k-DISTANT CROSSINGS AND NESTINGS OF MATCHINGS AND PARTITIONS

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ABSTRACT. We define and consider k-distant crossings and nestings for matchings and set partitions, which are a variation of crossings and nestings in which the distance between vertices is important. By modifying an involution of Kasraoui and Zeng (Electronic J. Combinatorics 2006, research paper 33), we show that the joint distribution of k-distant crossings and nestings is symmetric. We also study the numbers of k-distant noncrossing matchings and partitions for small k, which are counted by well-known sequences, as well as the orthogonal polynomials related to k-distant noncrossing matchings and partitions. We extend Chen et al.'s r-crossings and enhanced r-crossings.

1. INTRODUCTION

A *(set) partition* of $[n] = \{1, 2, \ldots, n\}$ is a set of disjoint subsets of $[n]$ whose union is $[n]$. Each element of a partition is called a *block*. We will write a partition as a sequence of blocks, for instance, $\{1, 4, 8\}\{2, 5, 9\}\{3\}\{6, 7\}$. Let Π_n denote the set of partitions of $[n]$.

Let π be a partition of $[n]$. A *vertex* of π is an integer $i \in [n]$. An *edge* of π is a pair (i, j) of vertices satisfying either (1) $i < j$, and i and j are in the same block with no vertex between them in that block, or (2) $i = j$ and the block containing i has no other vertex. Thus when we arrange vertices of $\pi = \{1, 5\}\{2, 4, 9\}\{3\}\{6, 12\}\{7, 10, 11\}\{8\}$, in a line in increasing order and draw edges we get Figure 1.

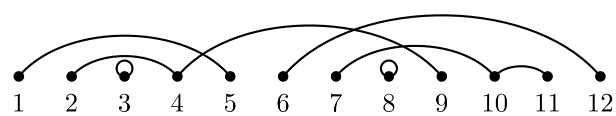

FIGURE 1. Diagram for $\{1, 5\}\{2, 4, 9\}\{3\}\{6, 12\}\{7, 10, 11\}\{8\}$.

A vertex v of π is called an *opener* (resp. *closer*) if v is the smallest (resp. largest) element of a block consisting of at least two integers. A vertex v is called a *singleton* if v itself makes a block. A vertex v is called a *transient* if there are two edges connected to v. Let $\mathcal{O}(\pi)$ (resp. $\mathcal{C}(\pi)$, $\mathcal{S}(\pi)$, $\mathcal{T}(\pi)$) be the set of openers (resp. closers, singletons, transients) of π. Let $\text{type}(\pi) = (\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi), \mathcal{T}(\pi))$ and $\text{type}'(\pi) = (\mathcal{O}(\pi), \mathcal{C}(\pi), \mathcal{S}(\pi) \cup \mathcal{T}(\pi))$. For the partition in Figure 1, the type of π is $\text{type}(\pi) = (\{1, 2, 6, 7\}, \{5, 9, 11, 12\}, \{3, 8\}, \{4, 10\})$.

A *(complete) matching* is a partition without singletons or transients; this is the same thing as a partition in which all blocks have size 2.

Now we can define the main object of our study.

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Definition. Let k be a nonnegative integer. A *k-distant crossing* of π is a pair of edges (i_1, j_1) and (i_2, j_2) of π satisfying $i_1 < i_2 \le j_1 < j_2$ and $j_1 - i_2 \ge k$. A *k-distant nesting* of π is a set of two edges (i_1, j_1) and (i_2, j_2) of π satisfying $i_1 < i_2 \le j_2 < j_1$ and $j_2 - i_2 \ge k$.

Let $\mathrm{dcr}_k(\pi)$ (resp. $\mathrm{dne}_k(\pi)$) denote the number of k-distant crossings (k-distant nestings) in π. Thus $\mathrm{dcr}_1(\pi)$ is the number of usual crossings of π.

For example, in the partition in Figure 1, the edges $(4, 9)$ and $(6, 12)$ form a 3-distant crossing (as well as an i-distant crossing for $i = 0, 1, 2$), the edges $(1, 5)$ and $(2, 4)$ form a 2-distant nesting, the edges $(2, 4)$ and $(4, 9)$ form a 0-distant crossing, and the edges $(7, 10)$ and $(8, 8)$ form a 0-distant nesting. That partition has $\mathrm{dcr}_0(\pi) = 5$, $\mathrm{dcr}_2(\pi) = 2$, and $\mathrm{dne}_2(\pi) = 2$.

Kasraoui and Zeng [5] found an involution $\varphi : \Pi_n \to \Pi_n$ such that $\mathrm{type}(\varphi(\pi)) = \mathrm{type}(\pi)$ and $\mathrm{dcr}_1(\varphi(\pi)) = \mathrm{dne}_1(\pi), \mathrm{dne}_1(\varphi(\pi)) = \mathrm{dcr}_1(\pi)$. Modifying this involution, for $k \ge 0$, we find an involution $\varphi_k : \Pi_n \to \Pi_n$ such that $\mathrm{dcr}_k(\varphi_k(\pi)) = \mathrm{dne}_k(\pi)$, $\mathrm{dne}_k(\varphi_k(\pi)) = \mathrm{dcr}_k(\pi)$ and $\mathrm{type}(\varphi_k(\pi)) = \mathrm{type}(\pi)$ if $k \ge 1$; $\mathrm{type}'(\varphi_k(\pi)) = \mathrm{type}'(\pi)$ if $k = 0$.

Noncrossing partitions and matchings are interesting and pervasive objects that arise frequently in diverse areas of mathematics; see [10] and [11] and the references therein for an introduction to noncrossing partitions. A partition π is called *k-distant noncrossing* if π has no k-distant crossing. Let $NCM_k(n)$ denote the number of k-distant noncrossing matchings of $[n]$. Let $NCP_k(n)$ denote the number of k-distant noncrossing partitions of $[n]$.

