leq 7$.

As demonstrated in Figure 3, the number of spotlight tiles in a particular spotlight tiling of $R_{m,n}$ is not fixed. For example, a spotlight tiling of $R_{2,3}$ can consist of 2, 3, or 4 tiles. Therefore, to better understand spotlight tilings of rectangles, it is important to understand how many tiles may (likewise, "must" and "can") be used in a spotlight tiling of $R_{m,n}$, and how many spotlight tilings of the rectangle use exactly r tiles. There are additional aspects of spotlight tilings using the minimal or maximal number of tiles that are of interest as well.

Definition 3.2. For a spotlight tiling t of a region R, let $|t|$ be the number of spotlight tiles used in t, known as the *size* of t.

Definition 3.3. Let $t^-_{m,n}$ denote the minimum number of spotlight tiles needed in a spotlight tiling of $R_{m,n}$, and let $t^+_{m,n}$ denote the maximum number of spotlight tiles that can be used in a spotlight tiling of $R_{m,n}$. That is,

$$t^-_{m,n} = \min_{t \in \mathcal{T}_{m,n}} |t|$$

$$t^+_{m,n} = \max_{t \in \mathcal{T}_{m,n}} |t|$$

An element of $\mathcal{T}_{m,n}$ using $t^-_{m,n}$ spotlight tiles is a *minimal* spotlight tiling, while one that uses $t^+_{m,n}$ spotlight tiles is a *maximal* spotlight tiling.

Proposition 3.4. *For all $m, n \geq 1$,*

$$t^-_{m,n} = \min\{m, n\}; \tag{5}$$

$$t^+_{m,n} = m + n - 1. \tag{6}$$

Proof. By the definition of spotlight tilings, it is clear that the minimum number of tiles needed depends on the minimum dimension of $R_{m,n}$. Suppose, without loss of generality, that $m \leq n$. If fewer than m tiles are placed in $R_{m,n}$, then at least one row and at least one column are not completely tiled. Thus, $t^-_{m,n}$ can be no less than m. Additionally, one spotlight tiling of the rectangle consists of m horizontal tiles, so $t^-_{m,n} = m$, which proves equation (5).

Equation (1) implies that $t^+_{m,n} = \max\{1 + t^+_{m-1,n}, 1 + t^+_{m,n-1}\}$. Then, since $t^+_{1,1} = 1$ and $t^+_{m,1} = m$, the rest of the proof of equation (6) follows inductively. □

Note that $t^-_{m,n} = t^+_{m,n}$ if and only if $m = n = 1$. Therefore, in anything larger than a 1×1 square, there will be variation in the number of spotlight tiles used.

The number of minimal spotlight tilings of an $m \times n$ rectangle is necessarily 1 or 2, depending on whether $m \neq n$ or $m = n$. This will be proved in a more general argument in Theorem 3.7.

The number of maximal spotlight tilings, on the other hand, is a somewhat more interesting case and must be treated independently.

Theorem 3.5. *The number of maximal spotlight tilings of $R_{m,n}$ is*

$$\binom{m+n-2}{m-1}.$$

Proof. Because of equations (1) and (6), once the first spotlight tile has been placed in the rectangle, this can (and, in fact, must) be completed to a maximal tiling of the rectangle by finding a maximal spotlight tiling of the resulting sub-rectangle ($R_{m-1,n}$ or $R_{m,n-1}$, depending on whether the first tile was horizontal or vertical). Therefore, whatever the directions of the first k spotlight tiles, a maximal tiling of the remaining sub-rectangle will yield a maximal tiling of the original rectangle.

There is a single element in the set $\mathcal{T}_{1,1}$, and it consists of a single HV-tile. Therefore, by induction (specifically, using equation (1)), the last tile placed in a maximal spotlight tiling must be an HV-tile. In fact, if m and n are not both equal to 1, then the penultimate tile placed in a maximal spotlight tiling of $R_{m,n}$ must also have length 1, although this will not be an HV-tile since its direction must be specified.

