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TOWARDS A HUMAN PROOF OF GESSEL'S CONJECTURE

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ABSTRACT. We interpret walks in the first quadrant with steps $\{(1,1),(1,0),(-1,0),(-1,-1)\}$ as a generalization of Dyck words with two sets of letters. Using this language, we give a formal expression for the number of walks in the steps above beginning and ending at the origin. We give an explicit formula for a restricted class of such words using a correspondance between such words and Dyck paths. This explicit formula is exactly the same as that for the degree of the polynomial satisfied by the square of the area of cyclic n-gons conjectured by Dave Robbins although the connection is a mystery. Finally we remark on another combinatorial problem in which the same formula appears and argue for the existence of a bijection.

1. INTRODUCTION

Ever since Gessel conjectured his formula for the number of walks in the steps $\{(1,1),(1,0),(-1,0),(-1,-1)\}$ (which we will call Gessel steps) starting and ending at the origin in $2n$ steps constrained to lie in the first quadrant, there has been much interest in studying lattice walks in the quarter plane. There have been conjectures for lattice walks with Gessel steps terminating at other points [1], as well as conjectures for the number of walks ending at the origin with other sets of steps, most of which have been proven [2]. In a remarkable *tour de force*, Gessel's original conjecture has been finally proven using computer algebra techniques [3]. Even so, it is important to consider walks on the quarter plane from a human point of view because newer approaches tend to open up interesting mathematical avenues.

In this article, we count a considerably restricted number of walks with Gessel steps starting and ending at the origin by rephrasing the problem using words with an alphabet consisting of four letters — 1, 2, $\bar{1}$ and $\bar{2}$ which obey certain conditions. We first show that the restatement of Gessel's conjecture in this context can be interpreted using

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Dyck paths. This gives a formal solution to the conjecture. Unfortunately, the solution is so formal as to be even computationally intractable![1]. We give a closed-form expression for the restricted problem and hope a generalization of this method will give a better understanding of Gessel's conjecture. Admittedly this result is a long way from a solution of the problem, but one hopes that this technique can be generalized to obtain a complete proof of Gessel's conjecture.

In Section 2 we start with the preliminaries by defining the alphabet and stating the main theorem. In Section 3, we make the connection to Dyck paths and give a formal expression for the number of walks beginning and ending at the origin using Gessel steps. Section 4 contains the proof which involves summations of hypergeometric type. In principle, such sums can be tackled by computer packages, but a certain amount of manipulation is needed before they are summable. Lastly, we comment on related problems in Section 5.

2. Gessel Alphabet

To rephrase the problem in the notation of formal languages, we need some definitions.

Definition 1. *The* Gessel alphabet *consists of a set of letters* $S = \{1, 2, \dots\}$ *with an order* $<$ *(*$1 < 2 < \dots$*) along with their complements which we denote* $\bar{S} = \{\bar{1}, \bar{2}, \dots\}$*. The order on the complement set is irrelevant.*

Definition 2. *Let* $S = [n]$*. Denote by* $N_\alpha(w)$ *the number of occurences of the letter* $\alpha \in S \cup \bar{S}$ *in the word* w*. (For example,* $N_2(2\bar{2}) = N_{\bar{2}}(2\bar{2}) = 1, N_1(2\bar{2}) = 0$*.) Then a* Gessel word w *is a word such that every prefix of the word satisfies*

$$\sum_{i=1}^{k} \left(N_{n+1-i}(w) - N_{\overline{n+1-i}}(w) \right) \geq 0$$

for each $k \in [1, n]$*.*

In words, this means that in each prefix, n has to occur more often than $\bar{n}$, the number of occurrences of n and n-1 must be at least equal to the number of occurrences of their barred counterparts and so on. For example, $2\bar{1}$ is a valid Gessel word but $1\bar{2}$ is not.