Table 1 and Table 2 show $NCM_k(n)$ and $NCP_k(n)$ for small values of n and k. We use $k = \infty$ to indicate that i-distant crossing is allowed for any positive integer i, so that $NCM_\infty(n)$ and $NCP_\infty(n)$ equal the total number of matchings of $[2n]$ and partitions of $[n]$, respectively. A matching or partition cannot have a k-distant crossing for $k > n - 3$, so for fixed n, $NCM_k(n)$ and $NCP_k(n)$ will "converge" to the number of matchings and number of partitions, respectively; for readability we omit those numbers in the tables. The $n = 0$ column is all 1's for both tables, of course.

It is well known that noncrossing matchings of $[2n]$ and noncrossing partitions of $[n]$ are counted by the Catalan number C_n. Thus $NCM_0(2n) = NCM_1(2n) = NCP_1(n) = C_n$. We will show that $NCM_2(2n) = s_n$ and $NCP_0(n) = M_n$, where s_n and M_n are the little Schröder numbers (A001003 in [12]) and the Motzkin numbers (A001006 in [12]) respectively. We will also find the generating functions for $NCP_2(n)$ and $NCM_3(2n)$.

Throughout this paper we will frequently refer to sequences in the Online Encyclopedia of Integer Sequences [12] using their "A number"; we will usually omit the citation to [12] and consider it understood that things like "A000108" are a reference to the corresponding sequence in the OEIS.

The rest of this paper is organized as follows. In section 2, we modify Kasraoui and Zeng's involution to prove the joint distribution of k-distant crossings and nestings is symmetric. In section 3, we review a bijection between partitions and Charlier diagrams. In section 4 and section 5, we study the number of k-distant noncrossing matchings and partitions, and, in section 6, we consider the orthogonal polynomials related to these numbers. In section 7, we extend r-crossings and enhanced r-crossings of Chen et al. [1]. We include an appendix of Sage code used to compute the entries of Table 1 and Table 2.

2. Modification of the involution of Kasraoui and Zeng

Kasraoui and Zeng [5] found an involution $\varphi : \Pi_n \to \Pi_n$ such that $\mathrm{dcr}_1(\varphi(\pi)) = \mathrm{dne}_1(\pi)$, $\mathrm{dne}_1(\varphi(\pi)) = \mathrm{dcr}_1(\pi)$ and $\mathrm{type}(\varphi(\pi)) = \mathrm{type}(\pi)$. In this section, for fixed

$k \backslash n$	2	4	6	8	10	12	14	16	18	20
1	1	2	5	14	42	132	429	1430	4862	16796
2		3	11	45	197	903	4279	20793	103049	518859
3			14	71	387	2210	13053	79081	488728	3069007
4			15	91	581	3906	27189	194240	1416168	10494328
5				102	753	5752	45636	372360	3101523	26266917
6				105	873	7541	66690	607128	5657520	53631564
7					930	8985	88450	885394	9067611	94719138
8					945	9885	107847	1187376	13233511	150234570
9						10290	122115	1476948	17933348	219754737
10						10395	130515	1715475	22701570	300724081
11							134190	1881495	26969370	386669322
12							135135	1975995	30306045	468680940
13								2016630	32546745	538581120
14								2027025	33794145	591287445
15									34324290	625810185
16									34459425	652702050
17										644729085
18										654729075
∞	1	3	15	105	945	10395	135135	2027025	34459425	654729075

TABLE 1. k-distant noncrossing matchings. The $k = 0$ row is omitted because, as matchings have no transient vertices, the $k = 0$ row is the same as $k = 1$ row; both, of course, are counted by the Catalan numbers (A000108). The $k = 2$ row is the little Schröder numbers (A001003).

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12
0	1	2	4	9	21	51	127	323	835	2188	5798	15511
1			5	14	42	132	429	1430	4862	16796	58786	208012
2				15	51	188	731	2950	12235	51822	223191	974427
3					52	201	841	3726	17213	82047	400600	1993377
4						203	872	4037	19796	101437	537691	2926663
5							877	4125	20802	110950	618777	3575688
6								4140	21095	114663	657698	3943294
7									21147	115772	673019	4118232
8										115975	677693	4187838
9											678570	4209457
10												4213597
∞	1	2	5	15	52	203	877	4140	21147	115975	678570	4213597

TABLE 2. k-distant noncrossing partitions. The $k = 0, 1,$ and 2 rows are counted by Motzkin numbers (A001006), the Catalan numbers, and A007317, respectively.

$k \geq 0$, we find an involution $\varphi_k : \Pi_n \to \Pi_n$ such that $\mathrm{dcr}_k(\varphi_k(\pi)) = \mathrm{dne}_k(\pi)$ and $\mathrm{dne}_k(\varphi_k(\pi)) = \mathrm{dcr}_k(\pi)$. Since complete matchings can be thought of as set partitions with blocks all of size two, this involution will also show that the distribution of dcr_k and dne_k is symmetric.

We will follow Kasraoui and Zeng's notations. We will identify a partition π to its diagram as shown in Figure 1.

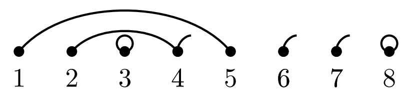

FIGURE 2. The 8-th trace $T_8(\pi)$ of π in Figure 1. The vacant vertices are $4, 6$ and 7.

