On a symmetric space attached to polyzeta values.

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ABSTRACT

Quickly converging series are given to compute polyzeta numbers $\zeta(r_1, \dots, r_k)$. The formulas involve an intricate combination of (generalized) polylogarithms at 1/2. However, the combinatoric has a very simple geometric interpretation: it corresponds with the map $p \mapsto p^2$ on a certain symmetric space P.

Introduction:

Let $k \geq 1$. For a k-uple $(r_1,\, r_2,\, \dots,\, r_k)$ of positive integers, set $\zeta(r_1, \dots, r_k) = \sum_{0<n_1 \dots <n_k} 1/n_1^{r_1} \dots n_k^{r_k}$. We have $\zeta(r_1, \dots, r_k) < \infty$ if and only if $r_k \geq 2$. By definition, the *polyzeta values* are the **Q**-linear combinations of the finite numbers $\zeta(r_1, \dots, r_k)$.

Using the definition of $\zeta(r_1, \dots, r_k)$, the evaluation of a polyzeta value up to the N^{th} digit requires to take into account something like $O(10^N)$ terms. Therefore it is a very slow computation. A similar computational problem arises with the classical series $\log 2 = -\sum_{n>0} (-1)^n/n$ and $\pi/4 = \sum_{n \geq 0} (-1)^n 1/(2n+1)$, which converge very slowly.

However, we easily notice that:

$$\log 2 = -\log(1 - 1/2) = \sum_{n>0} 2^{-n}/n.$$

A remarkable series for π has been discovered by Bailey, Borwein and Plouffe [**BBP**]:

$$\pi = \sum_{n \geq 0} 1/2^{4n} [4/(8n+1) - 2/(8n+4) - 1/(8n+5) - 1/(8n+6)]$$

Now to evaluate $\log 2$ or π up to the N^{th} digit, one only needs the first $O(N)$-terms of the series and therefore $\log 2$ and π can be computed very quickly. The goal of the paper is to provide similar identities for all polyzeta values.

To do so, one needs to use the functions $L_{r_1,\dots,r_k}(z) = \sum_{0<n_1 \dots <n_k} 1/n_1^{r_1} \dots n_k^{r_k} z^{n_k}$, where $r_1, \dots, r_k$ are positive integers. By definition, a **Q**-linear combinations of the functions $L_{r_1,\dots,r_k}(z)$ is called a *polylogarithmic function*. The obvious identity $\zeta(r_1, \dots, r_k) = L_{r_1,\dots,r_k}(1)$ does not help to quickly evaluate polyzeta values. However, the series defining polylogarithms at 1/2 converges very quickly: to evaluate $L_{r_1,\dots,r_k}(1/2)$ up to the N^{th} digit, one only needs to sum $O(N^k)$-terms, and this can be done in polynomial time. This remark suggests the following result:

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MAIN STATEMENT: *Any polyzeta value is the value of a certain polylogarithmic function at* $1/2$.

In order to get a useful statement, the corresponding polylogarithmic function is described explicitly: see Theorem 7 for a precise statement. At first glance, the combinatorics involved in Theorem 7 looks intricated and therefore no details are given in the introduction. However, we can precisely formulate the main statement in terms of very simple geometric notions.

Let $F(2)$ be the free group on two generators α and β and let s be the involution exchanging the generators. Let $\Gamma = \mathbf{Q}\otimes F_2$ be the Malcev completion of Γ (see Section 4 for an alternative definition of Γ). The involution s extends to Γ and there is a decomposition $\Gamma = P.K$ where K is the subgroup of fixed points of s and where $P = \{g \in \Gamma | s(g) = g^{-1}\}$. The group Γ is proalgebraic over $\mathbf{Q}$ and the symmetric space P is a pro-algebraic variety over $\mathbf{Q}$.

In section (4.8), all polyzeta values are naturally indexed by rational functions on P. Similarly, some polylogarithmic functions are naturally indexed by rational functions on P. So for $\phi \in \mathbf{Q}[P]$, denote by $\zeta(\phi)$ and $L_\phi(z)$ the corresponding polyzeta value and polylogarithmic function.

Now the square map $\square : P \to P, p \mapsto p^2$ induces an algebra morphism $\square : \mathbf{Q}[P] \to \mathbf{Q}[P]$. The geometric formulation of the main result is as follows:

MAIN THEOREM: *For any* $\phi \in \mathbf{Q}[P]$, $\zeta(\phi) = L_{\square\phi}(1/2)$.

We also express polyzeta values as values of polylogarithmic functions at $\rho^{\pm 1} = \exp \pm i\pi/3$. The geometric interpretation of this case is a bit more complicate because it involves an order 3 automorphism of Γ, see section 4, Theorem 18.

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Summary:

1. Polylogarithms and polyzeta values.

This section is devoted to main definitions and conventions. The definitions of *polyzeta values* and *polylogarithmic functions* are not standard: see the subsections (1.14) for more comments. Moreover in this section we adopt some conventions to renormalize infinite quantities like $\zeta(1)$ or $\int_0^z dt/t$.

(1.1) Shuffles: For $N \geq 0$, denote by S_N the symmetric group, i.e. the set of all bijections $\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$. Given n and m two non-negative integers, let

$S_{n,m}$ be the set of all $\phi \in S_{n+m}$ such that ϕ is increasing on the subset $\{1,\dots,n\}$ and on the subset $\{n+1,\dots,n+m\}$. The elements of $S_{n,m}$ are called *schuffles.*

(1.2) Shuffle product: Let $\mathcal{W}$ be the set of words into the letters a and b. By convention, $\mathcal{W}$ contains the empty word $\emptyset$. Set $\mathcal{H} = \mathbf{Q}\mathcal{W}$, i.e. $\mathcal{H}$ is the $\mathbf{Q}$-vector space with basis $\mathcal{W}$. For any two words $w = x_1 \dots x_n$ and $w' = x_{n+1} \dots x_{n+m}$, where each $x_i \in \{a, b\}$ is a letter, define the product $w * w' \in \mathcal{H}$ by the formula:

$$w * w' = \sum_{\sigma \in S_{n,m}} x_{\sigma(1)} \dots x_{\sigma(n+m)}$$

By convention, we have $\emptyset * w = w * \emptyset = w$ for all word w. The product $*$ is called the shuffle product. With respect to this product, $\mathcal{H}$ is a commutative associative algebra, and $\emptyset$ is its unit.

(1.3) Subalgebras of $\mathcal{H}$: Let $\mathcal{W}^+$ be the set of words whose first letter is not b. Equivalently, a word w belongs to $\mathcal{W}^+$ if $w = \emptyset$ or if w starts with a. Similarly, let $\mathcal{W}^{++}$ be the set of words whose first letter is not b and the last letter is not a. Set $\mathcal{H}^+ = \mathbf{Q}\mathcal{W}^+$ and $\mathcal{H}^{++} = \mathbf{Q}\mathcal{W}^{++}$. It is easy to prove that $\mathcal{H}^+$ and $\mathcal{H}^{++}$ are subalgebras of $\mathcal{H}$.

LEMMA 1: *There are isomorphisms of algebras:* $\mathcal{H} = \mathcal{H}^+[b]$ *and* $\mathcal{H} = \mathcal{H}^{++}[a, b]$.

Proof: For each $n \geq 0$, let $\mathcal{W}_n$ be the set of words of the form $b^n w$ with $w \in \mathcal{W}^+$, and set $\mathcal{H}_n = \oplus_{0 \leq k \leq n} \mathbf{Q}\mathcal{W}_i$. We have $b * \mathcal{H}_n \subset \mathcal{H}_{n+1}$. Moreover we have $b * w = (n+1)bw$ modulo $\mathcal{H}_n$ for any $w \in \mathcal{W}_n$. It follows easily by induction that $\mathcal{H}_n = \oplus_{0 \leq k \leq n} \mathcal{H}^+ * b^k$, i.e. $\mathcal{H}_n$ is the space of all polynomials in b wih coefficients in $\mathcal{H}^+$ and degree $\leq n$. Therefore the first assertion follows.

The proof of the second assertion is similar.

(1.4) The bijection $\lambda : \mathcal{W}^+ \to \Lambda$:

Let $\mathbf{N}$ be the set of positive integer. For clarity, a word into the letters $1, 2, \dots \in \mathbf{N}$ will be called a *sequence of positive integers.* Let Λ the set of sequence $(r_1 \dots r_k)$ of positive integers. By convention, Λ contains the empty sequence $\emptyset$.

