Card deals, lattice paths, abelian words and combinatorial identities

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Abstract

We give combinatorial interpretations of several related identities associated with the names Barrucand, Strehl and Franel, including one for the Apéry numbers, $\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$. The combinatorial constructs employed are derangement-type card deals as introduced in a previous paper on Barrucand's identity, labeled lattice paths and, following a comment of Jeffrey Shallit, abelian words over a 3-letter alphabet.

1 Introduction

The purpose of this paper is to give simple direct combinatorial interpretations of two identities of Strehl [1], for the Franel and Apéry numbers respectively,

$$\sum_{k=0}^{n}\binom{n}{k}^3=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}, \tag{1}$$

and

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2, \tag{2}$$

and of the following curious sequence of identities involving powers of successively larger integers,

$$\sum_{k=0}^{n}\binom{n}{k}\binom{2k}{k}2^k=\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n}3^k=\sum_{k=0}^{n}\binom{n}{k}^2 4^k=\sum_{k=0}^{n/2}\binom{n}{2k}\binom{2k}{k}4^k5^{n-2k}. \tag{3}$$

The first three of these expressions are equated in [2, Eqs. 34, 35], and all give sequence A084771 in OEIS.

The combinatorial constructs employed are (generalizations of) the derangement-type card deals introduced in a previous paper on Barrucand's identity [3], the labeled lattice paths cited by Nour-Eddine Fahssi in A084771, and, following a comment of Jeffrey Shallit [4], abelian words over a 3-letter alphabet.

Section 2 reviews the card deals and abelian words/matrices. Section 3 presents a 1-to-1 correspondence between them and reinterprets Barrucand's identity,

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{j=0}^{k} \binom{k}{j}^3 = \sum_{k=0}^{n} \binom{n}{k}^2 \binom{2k}{k} \tag{4}$$

in terms of abelian matrices. Section 4 gives interpretations for (1) and Section 5 for (2). Section 6 presents three equinumerous combinatorial constructs involving lattice paths, card deals and matrices respectively, and Section 7 uses them to interpret (3).

2 Card deals and abelian words/matrices

A *Barrucand* n*-deal* [3] is formed as follows. Start with a deck of $3n$ cards, n each colored red, green and blue, in denominations 1 through n, choose an arbitrary subset of the denominations and deal all cards of the chosen denominations into three equal-size hands to players designated red, green and blue in such a way that no player receives a card of her own color. Let $\mathcal{B}_n$ denote the set of Barrucand n-deals.

The left side of (4) counts $\mathcal{B}_n$ by total number of cards, k, in red's hand and number of green cards, j, in red's hand: first, there are $\binom{n}{k}$ ways to choose the denominations in the deal; next, j green cards in red's hand implies both j blue cards in green's hand and j red cards in blue's hand, and these cards determine the deal. For each hand there are $\binom{k}{j}$ ways to choose the determining cards, so $\binom{k}{j}^3$ choices in all. As shown in [3], the right side counts $\mathcal{B}_n$ by number of distinct denominations, k, in red's hand; another approach to establishing this count is given below.

Serendipitously, on the day [3] was published, the editor emailed me that the counting sequence for $\mathcal{B}_n$ also arose in his recently posted paper [5] counting abelian squares. An *abelian square* (over an alphabet) is a word of the form ww' where w' is a rearrangement of w. Its *size* is the number of letters in w (= number of letters in w'). As easily seen, the number of abelian squares over a three-letter alphabet, say $\{1, 2, 3\}$, of size n with $n-k$ 1s in w is $\binom{n}{k}^2\binom{2k}{k}$ [5], the summand on the right in (4). This raises the questions of a bijection from $\mathcal{B}_n$ to abelian squares over $\{1, 2, 3\}$ and of an abelian squares interpretation for the left side of (4). It is convenient to represent an abelian square ww' of size n as a $2 \times n$ matrix $\binom{w}{w'}$, a so-called *abelian matrix*, so that we can refer to its columns.