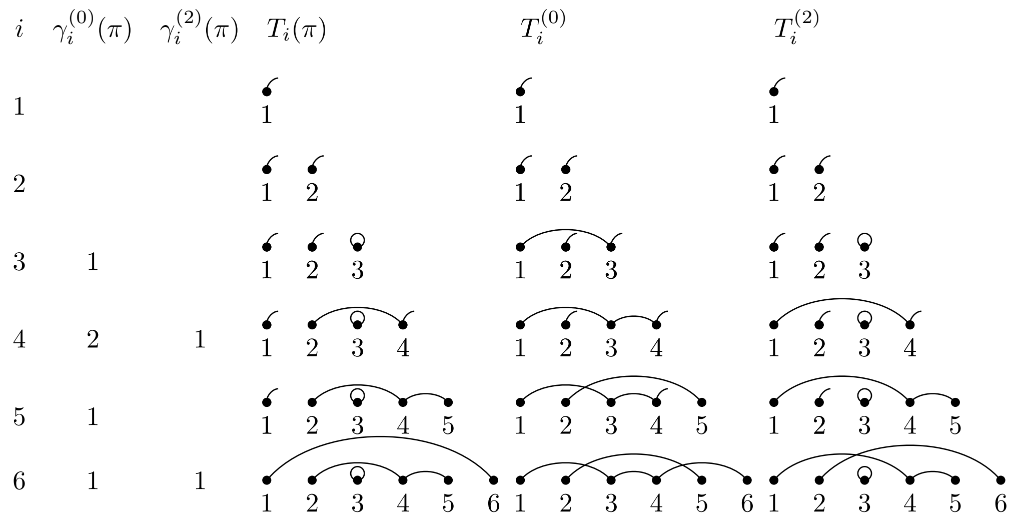

FIGURE 3. Construction of $\varphi_0(\pi) = T_6^{(0)}$ and $\varphi_2(\pi) = T_6^{(2)}$ for $\pi = \{1,6\}\{2,4,5\}\{3\}$.

The *i-th trace* $T_i(\pi)$ of π is the diagram obtained from π by removing vertices greater than i. If a vertex $v \le i$ is connected to $u > i$ in π then make a *half edge* from v in $T_i(\pi)$. Each vertex with a half edge is called *vacant* vertex. For an example, see Figure 2.

Let k be a fixed nonnegative integer. We define $\varphi_k : \Pi_n \to \Pi_n$ as follows.

(1) Set $T_0^{(k)} = \emptyset$.
(2) For $1 \le i \le n$, $T_i^{(k)}$ is obtained as follows.
   (a) Let $T_i^{(k)}$ (resp. $T_i'(\pi)$) be $T_{i-1}^{(k)}$ (resp. $T_{i-1}(\pi)$) with new vertex i.
   (b) If $i \in \mathcal{O}(\pi) \cup \mathcal{S}(\pi) \cup \mathcal{T}(\pi)$, then make a half edge from i both in $T_i^{(k)}$ and $T_i'(\pi)$.
   (c) If $i \in \mathcal{C}(\pi) \cup \mathcal{S}(\pi) \cup \mathcal{T}(\pi)$, let j be the vertex connected to i in π.
      (i) If $i - j < k$, then j must be a vacant vertex in $T_i^{(k)}$. Remove the half edge from j and add an edge (i,j) in $T_i^{(k)}$.
      (ii) If $i - j \ge k$, then let U (resp. V) be the set of all vacant vertices v in $T_i^{(k)}$ (resp. $T_i'(\pi)$) such that $i - v \ge k$. Let $\gamma_i^{(k)}(\pi)$ denote the integer r such that j is the r-th largest element of V. Let j' be the $\gamma_i^{(k)}(\pi)$-th smallest element of U. Remove the half edge from j' and add an edge (j', i) in $T_i^{(k)}$.
(3) Set $\varphi_k(\pi) = T_n^{(k)}$.

For example, see Figure 3. Using the same argument as in [5], we can prove that φ_k is an involution and satisfies $\mathrm{dcr}_k(\varphi_k(\pi)) = \mathrm{dne}_k(\pi)$, $\mathrm{dne}_k(\varphi_k(\pi)) = \mathrm{dcr}_k(\pi)$, $\mathrm{type}(\varphi_k(\pi)) = \mathrm{type}(\pi)$ if $k \ge 1$; $\mathrm{type}'(\varphi_k(\pi)) = \mathrm{type}'(\pi)$ if $k = 0$. Thus we have the following.

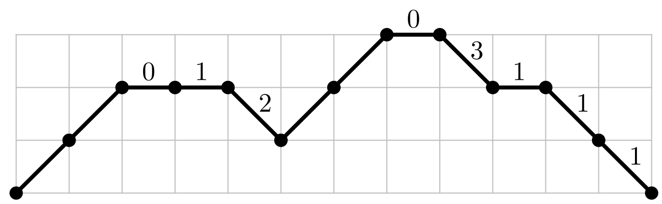

FIGURE 4. The Charlier diagram for the partition of Figure 1. The label e_i is written above the horizontal and down steps.

Theorem 2.1. *Let k be a nonnegative integer. Then*

$$\sum_{\pi\in\Pi_n} x^{\mathrm{dcr}_k(\pi)} y^{\mathrm{dne}_k(\pi)} = \sum_{\pi\in\Pi_n} x^{\mathrm{dne}_k(\pi)} y^{\mathrm{dcr}_k(\pi)}.$$

3. MOTZKIN PATHS AND CHARLIER DIAGRAMS

In this section, we recall a bijection between partitions and Charlier diagrams [4, 5].

A *step* is a pair (p, q) of points with p and q in $\mathbb{Z}\times\mathbb{Z}$. The *height* of a step (p, q) is the second component of p, i.e, if $p = (a, b)$ then the height of the step (p, q) is b. A step (p, q) is called an *up* (resp. *down*, *horizontal*) step if the component-wise difference $q - p$ is $(1, 1)$ (resp. $(1, -1)$, $(1, 0)$). A *path* of length n is a sequence $(p_0, p_1, p_2, \ldots, p_n)$ of $n+1$ points in $\mathbb{Z}\times\mathbb{Z}$. The *i-th step* of a path $(p_0, p_1, p_2, \ldots, p_n)$ is (p_{i-1}, p_i). A *Motzkin path* of length n is a path from $(0, 0)$ to $(n, 0)$ consisting of up steps, down steps, and horizontal steps that never goes below the x-axis. A *Charlier diagram* of length n is a pair (M, e) where $M = (p_0, p_1, \ldots, p_n)$ is a Motzkin path of length n and $e = (e_1, e_2, \ldots, e_n)$ is a sequence of integers such that:

(1) if the i-th step is an up step then $e_i = 0$,
(2) if the i-th step is a down step of height h then $1 \le e_i \le h$,
(3) if the i-th step is a horizontal step of height h then $0 \le e_i \le h$.

