

THE OPERAD LIE IS FREE

PAOLO SALVATORE AND ROBERTO TAURASO

ABSTRACT. We show that the operad *Lie* is free as a non-symmetric operad. Then we study the generating series counting the operadic generators, finding a recursive formula for its coefficients, and showing that the asymptotic density of the operadic generators is $1/e$.

1. INTRODUCTION

The operad *Lie* is the symmetric operad encoding the structure of a Lie algebra. It has an antisymmetric binary generator, the bracket, and a ternary relation, the Jacobi identity. For standard facts and notations on operads we refer to [5]. Over a commutative ring R , the k -term of the operad $Lie(k)$ is the subgroup of the free Lie algebra on k generators x_1, \dots, x_k spanned by the words where each generator appears exactly once. The action of the symmetric groups on k letters exchanges the indices. It is well known that $Lie(k)$ is a free R -modules of rank $(k-1)!$ [6]. The algebras over *Lie* in the operadic sense are exactly Lie algebras in the usual sense, unless R has 2-torsion.

A topological interpretation of *Lie* was discovered by Fred Cohen. Let D_n be the little n -discs operad. The top homology group of $D_n(k)$ is

$$H_{(k-1)(n-1)}(D_n(k), R) \cong R^{(k-1)!}$$

Cohen proved that for $n > 2$ odd the induced operad structure on the top homology groups of D_n is isomorphic to *Lie*. This story is well explained by D. Sinha in [6].

If we forget about the action of the symmetric groups, we can regard *Lie* as a non-symmetric operad. A motivation to study *Lie* in this sense comes from knot theory. Lambrechts Turchin and Volic proved in [4] that the rational homology of the space of long knots in \mathbb{R}^n is the value on the operad $H_*(D_n)$ of a functor (Hochschild homology) defined naturally on non-symmetric operads with multiplication. The main result of this note is the following.

Theorem 1. *The operad Lie is a free non-symmetric operad.*

Section 2 is devoted to the proof of the theorem. We start by constructing an operad L in the category of sets that spans linearly *Lie*. Its elements are those iterated brackets in *Lie* such that the smallest index and the largest index inside each bracket lie respectively on the left and on the right hand side. We recall an explicit construction of the free non-symmetric operad. Then we show that the operad L is free, generated by a sequence of sets $P = (P(n))$. The elements of P , that we call *primes*, are those iterated brackets such that the indices inside each (non outer) bracket do *not* form an interval of consecutive integers.

In section 3 we study the sequence of the cardinalities $b_n = |P(n)|$ of the sets of prime generators. We show in Theorem 9 that the following recursive formula holds:

$$b_2 = 1, \quad b_n = \sum_{k=2}^{n-2} ((k+1)b_{k+1} + b_k)b_{n-k} \quad \text{for } n \geq 3.$$

The formula suggests that there might be an explicit inductive way to construct the prime generators.

We also study the asymptotic density of the prime generators showing in Theorem 10 that

$$\frac{|P(n)|}{|L(n)|} = \frac{b_n}{(n-1)!} = \frac{1}{e} \left(1 - \frac{3}{n} - \frac{5}{2n^2} + O\left(\frac{1}{n^3}\right) \right).$$

The counting sequence of our prime generators is closely related to the counting sequence of the SIF permutations studied by Callan in [2]. We derive in Corollary 11 a formula for the asymptotic density of the SIF permutations, that was conjectured in [2].

2. PROOF OF THE MAIN THEOREM

We will define, as stated in the introduction, an operad L in the category of sets.

Definition 2. *Let $L(k)$ be the set of formal expressions obtained by applying iterated binary brackets to k symbols x_1, \dots, x_k , such that*

- 1) *Each symbol appears exactly once*
- 2) *The smallest index inside a bracket appears on the left hand side, and the largest index appears on the right hand side.*

For example the expression $[x_1, [x_2, x_3]]$ is in $L(3)$, but $[x_2, [x_1, x_3]]$ is not in $L(3)$ because x_1 is not on the left hand side of the outer bracket.