By nature of spotlight tiling, there cannot be more than m horizontal tiles or n vertical tiles in an element of $\mathcal{T}_{m,n}$. If the last tile is an HV-tile, than of the previous $m+n-2$ tiles in a maximal spotlight tiling, at most $m-1$ can be horizontal and at most $n-1$ can be vertical. Consequently, of these $m+n-2$ spotlight tiles, exactly $m-1$ are horizontal and exactly $n-1$ are vertical.

Because any initial set of spotlight tiles in $R_{m,n}$ can be completed to a maximal tiling, the number of maximal spotlight tilings depends only on which $m-1$ of the first $m+n-2$ tiles are horizontal, and thus is

$$\binom{m+n-2}{m-1}.$$

□

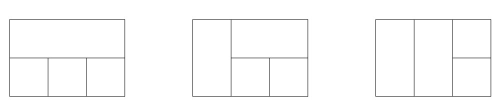


FIGURE 4. The three maximal spotlight tilings of a 2×3 rectangle. These are the tilings of Figure 3 which contain HV-tiles.

Definition 3.6. Let $t^r_{m,n}$ be the number of spotlight tilings of $R_{m,n}$ that use r spotlight tiles. That is, $t^r_{m,n} = |\{t \in \mathcal{T}_{m,n} \mid |t| = r\}|$.

Theorem 3.7. *For all integers $r \in [\min\{m,n\}, m+n-1)$,*

$$t^r_{m,n} = \binom{r-1}{m-1} + \binom{r-1}{n-1}.$$

Proof. As with some of the preceding remarks, this proof relies heavily on the recursive aspect of spotlight tilings. Suppose that $k < r$ spotlight tiles have been placed in the rectangle, and that the remaining untiled region is an $(m-a)\times(n-b)$ rectangle, where $a+b=k$. Then this tiling of $R_{m,n}$ must be completed with an element of $\mathcal{T}_{m-a,n-b}$ which uses $r-k$ tiles.

Once the remaining untiled region is $R_{1,p}$ or $R_{p,1}$, there is exactly one tiling of it by any number of tiles less than or equal to p, and there are no tilings of it using more than p tiles. Therefore, consider having already placed $k < r$ tiles in some way so that what remains is a $1\times(m+n-1-k)$ rectangle or a $(m+n-1-k)\times 1$ rectangle. Since $r < m+n-1$, and $k < r$, the inequality $m+n-1-k > 1$ holds, meaning that there is no over-counting. Likewise, because $r-k$ (the number of spotlight tiles as yet unplaced) is less than $m+n-1-k$ (the area of the untiled rectangle), $t^{r-k}_{m+n-1-k,1} = t^{r-k}_{1,m+n-1-k} = 1$.

It remains to count the number of ways to place k spotlight tiles in the rectangle so as to leave an untiled sub-rectangle with dimensions $1\times(m+n-1-k)$ or $(m+n-1-k)\times 1$.

To obtain a $1\times(m+n-1-k)$ untiled sub-rectangle using k spotlight tiles, without over-counting, the last of the k tiles must be horizontal. Of the preceding $k-1$ tiles, exactly $m-2$ must be horizontal, and the horizontal and vertical tiles can be placed in any order. Therefore, there are $\binom{k-1}{m-2}$ ways to obtain an untiled sub-rectangle with dimensions $1\times(m+n-1-k)$. Additionally, there are bounds on the value of k: $k \geq m-1$ and $k \leq r-1$. Combining this with analogous statements about obtaining $R_{m+n-1-k,1}$, and using elementary binomial identities, yields

$$\begin{aligned} t^r_{m,n} &= \sum_{k=m-1}^{r-1}\binom{k-1}{m-2} + \sum_{k=n-1}^{r-1}\binom{k-1}{n-2} \\ &= \binom{r-1}{m-1} + \binom{r-1}{n-1}. \end{aligned}$$

□

Therefore, Theorems 3.5 and 3.7 and Proposition 3.4 can be combined in the following equation:

$$t^r_{m,n} = \begin{cases} 0 & : \ r < \min\{m,n\} \text{ or } r > m+n-1; \\ \binom{r-1}{m-1} + \binom{r-1}{n-1} & : \ \min\{m,n\} \leq r < m+n-1; \\ \binom{m+n-2}{m-1} & : \ r = m+n-1. \end{cases}$$