We will identify a Charlier diagram (M, e) with the sequence $(s_1, s_2, \ldots, s_n)$ of labeled letters in $\{U, D_1, D_2, \ldots, H_0, H_1, H_2, \ldots\}$ such that $s_i = U$ (resp. $s_i = D_{e_i}$, $s_i = H_{e_i}$) if the i-th step of M is an up (resp. down, horizontal) step.

Let π be a partition of $[n]$. Recall that in the previous section, if i is a closer or transient, then $\gamma_i^{(1)}(\pi)$ is the integer r such that i is connected to the r-th largest integer in $T_{i-1}^{(1)}(\pi)$.

The corresponding Charlier diagram $\mathrm{Ch}(\pi) = (s_1, s_2, \ldots, s_n)$ is defined as follows:

(1) if i is an opener in π then $s_i = U$,
(2) if i is a closer in π and $\gamma_i^{(1)}(\pi) = r$ then $s_i = D_r$,
(3) if i is a singleton in π then $s_i = H_0$,
(4) if i is a transient in π and $\gamma_i^{(1)}(\pi) = r$ then $s_i = H_r$.

For example, see Figure 4.

It is easy to see that if there is a step D_ℓ or H_ℓ with $\ell \ge 2$ in $\mathrm{Ch}(\pi)$, than π has an $(\ell - 1)$-distant crossing.

4. k-DISTANT NONCROSSING MATCHINGS

In this section we will find the number of k-distant noncrossing matchings for $k = 0, 1, 2$ and 3. Note that since there is no matching of $[2n+1]$ we have $NCM_k(2n+1) = 0$ for all n and k. Thus we will only consider $NCM_k(2n)$.

4.1. **0- and 1-distant noncrossing matchings.** Since matchings have no transient vertices, being 0-distant crossing is equivalent to being 1-distant crossing.

We can easily see that a matching π is 1-distant noncrossing if and only if $\mathrm{Ch}(\pi)$ consists of U and D_1. Thus a 1-distant noncrossing matching corresponds to a Dyck path.

Theorem 4.1. *We have*

$$NCM_0(2n) = NCM_1(2n) = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

4.2. **2-distant noncrossing matchings.** Let π be a 2-distant noncrossing matching. Then $\mathrm{Ch}(\pi)$ consists of U, D_1, and D_2. By definition of $\mathrm{Ch}(\pi)$, D_2 is of height at least 2. Moreover, since π has no 2-distant crossing, D_2 must immediately follow U. Thus we can consider $\mathrm{Ch}(\pi)$ as a nonnegative path consisting of the three steps $U = (1, 1)$, $D_1 = (1, -1)$ and $UD_2 = (2, 0)$ such that UD_2 never touches the x-axis. This is exactly the definition of a *little Schröder path*, see [13]. Thus we get the following theorem.

Theorem 4.2. *We have*

$$NCM_2(2n) = s_n,$$

where s_n is the little Schröder number (A001003).

4.3. **3-distant noncrossing matchings.** Let π be a 3-distant noncrossing matching. One can check that $\mathrm{Ch}(\pi)$ consists of U, D_1, D_2, and D_3 satisfying the following.

(1) D_ℓ is of height at least ℓ for $\ell = 1, 2, 3$.
(2) D_3 can only occur after two consecutive U, and
(3) D_2 can only occur after U or after either D_2 or D_3 which follows U.

Thus we can consider $\mathrm{Ch}(\pi)$ as a path consisting of 6 kinds of steps: U and D_1, which can appear at any height; and UD_2, UUD_3, UD_2D_2, and UUD_3D_2, which can only appear above the line $y = 1$. Let $g(n)$ be the number of labeled Motzkin paths of length n consisting of U, D_1, UD_2, UUD_3, UD_2D_2, and UUD_3D_2 with no restriction on height—these are not properly Charlier diagrams since, for example, down steps at height 1 may have label 2. Let $F(x) = \sum_{n\ge0} NCM_3(2n)x^n$ and $G(x) = \sum_{n\ge0} g(2n)x^n$.

Decomposing Charlier diagrams as in Table 3, we can easily see that $G(x) = 1 + (x + x^2)G(x) + (x^{1/2} + x^{3/2})^2 G(x)^2$, which means

$$G(x) = \frac{1 - x - x^2 - \sqrt{1 - 6x - 9x^2 - 2x^3 + x^4}}{2x(x+1)^2}.$$

Since $F(x) = 1 + xG(x)F(x)$, we get the generating function for $NCM_3(2n)$.

Theorem 4.3. *We have*

$$\begin{aligned}\sum_{n\ge0} NCM_3(2n)x^n &= \frac{2(x+1)^2}{1 + 5x + 3x^2 + \sqrt{1 - 6x - 9x^2 - 2x^3 + x^4}} \\ &= 1 + x + 3\,x^2 + 14\,x^3 + 71\,x^4 + 387\,x^5 + 2210\,x^6 + 13053\,x^7 + \cdots.\end{aligned}$$

Path type	Weight
UD_2 followed by any path	$xG(x)$
UUD_3D_2 followed by any path	$x^2G(x)$
U, any path, D_1, any path	$xG(x)^2$
UUD_3, any path, D_1, any path	$x^2G(x)^2$
U, any path, UD_2D_2, any path	$x^2G(x)^2$
UUD_3, any path, UD_2D_2, any path	$x^3G(x)^2$

TABLE 3. The six possible first-return decompositions for non-empty paths counted by the generating function $G(x)$ of section 4.3.

5. k-DISTANT NONCROSSING PARTITIONS

5.1. **0-distant noncrossing partitions.** Let π be a 0-distant noncrossing partition. Then $\mathrm{Ch}(\pi)$ consists of U, D_1, H_0. Thus $\mathrm{Ch}(\pi)$ is a Motzkin path.