By induction each expression in k symbols must involve exactly $k-1$ brackets. For example $L(1)$ contains only the unbracketed expression x_1 , $L(2)$ contains only the bracket $[x_1, x_2]$, and $L(3)$ contains the elements $[x_1, [x_2, x_3]]$, $[[x_1, x_2], x_3]$.

Proposition 3. *The set $L(k)$ has cardinality $(k-1)!$*

Proof. By induction on k . This is true for $k=2$. Suppose that this is true for $k < n$. An expression $A \in L(n)$ has the form $A = [A_1, A_2]$, where A_1 is an expression involving symbols with indexes $1 = a_1 < \dots < a_j$ and A_2 involves symbols with indexes $b_1 < \dots < b_{n-j} = n$, for some $1 \leq j \leq n-1$. If we replace each a_i by i in the expression A_1 we get an element $A'_1 \in L(j)$, and similarly replacing each b_i by i in A_2 gives $A'_2 \in L(n-j)$.

How many expressions do we obtain in this way in $L(n)$ for fixed j, A'_1 and A'_2 ? There are $\binom{n-2}{j-1}$ ways of shuffling the a'_i s and the b'_i s into two disjoint sets of cardinality j and $n-j$, with 1 belonging to the first and n to the second. Then, by inductive hypothesis, if we fix j and let A'_1, A'_2 vary we have

$$\binom{n-2}{j-1} (j-1)! (n-j-1)! = (n-2)!$$

expressions, and summing over j we get $(n-1)(n-2)! = (n-1)!$ expressions. \square

The collection $L(k)$ forms a non-symmetric operad in the category of sets, where the composition operation $A \circ_i B$ replaces the variable x_i in A by the formal expression B , with its indices shifted by $i - 1$, and shifts by $b - 1$ the indices of A larger than i , where b is the number of symbols in B .

For example

$$[[x_1, x_3], [x_2, x_4]] \circ_3 [x_1, x_2] = [[x_1, [x_3, x_4]], [x_2, x_5]]$$

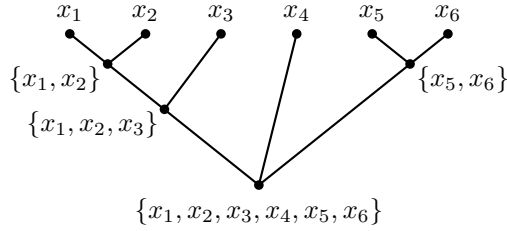
The unit of the operad is the expression x_1 .

We recall the definition of a free non-symmetric operad in the category of sets and R -modules. It will be sufficient for our purpose to consider reduced operads with no generators in degree (or arity) 0 and 1.

Definition 4. A reduced tree T on k leaves is a collection of subsets of $\{x_1, \dots, x_k\}$, the vertices, such that

- (1) The indexes of the symbols of a vertex form a sequence of consecutive numbers $\{i + 1, \dots, i + k\}$ of cardinality $k \geq 2$;
- (2) The full set $\{x_1, \dots, x_k\}$ is a vertex;
- (3) Any symbol x_i belongs to a vertex of T ;
- (4) Given two distinct vertices v, w of T , either $v \cap w = \emptyset$, or $v \subset w$, or $w \subset v$.

The *valence* $|v|$ of a vertex v is its cardinality. The terminology is justified because we can associate to each tree a planar directed graph that is a tree in the sense that it has no cycles, and each vertex in our sense yields a vertex of that graph, with a number of incoming edges equal to the valence. An example is given in the figure.



Let M_k be the set of all reduced trees on k leaves.

