Counting One-Vertex Maps*

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Abstract

The number of distinct maps (pre-maps) with a single vertex and valence d is computed for any value of d. The types of maps (pre-maps) that we consider depend on whether the underlaying graph (pre-graph) is signed or unsigned and directed or undirected.

1   Introduction

The motivation for this note lies in the fact that each orientable Cayley map of valence d is obtained via a regular covering construction from an orientable map with one vertex, k loops and $d - 2k$ half-edges (see [2]; for Cayley maps in general, see for instance [7]. For basic definitions of combinatorial maps see [5]). In this context, the problem of determining the number $\pi(d)$ of all non-isomorphic one-vertex d-valent maps arises naturally. For more detailed analysis of these embeddings see [1]. Coverings of graphs and pregraphs are combinatorially described in [4].

As we show in this paper, the number $\pi(d)$ equals the number of all essentially distinct matchings in the complete graph K_d with vertices arranged as in the regular d-gon, where two matchings are considered essentially the same whenever one can be obtained from the other by a rotation or a reflection of the d-gon. The latter can be obtained by the formula

$$\pi(d) \;=\; \frac{1}{2d}\left(F(d) + R(d)\right)$$

where

$$F(d) = \begin{cases} \dfrac{d}{2}\left(f\left(\dfrac{d}{2}\right) + 2f\left(\dfrac{d}{2} - 1\right)\right), & d \text{ even,} \\[2ex] d\,f\left(\dfrac{d-1}{2}\right), & d \text{ odd,} \end{cases}$$

$$f(n) = n! \sum_{0 \le 2j \le n} \frac{2^{n-2j}}{(n-2j)!j!},$$

$$R(d) = \sum_{r|d} \varphi\left(\frac{d}{r}\right) \sum_{0 \le 2j \le r} \binom{r}{2j}(2j-1)!! \left(\frac{d}{r}\right)^j w(r)^{r-2j},$$

and

$$w(r) = \left\{ \begin{array}{ll} 2, & r \mid (d/2) \\ 1, & r \nmid (d/2) \end{array} \right.$$

(here we assume that $r \nmid (d/2)$ when d is odd). Similar formulae are obtained for one-vertex maps without half-edges, non-orientable maps, and directed maps.

We introduce a structure that we call a pre-graph; see also [2]. A *pre-graph* G is a quadruple $G = (V, S, i, r)$ where V is the set of vertices, S is the set of *arcs* (also known as semi-edges, darts, sides, ...), i is the *initial mapping* $i : S \rightarrow V$, specifying the origin or initial vertex for each arc, while r is the *reversal involution*: $r : S \rightarrow S$, $r^2 = 1$. We may also define the *terminal mapping* $t : S \rightarrow V$ as $t(s) := i(r(s))$, specifying the terminal vertex for each arc. An arc s forms an edge $e = \{s, r(s)\}$, which is called *proper* if $|e| = 2$ and is called a *half-edge* if $|e| = 1$. Define $\partial(e) = \{i(s), t(s)\}$. A pre-graph without half-edges is called a *(general) graph*. Note that G is a graph if and only if the reversal involution has no fixed points. A proper edge e with $|\partial(e)| = 1$ is called a *loop* and two edges e, e' are *parallel* if $\partial(e) = \partial(e')$. A graph without loops and parallel edges is called *simple*. The *valence* of a vertex v is defined as $val(v) = |\{s \in S | i(s) = v\}|$. All pre-graphs in this note are connected unless stated otherwise.

Topologically, an oriented map is a 2-cell embedding of a graph into an orientable surface. However, in this paper we will operate with the following combinatorial description. For us an oriented map is a triple (S, r, R) where S is a finite non-empty set and r, R are permutations on S such that $r^2 = 1$ and $\langle r, R \rangle$ acts transitively on S (see [2]). Note that the vertices of the map correspond to the cycles in the cyclic decomposition of R. An isomorphism between two pre-maps (S, r, R) and (S', r', R') is any bijection $\pi : S \rightarrow S'$ for which $\pi R = R'\pi$ and $\pi r = r'\pi$ holds. An automorphism of a map (S, r, R) is thus a permutation of S which commutes with both R and r.