Any word $w \in \mathcal{W}^+$ can be uniquely factorized as: $w = ab^{t_1}ab^{t_2} \dots ab^{t_k}$, where k is the number of occurence of a in w and where the t_i are non-negative integers. Then, the map $w \in \mathcal{W}^+ \mapsto (1 + t_1, 1 + t_2, \dots, 1 + t_k) \in \Lambda$ defines a natural bijection $\lambda : \mathcal{W}^+ \to \Lambda$.

(1.5) Polylogarithmic functions and polyzeta values:

Let $k \geq 1$ and let $r_1 \dots r_k$ be a sequence of k positive integers. Consider the following series in the complex variable z:

$$L_{r_1,\dots,r_k}(z) = \sum_{0<n_1<\dots<n_k} n_1^{-r_1} \dots n_k^{-r_k} \, z^{n_k}$$

In the infinite sum, the indices $n_1, \dots n_k$ are integers. The functions $L_{r_1,\dots,r_k}(z)$ are called *polylogarithms.* Set $D = \{z \in \mathbf{C} | \, |z| < 1\}$, $\overline{D} = \{z \in \mathbf{C} | \, |z| \leq 1\}$. The two points of interest for the paper are the following:

(i) if $r_k \geq 2$, the series is absolutely convergent on $\overline{D}$ and therefore $L_{r_1,\dots,r_k}(z)$ extends to a continous function on $\overline{D}$.

(ii) if $r_k = 1$, the series converges on D and $L_{r_1,\dots,r_k}(z)$ extends to a continous function on $\overline{D} \setminus \{1\}$.
Indeed $L_{r_1,\dots,r_k}(z)$ extends to a multivalued function, see e.g. **[C]** and Proposition 2 below. For $r_k \geq 2$, set

$$\zeta(r_1,\dots,r_k) = \sum_{0<n_1<\dots<n_k} n_1^{-r_1}\dots n_k^{-r_k}$$

In the paper, the numbers $\zeta(r_1,\dots,r_k)$ will be called polyzeta values. Indeed the polyzeta value is both the value at $z=1$ of the polylogathm $L_{r_1,\dots,r_k}(z)$ and a value of the polyzeta function $\zeta(s_1,\dots,s_k) = \sum_{0<n_1<\dots<n_k} n_1^{-s_1}\dots n_k^{-s_k}$.

(1.6) New notations: Let $w \in \mathcal{W}^+$ and set $(r_1,\dots,r_k) = \lambda(w)$. It is convenient to denote the function $L_{(r_1,\dots,r_k)}(z)$ by $L_w(z)$. Similarly set $\zeta(w) = \zeta(r_1,\dots,r_k)$ if $w \in \mathcal{W}^{++}$.

(1.7) The one-forms ω_a and ω_b: Define the following one-forms on $\mathbf{C}$:

$$\omega_a(z) = \frac{\mathrm{d}z}{1-z} \quad \text{and} \quad \omega_b(z) = \frac{\mathrm{d}z}{z}$$

For an element $c = xa + yb \in \mathbf{Q}a \oplus \mathbf{Q}b$, set $\omega_c(z) = x\omega_a(z) + y\omega_b(z)$. Given a smooth path $\gamma : [0,1] \to \mathbf{C}, t \mapsto \gamma(t)$, recall that $\gamma^*\omega_a(t) = \frac{\gamma'(t)}{1-\gamma(t)}\,\mathrm{d}t$, $\gamma^*\omega_b(t) = \frac{\gamma'(t)}{\gamma(t)}\,\mathrm{d}t$. and $\gamma^*\omega_c(t) = x\gamma^*\omega_a(t) + y\gamma^*\omega_b(t)$.

(1.8) Kontsevitch formula: For a positive integer n, set $\Delta_n = \{(x_1, x_2 \dots, x_n) \in \mathbf{R}^n | 0 \leq x_1 \leq x_2 \dots \leq x_n \leq 1\}$. Let $w = c_1 \dots c_n \in \mathcal{W}^+$ be a word, where each c_i is a letter. The following formula is due to Kontsevitch (see **[Z]**).

PROPOSITION 2: *Let $w = c_1 \dots c_n \in \mathcal{W}^+$ be a word, let $z \in \overline{D}$, and let $\gamma : [0,1] \to \overline{D}$ be a path with $\gamma(0) = 0$ and $\gamma(1) = z$.*

Assume that $w \in \mathcal{W}^{++}$ or that γ does not meet 1. Then we have:

$$L_w(z) = \int_{\Delta_n} \gamma^*\omega_{c_1}(x_1)\gamma^*\omega_{c_2}(x_2)\dots\gamma^*\omega_{c_n}(x_n)$$

.

In **[Z]**, Kontsevitch formula is stated for the straight path $t \mapsto zt$, but it is easy to see that the integral is homotopy invariant as long γ stay in $\overline{D}$ (and γ stay $\overline{D} \setminus \{1\}$ if $w \notin \mathcal{W}^{++}$).

(1.9) Products: Let n, m be non negative integers and let $c_1, c_2, \dots, c_{n+m} \in \{a, b\}$ be letters with $c_1 = c_{n+1} = a$. Set $u = c_1 \dots c_n$ and $v = c_{n+1} \dots c_{n+m}$. For $\sigma \in S_{n,m}$, set $w_\sigma = c_{\sigma(1)} \dots c_{\sigma(n+m)}$.

COROLLARY 3: *For u, $v \in \mathcal{W}^+$, we have $L_u(z)L_v(z) = \sum_{\sigma\in S_{n,m}} L_{w_\sigma}(z)$ for all $z \in D$. Moreover for u, $v \in \mathcal{W}^{++}$, we have $\zeta(u)\zeta(v) = \sum_{\sigma\in S_{n,m}} \zeta(w_\sigma)$.*

Proof: Set

$$\omega' = \omega_{c_1}(zx_1)\omega_{c_2}(zx_2)\dots\omega_{c_n}(zx_n),$$
$$\omega" = \omega_{c_{n+1}}(zx_{n+1})\omega_{c_{n+2}}(zx_{n+2})\dots\omega_{c_{n+m}}(zx_{n+m}),$$
$$\Delta_m = \{(x_{n+1}, x_{n+2} \dots, x_{n+m}) \in \mathbf{R}^n | 0 \leq x_1 \leq x_2 \dots \leq x_n \leq 1\},$$

and for $\sigma \in S_{n,m}$, set

$$\Delta_\sigma = \{(x_1, \ldots, x_{n+m}) \in \mathbf{R}^{n+m} | x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n+m)}\}.$$

Since $\Delta_n \times \Delta_m = \cup_{\sigma \in S_{n,m}} \Delta_\sigma$, we get $\int_{\Delta_n} \omega' \int_{\Delta_m} \omega" = \int_{\Delta_n \times \Delta_m} \omega' \wedge \omega" = \sum_\sigma \int_{\Delta_\sigma} \omega' \wedge \omega"$. By Proposition 2, this identity is equivalent to $L_u(z) L_v(z) = \sum_{\sigma \in S_{n,m}} L_{w_\sigma}(z)$. At $z = 1$, one gets the second identity $\zeta(u)\zeta(v) = \sum_{\sigma \in S_{n,m}} \zeta(w_\sigma)$. Q.E.D.

(1.10) Final definitions and notations for polylogarithmic functions: Up to now, the polylogarithms $L_w(z)$ are defined for $w \in \mathcal{W}^+$. In order to extend the definition to all $w \in \mathcal{W}$, a renormalization procedure is used.

Set $\Omega = D \backslash]-1, 0]$ and let $Hol(\Omega)$ be the algebra of holomorphic functions on Ω. Since ω is simply connected, let denote by $\log z$ the holomorphic function on Ω whose restriction to $]0, 1[$ is the usual logarithmic function.

LEMMA 4: *There is a unique algebra morphism* $\Phi : \mathcal{H} \to Hol(\Omega)$ *such that* $\Phi(w) = L_w(z)$ *for* $w \in \mathcal{H}^+$ *and* $\Phi(b) = \log z$.

Proof: This follows from Lemmma 1 and corollary 3. Q.E.D.

For any $h \in \mathcal{H}$, set $L_h(z) = \Phi(h)$. When h is a word w in $\mathcal{W}^+$, this new notation agrees with the previous one. Corollary 3 can be restated as: $L_u(z) L_v(z) = L_{u*v}(z)$. By definition the *polylogarithmic functions* are the functions $L_h(z)$ whith $h \in \mathcal{H}$.