Definition 3. *A* complete Gessel word *is a Gessel word* w *where* $N_i(w) = N_{\bar{i}}(w)$ *for all letters* $i \in S$*. In other words, the number of*

[1] Computing the nth term in the sequence involves $2n$ sums of binomial coefficients

times the letter i appears equals the number of times $\bar{i}$ appears for each $i \in [n]$.

As an example, for $n = 3$ both $3\bar{2}21\bar{1}\bar{3}$ and $12\bar{1}\bar{2}1$ are Gessel words but $21\bar{3}23\bar{2}\bar{1}$ is not because the prefix consisting of three letters fails the criterion in Definition 2. Among the other two, the first one is a complete Gessel word.

Remark 1. *The number $G^{(d)}(n)$ of d dimensional lattice walks in the first 2^d-ant with steps*

$$\begin{aligned} &\{(1,\ldots,1),(1,\ldots,1,0),\cdots,(1,0,\ldots,0)\\ &(-1,\ldots,-1),(-1,\ldots,-1,0),\cdots,(-1,0,\ldots,0)\}\end{aligned} \tag{2.1}$$

starting at the origin and returning in $2n$ steps is the same as the number of complete Gessel words of length $2n$ in d letters. None of these sequences seem to be present in [4] for dimensions higher than two and it would be interesting to see if they are holonomic. Furthermore, none of these higher dimensional sequences seem to have the property of small factors which is present for the Gessel case.

$G^{(2)}(n)$ is conjectured by Gessel to be given by the closed form expression

$$16^n \frac{(5/6)_n(1/2)_n}{(2)_n(5/3)_n}, \tag{2.2}$$

where

$$(a)_n = a(a+1)\ldots(a+n-1) \tag{2.3}$$

is the Pochhammer symbol or rising factorial. The first few terms are the sequence A135404 in [4]. For the remainder of the paper, we implicitly assume $d = 2$ and omit the superscripts in defining various constrained Gessel numbers.

We express the number of walks of length $2n$ as a number triangle based on the number of times 2 and $\bar{2}$ appear increasing from left to right.

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 2 & & 7 & & 2 & & \\ & 5 & & 37 & & 38 & & 5 & \\ 14 & & 177 & & 390 & & 187 & & 14 \end{array} \tag{2.4}$$

One immediately notices that the leftmost and rightmost entries are the Catalan numbers. This is because the number of complete Gessel words with $N_2(w) = 0$ ($N_1(w) = 0$) in a word w of length $2n$ is in immediate bijection with the number of Dyck paths ending at $(2n, 0)$ because $N_1(x) > N_{\bar{1}}(x)$ ($N_2(x) > N_{\bar{2}}(x)$) for each prefix x of the word w.

What is more interesting, and the main result of the paper is the next-to-rightmost sequence beginning $1, 7, 38, 187$. Strangely enough, this sequence is already present in the OEIS as A000531 [4]. It turns out to be exactly the one conjectured by Dave Robbins [5] to be the degree of the polynomial satisfied by $16K^2$, where K is the area of a cyclic n-gon and proved in [6, 7]. As far as we know, this result is a coincidence without any satisfactory explanation. For a recent review of the subject, see [8]. This is also related to Simon Norton's conjecture on the same page in the OEIS. We comment on this in Section 5.

Theorem 1. *The number of complete Gessel words $G_1(n)$ in two letters with $n-1$ 2's and $\bar{2}$'s, and one 1 and $\bar{1}$, is given by*

$$G_1(n) = \frac{(2n+1)}{2}\binom{2n}{n} - 2^{2n-1}. \tag{2.5}$$

The proof uses the idea that the number of Gessel words with n_2 2's and $\bar{2}$'s and n_1 1 and $\bar{1}$'s can be calculated using a bijection with Dyck paths. The answer can be written as a sum of products of expressions counting the number of Dyck paths between two different heights. The summation can be done explicitly when $n_1 = 1$.

3. Complete Gessel words and Dyck paths

We consider Dyck paths to be paths using steps $\{(1,1),(1,-1)\}$ starting at the origin, staying on or above the x-axis and ending on the x-axis. In this section we exhibit a bijection between complete Gessel words (the counting of which is stated by the conjecture of Gessel) and a set of restricted Dyck paths which will be useful in the proof of Theorem 1.