In addition to oriented maps we will also consider general (possibly non-orientable) (pre-)maps, which can be defined as $M = (S, r, R, \lambda)$, where S, r and R are as in the definition of oriented maps, and λ is a *sign mapping* assigning either 1 or -1 to each proper edge of the underlying graph of the map M. Recall that each cycle $C = (s_1, \ldots, s_k)$ of R corresponds to a vertex of the map. Substituting the cycle C in R with the reverse cycle and inverting the λ-value of proper edges underlying the darts $s_1, \ldots, s_k$ other than loops, results in a new map M', which is said to be obtained from M by a *local orientation change*. Two general maps $M_1 = (S_1, r_1, R_1, \lambda_1)$ and $M_2 = (S_2, r_2, R_2, \lambda_2)$ are *isomorphic* if there exists a general map $M' = (S_1, r_1, R', \lambda')$ obtained from M_1 by a series of local orientation changes and a bijection $\pi : S_1 \rightarrow S_2$, such that $\pi r_1 = r_2\pi$, $\pi R_1 = R'\pi$ and $\pi\lambda_1 = \lambda'\pi$.

2 Counting one-vertex graphs and pre-graphs

Let $p(d)$ denote the number of one-vertex pre-graphs of valence d. Since each of them is determined by the number of loops, $p(d)$ can be computed using the formula: $p(d) = 1 + \lfloor d/2 \rfloor$. This gives rise to the generating function $P(x) = 1/((1-x)^2(1+x))$.

If there are no pending edges, the situation becomes much simpler. Let $g(d)$ denote the number of one-vertex graphs. Then $g(d) = 0$, for d odd, and $g(d) = 1$, for d even. The corresponding generating function $G(x)$ is $G(x) = 1/(1-x^2)$.

3 Counting one-vertex maps and pre-maps

When counting (oriented) pre-maps the same pre-graph may give rise to more than one pre-map.

Let $\pi(d)$ denote the number of oriented pre-maps whose underlying graphs are single vertex pre-graphs, and let $\gamma(d)$ denote the number of oriented maps whose underlying graphs are single vertex graphs.

Let $\pi_\tau(d)$ denote the number of non-isomorphic single vertex pre-maps of type τ and valence d. The types of pre-maps that we consider are denoted by $\bar{S}\bar{D}\bar{G}, S\bar{D}\bar{G}, \bar{S}D\bar{G}, SD\bar{G}, \bar{S}\bar{D}G, S\bar{D}G, \bar{S}DG$, and SDG, indicating whether the underlying pre-graphs are signed or unsigned (S resp. $\bar{S}$), directed or undirected (D resp. $\bar{D}$), graphs or pre-graphs (G resp. $\bar{G}$). As it turns out, $\pi_\tau(d)$ and the various auxiliary functions can be written in the same general form for all τ, but with different values of parameters (cf. Table 1). If necessary, we refer to the three symbols composing τ by τ_1, τ_2 and τ_3.

Each oriented one-vertex pre-map is isomorphic to one of the form (S, r, R) where $S = \{1, \ldots, d\}$, $R = (1, 2, \ldots, d)$. Such a pre-map can be represented by a matching in the complete graph K_d (possibly an empty one) in which two vertices $i, j \in \{1, \ldots, d\}$ are matched whenever $r(i) = j$. Hence the number of all one-vertex pre-maps (S, r, R) with a given rotation R is the same as the number of all matchings (including the empty one) in K_d. This number is easily computed to be

$$i(d) = \sum_{0 \le 2k \le d} \binom{d}{2k} (2k-1)!!,$$

where $(-1)!! = 1$. Note that $i(d) = i(d-1) + (d-1)i(d-2)$ for all $d \ge 2$, and the exponential generating function of this sequence is $\sum_{n=0}^{\infty} i(n)x^n/n! = \exp(x + x^2/2)$.

Of course, many of the above pre-maps are isomorphic. To compute the number of non-isomorphic ones, let I_d denote the set of all permutations r on S with $r^2 = 1$, and recall that two pre-maps $M = (S, r, R)$ and $M' = (S, r', R)$ are isomorphic if and only if there exists a permutation on S which centralizes R and conjugates r to r'. Since the permutations of S that centralize R form the dihedral group D_d of order $2d$, it follows that the number of orientable non-isomorphic one-vertex pre-maps of valence d equals the number of orbits of D_d in its action on the set I_d by conjugation. If the pre-maps are represented by the matchings in K_d, as described above, then the action of D_d on I_d corresponds to the natural action of D_d on the set of all matchings in K_d. In general, the number $\pi_\tau(d)$ of non-isomorphic one-vertex pre-maps of type τ and valence d

equals the number of orbits of D_d in its natural action on the set of matchings of type τ in K_d. The latter can be obtained by the well-known Cauchy-Frobenius Lemma (also known as Burnside's Lemma):

$$\pi_\tau(d) \;=\; \frac{1}{|D_d|} \sum_{\sigma \in D_d} |\mathrm{Fix}_\tau(\sigma)| \tag{1}$$

where $\mathrm{Fix}_\tau(\sigma)$ denotes the set of matchings of type τ in K_d invariant under σ. For more information on this method, see [6].