Definition 5. For given trees $T_1 \in M_k$, $T_2 \in M_l$ and $1 \leq i \leq k$, we define a new tree $T_1 \circ_i T_2 \in M_{k+l-1}$ with the following vertices:

- 1) For each vertex $v \in T_1$ a vertex $v' \in T_1 \circ_i T_2$ containing
 - the symbol x_j if $x_j \in v$ and $j < i$
 - the symbol x_{j+i-1} if $x_j \in v$ and $j > i$
 - the symbols x_{j+i-1} for $j = 1, \dots, l$ if $x_i \in v$
- 2) For each vertex $u \in T_2$ a vertex $u'' \in T_1 \circ_i T_2$ containing
 - the symbol x_{j+i-1} if $x_j \in u$.

This gives a natural bijection of collections $T_1 \amalg T_2 \cong T_1 \circ_i T_2$.

Given a sequence of sets $X = (X_n)_{n>1}$, the free operad $F(X)$ is defined in degree (or arity) $k > 1$ by

$$F(X)(k) = \prod_{T \in M_k} \prod_{v \in T} X_{|v|}$$

We call the summand indexed by a tree $T \in M_k$ the *stratum* $F(X)_T$.

The \circ_i composition $F(X)(k) \times F(X)(l) \rightarrow F(X)(k+l-1)$ is the inclusion that identifies the product $F(X)_{T_1} \times F(X)_{T_2}$ of the strata indexed respectively by $T_1 \in M_k$ and $T_2 \in M_l$ to the stratum $F(X)_{T_1 \circ_i T_2}$ indexed by $T_1 \circ_i T_2 \in M_{k+l-1}$.

The construction of a free operad in R -modules is entirely similar, except that the disjoint union and the cartesian product of sets are replaced respectively by the direct sum and the tensor product of R -modules .

In particular the free R -module functor $X \mapsto R[X]$ sending sequences of sets to sequences of R -modules commutes with the free operad construction.

Proposition 6. *The operad L is free (up to isomorphism).*

Proof. Let us be given a formal expression $A \in L(k)$. We say that a bracket b of A is *connected* if the set of indices inside b is an interval of consecutive integers. The indices do not need to appear in increasing order from left to right in the bracket. We say that a formal expression A is *prime* if the outer bracket is the only connected bracket it contains. For example

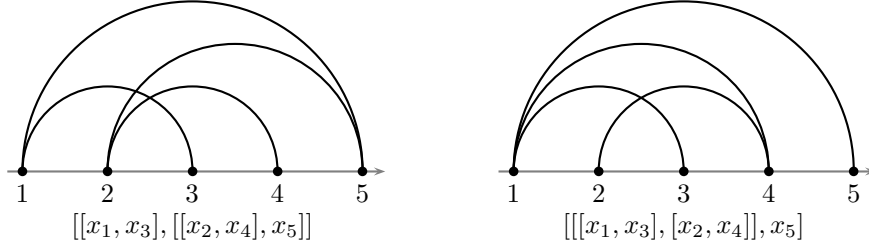
$$[[x_1, x_3], [x_2, x_4], x_5]$$

is prime but

$$(1) \quad [[[x_1, x_3], [x_2, x_4]], x_5]$$

is not prime because the bracket $[[x_1, x_3], [x_2, x_4]]$ is connected.

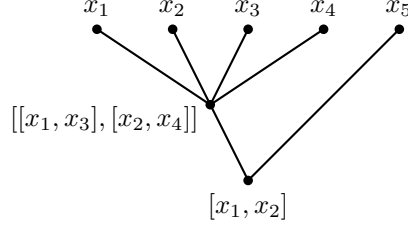
One can associate to each element of $L(k)$ a chord diagram, drawing the indices $1, \dots, k$ on a line, and drawing for each bracket a chord from the smallest index inside the bracket to the largest index inside the bracket.