Definition 4. *Let $P = (P_1, \dots, P_m)$ be an increasing list of positive integers and $H = (H_1, \dots, H_m)$ be a list of nonnegative integers of the same length. We define a (P,H)-*Dyck path *to be a Dyck path of length greater than m which satisfies the constraint that between positions P_i and P_{i+1} (both inclusive), the ordinate of the path is greater than or equal to H_i for $i = 1, \dots, m-1$.*

Notice that this forces the ordinates of the path at positions P_i to be greater than or equal to the heights $\max\{H_{i-1}, H_i\}$.

We now associate to every complete Gessel word w in two letters lists P and H using the following algorithm.

(1) Construct the list S of length $2n_1$ of letters 1 or $\bar{1}$ as they occur in the word.
(2) From the list S, construct the list T by replacing 1 by 1 and $\bar{1}$ by -1.
(3) Construct the list $\tilde{P}$ whose elements are positions of the letter S_i in w. Similarly, construct the elements of the list P as $\tilde{P}_i - i$.
(4) Finally, each element of the list H is given by

$$H_i = \max\Big\{ - \sum_{k=1}^{i} T_k, 0\Big\}. \tag{3.1}$$

Clearly, S and H determine each other and similarly, so do P and $\tilde{P}$. Therefore one can also associate a complete Gessel word to a (P, H)-Dyck path and vice versa. As an example, consider the Gessel word $w = 2\bar{1}21\bar{2}\bar{2}$. For this word, $S = (\bar{1}, 1), T = (-1, 1), \tilde{P} = (2, 4), P = (1, 2)$ and $H = (1, 0)$. Also, given this P and H, there is exactly one such (P, H)-Dyck path of length four, namely $(\nearrow, \nearrow, \searrow, \searrow)$, just as w is the only complete Gessel word of length six with $\bar{1}$ at position two at 1 at position four.

Lemma 2. *Complete Gessel words of length $2(n_1 + n_2)$ in two letters with positions of 1 and $\bar{1}$ given by the lists $\tilde{P}, S$ are in bijection with (P, H)-Dyck paths of length $2n_2$ where the pairs of lists $(\tilde{P}, S)$ and (P, H) are related by the algorithm described above.*

Proof. Starting with the complete Gessel word, one replaces each occurence of the letter 2 by the step $(1, 1)$ and that of $\bar{2}$ by the step $(1, -1)$. The constraint defining the (P, H)-Dyck path is simply another way of expressing the inequality in Definition 2. □

One could generalize this bijection to include paths not ending on the x-axis and Gessel words which are not complete, but this is sufficient for our purposes.

One of the main tools in the proof of Theorem 1 is an expression for the number of Dyck paths between two different heights, which can be readily obtained from the reflection principle [9].

Lemma 3. *The number of Dyck paths $a_{i,j}(k)$ that stay above the x-axis starting at the position $(0, i)$ and end at position (k, j) is given by*

$$a_{i,j}(k) = \begin{cases} \binom{k}{(k+i-j)/2} - \binom{k}{(k+i+j)/2+1} & \text{if } (k+i+j) \equiv 0 \mod 2, \\ 0 & \text{if } (k+i+j) \equiv 1 \mod 2. \end{cases} \tag{3.2}$$

We now use the bijection in Lemma 2 and the formula in Lemma 3 to write an expression for the number of complete Gessel words of length $2n$ for fixed positions of $1, \bar{1}$.

