9 Divides no Odd Fibonacci

Tanya Khovanova

December 20, 2007

Abstract

I discuss numbers that divide no odd Fibonacci. Number 9 plays a special role among such numbers.

1   Introduction

I stumbled upon the following sentence in the MathWorld article on the Fibonacci numbers [2]: "No odd Fibonacci number is divisible by 17." I started wondering if there are other, similar numbers. Of course there are — no odd Fibonacci number is divisible by 2. But then, an odd number need not be a Fibonacci number in order not to be divisible by 2.

So, let us forget about 2 and think about odd numbers. How do we know that the infinite Fibonacci sequence never produces an odd number that is divisible by 17? Is 17 the only such odd number? Is 17 the smallest such odd number? If there are many such odd numbers, how do we calculate the corresponding sequence?

2   No odd Fibonacci is divisible by 17

We will start with a general question: How can we approach puzzles about the divisibility of Fibonacci numbers? Suppose K is an integer. Consider the sequence $a_K(n) = F_n \pmod{K}$, of Fibonacci numbers modulo K. The cool thing about this sequence is that it is periodic. If this is not immediately obvious to you, think of what happens when a pair of consecutive numbers in the sequence $a_K(n)$ gets repeated. As a bonus for thinking you will get an upper bound estimate for this period.

Let us denote the period of $a_K(n)$ by P_K. By the way, this period is called a Pisano period (see wiki [5]). From the periodicity and the fact that $a_K(0) = 0$, we see right away that there are infinitely many Fibonacci numbers divisible by K. Are there odd numbers among them? If we trust MathWorld, then all of the infinitely many Fibonacci numbers divisible by 17 will be even.

How do we examine the divisibility by K for odd Fibonacci numbers? Let us look at the Fibonacci sequence modulo 2. As we just proved, this sequence is periodic. Indeed, every third Fibonacci number is even. And the evenness of a Fibonacci number is equivalent to this number having an index divisible by 3.

Now that we know the indices of even Fibonacci numbers we can come back to the sequence $a_K(n)$. In order to prove that no odd Fibonacci number is divisible by K, it is enough to check that all the zeroes in the sequence $a_K(n)$ have indices divisible by 3. We already have one zero in this sequence at index 0, which is divisible by 3. Because the sequence $a_K(n)$ is periodic, it will start repeating itself at $a_K(P_K)$. Hence, we need to check that P_K is divisible by 3 and all the zeroes up to $a_K(P_K)$ have indices divisible by 3. When $K = 17$ it is not hard to do the calculations manually. If you like, try this exercise. To encourage (or perhaps to discourage) you, here's an estimate of the scope of the work for this exercise: the Pisano period for $K = 17$ is 36.

After I checked that no odd Fibonacci number is ever divisible by 17, I wanted to find the standard solution for this statement and followed the trail in MathWorld. MathWorld sent me on a library trip where I found the proof of the statement in the book Mathematical Gems III by Ross Honsberger [1]. There was a proof there alright, but it was tailored to 17 and didn't help me with my questions about other such odd numbers.

3 Non-divisibility of odd Fibonacci numbers

The method we developed for 17 can be used to check any other number. I trusted this task to my computer. To speed up my program, I used the fact that the Pisano period for K is never more than $6K$. Here is the sequence calculated by my trustworthy computer, which I programmed with, I hope, equal trustworthiness:

- A133246: Odd numbers n with the property that no odd Fibonacci

number is divisible by n.

9, 17, 19, 23, 27, 31, 45, 51, 53, 57, 61, 63, 69, 79, 81, 83, 85, 93, 95, 99, ...

The sequence shows that 9 is the smallest odd number that no odd Fibonacci is ever divisible by, and 17 is the smallest odd prime with this same property. Here is a trick question for you: Why is this property of 17 more famous than the same property of 9?

Let us look at the sequence A133246 again. Is this sequence infinite? Obviously, it should include all multiples of 9 — hence, it is infinite. What about prime numbers in this sequence? Is there an infinite number of primes such that no odd Fibonacci number is divisible by them? While I do not know the answer, it's worth investigating this question a little bit further.

4 Non-divisibility of odd Fibonacci numbers by primes

From now on, let K be an odd prime. Let us look at the zeroes of the sequence $a_K(n)$ more closely. Suppose a zero first appears at the m-th place of $a_K(n)$. Then $a_K(m + 1) = a_K(m + 2) = a$, where $a \neq 0$. In this case the sequence starting from the m-th place is proportional modulo K to the sequence $a_K(n)$ starting from the 0-th index. Namely, $a_K(n+m) = a*a_K(n)$ (mod K). As a is mutually prime with K, then $a_K(n+m) = 0$ iff $a_K(n) = 0$. From here, for any index g that is a multiple of m, $a_K(g) = 0$. Furthermore, there are no other zeroes in the sequence $a_K(n)$. Hence, the appearances of 0 in the sequence $a_K(n)$ are periodic with period m.

By the way, m is called a fundamental period; and we just proved that the Pisano period is a multiple of the fundamental period for prime K. Hence, the fact that no odd Fibonacci number is divisible by K is equivalent to the fact that the fundamental period is not divisible by 3. This is like saying that if the smallest positive Fibonacci number divisible by an odd prime K is even, then no odd Fibonacci number is divisible by K. In particular, the first Fibonacci number divisible by 17 is $F_9 = 34$; and it is even. Thus we get another proof that 17 divides no odd Fibonacci.