In fact, if $(m,n) \neq (1,1)$, then $t^{m+n-2}_{m,n} = t^{m+n-1}_{m,n}$, and the values $t^r_{m,n}$ are strictly increasing on the interval $r \in [\min\{m,n\}, m+n-2]$. More specifically, for $r \in [1+\min\{m,n\}, m+n-2]$,

$$\begin{aligned} t^r_{m,n} - t^{r-1}_{m,n} &= \binom{r-1}{m-1} + \binom{r-1}{n-1} - \binom{r-2}{m-1} - \binom{r-2}{n-1} \\ &= \binom{r-2}{m-2} + \binom{r-2}{n-2} = t^{r-1}_{m-1,n-1}. \end{aligned}$$

Moreover, it is straightforward to check that

$$\sum_{r\geq 1} t^r_{m,n} = \binom{m+n}{m} - \binom{m+n-2}{m-1},$$

confirming Theorem 3.1.

Given Theorems 3.5 and 3.7, it is straightforward now to compute the average number of spotlight tiles used in a tiling of an $m \times n$ rectangle.

Corollary 3.8. *The average number of tiles used in a spotlight tiling of $R_{m,n}$, that is, the average size of an element of $\mathcal{T}_{m,n}$, is*

$$\frac{mn(m+n-1)}{(m+n)(m+n-1)-mn}\left(1+\frac{n-1}{m+1}+\frac{m-1}{n+1}\right). \tag{7}$$

Proof. This average is computed by evaluating

$$\begin{aligned}
\frac{\sum_{r=1}^{m+n-1} r\cdot t^r_{m,n}}{\binom{m+n}{m}-\binom{m+n-2}{m-1}} &= \frac{(m+n-1)\binom{m+n-2}{m-1} + \sum_{r=1}^{m+n-2}\left[r\binom{r-1}{m-1}+r\binom{r-1}{n-1}\right]}{\binom{m+n}{m}-\binom{m+n-2}{m-1}} \\
&= \frac{(m+n-1)\binom{m+n-2}{m-1} + m\binom{m+n-1}{m+1} + n\binom{m+n-1}{n+1}}{\binom{m+n}{m}-\binom{m+n-2}{m-1}} \\
&= \frac{mn(m+n-1)}{(m+n)(m+n-1)-mn}\left(1+\frac{n-1}{m+1}+\frac{m-1}{n+1}\right).
\end{aligned}$$

□

Admittedly, the expression in (7) is not particularly elegant, but it gives a closed formula for the expected number of tiles in a random spotlight tiling, and demonstrates how this is related to the dimensions of the rectangle.

In a maximal spotlight tiling of $R_{m,n}$, there are $m-1$ horizontal tiles, $n-1$ vertical tiles, and 1 HV-tile. Moreover, a spotlight tiling $t \in \mathcal{T}_{m,n}$ contains an HV-tile if and only if t is maximal. The breakdown of tile directions is immediate for maximal tilings, but the question is more subtle for non-maximal tilings.

Definition 3.9. For a spotlight tiling t with no HV-tiles, let $h(t)$ be the number of horizontal tiles in t, and let $v(t)$ be the number of vertical tiles in t.

Definition 3.10. Define the generating function

$$G_{m,n}(H,V) = \sum_{\substack{\text{non-maximal} \\ t\in\mathcal{T}_{m,n}}} H^{h(t)}V^{v(t)}.$$

Notice that the function $G_{1,1}(H,V)$ is not defined, since the only tiling of a 1×1 rectangle is maximal.

Theorem 3.11. *For all $m,n \geq 1$, where $(m,n)\neq(1,1)$,*

$$G_{m,n}(H,V) = H^m\sum_{r=0}^{n-2}\binom{r+m-1}{m-1}V^r + V^n\sum_{r=0}^{m-2}\binom{r+n-1}{n-1}H^r.$$

Proof. This proof is similar to the proof of Theorem 3.7. Consider a non-maximal tiling of $R_{m,n}$ using r tiles. In the successive iterations of the spotlight tiling procedure, the last untiled sub-rectangle will be covered either by a horizontal or by a vertical tile. Thus, after placing the first $r-1$ tiles, what remains must be a rectangle of dimensions $1 \times (m+n-r)$ or $(m+n-r) \times 1$. In the former case, the final tile is horizontal, and in the latter case the final tile is vertical.