3.1 Fixed points of reflections

In order to compute the sum in (1), assume first that σ is a reflection. We distinguish two cases.

If d is even there are two types of reflections: either across a median or across a main diagonal. Let σ be the reflection across a median, and let L denote the set of $n = d/2$ vertices of K_d on one side of the median. For each $u \in L$, denote by u' its mirror image across the median. We will derive a recurrence satisfied by $f_\tau(n) := |\mathrm{Fix}_\tau(\sigma)|$, using the so-called *method of distinguished element*. Assume that $n \geq 2$, pick any vertex $u \in L$, and partition $\mathrm{Fix}_\tau(\sigma)$ into subsets A and B where A contains those matchings in which u is matched with u' or is left unmatched, and B contains those matchings in which u is matched with v or v' where v is one of the remaining $n-1$ vertices in L. Denote by s_τ the number of ways in which u can be matched with u' (including leaving it unmatched), and by t_τ the number of ways in which u can be matched with v. Because of symmetry, the number of ways in which u can be matched with v' is also t_τ. The values of s_τ and t_τ depend on the type τ of the problem considered, and are shown in Table 1. Then $|A| = s_\tau f_\tau(n-1)$ and $|B| = 2t_\tau(n-1)\,f_\tau(n-2)$,

τ	$\bar{S}\bar{D}\bar{G}$	$S\bar{D}\bar{G}$	$\bar{S}D\bar{G}$	$SD\bar{G}$	$\bar{S}\bar{D}G$	$S\bar{D}G$	$\bar{S}DG$	SDG
s_τ	2	3	1	1	1	2	0	0
t_τ	1	2	2	4	1	2	2	4
m_τ	2	3	3	5	1	2	2	4

s_τ ... the number of ways to match $u \in L$ with u'
t_τ ... the number of ways to match $u \in L$ with $v \in L \setminus \{u\}$
m_τ ... the number of ways to match the two vertices on the mirror

Table 1: The values of parameters s_τ, t_τ, m_τ for the types of pre-maps considered

hence

$$f_\tau(n) = s_\tau f_\tau(n-1) + 2t_\tau(n-1)f_\tau(n-2), \quad \text{for } n \geq 2, \tag{2}$$

with $f_\tau(0) = 1$, $f_\tau(1) = s_\tau$. To solve (2), let $G_\tau(x) = \sum_{n=0}^{\infty} f_\tau(n)x^n/n!$ be the exponential generating function of the sequence $\langle f_\tau(n)\rangle_{n=0}^{\infty}$. Then it follows from (2) and the initial values that $G_\tau(x)$ satisfies the differential equation

$$G'_\tau(x) = (s_\tau + 2\,t_\tau\,x)G_\tau(x), \quad G_\tau(0) = 1,$$

whence

$$\sum_{n=0}^{\infty} f_\tau(n)\frac{x^n}{n!} \;=\; \exp(s_\tau\,x + t_\tau\,x^2). \tag{3}$$

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By expanding the right-hand side into power series and comparing coefficients we find the solution

$$f_\tau(n) = n! \sum_{0 \leq 2j \leq n} \frac{s_\tau^{n-2j} t_\tau^j}{(n-2j)! j!} = \sum_{0 \leq 2j \leq n} s_\tau^{n-2j} (2t_\tau)^j \binom{n}{2j} (2j-1)!! \quad (4)$$

where $0^0 = 1$. Note that (4) can also be obtained by a counting argument: To construct a matching M which is invariant under σ, select $2j$ vertices from among the n vertices in L, then construct a perfect matching on these $2j$ vertices. This can be done in $\binom{n}{2j}(2j-1)!!$ ways. As above, there are $2t_\tau$ ways to match the two elements in each of the j pairs, yielding the factor $(2t_\tau)^j$, and s_τ ways to match each of the remaining $n - 2j$ vertices to its mirror image, yielding the factor s_τ^{n-2j}.