Theorem 5.1. *The number of 0-distant noncrossing partitions of $[n]$ is equal to the number of Motzkin paths of length $[n]$ (A001006).*

5.2. **1-distant noncrossing partitions.** Let π be a 1-distant noncrossing partition. Then π is a usual noncrossing partition. It is well known that the number of noncrossing partitions of $[n]$ is the Catalan number C_n.

Theorem 5.2. *We have*

$$NCP_1(n) = C_n.$$

5.3. **2-distant noncrossing partitions.** Let π be a 2-distant noncrossing partition. Then $\mathrm{Ch}(\pi)$ consists of U, D_1, D_2, H_0, H_1, and H_2 and satisfies

(1) D_ℓ and H_ℓ are of height at least ℓ,
(2) H_2 and D_2 can only occur after U, H_1, or H_2.

Thus we can consider $\mathrm{Ch}(\pi)$ as a path with the following steps: UH_2^k, $UH_2^kD_2$, $H_1H_2^k$, $H_1H_2^kD_2$, H_0, and D_1, where k is a nonnegative integer and H_2^k means k consecutive H_2 steps.

Let $a(n)$ (resp. $b(n)$) denote the number of Charlier diagrams of length n consisting of the above steps such that D_ℓ and H_ℓ is of height at least $\ell-2$ (resp. at least $\ell-1$). In fact, the height condition is unnecessary for $a(n)$ since every step is of height at least 0. Let $F(x)=\sum_{n\ge0} NCP_2(n)x^n$, $A(x)=\sum_{n\ge0} a(n)x^n$, and $B(x)=\sum_{n\ge0} b(n)x^n$.

Note that the steps which increase the y-coordinate by 1 are UH_2^k; the steps which do not change the y-coordinate are H_0, $H_1H_2^k$, and $UH_2^kD_2$; and the steps which decrease y-coordinate by 1 are D_1 and $H_1H_2^kD_2$. By decomposing Charlier diagrams as we did with $G(x)$ in section 4.3, we get

$$A(x) = 1 + \left(x + \frac{x}{1-x} + \frac{x^2}{1-x}\right)A(x) + \frac{x}{1-x}\cdot\left(x+\frac{x^2}{1-x}\right)A(x)^2,$$

$$B(x) = 1 + \left(2x + \frac{x^2}{1-x}\right)B(x) + \frac{x}{1-x}\cdot\left(x+\frac{x^2}{1-x}\right)A(x)B(x), \text{ and}$$

$$F(x) = 1 + xF(x) + x^2B(x)F(x).$$

Solving these equations, we get the following theorem.

Theorem 5.3. *We have*

$$\sum_{n\ge 0} NCP_2(n)x^n = \frac{3-3x-\sqrt{1-6x+5x^2}}{2(1-x)} = \frac{3}{2}-\frac{1}{2}\sqrt{\frac{1-5x}{1-x}}$$
$$= 1 + x + 2\,x^2 + 5\,x^3 + 15\,x^4 + 51\,x^5 + 188\,x^6 + 731\,x^7 + 2950\,x^8 + \cdots .$$

This sequence is A007317. Mansour and Severini [9] proved that the generating function for the number of 12312-avoiding partitions is equal to that in 5.3. Thus the number of 2-distant noncrossing partitions of $[n]$ is equal to the number of 12312-avoiding partitions of $[n]$. Yan [17] found a bijection from 12312-avoiding partitions of $[n]$ to UH-free Schröder paths of length $2n-2$. Composing several bijections including Yan's bijection, Kim [7] found a bijection between 2-distant noncrossing partitions and 12312-avoiding partitions.

6. Orthogonal polynomials

Given a sequence $\{\mu_n\}_{n\ge 0}$, one may try to define a sequence of polynomials $\{P_n(x)\}_{n\ge 0}$ that are orthogonal with respect to $\{\mu_n\}$; that is, if we define a measure with $\mu_n = \int x^n\,\mathrm{d}\mu$, then

$$\int P_n(x)P_m(x)\,\mathrm{d}\mu = 0$$

whenever $n \ne m$. Any sequence of polynomials satisfying the above orthogonality relation must satisfy a three-term recurrence relation of the form

$$P_{n+1}(x) = (x-b_n)P_n(x) - \lambda_n P_{n-1}(x), \tag{1}$$

with $P_0(x) = 1$ and $P_1(x) = x - b_0$. Viennot showed [15, 16] that for any sequence $\{\mu_n\}$—which are called the *moments*—one can interpret the moment μ_n as the generating function for weighted Motzkin paths of length n in which up steps have weight 1, horizontal steps of height k have weight b_k, and down steps of height k have weight λ_k; then the polynomials in (1) will be orthogonal with respect to $\{\mu_n\}_{n\ge 0}$.

Many classical combinatorial sequences have been interpreted as the moment sequences for a set of orthogonal polynomials, and the corresponding orthogonality relation proved with a sign-reversing involution. In particular, it is known that:

- If $\mu_{2n+1} = 0$ and $\mu_{2n} = C_n$, the Catalan number, then $b_n = 0$ and $\lambda_n = 1$; the corresponding polynomials are Chebyshev polynomials of the second kind [2], which may be defined by $U_{n+1}(x) = xU_n(x) - U_{n-1}(x)$, with $U_0(x) = 1$ and $U_1(x) = x$. These moments are $NCM_0(n)$ (and $NCM_1(n)$).
- If $\mu_{2n+1} = 0$ and $\mu_{2n} = (2n-1)!!$, then $b_n = 0$ and $\lambda_n = n$; the corresponding polynomials are Hermite polynomials [15]. These moments are $NCM_\infty(n)$.
- If $\mu_n = M_n$, the n-th Motzkin number, then $b_n = 1$, $\lambda_n = 1$; the corresponding polynomials are shifted Chebyshev polynomials of the second kind: $U_n(x-1)$. See [3, section 4.1]. These moments are $NCP_0(n)$.
- If $\mu_n = B_n$, the number of partitions of $[n]$, then $b_n = n+1$ and $\lambda_n = n$; the corresponding polynomials are Charlier polynomials (with $a = 1$) [15]. These moments are $NCP_\infty(n)$.