In the former case, there are $m-1$ of the first $r-1$ tiles which are horizontal, and the remaining $r-m$ are vertical. The recursive nature of spotlight tiling means that these horizontal and vertical spotlight tiles can occur in any order. Thus there are $\binom{r-1}{m-1}$ ways for the last tile to be horizontal in a non-maximal element of $\mathcal{T}_{m,n}$ with r tiles. Similarly, there are $\binom{r-1}{n-1}$ ways for the last tile to be vertical in a non-maximal element of $\mathcal{T}_{m,n}$ with r tiles.

Therefore,

$$\begin{aligned} G_{m,n}(H,V) &= \sum_{\substack{\text{non-maximal} \\ t \in \mathcal{T}_{m,n}}} H^{h(t)} V^{v(t)} \\ &= \sum_{r=\min\{m,n\}}^{m+n-2} \binom{r-1}{m-1} H^{m-1} V^{r-m} \cdot H \\ &\qquad + \sum_{r=\min\{m,n\}}^{m+n-2} \binom{r-1}{n-1} V^{n-1} H^{r-n} \cdot V \\ &= \sum_{r=m}^{m+n-2} \binom{r-1}{m-1} H^m V^{r-m} + \sum_{r=n}^{m+n-2} \binom{r-1}{n-1} V^n H^{r-n} \\ &= H^m \sum_{r=0}^{n-2} \binom{r+m-1}{m-1} V^r + V^n \sum_{r=0}^{m-2} \binom{r+n-1}{n-1} H^r. \end{aligned}$$

□

One consequence of Theorem 3.11 is that in any non-maximal spotlight tiling of $R_{m,n}$, there are either exactly m horizontal tiles or exactly n vertical tiles. In the former case, there can be between 0 and $n-2$ vertical tiles, and in the latter case there can be between 0 and $m-2$ horizontal tiles.

Substituting x for both H and V in $G_{m,n}(H,V)$ gives the generating function for the numbers $t^r_{m,n}$ when $r < m+n-1$, and in fact the coefficient $[x^r]G_{m,n}(x,x)$ is equal to $\binom{r-1}{m-1} + \binom{r-1}{n-1}$, confirming Theorem 3.7.

4. Spotlight tilings of rectangles with missing corners

The highly recursive nature of spotlight tilings means that enumerating the spotlight tilings of certain families of regions can be done without difficulty. For the most part, the regions considered in this section are variations on rectangles, in particular rectangles missing squares at the corners. Because the northwest corner is specified in spotlight tilings, the enumeration of the spotlight tilings of these regions depends on which corner was removed.

It should be noted that it is possible to obtain formulae for the number of spotlight tilings of other regions as well, due to the iterative definition of this method of tiling. For example, the number of spotlight tilings of a rectangle with a single square removed from somewhere in the interior is not difficult to obtain, particularly if this square is parameterized by its position relative to the southeast corner of the rectangle, which does not change when spotlight tiles are placed.

Definition 4.1. Fix integers $m, n \geq 2$. Let $R^{\mathsf{NW}}_{m,n}$ (respectively, $R^{\mathsf{NE}}_{m,n}$, $R^{\mathsf{SW}}_{m,n}$, and $R^{\mathsf{SE}}_{m,n}$) be an $m \times n$ rectangle whose northwest (respectively, northeast, southwest, and southeast) corner has been removed. The set $\mathcal{T}^*_{m,n}$ consists of all spotlight tilings of the region $R^*_{m,n}$, and $T^*_{m,n} = |\mathcal{T}^*_{m,n}|$.

The most difficult of these spotlight tilings to enumerate, and the one with the least elegant answer, is for the region $R^{\mathsf{NW}}_{m,n}$. That this case differs from the others is no surprise, since there are two northwest corners in the new region, and thus spotlight tiles can start from two different squares.