Let $P(n) \subset L(n)$ be the set of all prime expressions in n symbols, and let us consider the sequence of sets $P = (P(n))$. We claim that $F(P)$ is isomorphic to L . The universal property defines an operad map $\theta : F(P) \rightarrow L$ out of the inclusion $P \subset L$. We show that θ is an isomorphism constructing its inverse ψ . Given $A \in L(k)$, let $T_A \in M_k$ be the tree that has exactly a vertex $v_b = \{x_i, \dots, x_j\}$ for each connected bracket b of A with set of indices $\{i, \dots, j\}$.

To each such vertex we associate a prime expression p_b obtained as follows: let k' be the number of maximal connected brackets contained properly in b . Let k'' be the number of symbols in b that are not contained in those k' maximal brackets. Then there is a unique monotone surjective map $\phi : \{i, \dots, j\} \rightarrow \{1, \dots, k' + k''\}$ that is constant on the set of indices of each maximal bracket. The prime expression p_b is obtained from b replacing each index t outside a maximal bracket by $\phi(t)$, and each maximal bracket b' by a single symbol indexed by $\phi(u)$, where u is any index inside b' . For example the expression (1) corresponds to the tree with a

vertex $v = \{x_1, x_2, x_3, x_4, x_5\}$ labelled by $[x_1, x_2]$ and a vertex $v' = \{x_1, x_2, x_3, x_4\}$ labelled by $[[x_1, x_3], [x_2, x_4]]$.



The collection $\{p_b\}_{b \in T(A)}$ defines an element $\psi_k(A) \in F(P)(k)$. By construction $\psi = (\psi_k)$ is exactly the inverse of θ . \square

There is a tautological map of operads of R -modules $\beta : R[L] \rightarrow Lie$, sending a formal expression to the same expression inside Lie .

Lemma 7. *The map β is an isomorphism.*

Proof. It is sufficient to establish the lemma for $R = \mathbb{Z}$. We prove first the surjectivity. We say that a bracket of an expression of iterated brackets has concentric index i if it is properly contained in exactly $i - 1$ brackets. For example the outer bracket has concentric index 1. Let us filter $Lie(k)$ by submodules $Lie_i(k)$ so that $Lie_i(k)$ is linearly generated by iterated brackets where in each bracket of index $\leq i$ the smallest index appears on the left and the largest on the right. In particular $Lie_0(k) = Lie(k)$, $Lie_{i+1}(k) \subseteq Lie_i(k)$ and $Lie_{k-1}(k) = Im(\beta_k)$. We prove first that $Lie(k) = Lie_1(k)$. Let $A \in Lie(k)$ be a linear generator, a combination of k symbols using $k - 1$ brackets. We say that A has weight i if the smallest index 1 and the largest index k share exactly i (concentric) brackets. For $i > 1$ an application of the Jacobi identity shows that A is the linear combination of two elements of weight $i - 1$. For example $[[x_1, x_3], x_2] = -[x_3, [x_2, x_1]] - [x_2, [x_1, x_3]]$ reduces an element of weight 1 as a combination of two elements of weight 0. By induction A is a linear combination of elements of weight 0, and by antisymmetry it is a linear combination of elements with the smallest index 1 on the left hand side of the outer bracket, and the largest index k on the right hand side of the outer bracket. Thus $Lie(k) = Lie_1(k)$.

The same principle applied to brackets of concentric index i shows that $Lie_i(k) = Lie_{i+1}(k)$ for $i = 1, \dots, k - 2$.

Thus $Lie(k) = Lie_{k-1}(k) = Im(\beta_k)$ and β_k is surjective. But the domain and the range of β_k have both rank $(k - 1)!$ in degree k and so β_k is an isomorphism for any k . \square

Theorem 1 follows from Proposition 6 and Lemma 7.

3. COUNTING THE PRIME GENERATORS

In this section we study the formal power series counting the number of prime generators.