With these observations in mind, it is natural to try to use, say, $NCM_k(n)$ as a sequence of moments. Letting k go from 0 to infinity would then allow us to interpolate between Chebyshev polynomials and Hermite polynomials; using $NCP_k(n)$ would give the corresponding interpolation between shifted Chebyshev and Charlier polynomials.

What happens if we use $NCM_2(n)$ for the moments? We know that $NCM_2(2n+1)$ is zero, and $NCM_2(2n)$ equals the little Schröder number s_n, which means the corresponding sequence of b_n's is all zeros. We need only find the λ_n's.

Theorem 6.1. *If $\lambda_{2n+1} = 1$ and $\lambda_{2n} = 2$, then the corresponding weighted Dyck paths are counted by the little Schröder numbers.*

Proof. Both sequences have the same generating function: if we weight the upsteps and downsteps of little Schröder paths by x and double horizontal steps by x^2, then the generating function of such paths is

$$\frac{1 + x^2 - \sqrt{x^4 - 6x^2 + 1}}{4x^2}.$$

See Stanley [13, p. 178]. On the other hand, consider the following weightings for Dyck paths:

$$\lambda_n = \begin{cases} 1 & n \text{ odd} \\ 2 & n \text{ even,} \end{cases} \quad \text{and} \quad \lambda_n = \begin{cases} 2 & n \text{ odd} \\ 1 & n \text{ even,} \end{cases}$$

with upsteps all weighted 1. Let $A(x)$ and $B(x)$ respectively denote the generating functions of such paths. By decomposing paths by their first return to the x-axis, we have

$$A(x) = \frac{1}{1 - x^2 B(x)} \quad \text{and} \quad B(x) = \frac{1}{1 - 2x^2 A(x)};$$

by substituting the expression for $B(x)$ into that for $A(x)$ and solving, we find that

$$A(x) = \frac{1 + x^2 \pm \sqrt{x^4 - 6x^2 + 1}}{4x^2}.$$

Using "$-$" yields the correct generating function, which coincides with the known generating function for the little Schröder numbers. □

This means that the orthogonal polynomials corresponding to $NCM_2(n)$ have $b_n = 0$, $\lambda_{2n+1} = 1$, and $\lambda_{2n} = 2$. They are a special case of polynomials studied by Kim and Zeng [6]: use $U_n(x, 2)$ in their paper. Vauchassade de Chaumont and Viennot [14] also studied these polynomials, although they use a different normalization for the moments and instead get the big Schröder numbers.

If we attempt to do the same with $NCM_3(n)$, we get stuck: since $NCM_3(2n+1) = 0$, we know that $b_n = 0$, but the λ_n sequence starts with

$$1, 2, \frac{5}{2}, \frac{3}{10}, \frac{76}{5}, -\frac{680}{57}, -\frac{2311}{7752}, \frac{1246001}{314296}, \frac{114710016}{151553069}, \ldots. \tag{2}$$

Not only are some λ_n's fractions, but some are negative, which means prospects for polynomials with nice combinatorics are dim.

Let us try the same line of attack with k-distant noncrossing partitions. Using $NCP_1(n)$—Catalan numbers—for a set of moments, we get a shifted version of Chebyshev polynomials of the second kind: $b_0 = 1$, all other $b_n = 2$, and all $\lambda_n = 1$. These polynomials can be written $U_n(x-2)$, with slightly different initial conditions: $U_0(x) = 1$ and $U_1(x) = x - 1$. The easiest way to see why these recurrence coefficients and initial conditions are orthogonal with respect to the Catalan numbers is with a bijection between Motzkin paths of length n with the above weighting and Dyck paths of length $2n$: take each up step U and make it UU, take each down step D and make it DD, and take each horizontal step H and make it either UD or DU—except for the horizontal step at height zero, which can only be made into UD. This process turns a weighted Motzkin path of length n into a Dyck path of length $2n$ and is easily shown to be a bijection.

When using $NCP_2(n)$ and $NCP_3(n)$ as the moments, we again get some fractional coefficients, but they seem much nicer. We have computed the following with Maple: if $\mu_n = NCP_2(n)$ then

$$\{b_n\}_{n\ge 0} = \left\{1, 3-1, 3-\frac{1}{2}, 3-\frac{1}{10}, 3-\frac{1}{65}, 3-\frac{1}{442}, 3-\frac{1}{3026}, \dots\right\} \text{ and}$$
$$\{\lambda_n\}_{n\ge 1} = \left\{1, 1+1, 1+\frac{1}{4}, 1+\frac{1}{25}, 1+\frac{1}{169}, 1+\frac{1}{1156}, 1+\frac{1}{7921}, \dots\right\};$$

if $\mu_n = NCP_3(n)$ then

$$\{b_n\}_{n\ge 0} = \{1, 2, 3, 3, 3, \dots\} \quad \text{and} \quad \{\lambda_n\}_{n\ge 1} = \{1, 2, 2, 2, 2, \dots\}.$$

The first case is very interesting. The sequences of denominators of b_n's and λ_n's appear in A064170 and A081068 respectively. Based on the above evidence, we make the following conjecture.

Conjecture 6.1. *If $\mu_n = NCP_2(n)$ then $b_0 = b_1 = \lambda_1 = 1$, and for $n \ge 2$*

$$b_n = 3 - \frac{1}{F_{2n-1}F_{2n-3}} \quad \textit{and} \quad \lambda_n = 1 + \frac{1}{(F_{2n-3})^2},$$

where F_n is the n-th Fibonacci number, i.e., $F_{n+1} = F_n + F_{n-1}$ and $F_1 = F_2 = 1$.