Corollary 4.2. *For all $m, n \geq 2$,*

$$\begin{aligned} T^{\mathsf{NW}}_{m,n} &= T_{m-1,n-1} + (n-1)T_{m-2,n} + (m-1)T_{m,n-2} \\ &= \binom{m+n-2}{m-1}\left[1 + (m-1)(n-1)\left(\frac{1}{m} + \frac{1}{n} - \frac{1}{m+n-2}\right)\right] \end{aligned}$$

Just as Corollary 4.2 computes $T^{\mathsf{NW}}_{m,n}$, the spotlight tilings of $R^{\mathsf{NE}}_{m,n}$, $R^{\mathsf{SW}}_{m,n}$, and $R^{\mathsf{SE}}_{m,n}$ can also be enumerated. In fact, these enumerations are significantly more elegant, due to the fact that the missing corner does not affect where spotlight tiles may begin. The proofs of these results are inductive, and use the recursion inherent to spotlight tilings.

Corollary 4.3. *For all $m, n \geq 2$, the number of spotlight tilings of an $m \times n$ rectangle missing either its northeast or its southwest corner is*

$$\begin{aligned} T^{\mathsf{NE}}_{m,n} = T^{\mathsf{SW}}_{m,n} &= T_{m,n} - 1 \\ &= \binom{m+n}{m} - \binom{m+n-2}{m-1} - 1. \end{aligned}$$

Corollary 4.4. *For all $m, n \geq 2$, the number of spotlight tilings of an $m \times n$ rectangle missing its southeast corner is*

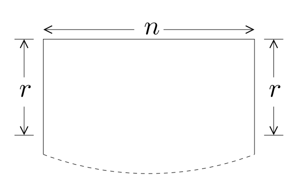
$$\begin{aligned} T^{\mathsf{SE}}_{m,n} &= T_{m,n} - \binom{m+n-2}{m-1} \\ &= \binom{m+n}{m} - 2\binom{m+n-2}{m-1}. \end{aligned}$$

The numbers described in Corollary 4.4 are sequence A051601 in [2].

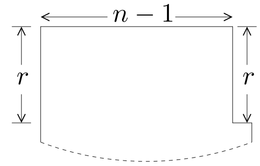
While the symmetry $T^{\mathsf{NE}}_{m,n} = T^{\mathsf{SW}}_{n,m}$ in Corollary 4.3 is not surprising, the fact that $T^{\mathsf{NE}}_{m,n}$ (and $T^{\mathsf{SW}}_{m,n}$) is symmetric with respect to m and n is intriguing. Similarly, the fact that the results of Corollaries 4.3 and 4.4 are so similar to $T_{m,n}$ indicates that removing one of these corners does not drastically alter the spotlight tilings of the original rectangle.

In fact, Corollary 4.3 speaks to a more general trend in spotlight tilings, related to the northeast and southwest corners of a region.

Definition 4.5. Suppose that R is a region as in the following figure, where the only requirement of R in the dashed portion is that it have no northwest corners there.



Let $R[r]$ be the region obtained from R be removing the top r squares in the rightmost column specified in R. That is, $R[r]$ is the region displayed below.



The column of r squares which gets removed from R to form $R[r]$ is the *difference column.*

By this definition, $R^{\mathsf{NE}}_{m,n} = R_{m,n}[1]$.

Proposition 4.6. *Let R and $R[r]$ be regions defined as in Definition 4.5, keeping the meaning of r and n. Then*

$$\#\{\textit{spotlight tilings of } R[r]\} = \#\{\textit{spotlight tilings of } R\} - \sum_{k=0}^{r-1} \binom{n-1}{k}.$$

Proof. Consider the ways that the difference column might be tiled in R. It can consist of the ends of r horizontal tiles, or the ends of k horizontal tiles atop a vertical tile, where $0 \leq k \leq r-1$. If a vertical tile is involved, then this spotlight tile would continue down below the difference column into $R[r] \subset R$. Additionally, if a vertical tile is used to tile the difference column, then there must be $n-1$ other vertical tiles positioned to the left of the difference column in R. The placement of these $n-1$ vertical tiles and the k horizontal tiles can be done in any order.