By comparing (3) to the generating function of Hermite polynomials

$$\sum_{n=0}^{\infty} H_n(z) \frac{x^n}{n!} = \exp(2z\,x - x^2)$$

we can also express $f_\tau(n)$ in terms of the n-th Hermite polynomial as

$$f_\tau(n) = \left(i\sqrt{t_\tau}\right)^n H_n\left(\frac{s_\tau}{2i\sqrt{t_\tau}}\right). \quad (5)$$

If σ is the reflection across a main diagonal then $|\mathrm{Fix}_\tau(\sigma)| = m_\tau f(d/2 - 1)$ where m_τ is the number of ways in which it is possible to match the two vertices on the mirror with each other. The value of m_τ depends on the type τ of the problem considered, and is shown in Table 1.

If d is odd there is only one type of reflections, and $|\mathrm{Fix}_\tau(\sigma)| = f_\tau((d-1)/2)$ for pre-maps and 0 for maps. Thus the total contribution $F_\tau(d)$ of the d reflections to the sum in (1) is

$$F_\tau(d) = \begin{cases} \dfrac{d}{2}\left(f_\tau\left(\dfrac{d}{2}\right) + m_\tau f_\tau\left(\dfrac{d}{2}-1\right)\right), & d \text{ even,} \\[2ex] d\,f_\tau\left(\dfrac{d-1}{2}\right), & d \text{ odd and } \tau_3 = \bar{G}, \\[2ex] 0, & d \text{ odd and } \tau_3 = G, \end{cases} \quad (6)$$

where f_τ is given by any of (2), (3), (4), or (5).

3.2   Fixed points of rotations

Now assume that σ is the counter-clockwise rotation of $2\pi k_\sigma/d$ where $0 \leq k_\sigma < d$. In how many ways can we construct a matching M of K_d which is invariant under σ?

Let $r = \gcd(d, k_\sigma)$. Then σ has r orbits in $V(K_d)$, each containing d/r vertices. Let C denote a set of r consecutive vertices of K_d. Since C contains one representative from each orbit, it suffices to define M on C, and to extend it to $V(K_d) \setminus C$ by symmetry. Hence we can also think of M as a matching of orbits. Assume that $2j$ of the r orbits are matched in pairs, while the rest remain unmatched or are matched with themselves (the latter is possible only if antipodal vertices belong to the same orbit, i.e., if d is even and $r \mid d/2$). There

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are $\binom{r}{2j}$ ways to select the $2j$ orbits, and $(2j-1)!!$ ways to group them into pairs. In each of the j pairs of orbits (α_i, β_i), $i = 1, 2, \ldots, j$, the vertex in $\alpha_i \cap C$ can be matched with any of the d/r vertices in β_i in t_τ ways, and each of the remaining $r - 2j$ orbits can be matched to themselves (or be left unmatched) in $w_\tau(r)$ ways where

$$w_\tau(r) \;=\; \begin{cases} s_\tau, & r \mid (d/2) \\ 0, & r \nmid (d/2) \text{ and } \tau_3 = G \\ 1, & r \nmid (d/2) \text{ and } \tau_3 = \bar{G} \end{cases}$$

(for the values of s_τ and t_τ, see Table 1). Now for each divisor r of d, there are $\varphi(d/r)$ rotations σ in D_d having $\gcd(d, k_\sigma) = r$. Hence the total contribution $R_\tau(d)$ of the d rotations to the sum in (1) is

$$R_\tau(d) \;=\; \sum_{r \mid d} \varphi\left(\frac{d}{r}\right) \sum_{0 \le 2j \le r} \binom{r}{2j} (2j-1)!! \left(\frac{t_\tau d}{r}\right)^j w_\tau(r)^{r-2j} \tag{7}$$

where, as before, $0^0 = 1$.

3.3 The master formula

From (1) it follows that the number of non-isomorphic single vertex pre-maps of valence d is

$$\pi_\tau(d) \;=\; \frac{1}{2d}\left(F_\tau(d) + R_\tau(d)\right) \tag{8}$$

where $F_\tau(d)$ resp. $R_\tau(d)$ are given by (6) resp. (7), and the values of parameters s_τ, t_τ, m_τ for each type τ of pre-maps considered are given in Table 1.