If $\mu_n = NCP_3(n)$ then $b_0 = b_1 = \lambda_1 = 1, \lambda_2 = 2$, and, for $n \ge 3$, $b_n = 3$ and $\lambda_n = 2$.

7. k-distant r-crossing

Chen et al. [1] considered a different kind of crossing number. Our definition of k-distant crossing can be applied to their definition.

Let $k \ge 0$ and $r \ge 2$ be integers. A *k-distant r-crossing* is a set of r edges $(i_1, j_1), (i_2, j_2), \dots, (i_r, j_r)$ such that $i_1 < i_2 < \cdots < i_r \le j_1 < j_2 < \cdots < j_r$ and $j_1 - i_r \ge k$. Similarly, a *k-distant r-nesting* is a set of r edges $(i_1, j_1), (i_2, j_2), \dots, (i_r, j_r)$ such that $i_1 < i_2 < \cdots < i_r \le j_r < j_{r-1} < \cdots < j_1$ and $j_r - i_r \ge k$. In [1], they defined an r-crossing and an enhanced r-crossing, which are a 1-distant r-crossing and a 0-distant r-crossing respectively.

Let $\mathrm{DCR}_k(\pi)$ (resp. $\mathrm{DNE}_k(\pi)$) be the maximal r such that π has a k-distant r-crossing (resp. k-distant r-nesting). Let $f_{n,S,T}(k; i, j)$ denote the number of partitions π of $[n]$ such that $\mathrm{DCR}_k(\pi) = i$, $\mathrm{DNE}_k(\pi) = j$, $\mathcal{O}(\pi) = S$ and $\mathcal{C}(\pi) = T$. Chen et al. [1] proved that $f_{n,S,T}(k; i, j) = f_{n,S,T}(k; j, i)$ for $k = 0, 1$. Krattenthaler [8] extended this result using growth diagrams.

Using Krattenthaler's growth diagram method, we can get the following theorem.