3 Bijection from Barrucand deals to abelian matrices

The following table describes a bijection from $\mathcal{B}_n$, the set of Barrucand n-deals, to $2 \times n$ abelian matrices over $\{1, 2, 3\}$ by specifying the locations of the 9 possible distinct columns in the matrix (R, G, B are short for red, green, blue respectively).

matrix column	locations given by denominations that are ...
$\frac{1}{1}$	in $[n]$, not in deal
$\frac{1}{2}$	in deal, not in red's hand and not on R in blue's hand
$\frac{1}{3}$	not in red's hand but do occur on R in blue's hand
$\frac{2}{1}$	in red's hand on G and B and also occur on R in blue's hand
$\frac{2}{2}$	in red's hand on G only and also occur on R in blue's hand
$\frac{2}{3}$	in red's hand on B only and also occur on R in blue's hand
$\frac{3}{1}$	in red's hand on G and B and don't occur on R in blue's hand
$\frac{3}{2}$	in red's hand on G only and don't occur on R in blue's hand
$\frac{3}{3}$	in red's hand on B only and don't occur on R in blue's hand

Bijection from deals to matrices

Table 1

Note, for example, that the denominations not in red's hand give the locations of 1s in the top row. It is straightforward to check that this mapping is a bijection as claimed and that its inverse is given by the following table.

player	denominations on . . . cards	given by locations of . . .
red	G and B	$\genfrac{}{}{0pt}{}{2}{1},\ \genfrac{}{}{0pt}{}{3}{1}$
	G only	$\genfrac{}{}{0pt}{}{2}{2},\ \genfrac{}{}{0pt}{}{3}{2}$
	B only	$\genfrac{}{}{0pt}{}{2}{3},\ \genfrac{}{}{0pt}{}{3}{3}$
green	B and R	$\genfrac{}{}{0pt}{}{1}{2},\ \genfrac{}{}{0pt}{}{3}{2}$
	B only	$\genfrac{}{}{0pt}{}{1}{3},\ \genfrac{}{}{0pt}{}{2}{2}$
	R only	$\genfrac{}{}{0pt}{}{3}{1},\ \genfrac{}{}{0pt}{}{3}{3}$
blue	R and G	$\genfrac{}{}{0pt}{}{1}{3},\ \genfrac{}{}{0pt}{}{2}{3}$
	R only	$\genfrac{}{}{0pt}{}{2}{1},\ \genfrac{}{}{0pt}{}{2}{2}$
	G only	$\genfrac{}{}{0pt}{}{1}{2},\ \genfrac{}{}{0pt}{}{3}{3}$

Bijection from matrices to deals

Table 2

For example, with $n = 5$ and subscripts referring to card color, the deal for which red's hand contains 2_G, 2_B, 4_B, 5_G, green's hand contains 1_B, 2_R, 4_R, 5_B, and blue's hand contains 1_G, 1_R, 4_G, 5_R corresponds to the abelian matrix $\left(\begin{smallmatrix} 1 & 3 & 1 & 3 & 2 \\ 3 & 1 & 1 & 3 & 2 \end{smallmatrix}\right)$.

Evidently, abelian matrices are somewhat more concise than Barrucand deals but, on the other hand, some statistics on $\mathcal{B}_n$ are more appealing than their counterparts for abelian matrices. For example,

$$\begin{array}{ccc} \#\,\text{cards in red's hand} & \leftrightarrow & n - \#\ \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)\ \text{columns} \\ \#\,\text{distinct denominations in red's hand} & \leftrightarrow & \text{total}\ \#\ \text{2s and 3s in top row} \\ \#\,\text{green cards in red's hand} & \leftrightarrow & \#\ \text{columns}\ \left(\begin{smallmatrix}p\\q\end{smallmatrix}\right)\ \text{with}\ p > 1\ \text{and}\ q < 3. \end{array}$$

In particular, using these correspondences and the second paragraph of Section 2, the left side of Barrucand's identity (4) counts abelian matrices of size n over $\{1, 2, 3\}$ by number, k, of columns $\left(\begin{smallmatrix}p\\q\end{smallmatrix}\right) \neq \left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ and number, j, of columns $\left(\begin{smallmatrix}p\\q\end{smallmatrix}\right)$ with $p > 1$ and $q < 3$. Summarizing these observations, we have the following alternative interpretation.

Proposition 1. *For Barrucand's identity (4), the right side of counts abelian words ww' of length $2n$ by number, $n - k$, of 1s in w while the left side counts them by number of positions, $n - k$, in which both w and w' have a 1.*

A generalization of Barrucand's identity (identity (37) in [2]),

$$\sum_{k=0}^{n}\binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}^2\binom{k}{j-a}=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k-a}, \tag{5}$$

can be treated similarly. Let $\mathcal{A}_{n,a}$ denote the set of $2\times n$ matrices with entries in $\{1,2,3\}$, the same number of 1s in each row, and a more 3s in the top row than in the bottom row. For example, $\left(\begin{smallmatrix}1&2&3\\2&1&2\end{smallmatrix}\right)\in\mathcal{A}_{3,1}$, and $a=0$ gives abelian matrices. Then the two sides of (5) count $\mathcal{A}_{n,a}$ by the very same statistics as the two sides of (4) count abelian matrices.