This definition is a slighty different from the introduction. However, we will only use polylogarithmic functions $L_h(z)$ with $h \in \mathcal{H}^+$, which are the polylogaritmic functions defined in introduction.

(1.11) Final definitions and notations for polyzeta values: Up to now, the polyzeta values $\zeta(w)$ are defined for $w \in \mathcal{W}^{++}$. In order to extend the definition to all $w \in \mathcal{W}$, we will use a renormalization procedure as follows.

LEMMA 5: *There are three algebra morphisms* ψ, ψ^+, $\psi^- : \mathcal{H} \to \mathbf{C}$ *uniquely defined by the following requirements:*

$\psi(w) = \psi^+(w) = \psi^-(w) = \zeta(w)$ *if* $w \in \mathcal{H}^{++}$

$\psi(a) = 0$, $\psi^+(a) = i\pi$, $\psi^-(a) = -i\pi$

$\psi(b) = \psi^+(b) = \psi^-(b) = 0$.

Proof: This follows from Lemmma 1 and corollary 3.

Similarly, this allows to define $\zeta(h) = \psi(h)$, $\zeta^\pm(h) = \psi^\pm(h)$ for any $h \in \mathcal{H}$. By definition the *polyzeta values* are the numbers $\zeta(h)$ whith $h \in \mathcal{H}$. Corollary 3 can be restated as: $\zeta(u)\zeta(v) = \zeta(u * v)$ and $\zeta^\pm(u)\zeta^\pm(v) = \zeta^\pm(u * v)$ for any u, $v \in \mathcal{H}$.

Set $\mathcal{Z} = \zeta(\mathcal{H})$ and $\mathcal{Z}^\pm = \zeta(\mathcal{H}^\pm)$. By definition, $\mathcal{Z}$ and $\mathcal{Z}^\pm$ are subrings of $\mathbf{C}$, and $\mathcal{Z}$ is the space of all polyzeta values. It is easy to compare the three algebras $\mathcal{Z}$ and $\mathcal{Z}^+$ and $\mathcal{Z}^-$.

LEMMA 6:

(i) As a $\mathbf{Q}$ *vector space,* $\mathcal{Z}$ *is generated by all* $\zeta(w)$ *with* $w \in \mathcal{W}^{++}$.

(ii) We have $\mathcal{Z} \subset \mathbf{R}$.

(iii) $\mathcal{Z}^\pm = \mathcal{Z} \oplus i\pi\mathcal{Z}$.

Proof: The assertions (i) and (ii) follow from Lemma 1. Moreover it follows that $\mathcal{Z}^\pm$ is the $\mathbf{Q}$-algebra generated by $\mathcal{Z}$ and $\zeta^\pm(b) = \pm i\pi$. However $(\zeta^\pm(b))^2 = -\pi^2 = -6\zeta(2)$, therefore $(\zeta^\pm(b))^2$ belongs to $\mathcal{Z}$ and assertion (6.3) follows.

(1.12) Hopf algebra structure: Define the linear maps $\eta : \mathcal{H} \to \mathbf{Q}$, $\iota : \mathcal{H} \to \mathcal{H}$ and $\Delta : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ as follows. For any word $w = c_1 \dots c_n \in \mathcal{W}$, set

$$\eta(w) = 1 \text{ if } w = \emptyset \text{ and } \eta(w) = 0 \text{ otherwise}$$

$$\iota(w) = (-1)^n \, c_n c_{n-1} \dots c_n$$

$$\Delta(w) = \sum_{0 \le i \le n} c_1 \dots c_i \otimes c_{i+1} \dots c_n$$

The map η, ι and Δ are algebra morphisms. Indeed $\mathcal{H}$ is a Hopf algebra with co-unit η, inverse map ι and coproduct Δ.

(1.13) Concatenation product: For two words $w = c_1 \dots c_n$ and $w' = c_{n+1} \dots c_{n+m}$, their concatenation is the word $ww' = c_1 \dots c_n c_{n+1} \dots c_{n+m}$. This induces another structure of algebra on $\mathcal{H}$, for which the product of two elements h, h' is simply denoted by $h\,h'$.

(1.14) Remarks on references and on the terminology:

In the classical litterature, only the functions $L_k(z) = \sum_{n>0} z^n/n^k$ are called polylogarithms, see **[L]** **[Oe]**. We did not find a standard name for the $L_{(r_1,\dots,r_k)}(z)$. They are defined in the Bourbaki's talk **[C]**, where the title suggests to call them again polylogarithms.

It seems that some polyzeta values, like $\zeta(1,3)$, were already known by Euler, see **[C]**. The general definition of $\zeta(r_1,\dots,r_k)$ appears explicitely around 1990 in **[H]** and **[Z]**. These numbers are also called multiple zeta values in **[Z]**, multiple harmonic sums in **[H]**, multizeta numbers in **[E]**, Euler-Zagier numbers in **[BB]** and polyzetas numbers in **[C]**.

The fact that polyzeta values are naturally indexed by words has been observed by many authors, see **[H]**, **[H-P]** and **[C]**. Lemma 1 and corollary 3 are well-known. Proofs are given for the convenience of the reader.

2. Polylogaritmic functions at 1/2 and at $\rho^{\pm 1}$:

Define two linear maps $\sigma, \tau : \mathcal{H} \to \mathcal{H}$ as follows. First set $\sigma(a) = b$, $\sigma(b) = a$, $\tau(a) = a + b$ and $\tau(b) = -a$. For any word $w = c_1 \dots c_n \in \mathcal{W}$, set $\sigma(w) = \sigma(c_n) \dots \sigma(c_1)$ and $\tau(w) = \tau(c_n) \dots \tau(c_1)$. It is easy to see that σ and τ are algebra morphisms relative to the shuffle product $*$ (they are anti-morphism relative to the concatenation product).

Define now the two operators $\square, \nabla : \mathcal{H} \to \mathcal{H}$ as the following composite maps:

$$\square : \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{id \otimes \sigma} \mathcal{H} \otimes \mathcal{H} \xrightarrow{*} \mathcal{H}$$

$$\nabla : \mathcal{H} \xrightarrow{\Delta} \mathcal{H} \otimes \mathcal{H} \xrightarrow{id \otimes \tau} \mathcal{H} \otimes \mathcal{H} \xrightarrow{*} \mathcal{H}$$

Set $\rho = e^{i\pi/3}$.

THEOREM 7: *For any $h \in \mathcal{H}$, we have:*

$$\zeta(h) = L_{\square(h)}(1/2)$$

$$\zeta^{\pm}(h) = L_{\nabla(h)}(\rho^{\pm 1})$$

Proof: Note that $\square(a) = \square(b) = a+b$, $L_a(z) = \log 1/(1-z)$ and $L_b(z) = \log z$, therefore $L_a(1/2) + L_b(1/2) = 0 = \zeta(a) = \zeta(b)$. Similarly, $\nabla(a) = 2a + b$ and $\nabla(b) = 0$, and we have $L_\nabla(a)(\rho^{\pm 1}) = -2\log(1-\rho^{\pm 1}) + \log(\rho^{\pm 1}) = \pm i\pi = \zeta^{\pm}(a)$ and $L_\nabla(b)(\rho^{\pm 1}) = 0 = \zeta^{\pm}(b)$.

Since $\square$ and ∇ are algebra morphisms, and since the algebra $\mathcal{H}$ is generated by a, b and $\mathcal{W}^{++}$, it is enough to show the formulas when h is a non empty word w in $\mathcal{W}^{++}$. So let $w \in \mathcal{W}^{++}$ be a word of length $n \geq 2$. Set $w = c_1 \dots c_n$ where $c_i \in \{a, b\}$ are letters with $c_1 = a$ and $c_n = b$.

Let $\gamma : [0,1] \to \overline{D}$, $t \mapsto t$ be the straight path from 0 to 1. Choose two smooth paths $\gamma_\pm : [0,1] \to \overline{D}$ with the following properties: $\gamma_\pm(0) = 0$, $\gamma_\pm(1) = 1$, $\gamma_\pm(1/2) = \rho^{\pm 1} = 1/2 \pm i\sqrt{3}/2$ and $\mathrm{Re}\,\gamma_\pm(t) \geq 1/2$ for all $t \in [1/2, 1]$. Set $\Delta_n = \{(x_1, x_2 \dots, x_n) \in \mathbf{R}^n | 0 \leq x_1 \leq x_2 \dots \leq x_n \leq 1\}$. By Proposition 2, we have:

$$\zeta(w) = \int_{\Delta_n} \eta^*\omega_{c_1}(x_1)\, \eta^*\omega_{c_2}(x_2) \dots \eta^*\omega_{c_n}(x_n)$$

where η is the path γ, or γ^+ or γ^-.