A given spotlight tiling of $R[r]$ can be extended to a spotlight tiling of R by filling the difference column with horizontal spotlight tiles (if the tiling of $R[r]$ included a horizontal terminating at the difference column in some row, then glue an extra square to the end of this tile). This will yield all spotlight tilings of R except those which cover some portion of the difference column with a vertical tile. This concludes the proof. □

Notice that Proposition 4.6 agrees with Corollary 4.3 by setting $r = 1$.

Also notice that the symmetry of spotlight tilings indicates that Proposition 4.6 would also be true if the figures in Definition 4.5 were reflected across the northwest-southeast diagonal.

One specific corollary to Proposition 4.6 is presented below, although this could also have been shown in a straightforward proof using the recursion inherent to spotlight tilings.

Definition 4.7. Fix integers $m, n \geq 3$. Let $R^{\mathsf{NE,SE}}_{m,n}$ be the region obtained from $R_{m,n}$ by removing the northeast and southeast corners. Likewise, $R^{\mathsf{NE,SW,SE}}_{m,n}$ is an $m \times n$ rectangle whose northeast, southwest, and southeast corners have been removed. Other regions are defined analogously, and $\mathcal{T}^*_{m,n}$ and $T^*_{m,n}$ have their customary definitions.

Corollary 4.8. *For all $m, n \geq 3$*

$$\begin{aligned} T^{\mathsf{NE,SW}}_{m,n} &= T_{m,n} - 2 \\ &= \binom{m+n}{m} - \binom{m+n-2}{m-1} - 2; \\ T^{\mathsf{NE,SE}}_{m,n} = T^{\mathsf{SW,SE}}_{m,n} &= T^{\mathsf{SE}}_{m,n} - 1 \\ &= \binom{m+n}{m} - 2\binom{m+n-2}{m-1} - 1; \\ T^{\mathsf{NE,SW,SE}}_{m,n} &= T^{\mathsf{SE}}_{m,n} - 2 \\ &= \binom{m+n}{m} - 2\binom{m+n-2}{m-1} - 2. \end{aligned}$$

There are several regions $R^*_{m,n}$ whose spotlight tilings have not yet been enumerated. In these, the northwest corner has been removed, along with at at least one other corner. Six of these seven cases are treated in Corollary 4.9, and the remaining case (when all four corners have been removed) appears independently. The results of Corollary 4.9 are not written in closed form, although it would not be hard to do so.

Corollary 4.9. *For $m, n \geq 3$,*

$$T^{\mathsf{NW,SE}}_{m,n} = T^{\mathsf{SE}}_{m-1,n-1} + (n-1)T^{\mathsf{SE}}_{m-2,n} + (m-1)T^{\mathsf{SE}}_{m,n-2};$$

$$T^{\mathsf{NW,NE}}_{m,n} = T^{\mathsf{NW,SW}}_{n,m} = T_{m-1,n-1} + (n-2)T_{m-2,n} + (m-1)T_{m,n-2} - m + 1;$$

$$T^{\mathsf{NW,NE,SE}}_{m,n} = T^{\mathsf{NW,SW,SE}}_{n,m} = T^{\mathsf{SE}}_{m-1,n-1} + (n-2)T^{\mathsf{SE}}_{m-2,n} + (m-1)T^{\mathsf{SE}}_{m,n-2} - m + 1;$$

$$T^{\mathsf{NW,NE,SW}}_{m,n} = T_{m-1,n-1} + (n-2)T_{m-2,n} + (m-2)T_{m,n-2} - m - n + 4.$$

Definition 4.10. For $m, n \geq 3$, let $R^\circ_{m,n}$ be the region obtained from $R_{m,n}$ by removing the northwest, northeast, southwest, and southeast corner squares. Let $\mathcal{T}^\circ_{m,n}$ be the set of spotlight tilings of $R^\circ_{m,n}$, and $T^\circ_{m,n} = |\mathcal{T}^\circ_{m,n}|$.

The following formula for $T^\circ_{m,n}$ is not difficult to compute, using the inductive definition of spotlight tilings.