4 Franel numbers, $\sum_{k=0}^{n}\binom{n}{k}^3=\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{n}$

A *Franel n-deal* is a Barrucand n-deal in which *all* the cards are dealt to the players. Let $\mathcal{F}_n$ denote the set of Franel n-deals. As observed in Section 2, the left side of the identity for the Franel numbers counts $\mathcal{F}_n$ by number, k, of green cards in red's hand. Translated to abelian matrices, the left side counts $\mathcal{F}'_n$, the abelian matrices of size n over $\{1,2,3\}$ with no $\left(\begin{smallmatrix}1\\1\end{smallmatrix}\right)$ columns, by number, k, of columns $\left(\begin{smallmatrix}p\\q\end{smallmatrix}\right)$ with $p>1$ and $q<3$.

As for the right side, let us count $\mathcal{F}'_n$ by number, j, of 1s in each row: $\binom{n}{j}$ [place 1s in top row] $\times\binom{n-j}{j}$ [place 1s in bottom row] $\times\binom{2n-2j}{n-j}$ [choose $n-j$ of the remaining $2n-2j$ positions; place 2s in the chosen positions in the top row and fill out the top row with 3s; place 3s in the chosen positions in the bottom row and fill out the bottom row with 2s]. (The latter clever argument is due to Richmond and Shallit [5].) Thus, with $k:=n-j$, the number of abelian matrices in $\mathcal{F}'_n$ with a total of k 2s and 3s in each row is $\binom{n}{n-k}\binom{k}{n-k}\binom{2k}{k}=\binom{n}{k}^2\binom{2k}{n}$. Translated back to card deals, the right side counts $\mathcal{F}_n$ by number of distinct denominations in red's hand.

5 Apéry numbers, $\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$

The counting sequence for this identity, A005259, cropped up in Roger Apéry's celebrated proof of the irrationality of $\zeta(3)$ [6] and the identity inspired a survey paper by Volker Strehl [2] in which he offers six different proofs including a combinatorial proof of a substantial generalization and, indeed, proves most of the other identities in this paper. Still, simple direct fully bijective proofs may be of interest.

Let $\mathcal{B}_{n,k}$ denote the set of deals in $\mathcal{B}_n$ with k cards in red's hand, equivalently, k denominations in the deal. Thus $|\,\mathcal{B}_{n,k}\,| = \binom{n}{k}\sum_{j=0}^{k}\binom{k}{j}^3$. To get the left side of Apéry (2), we need an additional factor of $\binom{n+k}{k}$ on the left side of Barrucand (4). This motivates us to consider a simple construction and define $\mathcal{A}_{n,k}$ to be the set of pairs (D,i) where $D \in \mathcal{B}_{n,k}$ and $1 \le i \le \binom{n+k}{k}$. Thus $|\,\mathcal{A}_{n,k}\,| = \binom{n+k}{k}|\,\mathcal{B}_{n,k}\,| = \binom{n}{k}\binom{n+k}{k}\sum_{j=0}^{k}\binom{k}{j}^3$ and $\mathcal{A}_n := \bigcup_{k=0}^{n}\mathcal{A}_{n,k}$ is counted by the left side of Apéry.

Proposition 2. *Just as for Barrucand, the right side of Apéry counts $\mathcal{A}_n$ by number of distinct denominations in the red player's hand in the associated deal.*

The proof needs the identity

$$\sum_{a\ge 0}\binom{k}{a}\binom{n-k}{a}\binom{n+k+a}{n} = \binom{n+k}{k}\binom{n+k}{n-k}, \tag{6}$$

proved combinatorially by George Andrews [7] in a more general form (see also [2, Eqs. (19) and (20)]). Applied to (6), his proof shows that the right side counts pairs (K,L) where K is a k-element subset of $[n+k]$ and L is an $(n-k)$-element subset of $[n+k]$ while the left side counts these pairs by "intermingling coefficient" a: the number of elements in L among the k smallest elements of $K \cup L$.

A proof of Prop. 2 can now be devised following the analysis of $\mathcal{B}_n$ in [3] but it is a little simpler to translate to abelian matrices and prove the following equivalent result.