If the remainder of the fundamental period modulo 3 were random, we would expect that about every third prime number would not divide any odd Fibonacci. In reality there are 561 such primes among the first 1,500 primes

(including 2). This is somewhat more than one third. This gives me hope that there is a non-random reason for such primes to exist. Consequently, it may be possible to prove that the sequence of prime numbers that do not divide odd Fibonacci numbers is infinite.

Can you prove that? Here is the start of this sequence:

- A133247: Prime numbers p with the property that no odd Fibonacci number is divisible by p.
  2, 17, 19, 23, 31, 53, 61, 79, 83, 107, 109, 137, 167, 173, 181, 197, ...

5   Base sequence

When I proudly showed to my sons, Alexey Radul and Sergei Bernstein, the two sequences A133246 and A133247 above that I have submitted to the Online Encyclopedia of Integer Sequences (OEIS) [3] they both told me (independently of each other): "Now you should calculate the base sequence."

The base sequence means that out of all the numbers that no odd Fibonacci divides we remove multiples of other such numbers. In particular, the base sequence contains all the primes. When I calculated this sequence the result was the following:

- 2, 9, 17, 19, 23, 31, 53, 61, 79, 83, 107, 109, 137, 167, 173, 181, 197, ...

If you compare the base sequence with the prime sequence A133247, you will see that the only difference is that number 9 belongs to the base sequence.

Theorem 5.1. *Number 9 is the only composite in the base sequence.*

Proof. Let us denote the fundamental period corresponding to a number n as $fun(n)$. We proved before that for prime n all the Fibonacci numbers that are divisible by n have indices that are multpiples of $fun(n)$. This fact is also true for any n (see Wall [4] for the proof). Suppose two numbers n and m are mutually coprime. Then the Fibonacci numbers that are divisible by nm have indices that are multiples of both $fun(n)$ and $fun(m)$. The last statement is equivalent to

$$fun(nm) = \gcd(fun(n), fun(m)).$$

If no odd Fibonacci number is divisible by nm, then $fun(nm)$ is divisible by 3. Hence, $fun(n)$ or $fun(m)$ is divisible by 3. That means, if nm is in the complete sequence A133246 for coprime m and n, then m or n are in the sequence. Hence nm doesn't belong to the base sequence.

Now it remains to investigate the powers of prime numbers.

Lemma 5.2. *For a prime $p > 2$ and a positive integer t, $fun(p^{t+1})$ equals either $fun(p^t)$ or $p * fun(p^t)$.*

Proof. To prove the lemma I will use the explicit formula for the Fibonacci numbers. Let us denote the golden ratio by ϕ, then $F_n = (\phi^n - \phi^{-n})/\sqrt{5}$. We will also use Lucas numbers $L_n = \phi^n + \phi^{-n}$ as a helping tool in our calculations. You can check that $L_n = F_{n-1} + F_{n+1}$. The useful thing about L_n is that $\gcd(F_n, L_n)$ is either 1 or 2.

Suppose m is the fundamental period for p^t. We know that the fundamental period of p^{t+1} is a multiple of m. If $fun(p^{t+1}) = m$, we are done. Otherwise, we would like to look at Fibonacci numbers of the form F_{am}, where $a > 1$. Let us express F_{am} in terms of F_m and L_m (here I follow a similar calculation in Wall's paper [4]):

$$F_{am} = (\phi^{am} - \phi^{-am})/\sqrt{5} = 2^{-a}((\sqrt{5}F_m + L_m)^a - (-\sqrt{5}F_m + L_m)^a)/\sqrt{5}.$$

I am interested in the divisibility of F_{am} by p^{t+1}. That means I can ignore the coefficient 2^{-a}. Also, the non-trivial powers of F_m are divisible by p^{t+1}. Hence, I do not need to write the whole expansion formula for F_{am}. I am only interested in $((\sqrt{5}F_m + L_m)^a - (-\sqrt{5}F_m + L_m)^a)/\sqrt{5} \pmod{F_m^2}$, which is $2aF_m L_m^{a-1}$. This means, that if F_m is not divisible by p^{t+1}, then the smallest number divisible by p^{t+1} is F_{pm}. $\qquad\square$

Corollary 5.3. *For a prime $p > 2$, the $fun(p^t)/fun(p)$ is a power of p.*

Hence, if p is not 3, then 3 divides $fun(p^t)$ iff 3 divides $fun(p)$. In other words, no odd Fibonacci divides p^t iff no odd Fibonacci divides p. Hence, for $p \neq 3$, non-trivial powers of p can not belong to the base sequence.

At the conclusion of the proof of the theorem we can remember that there is an odd Fibonacci that divides 3 and no odd Fibonacci divides 9. $\qquad\square$

6 Acknowledgements

References

[1] Ross Honsberger, Mathematical Gems III, 1997.

[2] MathWorld article on Fibonacci Numbers. *http://mathworld.wolfram.com/FibonacciNumber.html*

[3] N. J. A. Sloane, Online Encyclopedia of Integer Sequences (OEIS). *http://www.research.att.com/∼njas/sequences/*

[4] D.D. Wall, Fibonacci Series Modulo m, The American Mathematica Monthly, Vol. 67, No. 6, (1960), 525-532.

[5] Wiki article on Pisano period. *http://en.wikipedia.org/wiki/Pisano_period*