Lemma 8. *Let (X_n) be a sequence of finite sets of cardinality $|X_n| = \beta_n$, for $n \geq 2$. Let FX be the free (reduced) operad generated by the sequence $X = (X_n)$. If $\alpha_n = |FX(n)|$ for $n \geq 1$, then the formal power series $\alpha(x) = \sum_{n=1}^{\infty} \alpha_n x^n$ and $\beta(x) = \sum_{n=2}^{\infty} \beta_n x^n$ satisfy the identity*

$$\beta(\alpha(x)) + x = \alpha(x)$$

Proof. Any element of FX is either the unit $e \in FX(1)$, or it can be written uniquely as operadic composition $x(y_1, \dots, y_m)$, for some $m \geq 2$, $x \in X_m$ and $y_i \in FX$, with $i = 1, \dots, m$. \square

Actually the lemma is a special case of the non-symmetric version of a result by Ginzburg and Kapranov, Theorem 3.3.2 in [3], relating the Poincaré series of a differential graded operad to the series of its dual, or bar construction.

We know that the formal power series of *Lie* is

$$F(x) = \sum_{n=1}^{\infty} (n-1)! x^n$$

Let b_n be the number of prime expressions in $L(n)$ and let

$$B(x) = -x + \sum_{n=2}^{\infty} b_n x^n.$$

The lemma applied to $\alpha(x) = F(x)$ and $\beta(x) = B(x) + x$ yields the identity $-B(F(x)) = x$ that is $B(x) = -F^{<-1>}(x)$, where the formal inverse $F^{<-1>}$ of F exists because $F(x) = x +$ (higher order terms).

Theorem 9. *The formal power series $B(x)$ counting the operadic generators of Lie satisfies the differential equation*

$$xB'(x) + (B'(x) + B(x))B(x) = 0$$

and the following recursive formula holds:

$$b_2 = 1, \quad b_n = \sum_{k=2}^{n-2} ((k+1)b_{k+1} + b_k)b_{n-k} \quad \text{for } n \geq 3.$$

Proof. By differentiating the identity $-B(F(x)) = x$ we obtain

$$-B'(F(x))F'(x) = 1.$$

Since $F'(x) = (F(x) - x)/x^2$ then

$$-B'(F(x))(F(x) - x) = x^2$$

and, by using $F(-B(x)) = x$, we get

$$xB'(x) + (B'(x) + B(x))B(x) = 0.$$

Interpreting this identity termwise yields

$$[x^n](xB'(x) + (B'(x) + B(x))B(x)) = nb_n + \sum_{k=0}^n ((k+1)b_{k+1} + b_k)b_{n-k} = 0$$

and, since $b_0 = 0$, $b_1 = -1$ and $b_2 = 1$, we have that

$$\sum_{k=2}^{n-2} ((k+1)b_{k+1} + b_k)b_{n-k} = -nb_n - (-1)b_n - (2-1)b_{n-1} - (nb_n + b_{n-1})(-1) = b_n.$$

□

We obtain thus

$$\sum_{n=2}^{\infty} b_n x^n = x^2 + x^4 + 4x^5 + 22x^6 + 144x^7 + 1089x^8 + 9308x^9 + 88562x^{10} + \dots$$

For more terms of $\{b_n\}_{n \geq 2}$ see the sequence A134988 in Sloane's OEIS [7].

Theorem 10. *The asymptotics density of the operadic generators of Lie is*

$$\frac{|P(n)|}{|\dim(\text{Lie}(n))|} = \frac{b_n}{(n-1)!} = e^{-1} \left(1 - \frac{3}{n} - \frac{5}{2n^2} + O\left(\frac{1}{n^3}\right) \right).$$