For $0 \leq i \leq n$, set $\Delta'_i = \{(x_1, x_2 \dots, x_i) \in \mathbf{R}^n | 0 \leq x_1 \leq x_2 \dots \leq x_i \leq 1/2\}$ and $\Delta"_i = \{(x_{i+1}, x_{i+2} \dots, x_n) \in \mathbf{R}^n | 1/2 \leq x_{i+1} \leq x_{i+2} \dots \leq x_n \leq 1\}$. From the decomposition: $\Delta = \cup_{0 \leq i \leq n} \Delta'_i \times \Delta"_i$, it follows that $\zeta(w) = \sum_{0 \leq i \leq n} L'_i \, L"_i = \sum_{0 \leq i \leq n} L'^{\pm}_i \, L"^{\pm}_i$, where the numbers L'_i, $L"_i$, $L'^{\pm}_i$, $L_i"^{\pm}$ are the following integrals:

$$L'_i = \int_{\Delta'_i} \gamma^*\omega_{c_1}(x_1)\, \gamma^*\omega_{c_2}(x_2) \dots \gamma^*\omega_{c_i}(x_1)$$

$$L"_i = \int_{\Delta"_i} \gamma^*\omega_{c_{i+1}}(x_{i+1})\, \gamma^*\omega_{c_{i+2}}(x_{i+2}) \dots \gamma^*\omega_{c_n}(x_n)$$

$$L'^{\pm}_i = \int_{\Delta'_i} \gamma^*_\pm\omega_{c_1}(x_1)\, \gamma^*_\pm\omega_{c_2}(x_2) \dots \gamma^*_\pm\omega_{c_i}(x_1)$$

$$L"^{\pm}_i = \int_{\Delta"_i} \gamma^*_\pm\omega_{c_{i+1}}(x_{i+1})\, \gamma^*_\pm\omega_{c_{i+2}}(x_{i+2}) \dots \gamma^*_\pm\omega_{c_n}(x_n)$$

Using Kontsevitch formula, we get $L'_i = L_{w'_i}(1/2)$ and $L'^{\pm}_i = L_{w'_i}(\rho^{\pm 1})$, where $w'_i = c_1 \dots c_i$. To evaluate $L"_i$, one needs to introduce some new notations. Define by S, T : $\mathbf{C} \to \mathbf{C}$ the rational maps: $S(z) = 1 - z$ and $T(z) = 1 - 1/z$. Define the new paths δ, $\delta_\pm$: $[0, 1/2] \to \mathbf{C}$ by $\delta(t) = 1 - \gamma(1-t) = S \circ \gamma(1-t)$ and $\delta_\pm(t) = 1 - 1/\gamma_\pm(1-t) = T \circ \gamma(1-t)$. Clearly, δ is the straight path from 0 to 1/2. Since $T(\rho^{\pm 1}) = \rho^{\pm 1}$ and $T(1) = 0$, $\delta_\pm$ is a path from 0 to $\rho^{\pm 1}$. Since $\mathrm{Re}\,\gamma_{\pm}(t) \geq 1/2$ for all $t \in [1/2, 1]$, it follows that $\delta_{\pm}$ lies in $\overline{D} \setminus \{1\}$.

With the convention of (1.7), we get:

$$\gamma^* \omega_c(t) = \delta^* \omega_{\sigma(c)}(1-t) \text{ and } \gamma_{\pm}^* \omega_c(t) = \delta_{\pm}^* \omega_{\tau(c)}(1-t)$$

for any $c \in \mathbf{Q}a \oplus \mathbf{Q}b$. Using the new variables $y_j = 1 - x_j$, we thus get:

$$L"_i = \int_{\overline{\Delta"_i}} \delta^* \omega_{\sigma(c_n)}(y_n) \delta^* \omega_{\sigma(c_{n-1})}(y_{n-1}) \dots \delta^* \omega_{\sigma(c_{i+1})}(y_{i+1})$$

$$L"^{\pm}_i = \int_{\overline{\Delta"_i}} \delta_{\pm}^* \omega_{\tau(c_n)}(y_n) \delta_{\pm}^* \omega_{\tau(c_{n-1})}(y_{n-1}) \dots \delta_{\pm}^* \omega_{\tau(c_{i+1})}(y_{i+1})$$

where $\overline{\Delta"_i} = \{(y_n, y_{n-1}, \dots, y_{i+1}) \in \mathbf{R}^n | 0 \leq y_n \leq y_{n+1} \dots \leq y_{i+1} \leq 1/2\}$. It follows from Proposition 2 that $L"_i = L_{\sigma(w"_i)}(1/2)$ and $L"^{\pm}_i = L_{\tau(w"_i)}(\rho^{\pm 1})$ where $w"_i$ is the word $c_{i+1} \dots c_n$. Therefore we get

$$\zeta(w) = \sum_{0 \leq i \leq n} L_{w'_i}(1/2) L_{\sigma(w"_i)}(1/2), \text{ and}$$

$$\zeta(w) = \sum_{0 \leq i \leq n} L_{w'_i}(\rho^{\pm 1}) L_{\tau(w"_i)}(\rho^{\pm 1}).$$

Since $\Delta(w) = \sum_{0 \leq i \leq n} w'_i \otimes w"_i$, it is clear that $\square\,(w) = \sum_{0 \leq i \leq n} w'_i * \sigma(w"_i)$ and $\nabla(w) = \sum_{0 \leq i \leq n} w'_i * \tau(w"_i)$, and therefore the formula follows from Corollary 3. Q.E.D.

3. Explicit expressions for $\zeta(r)$.

Theorem 7 provides a combinatorial way to express any polyzeta value as a polylogarithmic functions at 1/2 or at ρ or at $\overline{\rho}$. In this section, Theorem 10 and Corollary 12 provided closed formulas for zeta values $\zeta(r)$, where $r \geq 2$ is a given integer. The formulas are derived from Theorem 7. However, for general polyzeta values $\zeta(r_1, \dots, r_k)$ the combinatorics seem too intricate to find a simple combinatorial formula.

The concatenation product hh', which is not commutative, should not be confused with the commutative schuffle product $h * h'$. The following conventions will be used. First, for $h \in \mathcal{H}$ and $n \geq 1$, the notation h^n will be the n^{th} power of h with respect to the concatenation product. Moreover, the concatenation product takes precedence of the schufle product. For example, the expression $hh' * h"$ should be understood as $(hh') * h"$.

LEMMA 8: We have:

$$\square\,(ab^{r-1}) = 2a^{r-1}(a+b) + \sum_{1 \leq j \leq r-2} a^j (a+b)^{r-j}$$

$$\nabla(ab^{r-1}) = (-1)^{r+1} 3a^r + \sum_{1 \leq j \leq r-2} (-1)^{j+1} a^j (b-a)^{r-j}$$

Proof: We have:

$\square\,(ab^{r-1}) = \sigma(ab^{r-1}) + \sum_{0\le j\le r-1} ab^j * \sigma(b^{r-1-j})$

$\quad = a^{r-1}b + \sum_{u+v=r-1} ab^u * a^v.$

Similarly, we have:

$\nabla(ab^{r-1}) = \tau(ab^{r-1}) + \sum_{0\le j\le r-1} ab^j * \tau(b^{r-1-j})$

$\quad = (-1)^{r+1}a^{r-1}(a+b) + \sum_{u+v=r-1} (-1)^v ab^u * a^v.$

Use now the formula, $cw * a^v = \sum_{i+j=v} a^j c(w * a^i)$, which holds for any word w and any letter c. Thus we have $ab^u * a^v = \sum_{i+j=v} a^{j+1}(b^u * a^i)$ and we get:

$\square\,(ab^{r-1}) = a^{r-1}b + \sum_{j+u+v=r-1} a^{j+1}(b^u * a^v)$, and

$\nabla(ab^{r-1}) = (-1)^{r+1}a^{r-1}(a+b) + \sum_{j+u+v=r-1} (-1)^{j+v}\, a^{j+1}(b^u * a^v).$

Using now the formulas:

$(a+b)^N = \sum_{u+v=N} a^u * b^v$ and $(b-a)^N = \sum_{u+v=N} (-1)^u\, a^u * b^v$

we get:

$\square\,(ab^{r-1}) = a^{r-1}b + \sum_{0\le j\le r-1} a^{j+1}(a+b)^{r-1-j}$

$\quad = a^{r-1}b + \sum_{1\le j\le r} a^j(a+b)^{r-j}$

$\quad = a^{r-1}b + a^r + \sum_{1\le j\le r-1} a^j(a+b)^{r-j}$

$\quad = a^{r-1}(a+b) + \sum_{1\le j\le r-1} a^j(a+b)^{r-j}$

$\quad = 2a^{r-1}(a+b) + \sum_{1\le j\le r-2} a^j(a+b)^{r-j}.$

We also get:

$\nabla(ab^{r-1}) = (-1)^{r+1}a^{r-1}(a+b) + + \sum_{0\le j\le r-1} (-1)^j a^{j+1}(b-a)^{r-1-j}$

$\quad = (-1)^{r+1}a^{r-1}(a+b) + \sum_{1\le j\le r} (-1)^{j+1}a^j(b-a)^{r-j}$

$\quad = (-1)^{r+1}a^{r-1}(a+b) + (-1)^{r+1}a^r + (-1)^r a^{r-1}(b-a)$

$\qquad + \sum_{1\le j\le r-2} (-1)^{j+1}a^j(b-a)^{r-j}$

$\quad = (-1)^{r+1}3a^r + \sum_{1\le j\le r-2}(-1)^{j+1}a^j(b-a)^{r-j}$. Q.E.D.