Lemma 4. *Let us fix the positions of* n_1 $1, \bar{1}$ *by the lists* $S, \tilde{P}$. *Calculate the lists* T *and* H *by the algorithm above and let* $G_{n_1}(S, \tilde{P}; 2n)$ *denote the number of such complete Gessel words. Then*

(3.3)

$$G_{n_1}(S,\tilde{P};2n) = \sum_{k_1=\delta_{(1-T_1)/2,0}}^{\tilde{P}_1-1} \sum_{k_2=H_2}^{\tilde{P}_2-1} \cdots \sum_{k_i=H_i}^{\tilde{P}_i-1} \cdots \sum_{k_{2n_1}=H_{2n_1}}^{\tilde{P}_{2n_1}-1}$$
$$a_{0,k_1}(\tilde{P}_1-1)\; a_{k_{2n_1},0}(2n-\tilde{P}_{2n_1}) \prod_{i=2}^{2n_1-1} a_{k_{i-1}-H_i,k_i-H_i}(\tilde{P}_i-\tilde{P}_{i-1}-1),$$

where the lower index of the sum k_1 *depends on the first element of the list* T.

Proof. The proof is straightforward, using the bijection of Lemma 2 to rewrite each Gessel word with the positions of $1, \bar{1}$ given by the lists $S, \tilde{P}$ as a Dyck path with heights at the points P_i (given by k_i) being not less than H_i and then the reflection principle in Lemma 3 to count the number of paths between position P_{i-1} and P_i for each i. □

Corollary 5. *For a given configuration of* $1, \bar{1}$, *replace each* $+1$ *in* T *by an upward Dyck step and each* -1 *by a downward Dyck step. If the whole of* T *forms an legal Dyck path, then* $G_{n_1}(S, \tilde{P}; 2n) = C_{n_1}$, *the* n_1*th Catalan number independent of the list* P.

Proof. Whenever the above condition is satisfied, $H_i = 0$ for all i, which means we simply count the number of Dyck paths of length $2n_1$ in (3.3) by definition. □

Now we obtain a formula for the number of complete Gessel words with n_1 $1, \bar{1}$'s using Lemma 4 and writing down all possibilities for $\tilde{P}$ and S. The number of ways of writing all possible $\tilde{P}$'s is simply $\binom{2n}{2n_1}$ because one has to choose $2n_1$ positions out of $2n$ positions. For each $\tilde{P}$, one has to choose n_1 positions for 1 and $\bar{1}$ each and therefore the number of such ways is $\binom{2n_1}{n_1}$.

Let us form the set

$$\mathcal{S} = \left\{ (S,\tilde{P}) \;\middle|\; \begin{array}{l} S \text{ is an ordered list of } n_1 \text{ 1's and } n_1 \text{ } \bar{1}\text{'s.} \\ \tilde{P} \text{ is an increasing list of} \\ \quad 2n_1 \text{ positions between 1 and } 2n, \end{array} \right\}, \tag{3.4}$$

whose cardinality is

$$\binom{2n}{2n_1}\binom{2n_1}{n_1} = \frac{(2n)!}{(n_1)!^2(2n-2n_1)!}. \tag{3.5}$$

Therefore the number of complete Gessel words with exactly n_1 $1, \bar{1}$'s is given by

$$G_{n_1}(n) = \sum_{(S,\tilde{P})\in\mathcal{S}} G_{n_1}(S, \tilde{P}; 2n), \tag{3.6}$$

and the number of complete Gessel words in $2n$ letters is

$$G(n) = \sum_{n_1=0}^{n} G_{n_1}(n). \tag{3.7}$$

Showing that $G(n)$ is equal to the expression (2.2) would be the ultimate (and possibly hopeless) aim of this line of approach.

We now have all the ingredients necessary to prove Theorem 1 which corresponds to the special case $n_1 = 1$. Before we go on to the proof, however, we make some observations about complete Gessel words with exactly one 1 and $\bar{1}$. Let $d_{i,j}$ be the number of times there is an 1 or a $\bar{1}$ at position i and its counterpart at position j. Then we draw the following triangle for a specific n,

(3.8)

$$\begin{array}{ccccccccc}
 & & & & d_{1,2n} & & & & \\
 & & & d_{1,2n-1} & & d_{2,2n} & & & \\
 & & d_{1,2n-2} & & d_{2,2n-1} & & d_{3,2n} & & \\
 & \iddots & & & & & & \ddots & \\
d_{1,2} & & d_{2,3} & & \cdots & & d_{2n-2,2n-1} & & d_{2n-1,2n}.
\end{array}$$