Proof. Since $B(x) = -F^{<-1>}(x)$, then by the Lagrange inversion formula

$$\begin{aligned} \frac{b_n}{(n-1)!} &= -\frac{1}{(n-1)!} [x^n] F^{<-1>}(x) = -\frac{1}{n!} [x^{n-1}] \left(\frac{x}{F(x)} \right)^n \\ &= -\frac{1}{n!} [x^{n-1}] (1 + A(x))^{-(n-1)-1} = -\frac{1}{n!} (\alpha e^{\alpha a_1 \gamma} (n-1) a_{n-1} + O(a_{n-1})) \\ &= -\frac{1}{n!} (-e^{-1} (n-1)(n-1)! + O((n-1)!)) = e^{-1} + O(1/n) \end{aligned}$$

where we applied Theorem 1(i) of [1] with $A(x) = \sum_{k=1}^{\infty} k!x^k$, $\alpha = -1$, $\beta = -1$, $\gamma = 1$. Let $p_n = b_n/(n-1)!$ and

$$p_n = e^{-1} \left(1 + \frac{c_1}{n} + \frac{c_2}{n^2} + O\left(\frac{1}{n^3}\right) \right).$$

By considering the recurrence, we will show that $c_1 = -3$ (c_2 can be found in a similar way). Note that $p_2 = 1$, $p_3 = 0$ and $p_4 = 1/6$, moreover

$$p_n = \sum_{k=2}^{n-2} \left((k+1)p_{k+1} + \frac{p_k}{k} \right) p_{n-k} \binom{n-1}{k}^{-1} \quad \text{for } n \geq 3.$$

Since $0 \leq b_n \leq (n-1)!$ then $0 \leq p_n \leq 1$ and

$$0 \leq \sum_{k=3}^{n-5} \left((k+1)p_{k+1} + \frac{p_k}{k} \right) p_{n-k} \binom{n-1}{k}^{-1} \leq \sum_{k=3}^{n-5} (k+2) \binom{n-1}{k}^{-1}.$$

Therefore

$$\sum_{k=3}^{n-5} \left((k+1)p_{k+1} + \frac{p_k}{k} \right) p_{n-k} \binom{n-1}{k}^{-1} = O\left(\frac{1}{n^3}\right)$$

because for some positive constant C

$$\begin{aligned} \sum_{k=3}^{n-5} (k+2) \binom{n-1}{k}^{-1} &\leq 5 \binom{n-1}{3}^{-1} + 6 \binom{n-1}{4}^{-1} \\ &\quad + Cn^2 \binom{n-1}{5}^{-1} + (n-3) \binom{n-1}{n-5}^{-1} \end{aligned}$$

Hence

$$\begin{aligned}
ep_n &= \left(3p_3 + \frac{p_2}{2}\right) (ep_{n-2}) \binom{n-1}{2}^{-1} + O\left(\frac{1}{n^3}\right) \\
&\quad + \sum_{k=n-4}^{n-2} \left((k+1)(ep_{k+1}) + \frac{1}{k}(ep_k) \right) p_{n-k} \binom{n-1}{k}^{-1} \\
&= \frac{1}{2} \binom{n-1}{2}^{-1} + \frac{1}{6} (n-3) \binom{n-1}{n-4}^{-1} \\
&\quad + \left((n-1) \left(1 + \frac{c_1}{n-1} + \frac{c_2}{(n-1)^2} \right) + \frac{1}{n-2} \right) \binom{n-1}{n-2}^{-1} + O\left(\frac{1}{n^3}\right) \\
&= 1 + \frac{c_1}{n-1} + \frac{c_2+3}{n^2} + O\left(\frac{1}{n^3}\right) = 1 + \frac{c_1}{n} + \frac{c_1+c_2+3}{n^2} + O\left(\frac{1}{n^3}\right)
\end{aligned}$$

that is

$$\frac{c_1+3}{n^2} = O\left(\frac{1}{n^3}\right)$$

which implies that $c_1 = -3$. \square

It is interesting to note that b_n , the number of prime expressions in $L(n)$, is related with the number a_n of *stabilized-interval-free* (SIF) permutations on $[n] = \{1, 2, \dots, n\}$ introduced by Callan in [2] (seq. A075834 in [7]). A permutation on $[n]$ is SIF if it does not stabilize any proper subinterval of $[n]$. The SIF permutations on $[n]$ for $2 \leq n \leq 4$ are as follows: $n = 2$: (12); $n = 3$: (123), (132) (the two 3-cycles); $n = 4$: (13)(24) and the six 4-cycles. The power series counting the SIF permutations is

$$A(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + x + x^2 + 2x^3 + 7x^4 + 34x^5 + 206x^6 + 1476x^7 + 12123x^8 + \dots$$