Let $\mathcal{W}_r$ be the set of words of lenght r. Any $w \in \mathcal{W}_r$ can be written as $w = a^j u$, where u does not start with a. Let k be the number of occurence of a in w. Set $j(w) = j$, $k(w) = k$ and define the numbers $c(w)$ and $c^{\pm}(w)$ as follows.

(i) If $w = a^r$, set $c(w) = r$ and $c^{\pm}(w) = (-1)^{r+1}(r+1)$.

(ii) If $w = a^{r-1}b$, set $c(w) = r$ and $c^{\pm}(w) = (-1)^r(r-2)$.

(iii) Otherwise, set $c(w) = j$ and $c^{\pm}(w) = (-1)^{k+1}j$.

LEMMA 9: We have:

$$\square\,(ab^{r-1}) = \sum_{w\in\mathcal{W}_r} c(w)w$$

$$\nabla(ab^{r-1}) = \sum_{w\in\mathcal{W}_r} c^{\pm}(w)w$$

Proof: The first identity of Lemma 8 can be written as:

$$\Box\,(ab^{r-1}) = a^{r-1}b + \sum_{1\leq i\leq r} a^i(a+b)^{r-i}$$

.

Since $(a+b)^{r-i} = \sum_{u\in\mathcal{W}_{r-i}} u$, we thus get

$$\Box\,(ab^{r-1}) = a^{r-1}b + \sum_{1\leq i\leq r}\sum_{u\in\mathcal{W}_{r-i}} a^i u$$

The word $w = a^{j(w)}v$ belongs to $a^i\mathcal{W}_{r-i}$ for all $i\leq j(w)$. Therefore $\sum_{1\leq i\leq r} a^i(a+b)^{r-i} = \sum_{w\in\mathcal{W}_r} j(w)w$. Since $c(a^{r-1}b) = j(a^{r-1}b)+1$ and $c(w) = j(w)$ otherwise, the formula $\Box\,(ab^{r-1}) = \sum_{w\in\mathcal{W}_r} c(w)w$ is now proved.

Set $\overline{a} = -a$ and $\overline{b} = b$. For a word $w = c_1\dots c_n$, set $\overline{w} = \overline{c_1}\dots\overline{c_r}$ and for a general element $h = \sum c_w w$ in $\mathcal{H}$ set $\overline{h} = \sum c_w\overline{w}$. Since the involution $h\mapsto\overline{h}$ is a morphism relative to the concatenation product, it follows from Lemma 8 that:

$-\overline{\nabla(ab^{r-1})} = 3a^r + \sum_{1\leq i\leq r-2} a^i(b+a)^{r-i}$

$= a^r - a^{r-1}b + \sum_{1\leq i\leq r} a^i(b+a)^{r-i}$

It follows from the previous proof that $\sum_{1\leq i\leq r} a^i(a+b)^{r-i} = \sum_{w\in\mathcal{W}_r} j(w)w$. Therefore, one gets:

$-\overline{\nabla(ab^{r-1})} = a^r - a^{r-1}b + \sum_{w\in\mathcal{W}_r} j(w)w$.

Note that $\overline{w} = (-1)^{k(w)}\,w$ for all words w. Thus:

$\nabla(ab^{r-1}) = (-1)^{r+1}a^r - (-1)^r a^{r-1}b + \sum_{w\in\mathcal{W}_r} (-1)^{(1+k(w))}j(w)w$

Since $c^{\pm}(a^r) = (-1)^{r+1}(j(a^r+1)$, $c^{\pm}(a^{r-1}b) = (-1)^r(j(a^{r-1}b)-1)$ and $c^{\pm}(w) = (-1)^{1+k(w)}\,j(w)$ otherwise, the formula $\nabla(ab^{r-1}) = \sum_{w\in\mathcal{W}_r} c^{\pm}(w)w$ is now proved. Q.E.D.

Let $r\geq 2$ be an integer. Let Λ_r be the set of all $\mathbf{m} = (m_1,\dots m_k)\in\Lambda$ with $m_1+\dots+m_k = r$. For $\mathbf{m} = (m_1,\dots m_k)\in\Lambda_r$, set $k(\mathbf{m}) = k$ and define the integers $j(\mathbf{m})$, $b(\mathbf{m})$ and $b^{\pm}(\mathbf{m})$ as follows.

(i) If $m_1 = m_2 = \dots = m_r = 1$, set $j(\mathbf{m}) = r$, $b(\mathbf{m}) = r$ and $b^{\pm}(\mathbf{m}) = (-1)^{r+1}(r+1)$. Otherwise, let $j(\mathbf{m}) = j$ be the index such that $m_1 = m_2 = \dots = m_{j-1} = 1$ and $m_j\geq 2$.

(ii) If $m_1 = m_2 = \dots = m_{r-2} = 1$ and $m_{r+1} = 2$, set $b(\mathbf{m}) = r$ and $b^{\pm}(\mathbf{m}) = (-1)^r(r-2)$.

(iii) Otherwise, set $b(\mathbf{m}) = j$ and $b^{\pm}(\mathbf{m}) = (-1)^{k+1}j$, where $j = j(\mathbf{m})$ and $k = k(\mathbf{m})$.

THEOREM 10: *For $r\geq 2$, we have:*

$$\zeta(r) = \sum_{\mathbf{m}\in\Lambda_r} b(\mathbf{m})L_{\mathbf{m}}(1/2)$$

$$\zeta(r) = \sum_{\mathbf{m} \in \Lambda_r} b^{\pm}(\mathbf{m}) L_{\mathbf{m}}(\rho^{\pm 1})$$

Proof: It is clear that $c(w)$ and $c^{\pm}(w)$ vanish if $w \notin \mathcal{W}^+$. Therefore it follows from Theorem 7 and Lemma 9 that

$$\zeta(r) = \sum_{w \in \mathcal{W}_r^+} c(w) L_w(1/2)$$

$$\zeta(r) = \sum_{w \in \mathcal{W}_r^+} c^{\pm}(w) L_w(\rho^{\pm 1})$$

where $\mathcal{W}_r^+ = \mathcal{W}^+ \cap \mathcal{W}_r$. Note that the map λ of section 1.4 provides a bijection $\lambda : \mathcal{W}_r^+ \to \Lambda_r$. It is easy to check that $j(w) = j(\lambda(w))$ and $k(w) = k(\lambda(w))$ for all $w \in \mathcal{W}_r^+$, and therefore

$$c(w) = b(\lambda(w)) \quad \text{and} \quad c^{\pm}(w) = b^{\pm}(\lambda(w))$$

for all $w \in \mathcal{W}_r^+$. Therefore Theorem 10 is proved. Q.E.D.

For $1 \leq i \leq r-1$, set

$$C_i = \{\mathbf{n} = (n_1 \dots n_k) \in \mathbf{Z}^r | 0 < n_1 < ... < n_i \leq n_{i+1} \leq \dots \leq n_k\}.$$

Also, set $C_r = C_{r-1}$.

LEMMA 11: *We have:*

$$\zeta(r) = \sum_{1 \leq i \leq r} \sum_{\mathbf{n} \in C_i} \frac{2^{-n_r}}{n_1 n_2 \dots n_r}$$

.