Proposition 3. *Let $\mathcal{A}'_n$ denote the set of pairs (A,i) with A a $2\times n$ abelian matrix over $\{1,2,3\}$ and $1 \le i \le \binom{n+j}{j}$ where $n-j$ is the number of $\binom{1}{1}$ columns in A.*

Then $\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2$ counts $\mathcal{A}'_n$ by total number, k, of 2s and 3s in the top row.

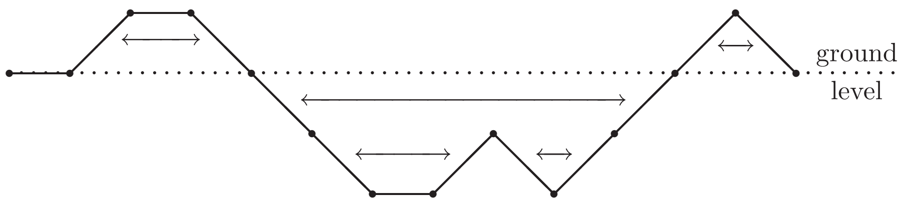
Proof Suppose $(A,i) \in \mathcal{A}'_n$ has k 2s and 3s, hence $n-k$ 1s, in the top row. Now count by number of $\binom{1}{1}$ columns, say $n-k-a$. Thus we have $\binom{n+k+a}{k+a}$ choices for the second member i of the pair (A,i) and choices for A as follows: place 1s in top row [$\binom{n}{n-k}$ choices], locate $\binom{1}{1}$ columns [$\binom{n-k}{n-k-a}$ choices], place a 1s in the bottom row not below 1s in the top row [$\binom{k}{a}$ choices], place 2s and 3s [$\binom{2k}{k}$ choices, as explained in Section 4]. All told, the number of choices for (A,i) is

$$\begin{aligned}\binom{n}{k}\binom{2k}{k}\sum_{a\ge 0}\binom{n-k}{a}\binom{k}{a}\binom{n+k+a}{n} &= \binom{n}{k}\binom{2k}{k}\binom{n+k}{k}\binom{n+k}{n-k}\\ &= \binom{n}{k}^2\binom{n+k}{k}^2,\end{aligned}$$

using (6) at the first equality. $\square$

6 Combinatorial constructs for (3)

A *Delannoy path* is a lattice path of upsteps $U = (1,1)$, downsteps $D = (1,-1)$, and flatsteps $F = (1,0)$ with an equal number of Us and Ds. The line joining its endpoints, necessarily horizontal, is *ground level* . Each upstep in a Delannoy path has a matching downstep (and conversely): given an upstep above ground level (resp. below ground level), travel directly east (resp. west) until you encounter a downstep.



matching step pairs in a Delannoy path

Thus the slanted steps (U and D) in a Delannoy path are partitioned into matching pairs of opposite-slope steps.

A *Hanna n-path* is a Delannoy path with n labeled steps: each slanted step gets one of two labels (colors), say 1 or 2, and each flat step gets one of five labels, say $1, 2, 3, 4$ or 5. As observed by Nour-Eddine Fahssi, Hanna n-paths are counted by A084771.

A *Hanna n-deal* is formed in the same way as a Barrucand deal except that the hands need not all be of equal size: if there are j denominations in the deal, only red's hand is required to contain its fair share of j cards and the remaining $2j$ cards are split arbitrarily between the green and blue players.

A *Hanna n-matrix* is a $2 \times n$ matrix with entries in $\{1, 2, 3\}$ and the same number of 1s in each row.

Hanna n-matrices, n-deals, and n-paths are equinumerous: the mapping in Table 1 of Section 3 (with a larger domain) is a bijection from the matrices to the deals, and there is a simple bijection from the matrices to the paths: transform each column in turn (subscripts denote step labels) according to the following table.

matrix column	$\genfrac{}{}{0pt}{}{1}{1}$	$\genfrac{}{}{0pt}{}{1}{2}$	$\genfrac{}{}{0pt}{}{1}{3}$	$\genfrac{}{}{0pt}{}{2}{1}$	$\genfrac{}{}{0pt}{}{2}{2}$	$\genfrac{}{}{0pt}{}{2}{3}$	$\genfrac{}{}{0pt}{}{3}{1}$	$\genfrac{}{}{0pt}{}{3}{2}$	$\genfrac{}{}{0pt}{}{3}{3}$
labeled step	F_1	U_1	U_2	D_1	F_2	F_3	D_2	F_4	F_5

In the next section, we use these constructs to give a combinatorial interpretation of the identities (3).