Corollary 4.11. *For all $m, n \geq 3$,*

$$T^\circ_{m,n} = T^{\mathsf{SE}}_{m-1,n-1} + (n-2)T^{\mathsf{SE}}_{m-2,n} + (m-2)T^{\mathsf{SE}}_{m,n-2} - m - n + 4.$$

The similarities between the results in Corollaries 4.9 and 4.11 are striking, and suggest that the iterative nature of spotlight tiling respects certain substructures of a region.

5. Spotlight tilings of frame-like regions

This section explores the spotlight tilings of a family of regions that are formed by making a large hole in the center of a rectangle. To give a flavor for these results, this discussion studies only those cases where the remaining region has width 1, although it would not be difficult to generalize to wider frames.

Definition 5.1. Fix $m, n \geq 3$. Let $F_{m,n}$ be the region formed by removing a centered $(m-2) \times (n-2)$ rectangle from the rectangle $R_{m,n}$. Let $f_{m,n}$ be the number of spotlight tilings of $F_{m,n}$.

In other words, the region $F_{m,n}$ looks like an $m \times n$ picture frame of width 1. To understand $f_{m,n}$, it is helpful first to enumerate the spotlight tilings of some related regions.

Definition 5.2. Fix $m, n \geq 1$. Let $C^{\mathsf{NW}}_{m,n}$ be the region of $m+n-1$ squares formed by overlapping the north-most square of a column of length m and the west-most square of a row of length n. Let $c^{\mathsf{NW}}_{m,n}$ be the number of spotlight tilings of $C^{\mathsf{NW}}_{m,n}$. The regions $C^{\mathsf{NE}}_{m,n}$, $C^{\mathsf{SW}}_{m,n}$, and $C^{\mathsf{SE}}_{m,n}$ and their enumerations are defined analogously.

Proposition 5.3. *For* $m, n \geq 1$,

$$\begin{aligned} c^{\mathsf{NW}}_{m,n} &= m+n-2 \\ c^{\mathsf{NE}}_{m,n} = c^{\mathsf{SW}}_{n,m} &= n(m-1)+1 \\ c^{\mathsf{SE}}_{m,n} &= 2(m-1)(n-1)+1 \end{aligned}$$

Proof. Each of these quantities can be computed by careful counting, together with the fact that $T_{1,p} = T_{p,1} = p$. □

Theorem 5.4. *For* $m, n \geq 3$,

$$f_{m,n} = 2(m-2)(n-2)(m+n-2) + (m-2)(m+1) + (n-2)(n+1).$$

Proof. Initially, there is only one northwest corner in the region $F_{m,n}$. This can be tiled with a horizontal spotlight tile of length n or a vertical spotlight tile of length m. Either way, the remaining region has two northwest corners, and careful applications of Proposition 5.3 and the inclusion-exclusion property give the answer. □

The values of $f_{m,n}$ for small m and n are displayed in Table 2. These values are sequence A132370 of [2].

$f_{m,n}$	$n=3$	4	5	6	7
$m=3$	16	34	58	88	124
4	34	68	112	166	230
5	58	112	180	262	358
6	88	166	262	376	508
7	124	230	358	508	680

TABLE 2. The number of spotlight tilings of $F_{m,n}$, for $m, n \leq 7$.

6. Further directions

The preceding sections have examined the spotlight tilings of several families of regions. In each case, the enumeration of these tilings had a concise and often illuminating form. For the rectangle, more refined analyses were also performed, and yielded results whose simplicity and elegance may not have been anticipated.

The recursive nature of spotlight tiling means that further enumerations of this tiling method for other families of regions should not be difficult. The obvious analogue of spotlight tiling in higher dimensions may also yield fruitful results. Additionally, the questions particular to spotlight tiling (such as the distribution of the number of tiles in a given tiling) may give rise to new aspects of this and other tilings methods which warrant further study.

References

[1] P. W. Kasteleyn, The statistics of dimers on a lattice, I. The number of dimer arrangements on a quadratic lattice, *Physica* 27 (1961) 1209–1225.

[2] N. J. A. Sloane, The on-line encyclopedia of integer sequences, published electronically at `http://www.research.att.com/~njas/sequences/`.

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