Proof: Set $c = a + b$. For any word $w = d_1 \dots d_r$ into the letters a, b and c, let C_w be the set of all $\mathbf{n} = (n_1 \dots n_k) \in \mathbf{Z}^r$ satisfying the following property:

$$0 \mathcal{R}_1 n_1 \mathcal{R}_2 n_2 \dots \mathcal{R}_r n_r$$

where $\mathcal{R}_i$ stands for the symbol $<$ if $d_i = a$, $\mathcal{R}_i$ stands for the symbol $=$ if $d_i = b$ and $\mathcal{R}_i$ stands for the symbol $\leq$ if $d_i = c$. So if w is a word into the letters a, b and c, we get $L_w(z) = \sum_{\mathbf{n} \in C_w} \frac{z^{n_r}}{n_1 \dots n_r}$.

By Lemma 8, we have:

$$\square\,(ab^{r-1}) = a^{r-1}(a+b) + \sum_{1 \leq j \leq r-1} a^j (a+b)^{r-j}$$

Since $C_i = C_{a^i c^{r-i}}$ for all $i \leq r-1$, and $C_r = C_{a^{r-1}c}$, we get

$$L_{\square(ab^{r-1})}(z) = \sum_{1\leq i\leq r} \sum_{\mathbf{n}\in C_i} \frac{z^{-n_r}}{n_1 n_2 \dots n_r}$$

Therefore, Lemma 11 follows from Theorem 7. Q.E.D.

Set

$$C = \{\mathbf{n} = (n_1 \dots n_r) \in \mathbf{Z}^r | 0 < n_1 \leq n_2 \dots \leq n_r\}.$$

For $\mathbf{n} = (n_1 \dots n_r) \in C$, define the number $a(\mathbf{n})$ as follows. If we have $0 < n_1 < \dots < n_{r-1}$ set $a(\mathbf{n}) = k$. Otherwise, there exist an index $i \leq r-2$ such that $0 < n_1 < n_2 \dots n_i = n_{i+1}$. In such a case, set $a(\mathbf{n}) = i$. Note that $a(\mathbf{n})$ does not depend on the last component n_r of $\mathbf{n}$, and the function $\mathbf{n} \mapsto a(\mathbf{n})$ takes value in the set $\{1, 2, \dots, r-2, r\}$

COROLLARY 12: *We have:*

$$\zeta(r) = \sum_{\mathbf{n}\in C} a(\mathbf{n}) \frac{2^{-n_r}}{n_1 n_2 \dots n_r}$$

.

Proof: It is easy to check that $a(\mathbf{n})$ is precisely the number of indices i, $1 \leq i \leq r$ such that $\mathbf{n}$ belongs to C_i. Therefore the formula of Corollary 7 follows from Lemma 11. Q.E.D.

Examples: For $r = 2$, then $a((m, n)) = 2$ for all $(m, n) \in C$. Therefore, we get

$$\zeta(2) = 2 \sum_{0<m\leq n} \frac{2^{-n}}{nm} = 2L_2(1/2) + \log^2 2$$

Accordingly to **[C]**, this formula is due to Euler.

For $r = 5$, we have $a((k, l, m, n, p)) = 1$ if $k = l$, $a((k, l, m, n, p)) = 2$ if $k < l = m$ $a((k, l, m, n, p)) = 3$ if $k < l < m = n$ and $a((k, l, m, n, p)) = 5$ if $k < l < m < n$. Therefore, we get the following expansion for $\zeta(5)$

$$\sum_{0<l\leq m\leq n\leq p} \frac{2^{-p}}{l^2 mnp} + 2 \sum_{0<l<m\leq n\leq p} \frac{2^{-p}}{lm^2 np} + 3 \sum_{0<l<m<n\leq p} \frac{2^{-p}}{lmn^2 p} + 5 \sum_{0<k<l<m<n\leq p} \frac{2^{-p}}{klmnp}$$

4. Geometric interpretation of Theorem 7.

Theorem 7 provides a combinatorial way to exress any polyzeta value as the value of a polylogarithmic functions at $1/2$ or at $\rho^{\pm 1}$. The combinatorics seem very intricate: e.g. the explicit formulas for zeta values $\zeta(r)$ of Section 3 are difficult to extend for general polyzeta values $\zeta(r_1, \dots, r_k)$.

In this section, Theorem 7 is reformulated in terms of simple geometric notions.

(4.1) First, the free pro-algebraic group on two generators Γ and its Lie algebra $\mathfrak{g}$ are defined.

Let F be the free Lie $\mathbf{Q}$-algebra with two generators α and β, let C^nF be its central descending series and set $\mathfrak{g} = \varprojlim F/C^nF$. Since F/C^nF is a nilpotent Lie algebra, the Campbell-Hausdorf series defines a structure of algebraic group on F/C^nF, denoted by Γ_n. Then $\Gamma = \varprojlim \Gamma_n$ is a proalgebraic group (an alternative definition of Γ is given in the introduction). As pro-algebraic varieties, $\mathfrak{g}$ and Γ are identical, and the corresponding isomorphism is denoted by $\exp : \mathfrak{g} \to \Gamma$.

Let $F = \oplus_{n\geq 1} F_n$ be the grading of F such that $F_1 = \mathbf{Q}\alpha \oplus \mathbf{Q}\beta$. Then we have $\mathfrak{g} = \prod_{n\geq 1} F_n$, so any $x \in \mathfrak{g}$ can be written as the series $x = \sum_{i>0} x_i$ where $x_i \in F_i$. The multiplicative group $\mathbf{Q}^*$ acts linearly on $\mathfrak{g}$ as follows: $t.x = \sum_{i>0} t^i\, x_i$, for any $t \in \mathbf{Q}^*$.

LEMMA 13: Let $\Phi : \mathfrak{g} \to \mathfrak{g}$ be a morphism of pro-algebraic varieties. Assume that Φ is $\mathbf{Q}^*$-invariant and that $\mathrm{d}\Phi_0$ is invertible, then Φ is an isomorphism.

Proof: One can assume that $\mathrm{d}\Phi_0$ is the identity. Then choose a basis of F consisting of homogenous elements $(e_n)_{n\geq 1}$ with $d_n \leq d_m$ if $n < m$, where d_n is the degree of e_n. Accordingly, we have $\Phi(\sum_{n\geq 1} x_n e_n) = \sum_{n\geq 1} \Phi_n(x) e_n$, where each Φ_n is a polynomial in $x = (x_1, x_2, \ldots)$. By hypothesis, the linear part of $\Phi_n(x)$ is x_n and for any monomial $x_{i_1} \ldots x_{i_k}$ occuring in $\Phi_n(x)$ we have $d_{i_1} + \ldots d_{i_k} = d_n$. It follows that $\Phi_n(x) - x_n$ depends only on $x_1, \ldots, x_{n-1}$, so we can write: $\Phi_n(x) = x_n + H_n(x_1, \ldots, x_{n-1})$. Since Φ is triangular, it is an isomorphism.

(4.2) There is an isomorphism of Hopf algebras $\mathbf{Q}[\Gamma] \simeq \mathcal{H}$, see **[P]**. A natural group isomorphism $\psi : \Gamma \to \operatorname{Spec} \mathcal{H}$ is now described.

For $t \in \mathbf{Q}$, define two points $\phi_a(t)$ and $\phi_b(t)$ in $\operatorname{Spec} \mathcal{H}$ as follows. Since words $w \in \mathcal{W}$ are functions on $\operatorname{Spec} \mathcal{H}$, one needs to evaluate w at the points $\phi_a(t)$ and $\phi_b(t)$. The rule is as follows:

$w(\phi_a(t)) = t^n/n!$ if $w = a^n$ and $w(\phi_a(t)) = 0$ if b occurs in w.

$w(\phi_b(t)) = t^n/n!$ if $w = b^n$ and $w(\phi_b(t)) = 0$ if a occurs in w.

Then it is clear that $\phi_a(t)$ and $\phi_b(t)$ are two one-parameter groups in $\operatorname{Spec} \mathcal{H}$. Since Γ is freely generated (as a proalgebraic group) by the two one-parameter groups $\exp \mathbf{Q}\alpha$ and $\exp \mathbf{Q}\beta$, the isomorphism ψ is prescribed by the requirements $\psi(\exp t\alpha) = \phi_a(t)$ and $\psi(\exp t\beta) = \phi_b(t)$ for all $t \in \mathbf{Q}$.