4 Additional formulæ and tables

Some of the sequences encountered in this paper can be found in the *The Online Encyclopedia of Integer Sequences* (OEIS, [9]).

sequence	OEIS ID number	exponential generating function
$\langle f_{\bar{S}\bar{D}\bar{G}}(n)\rangle_{n=0}^\infty$	A000898	$\exp(x^2 + 2x)$
$\langle f_{S\bar{D}\bar{G}}(n)\rangle_{n=0}^\infty$		$\exp(2x^2 + 3x)$
$\langle f_{\bar{S}D\bar{G}}(n)\rangle_{n=0}^\infty$	A115329	$\exp(2x^2 + x)$
$\langle f_{SD\bar{G}}(n)\rangle_{n=0}^\infty$		$\exp(4x^2 + x)$
$\langle f_{\bar{S}\bar{D}G}(n)\rangle_{n=0}^\infty$	A047974	$\exp(x^2 + x)$
$\langle f_{S\bar{D}G}(n)\rangle_{n=0}^\infty$		$\exp(2x^2 + 2x)$
$\langle f_{\bar{S}DG}(n)\rangle_{n=0}^\infty$		$\exp(2x^2)$
$\langle f_{SDG}(n)\rangle_{n=0}^\infty$		$\exp(4x^2)$
$\langle f_{\bar{S}DG}(2n-2)\rangle_{n=1}^\infty$	A052714	$(1 - \sqrt{1 - 8x})/4$
$\langle f_{SDG}(2n-2)\rangle_{n=1}^\infty$	A052734	$(1 - \sqrt{1 - 16x})/8$
$\langle \pi_{\bar{S}\bar{D}G}(2n)\rangle_{n=0}^\infty$	A054499	

Table 2: ID numbers and generating functions of some of our sequences

When $s_\tau = 0$, the formula giving $f_\tau(d)$ can be expressed in closed form. Thus, for d even,

$$
\begin{aligned}
f_{\bar{S}DG}(d) &= 2^d (d-1)!!, \\
f_{SDG}(d) &= (2\sqrt{2})^d (d-1)!!.
\end{aligned}
$$

Also, when $w_\tau(r) = 0$ for all r, the double sum in the formula giving $R_\tau(d)$ reduces to a single sum. For d even we thus have

$$
\begin{aligned}
R_{\bar{S}DG}(d) &= \sum_{r\mid d,\ r\,\text{even}} \varphi\left(\frac{d}{r}\right) (r-1)!! \left(\frac{2d}{r}\right)^{r/2}, \\
R_{SDG}(d) &= \sum_{r\mid d,\ r\,\text{even}} \varphi\left(\frac{d}{r}\right) (r-1)!! \left(\frac{4d}{r}\right)^{r/2}.
\end{aligned}
$$

In Tables 3 resp. 4 we list the numbers of non-isomorphic single vertex pre-maps resp. maps of valence d for small values of d. For instance, in [8] the six pre-maps of valence five are discussed in detail.

d	$\pi_{\bar{S}\bar{D}\bar{G}}(d)$	$\pi_{S\bar{D}\bar{G}}(d)$	$\pi_{\bar{S}D\bar{G}}(d)$	$\pi_{SD\bar{G}}(d)$
1	1	1	1	1
2	2	3	2	3
3	2	3	2	3
4	5	11	6	14
5	6	15	11	33
6	17	60	37	167
7	27	125	100	619
8	83	529	405	3686
9	185	1663	1527	18389
10	608	7557	6824	120075
11	1779	31447	30566	706851
12	6407	155758	151137	5032026
13	22558	763211	757567	33334033
14	87929	4089438	4058219	255064335
15	348254	22190781	22150964	1855614411
16	1456341	127435846	127215233	15129137658
17	6245592	745343353	745057385	119025187809
18	27766356	4549465739	4547820514	1026870988199
19	126655587	28308456491	28306267210	8640532108675
20	594304478	182435301597	182422562168	78446356190934

Table 3: The numbers of non-isomorphic one-vertex pre-maps

Using methods of [3] one can easily extend the counting to graphs with one-vertex connected components. Motivated by [1], it would be worthwhile to extend this analysis to dipoles or any two-vertex graphs or pre-graphs.

d	$\pi_{\bar{S}\bar{D}G}(d)$	$\pi_{S\bar{D}G}(d)$	$\pi_{\bar{S}DG}(d)$	$\pi_{SDG}(d)$
2	1	2	1	2
4	2	6	3	9
6	5	26	13	90
8	17	173	121	1742
10	79	1844	1538	48580
12	554	29570	28010	1776358
14	5283	628680	618243	79080966
16	65346	16286084	16223774	4151468212
18	966156	490560202	490103223	250926306726
20	16411700	16764409276	16761330464	17163338379388
22	312700297	639992710196	639968394245	1310654311464970
24	6589356711	26985505589784	26985325092730	110531845060209836

Table 4: The numbers of non-isomorphic one-vertex maps

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