7 Combinatorial interpretations for (3)

The summand in the first expression in (3), $\binom{n}{k}\binom{2k}{k}2^k$, is the number of Hanna n-deals with k cards in red's hand. To see this, expand 2^k as $\sum_{j=0}^{k}\binom{k}{j}$. Then the resulting summand, $\binom{n}{k}\binom{k}{j}\binom{2k}{k}$, is the number of Hanna n-deals with k cards in red's hand and j red cards in blue's hand: choose denominations in the deal [$\binom{n}{k}$ choices], choose red denominations in blue's hand [$\binom{k}{j}$ choices] and the remaining red cards are forced into green's hand, select red's hand from the green and blue cards [$\binom{2k}{k}$ choices] and the remaining green and blue cards are forced into the hand of opposite color. □

The least obvious statistic for the sums in (3) is the one for the second sum. Actually, it is a sum of two statistics. On Hanna n-paths, define the statistic X to be the number of matching pairs of slanted steps not both labeled 1, and define Y to be the number of flatsteps whose label exceeds 2. Then the summand in the second expression in (3), $\binom{n}{k}\binom{2n-k}{n}3^k$, is the number of Hanna n-paths for which $X + Y = k$. This is an immediate consequence of the following two propositions.

Proposition 4. *The number of Hanna n-paths with $X = i$ and $Y = j$ is*

$$\binom{n}{j}\binom{n-j}{i}\binom{2n-2i-2j}{n-j}3^{i+j}.$$

Proposition 5.

$$\sum_{\substack{i,j:\\ i+j=k}}\binom{n}{j}\binom{n-j}{i}\binom{2n-2i-2j}{n-j}3^{i+j} = \binom{n}{k}\binom{2n-k}{n}3^k.$$

Proof of Prop. (4) To form a Hanna n-path with $X = i$ and $Y = j$, choose locations in the path for flatsteps whose label exceeds 2 [$\binom{n}{j}$ choices], label these flatsteps [3^j choices], choose locations for the upsteps in matching pairs whose members are not both labeled 1 [$\binom{n-j}{i}$ choices], assign labels to these pairs [3^i choices, since each U-D pair may be labeled 1-2, 2-1, or 2-2]. Now consider the steps in the $n-i-j$ locations not yet filled (including the downsteps in the matching pairs). These steps form a path of Us, Ds, and Fs of length $n-i-j$ with i more Ds than Us. The labels on the slanted steps in this path are already determined and the flatsteps are bicolored (labeled 1 or 2). Expanding the path via the transformation rules $U \to UU$, $D \to DD$, $F_1 \to UD$, $F_2 \to DU$ (subscript denotes label), it becomes a path of Us and Ds of length $2n-2i-2j$ with $n-2i-j$ Us and $n-j$ Ds. There are $\binom{2n-2i-2j}{n-j}$ such paths, and the expansion is reversible. Thus all factors in the expression of Prop. (4) have been accounted for. □

Proof of Prop. (5)

$$\begin{aligned}\sum_{\substack{i,j:\\ i+j=k}} \binom{n}{j}\binom{n-j}{i}\binom{2n-2i-2j}{n-j}3^{i+j} &= \sum_j \binom{n}{j}\binom{n-j}{k-j}\binom{2n-2k}{n-j}3^k \\ &= \sum_j \binom{n}{k}\binom{k}{j}\binom{2n-2k}{n-j}3^k \\ &= \binom{n}{k}\binom{2n-k}{n}3^k,\end{aligned}$$

using the Chu-Vandermonde identity at the last equality. □

The summand in the third expression in (3), $\binom{n}{k}^2 4^k$, is the number of $2 \times n$ Hanna n-matrices with $n-k$ 1s in each row: place the 1s [$\binom{n}{n-k}^2 = \binom{n}{k}^2$ choices] and then fill the remaining $2k$ entries with 2s and 3s arbitrarily [2^{2k} choices]. Equivalently, it counts Hanna n-deals by number, $n-k$, of denominations appearing in red's hand. (Alternative interpretations of the other expressions in (3) are left to the reader.) □

The summand in the fourth expression in (3), $\binom{n}{2k}\binom{2k}{k}4^k 5^{n-2k}$, is the number of Hanna n-paths with k upsteps: choose locations for the slanted steps [$\binom{n}{2k}$ choices], insert Us and Ds into these locations [$\binom{2k}{k}$ choices], label the slanted steps [2^{2k} choices], and lastly, label the flatsteps [5^{n-2k} choices].

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