(4.3) From now on, we identify $\mathcal{H}$ and $\mathbf{Q}[\Gamma]$. Since $\mathcal{H} = \mathbf{Q}[\Gamma]$, any function $\phi \in \mathbf{Q}[\Gamma]$ defines a polylogarithmic function $L_\phi(z)$ and the polyzeta value $\zeta(\phi)$ and the numbers $\zeta^{\pm}(\phi)$.

(4.4) The maps $\sigma, \tau : \mathcal{H} \to \mathcal{H}$ are anti-isomorphisms of Hopf algebras, and therefore they induce two anti-isomorphisms of Γ and of its Lie algebra $\mathfrak{g}$. These are again denoted by σ and τ. They are uniquely characterized by the requirements:

$$\sigma \exp t\alpha = \exp t\beta \text{ and } \sigma \exp t\beta = \exp t\alpha$$

$$\tau \exp t\alpha = \exp t(\alpha+\beta) \text{ and } \tau \exp t\beta = \exp -t\alpha,$$

for all $t \in \mathbf{Q}$. We have $\sigma^2(g) = g$ and $\tau^3(g) = g^{-1}$ for any $g \in G$.

(4.5) Since $\mathcal{H} = \mathbf{Q}[\Gamma]$, the maps $\square$, ∇ occuring in Theorem 7 are now identified with some algebra morphisms $\square, \nabla : \mathbf{Q}[\Gamma] \to \mathbf{Q}[\Gamma]$.

LEMMA 14: *Let $\phi \in \mathbf{Q}[\Gamma]$. Then for any $g \in \Gamma$, we have*

$$\square\ \phi(g) = \phi(g\sigma(g)) \quad \text{and} \quad \nabla\, \phi(g) = \phi(g\tau(g))$$

Proof: Using their definitions, $\square$ and ∇ are the composition of the following maps:

$$\Gamma \overset{\text{diag}}{\to} \Gamma\times\Gamma \overset{id\times\sigma}{\longrightarrow} \Gamma\times\Gamma \overset{\mu}{\to} \Gamma$$

$$\Gamma \overset{\text{diag}}{\to} \Gamma\times\Gamma \overset{id\times\tau}{\longrightarrow} \Gamma\times\Gamma \overset{\mu}{\to} \Gamma$$

where $\text{diag}(g) = (g, g)$ and $\mu(g_1, g_2) = g_1.g_2$. Therefore we have
$\square\ \phi(g) = \phi(g.\sigma(g))$ and $\nabla\, \phi(g) = \phi(g.\tau(g))$.

(4.6) In this subsection, the symmetric space associated with σ is defined.

Set $\mathfrak{k} = \{x \in \mathfrak{g} | \sigma(x) = -x\}$, $K = \{g \in \Gamma | \sigma(g) = g^{-1}\}$, $\mathfrak{p} = \{x \in \mathfrak{g} | \sigma(x) = x\}$ $P = \{p \in \Gamma | \sigma(g) = g\}$.

Since σ is an anti-involution, K is a subgroup in Γ and $\mathfrak{k}$ is its a Lie algebra. Obviously we have $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Since Γ is a pro-unipotent group, we have $K = \exp\,\mathfrak{k}$, $P = \exp\,\mathfrak{p}$ and $\Gamma = P.K$. So any element $g \in \Gamma$ can be written as $g = p.k$, where $k \in K$ and $p \in P$. Moreover $P \simeq \Gamma/K$ is a symmetric space.

LEMMA 15: *Let $\phi \in \mathbf{Q}[\Gamma]$. Then for any $g = p.k \in \Gamma$, we have*

$$\square\ \phi(p.k) = \phi(p^2)$$

In particular, $\zeta(\phi) = 0$ if $\phi|_P \equiv 0$.

Proof: This follows from Lemma 14 and Theorem 7.

(4.7) Note that τ is not an involution, but an "ordrer three" anti-isomorphism, i.e. $\tau^3(g) = g^{-1}$. In this sub-section, we introduce a space Q which is analogous to a symmetric space.

Set $\mathfrak{l} = \{x \in \mathfrak{g} | \tau(x) = -x\}$, $L = \{g \in \Gamma | \tau(g) = g^{-1}\}$, $\mathfrak{q} = \{x \in \mathfrak{g} | \tau^2(x) - \tau(x) + x = 0\}$. Also define Q as the image of the map $g \in \Gamma \mapsto g\tau(g)$. Note that L is a subgroup with Lie algebra $\mathfrak{l}$.

LEMMA 16: The subset Q is a closed subvariety of Γ and the natural map: $Q \times L \to \Gamma$, $(q, l) \mapsto ql$ is a isomorphism of pro-algebraic varieties.

Proof It is easy to prove that $\Gamma = \exp \mathfrak{q}.L$. For $g = \exp q.l$, with $q \in \mathfrak{q}$ and $l \in L$, we have $g\tau(g) = \exp q \exp \tau(q)$, therefore Q is the set all $\exp q \exp \tau(q)$ for $q \in \mathfrak{q}$.

Let $\Phi : \mathfrak{q} \oplus \mathfrak{l} \to \Gamma$ be define by $\Phi(q, l) = \exp q \exp \tau(q) \exp l$. Note that $\mathrm{d}\Phi_0$ is the linear map from $\mathfrak{g}$ to $\mathfrak{g}$ which is the identity on $\mathfrak{l}$ and whose restriction to $\mathfrak{q}$ is $1 + \tau$. Therefore, $\mathrm{d}\Phi_0$ is invertible. By Lemma 13 that Φ is an isomorphism and Lemma 16 follows easily. Q.E.D.

The definition of Q is slighty more complicated than the definition of P because $Q \neq \exp \mathfrak{q}$. However, the map $\mathfrak{q} \to Q$, $q \mapsto \exp q\ \exp \tau(q)$ is an isomorphism from $\mathfrak{q}$ to Q. Any element $g \in \Gamma$ can be written as $g = q.l$, where $l \in L$ and $q \in Q$

LEMMA 17: *Let $\psi \in \mathbf{Q}[\Gamma]$. Then for any $g = q.l \in \Gamma$, we have*

$$\nabla \psi(q.l) = \psi(q\tau(q))$$

In particular, $\zeta(\psi) = 0$ if $\psi|_Q \equiv 0$.

Proof: This follows from Lemma 14 and Theorem 7.

(4.8) Let $\phi \in \mathbf{Q}[P]$ be a rational function on P. The notations $\zeta(\phi)$ and $L_\phi(z)$ are now defined. Set $\zeta(\phi) = \zeta(\hat{\phi})$ where $\hat{\phi}$ is any function on Γ extending ϕ. By Lemma 15, $\zeta(\phi)$ is well defined. Since $P \simeq \Gamma/K$, ϕ can be uniquely extended to a right K-invariant function Φ on Γ. Then set $L_\phi(z) = L_\Phi(z)$.

Similarly, for $\psi \in \mathbf{Q}[Q]$, the notations $\zeta^\pm(\psi)$ and $L_\psi(z)$ are defined as follows. Set $\zeta^\pm(\psi) = \zeta^\pm(\hat{\psi})$ where $\hat{\psi}$ is any function on Γ extending ψ. By Lemma 17, $\zeta^\pm(\psi)$ is well defined. By Lemma 16, we have $Q \simeq \Gamma/L$, therfore ψ can be uniquely extended to a right L-invariant function Ψ on Γ. Then set $L_\psi(z) = L_\Psi(z)$.

Define the algebra morphisms $\Box : \mathbf{Q}[P] \to \mathbf{Q}[P]$ and $\nabla : \mathbf{Q}[Q] \to \mathbf{Q}[Q]$ by

$$\Box\, \phi(p) = \phi(p^2), \quad \text{for} \phi \in \mathbf{Q}[P]$$

$$\nabla\psi(q) = \psi(q\tau(q)), \quad \text{for} \psi \in \mathbf{Q}[Q]$$

These operators are simply the restrictions to P and to Q of the already defined operators $\Box, \nabla : \mathbf{Q}[\Gamma] \to \mathbf{Q}[\Gamma]$. So using the same notations should not bring confusions.

THEOREM 18: *For any $\phi \in \mathbf{Q}[P]$ and $\psi \in \mathbf{Q}[Q]$, we have*

$$\zeta(\psi) = L_{\Box(\psi)}(1/2) \quad \text{and} \quad \zeta^\pm(\psi) = L_{\nabla(\psi)}(\rho^\pm)$$

Proof: It follows immediately from Theorem 7, and Lemmas 15 and 17.