For $n = 3$, the triangle is

$$\begin{array}{ccccccccc}
 & & & & 2 & & & & \\
 & & & 2 & & 2 & & & \\
 & & 2 & & 3 & & 2 & & \\
 & 2 & & 3 & & 3 & & 2 & \\
2 & & 4 & & 3 & & 4 & & 2,
\end{array} \tag{3.9}$$

and for $n = 4$, the triangle is

$$
\begin{array}{ccccccccccccc}
 & & & & & & 5 & & & & & & \\
 & & & & & 5 & & 5 & & & & & \\
 & & & & 5 & & 7 & & 5 & & & & \\
 & & & 5 & & 7 & & 7 & & 5 & & & \\
 & & 5 & & 8 & & 7 & & 8 & & 5 & & \\
 & 5 & & 8 & & 8 & & 8 & & 8 & & 5 & \\
5 & & 10 & & 8 & & 10 & & 8 & & 10 & & 5.
\end{array}
\tag{3.10}
$$

It is clear that if one stacks these triangles on top of one another, one gets a generalization of Pascal's pyramid, where each layer n fits in the vacancies of the layer $n-1$ above it.

Remark 2. *We note some properties of these triangles.*

(1) *The sum of the entries in the triangle are precisely what we claim are given by* (2.5).
(2) *One notices immediately that the extremal columns are Catalan numbers C_{n-1}. This follows immediately from Corollary 5 and the fact that a Gessel word cannot begin with $\bar{1}$ or $\bar{2}$, or end with a 1 or 2. The even entries in the last row are $2C_{n-1}$.*
(3) *Every number in the interior of the triangle occurs $4k$ times for k a positive integer. Furthermore, they are organized as rhombus-shaped blocks of size four. This turns out to be true for all n. We will need this fact in the proof later and we state it as Lemma 6.*

4. Proof of Theorem 1

One simply has to analyze all possibilities of occurences of 1 and $\bar{1}$ case by case. Suppose 1 occurs at position i and $\bar{1}$ occurs at position j in a word of length $2n$ and $i < j$. Then by Corollary 5, the number of such Gessel words is C_{n-1}. The number of possibilities of i, j such that $1 \leq i < j \leq 2n$ is $n(2n-1)$. Therefore, the number of Gessel words where the 1 occurs before the $\bar{1}$ is

$$(2n-1)\binom{2n-2}{n-1}. \tag{4.1}$$

We now use Lemma 4 to count the number of words where $\bar{1}$ occurs at site i before 1 at site j,

$$
\begin{aligned}
&G_1([i,j],[-1,1];2n) \\
&\quad = \sum_{k_1=1}^{i-1} \sum_{k_2=0}^{j-1} a_{0,k_1}(i-1)\, a_{k_1-1,k_2-1}(j-i-1)\, a_{k_2,0}(2n-j),
\end{aligned}
\tag{4.2}
$$

which, using (3.2) gives

(4.3)

$$G_1([i,j],[-1,1];2n) = \sum_{k_1=1}^{i-1}\sum_{k_2=0}^{j-1} C^{(i-1+k_1)/2}_{(i-1-k_1)/2}\, C^{(2n-j+k_2)/2}_{(2n-j-k_2)/2} \left[\binom{j-i-1}{(j-i-1+k_1-k_2)/2} - \binom{j-i-1}{(j-i-1+k_1+k_2)/2}\right]$$

where C_n^m is the Catalan triangle number given by $\frac{(m-n+1)}{(m+1)}\binom{m+n}{n}$ for $0 \le n \le m, m \ge 0$.

We now use the following result to simplify calculations. The proof of this assertion is easily verified by expanding (4.3) and noting that the answer is the same when i is replaced by either $2i$ or $2i+1$ and similarly for j.