Theorem 7.1. *Let $n \ge 1$ and $k \ge 0$ be integers. Then*

$$f_{n,S,T}(k; i, j) = f_{n,S,T}(k; j, i).$$

Appendix: Sage code

In this appendix we provide code used to compute the values in Table 1 and Table 2. This code is for use with the free open-source computer mathematics system Sage (`http://sagemath.org`). The source code is available as a separate file: select "source" from the "other formats" link on the abstract page for this preprint (arxiv.org/abs/0812.2725), and extract `sage-code-appendix.sage` from the file you download.

```
r"""
Sage code for computing k-distant crossing numbers.

This code accompanies the article arxiv:0812.2725; see
http://arxiv.org/abs/0812.2725.
```

```
Right now, this code only computes k-dcrossings. If you are only
interested in the distribution, this is good enough because the extended
Kasraoui-Zeng involution tells us the distribution of k-dcrossings and
k-dnestings is symmetric. It would be nice, though, to have a function
which actually performed that involution.

AUTHORS:
   -- Dan Drake (2008-12-15): initial version.

EXAMPLES:

The example given in the paper. Note that in this format, we omit fixed
points since they cannot create any sort of crossing.

   sage: dcrossing([(1,5), (2,4), (4,9), (6,12), (7,10), (10,11)])
   3

"""

#*****************************************************************************
# Copyright (C) 2008 Dan Drake <ddrake@member.ams.org>
#
# This program is free software: you can redistribute it and/or modify
# it under the terms of the GNU General Public License as published by
# the Free Software Foundation, either version 2 of the License, or (at
# your option) any later version.
#
# See http://www.gnu.org/licenses/.
#*****************************************************************************

def CompleteMatchings(n):
   """Return a generator for the complete matchings of the set [1..n].

   INPUT:
      n -- nonnegative integer

   OUTPUT:
      A generator for the complete matchings of the set [1..n], or,
      what is basically the same thing, complete matchings of the
      graph K_n. Each complete matching is represented by a list of
      2-element tuples.

   EXAMPLES:
   There are 3 complete matchings on 4 vertices:
      sage: [m for m in CompleteMatchings(4)]
      [[(3, 4), (1, 2)], [(2, 4), (1, 3)], [(2, 3), (1, 4)]]

   There are no complete matchings on an odd number of vertices; the
   number of complete matchings on an even number of vertices is a
   double factorial:
      sage: [len([m for m in CompleteMatchings(n)]) for n in [0..8]]
      [1, 0, 1, 0, 3, 0, 15, 0, 105]

   The exact behavior of CompleteMatchings(n) if n is not a nonnegative
   integer depends on what [1..n] returns, and also on what range(1,
   len([1..n])) is.
```

```
    """
    for m in matchingsset([1..n]): yield m

def matchingsset(L):
    """Return a generator for complete matchings of the sequence L.

    This is not really meant to be called directly, but rather by
    CompleteMatchings().

    INPUT:
        L -- a sequence. Lists, tuples, et cetera; anything that
        supports len() and slicing should work.

    OUTPUT:
        A generator for complete matchings on K_n, where n is the length
        of L and vertices are labeled by elements of L. Each matching is
        represented by a list of 2-element tuples.

    EXAMPLES:
        sage: [m for m in matchingsset(('a', 'b', 'c', 'd'))]
        [[('c', 'd'), ('a', 'b')], [('b', 'd'), ('a', 'c')], [('b', 'c'), ('a', 'd')]]

        There's only one matching of the empty set/list/tuple: the empty
        matching.

        sage: [m for m in matchingsset(())]
        [[]]
    """
    if len(L) == 0:
        yield []
    else:
        for k in range(1, len(L)):
            for m in matchingsset(L[1:k] + L[k+1:]):
                yield m + [(L[0], L[k])]

def dcrossing(m_):
    """Return the largest k for which the given matching or set
    partition has a k-distant crossing.

    INPUT:
       m -- a matching or set partition, as a list of 2-element tuples
       representing the edges. You'll need to call setp_to_edges() on
       the objects returned by SetPartitions() to put them into the
       proper format.

    OUTPUT:
       The largest k for which the object has a k-distant crossing.
       Matchings and set partitions with no crossings at all yield -1.

    EXAMPLES:
    The main example from the paper:
        sage: dcrossing(setp_to_edges(Set(map(Set, [[1,5],[2,4,9],[3],[6,12],[7,10,11],[8]]))))
        3

    A matching example:
```

```
        sage: dcrossing([(4, 7), (3, 6), (2, 5), (1, 8)])
        2

    TESTS:
    The empty matching and set partition are noncrossing:
        sage: dcrossing([])
        -1
        sage: dcrossing(Set([]))
        -1

    One edge:
        sage: dcrossing([Set((1,2))])
        -1
        sage: dcrossing(Set([Set((1,2))]))
        -1

    Set partition with block of size >= 3 is always at least
    0-dcrossing:
        sage: dcrossing(setp_to_edges(Set([Set((1,2,3))])))
        0
    """
    d = -1
    m = list(m_)
    while len(m) > 0:
        e1_ = m.pop()
        for e2_ in m:
            e1, e2 = sorted(e1_), sorted(e2_)
            if (e1[0] < e2[0] and e2[0] <= e1[1] and e1[1] < e2[1] and
                e1[1] - e2[0] > d):
                d = e1[1] - e2[0]
            if (e2[0] < e1[0] and e1[0] <= e2[1] and e2[1] < e1[1] and
                e2[1] - e1[0] > d):
                d = e2[1] - e1[0]
    return d

def setp_to_edges(p):
    """Transform a set partition into a list of edges.

    INPUT:
        p -- a Sage set partition.

    OUTPUT:
        A list of non-loop edges of the set partition. As this code just
        works with crossings, we can ignore the loops.

    EXAMPLE:
    The main example from the paper:
        sage: setp_to_edges(Set(map(Set, [[1,5],[2,4,9],[3],[6,12],[7,10,11],[8]])))
        [[7, 10], [10, 11], [2, 4], [4, 9], [1, 5], [6, 12]]
    """
    q = [ sorted(list(b)) for b in p ]
    ans = []
    for b in q:
        for n in range(len(b) - 1):
            ans.append(b[n:n+2])
    return ans
```

```
def dcrossvec_setp(n):
    """Return a list with the distribution of k-dcrossings on set partitions of [1..n].

    INPUT:
        n -- a nonnegative integer.

    OUTPUT:
        A list whose k'th entry is the number of set partitions p for
        which dcrossing(p) = k. For example, let L = dcrossvec_setp(3).
        We have L = [1, 0, 4]. L[0] is 1 because there's 1 partition of
        [1..3] that has 0-dcrossing: [(1, 2, 3)].

        One tricky bit is that noncrossing matchings get put at the end,
        because L[-1] is the last element of the list. Above, we have
        L[-1] = 4 because the other four set partitions are all
        d-noncrossing. Because of this, you should not think of the last
        element of the list as having index n-1, but rather -1.

    EXAMPLES:
        sage: dcrossvec_setp(3)
        [1, 0, 4]

        sage: dcrossvec_setp(4)
        [5, 1, 0, 9]

    The one set partition of 1 element is noncrossing, so the last
    element of the list is 1:
        sage: dcrossvec_setp(1)
        [1]
    """
    vec = [0] * n
    for p in SetPartitions(n):
        vec[dcrossing(setp_to_edges(p))] += 1
    return vec

def dcrossvec_cm(n):
    """Return a list with the distribution of k-dcrossings on complete matchings on n vertices.

    INPUT:
        n -- a nonnegative integer.

    OUTPUT:
        A list whose k'th entry is the number of complete matchings m
        for which dcrossing(m) = k. For example, let L =
        dcrossvec_cm(4). We have L = [0, 1, 0, 2]. L[1] is 1 because
        there's one matching on 4 vertices that is 1-dcrossing: [(2, 4),
        (1, 3)]. L[0] is zero because dcrossing() returns the *largest*
        k for which the matching has a dcrossing, and 0-dcrossing is
        equivalent to 1-dcrossing for complete matchings.

        One tricky bit is that noncrossing matchings get put at the end,
        because L[-1] is the last element of the list. Because of this, you
        should not think of the last element of the list as having index
        n-1, but rather -1.

        If n is negative, you get silly results. Don't use them in your
        next paper. :)
```

```
    EXAMPLES:
    The single complete matching on 2 vertices has no crossings, so the
    only nonzero entry of the list (the last entry) is 1:
        sage: dcrossvec_cm(2)
        [0, 1]

    Similarly, the empty matching has no crossings:
        sage: dcrossvec_cm(0)
        [1]

    For odd n, there are no complete matchings, so the list has all
    zeros:
        sage: dcrossvec_cm(5)
        [0, 0, 0, 0, 0]

        sage: dcrossvec_cm(4)
        [0, 1, 0, 2]
    """
    vec = [0] * max(n, 1)
    for m in CompleteMatchings(n):
        vec[dcrossing(m)] += 1
    return vec

def tablecolumn(n, k):
    """Return column n of Table 1 or 2 from the paper arxiv:0812.2725.

    INPUT:
        n -- positive integer.

        k -- integer for which table you want: Table 1 is complete
             matchings, Table 2 is set partitions.

    OUTPUT:
        The n'th column of the table as a list. This is basically just the
        partial sums of dcrossvec_{cm,setp}(n).

        table2column(1, 2) incorrectly returns [], instead of [1], but you
        probably don't need this function to work through n = 1.

    EXAMPLES:
    Complete matchings:
        sage: tablecolumn(2, 1)
        [1]

        sage: tablecolumn(6, 1)
        [5, 5, 11, 14, 15]

    Set partitions:
        sage: tablecolumn(5, 2)
        [21, 42, 51, 52]

        sage: tablecolumn(2, 2)
        [2]
    """
    if k == 1:
        v = dcrossvec_cm(n)
```

```
else:
    v = dcrossvec_setp(n)
i = v[-1]
return [i + sum(v[:k]) for k in range(len(v) - 1)]
```

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