5. Other expressions for zeta values.

In this section, we follow a suggestion of W. Zudilin

(5.1) Theorem 7 shows that any polyzeta value is the value of a polylogarithmic function at $1/2$ or at $\rho^\pm$. However, there is a much more simple way to express the zeta values $\zeta(r)$ as a value of polylogarithmic functions at $1/2$ or at $\rho^\pm$, see Corollary 20. It is surprizing that the two approaches give different expressions, except for $\zeta(2)$. Moreover, this simpler approach does not generalize to polyzeta values.

(5.2) Let $\sigma' : \mathcal{H} \to \mathcal{H}$ be the linear map defined as follows. Set $\sigma'(a) = -a$ and $\sigma'(b) = a + b$. For a word $w = c_1 \dots c_n$, where $c_i \in \{a, b\}$ are letters, set $\sigma'(w) = \sigma'(c_1) \dots \sigma'(c_n)$. It is easy to prove that σ' is an algebra morphism relative to the schuffle product and that $\sigma'(\mathcal{H}^+) = \mathcal{H}^+$.

LEMMA 19 *For any $h \in \mathcal{H}^+$, we have:*

$$L_h(-1) = L_{\sigma'(h)}(1/2) \text{ and } L_h(\overline{\rho}) = L_{\sigma'(h)}(\rho).$$

Proof: One can assume that h is a word $w = c_1 \dots c_n \in \mathcal{W}^+$. Set $F = \{z \in \mathbf{C} | \, |z| \leq 1 \, \text{and} \, Im z \leq 1/2\}$ and choose a two paths $\gamma, \gamma_- : [0,1] \to F$ with $\gamma(0) = \gamma_-(0) = 0$, $\gamma(1) = -1$ and $\gamma_-(1) = \overline{\rho}$. By Proposition 2, we have:

$$L_w(-1) = \int_{\Delta_n} \gamma^*\omega_{c_1}(x_1)\gamma^*\omega_{c_2}(x_2)\dots\gamma^*\omega_{c_n}(x_n)$$

$$L_w(\overline{\rho}) = \int_{\Delta_n} \gamma_-^*\omega_{c_1}(x_1)\gamma_-^*\omega_{c_2}(x_2)\dots\gamma_-^*\omega_{c_n}(x_n)$$

For $z \in F$, set $S'(z) = z/(z-1)$ and set $\delta = S' \circ \gamma$ and $\delta_- = S' \circ \gamma_-$. We have $S'^*\omega_{\sigma'(c)} = \omega_c$ for $c = a$ or b, and therefore we get $\gamma^*\omega_c = \delta_-^*\omega_{\sigma'(c)}$ and $\gamma_-^*\omega_c = \delta_-^*\omega_{\sigma'(c)}$ It follows that

$$L_w(-1) = \int_{\Delta_n} \delta^*\omega_{\sigma'(c_1)}(x_1)\delta^*\omega_{\sigma'(c_2)}(x_2)\dots\delta^*\omega_{\sigma'(c_n)}(x_n)$$

$$L_w(\overline{\rho}) = \int_{\Delta_n} \delta_-^*\omega_{\sigma'(c_1)}(x_1)\delta_-^*\omega_{\sigma'(c_2)}(x_2)\dots\delta_-^*\omega_{\sigma'(c_n)}(x_n)$$

We have $S'(F) = F$, $S'(0) = 0$, $S'(-1) = 1/2$ and $S'(\overline{\rho}) = \rho$. Therefore δ is a path from 0 to 1/2 and δ_- is a path from o to ρ. Thus, these integrals can be identified by Proposition 2, and we get $L_h(-1) = L_{\sigma'(h)}(1/2)$, and $L_h(\overline{\rho}) = L_{\sigma'(h)}(\rho)$. Q.E.D.

LEMMA 20: Let $r \geq 1$. We have

$$\zeta(r+1) = \frac{-1}{1-2^{-r}} L_{r+1}(-1)$$

$$\zeta(r+1) = \frac{1}{(1-2^{-r})(1-3^{-r})} \left[L_{r+1}(\rho) + L_{r+1}(\overline{\rho})\right]$$

Proof: For each positive integer a, set $\delta_a(n) = 1$ if a divides n and $\delta_a(n) = 0$ otherwise.

From the formula $(-1)^n = -\delta_1(n) + 2\delta_2(n)$, we get

$$\begin{aligned} L_{r+1}(-1) &= \sum_{n>0} (-1)^n/n^{r+1} \\ &= -\sum_{n>0} 1/n^{r+1} + 2\sum_{n>0} 1/(2n)^{r+1} \\ &= (-1+2^{-r})\,\zeta(r) \end{aligned}$$

from which the first formula follows.

From the formula $\rho^n + \overline{\rho}^n = \delta_1(n) - 2\delta_2(n) - 3\delta_3(n) + 6\delta_6(n)$ we get

$$\begin{aligned} L_{r+1}(\rho) + L_{r+1}(\overline{\rho}) &= \sum_{n>0} [\rho^n + \overline{\rho}^n]/n^{r+1} \\ &= \sum_{n>0} 1/n^{r+1} - 2\sum_{n>0} 1/(2n)^{r+1} - 3\sum_{n>0} 1/(3n)^{r+1} + 6\sum_{n>0} 1/(6n)^{r+1} \\ &= (1-2^{-r}-3^{-r}+6^{-r})\,\zeta(r+1) \\ &= (1-2^{-r})(1-3^{-r})\,\zeta(r+1), \end{aligned}$$

from which the second formula follows.

COROLLARY 21: *Let $r \geq 1$. We have*

$$\zeta(r+1) = \frac{2^r}{(2^r-1)} L_{a(a+b)^r}(1/2)$$

$$\zeta(r+1) = \frac{6^r}{(2^r-1)(3^r-1)} \left[L_{ab^r}(\rho) - L_{a(a+b)^r}(\rho)\right]$$

Proof: The corollary follows from Lemmas 18 and 19.

Examples:

$$\zeta(2) = 2 \sum_{0<m\leq n} \frac{2^{-n}}{nm}$$

$$\zeta(3) = 8/7 \sum_{0<l\leq m\leq n} \frac{2^{-n}}{lnm}$$

The expression for $\zeta(2)$ is the same as in section 3. However for all other zeta values $\zeta(r)$ with $r \geq 3$, the expressions are different: e.g., the formula of corollary 20 uses non integral coefficients. Moreover, this simpler approach only concerns zeta values but not the polyzeta values.

6. Conclusion: Polyzeta values are mixed periods [**G**], [**T**], [**Z**]. In the philosophy of motives, there is a proalgebraic group G and periods should be regular functions on G: more precisely, the algebra of periods should be a $\overline{\mathbf{Q}}$-form of $\mathbf{C}[G]$, modulo conjectures. Here polyzeta values are attached to some symmetric space. Does there is a motivic interpretation of this construction? Note however that the map $\mathbf{Q}[P] \to \mathcal{Z}$ is not injective.

Bibliography:

[**BBP**] Bailey, David, Borwein, Peter , and Plouffe, Simon: On the Rapid Computation of Various Polylogarithmic Constants. Mathematics of Computation 66 (1997) 903-913.

[**C**] P. Cartier: Polylogarithmes, polyzetas et groupes pro-unipotents. *Séminaire Bourbaki 2000-2001* Astérisque 282 (2002) 137-173

[**E**] Ecalle, Jean: La libre génération des multizêtas et leur décomposition canonici-explicite en irréductibles, Preprint (1999).

[**G**] Goncharov, Alexander: Multiple ζ-values, Galois groups, and geometry of modular varieties. *European Congress of Mathematics (Barcelona, 2000).* Progr. Math. 201 (2001) 361-392.

[**H**] M.E. Hoffman: Multiple harmonic series, J. of Algebra 194 (1992) 275-290

[**L**] Lewin, Leonard: *Polylogarithms and Associated Functions.* North-Holland (1981)

[**MP**] Minh, Hoang Ngoc, and Petitot, Michel Lyndon: Words, polylogarithms and the Riemann ζ function. Discrete Math. 217 (2000) 273–292.

[**O**] Oesterlé, Joseph: Polylogarithmes. *Séminaire Bourbaki.* Astérisque 216 (1993) 49-67.

[**P**] Pianzola, Arturo: Free group functors. J. Pure Appl. Algebra 140 (1999) 289-297.

[**T**] Terasoma, Tomohide: Mixed Tate motives and multiple zeta values. Invent. Math. 149 (2002) 339-369.

[**Z**] Zagier, Don: Values of zeta functions and their applications. First European Congress of Mathematics (Paris, 1992), Birkhäuser, Basel, Progr. Math., 120 (1994) 497-512.

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