Callan proved that

$$\frac{1}{n} [x^{n-1}] (A(x))^n = (n-1)! = [x^n] F(x)$$

which means, by the Lagrange inversion formula, that

$$A(x) = \frac{x}{F^{\langle -1 \rangle}(x)} = -\frac{x}{B(x)}.$$

Hence

$$xA'(x) - A(x) + x = -\frac{x}{B(x)} + \frac{x^2 B'(x)}{(B(x))^2} + \frac{x}{B(x)} + x = \frac{x^2 B'(x)}{(B(x))^2} + x.$$

and since $xB'(x) = -(B'(x) + B(x))B(x)$ then

$$xA'(x) - A(x) + x = -\frac{x(B'(x) + B(x))B(x)}{(B(x))^2} + x = -\frac{xB'(x)}{B(x)} = B'(x) + B(x).$$

This differential equation yields the following recurrence formula

$$(n-1)a_n = (n+1)b_{n+1} + b_n \quad \text{for } n \geq 2.$$

As a consequence we confirm a numerical estimate given at the end of [2].

Corollary 11. *The asymptotic density of the SIF permutations is given by*

$$\frac{a_n}{n!} = e^{-1} \left(1 - \frac{1}{n} - \frac{5}{2n^2} + O\left(\frac{1}{n^3}\right) \right).$$

Proof.

$$\begin{aligned} \frac{a_n}{n!} &= \frac{n+1}{n-1} p_{n+1} + \frac{1}{n(n-1)} p_n \\ &= \left(1 + \frac{2}{n} + \frac{2}{n^2} + O\left(\frac{1}{n^3}\right) \right) \frac{1}{e} \left(1 - \frac{3}{n+1} - \frac{5}{2} \frac{1}{(n+1)^2} + O\left(\frac{1}{n^3}\right) \right) \\ &\quad + \frac{1}{n^2} \frac{1}{e} \left(1 + O\left(\frac{1}{n}\right) \right) \\ &= \frac{1}{e} \left(\left(1 + \frac{2}{n} + \frac{2}{n^2} \right) \left(1 - \frac{3}{n} \left(1 - \frac{1}{n} \right) - \frac{5}{2n^2} \right) + \frac{1}{n^2} + O\left(\frac{1}{n^3}\right) \right) \\ &= e^{-1} \left(1 - \frac{1}{n} - \frac{5}{2n^2} + O\left(\frac{1}{n^3}\right) \right) \end{aligned}$$

□

REFERENCES

- [1] E. A. Bender and L. B. Richmond, *An asymptotic expansion for the coefficients of some power series II: Lagrange inversion*, Discrete Mathematics, 50 (1984), 135–141.
- [2] D. Callan, *Counting Stabilized-Interval-Free Permutations*, Journal of Integer Sequences, Article 04.1.8, 7 (2004).
- [3] V. Ginzburg and M. Kapranov, *Koszul duality for operads*, Duke Math.J. 76 (1994), 203–272.
- [4] P. Lambrechts, V. Turchin and Ismar Volić, *The rational homology of the space of long knots in codimension > 2*, arXiv:math/0703649
- [5] M. Markl, S. Shnider and J. Stasheff, *Operads in algebra, topology and physics*, Mathematical Surveys and Monographs, 96. American Mathematical Society.
- [6] D. Sinha, *The homology of the little disks operad*, arXiv:math/0610236
- [7] N. J. A. Sloane, *On-Line Encyclopedia of Integer Sequences*, <http://www.research.att.com/~njas/sequences/>