Lemma 6. *For* $1 \le i < j \le n-1$,

$$\begin{aligned} &G_1([2i,2j],[-1,1];2n) = G_1([2i,2j+1],[-1,1];2n) \\ &= G_1([2i+1,2j],[-1,1];2n) = G_1([2i+1,2j+1],[-1,1];2n). \end{aligned} \tag{4.4}$$

Then the total number of Gessel words with an $\bar{1}$ preceding an 1 is given by

(4.5)

$$\begin{aligned} \sum_{i=1}^{2n-1}\sum_{j=i+1}^{2n} G_1([i,j],[-1,1];2n) =& 4\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} G_1([2i,2j],[-1,1];2n) \\ &+ \sum_{i=1}^{n-1} G_1([2i,2i+1],[-1,1];2n) \\ &= 4(S_2 - S_3) + S_1. \end{aligned}$$

where we have split the sum in three parts, with

$$S_1 = \sum_{i=1}^{n-1} G_1([2i,2i+1],[-1,1];2n). \tag{4.6}$$

The remainder in (4.5) we split using (4.3), and using the variables $r = (2i - k_1 - 1)/2, s = (2n - 2j - k_2)/2$, as

$$\sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1} G_1([2i,2j],[-1,1];2n) = S_2 - S_3 \tag{4.7}$$

where

$$
\begin{aligned}
S_2 &= \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=0}^{i-1} \sum_{s=0}^{n-j-1} C_r^{2i-r-1} C_s^{2n-2j-s} \binom{2j-2i-1}{2j+s-n-r-1}, \\
S_3 &= \sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{r=0}^{i-1} \sum_{s=0}^{n-j-1} C_r^{2i-r-1} C_s^{2n-2j-s} \binom{2j-2i-1}{n-s-r-1}.
\end{aligned} \tag{4.8}
$$

We now estimate these three sums in turn.

4.1. **The sum S_2.** Replacing $r \to i-1-r$ and $s \to n-j-s-1$, substituting $k = j-i$ and rearranging the variables, we get

$$
S_2 = \sum_{r=0}^{n-3} \sum_{k=1}^{n-r-2} \sum_{i=r+1}^{n-1-k} \sum_{s=0}^{n-i-k-1} C_{i-r-1}^{i+r} C_{n-k-i-s-1}^{n-k-i+s+1} \binom{2k-1}{k-s+r-1}. \tag{4.9}
$$

Now replace $k \to k-1, i \to i-r-1$ to get

$$
S_2 = \sum_{r=0}^{n-3} \sum_{k=0}^{n-r-3} \sum_{i=0}^{n-r-k-3} \sum_{s=0}^{n-r-k-i-3} C_i^{i+2r+1} C_{n-k-i-r-s-3}^{n-k-i-r+s-1} \binom{2k+1}{k-s+r}. \tag{4.10}
$$

We now replace the r variable by $u = k+r$. Notice that the binomial coefficient term is independent of i for which we use the identity

$$
\sum_{i=0}^{C} C_i^{i+A} C_{C-i}^{B-i} = C_C^{A+B+1}, \tag{4.11}
$$

which means we are left with

$$
\begin{aligned}
S_2 &= \sum_{u=0}^{n-3} \sum_{k=0}^{u} \sum_{s=0}^{n-u-3} C_{n-u-s-3}^{u+n+s-2k+1} \binom{2k+1}{u-s} \\
&= \sum_{k=0}^{n-3} \sum_{u=0}^{n-k-3} \sum_{s=0}^{n-u-k-3} C_{n-u-s-k-3}^{u+n+s-k+1} \binom{2k+1}{u+1-s}.
\end{aligned} \tag{4.12}
$$

Let $A = n-k-3, v = u-s$ and $v' = s-u$. Then one easily verifies that

$$
\sum_{u=0}^{A} \sum_{s=0}^{A-u} = \sum_{v=0}^{A} \sum_{\substack{u=v \\ (s=u-v)}}^{A/2+v/2} + \sum_{v'=0}^{A} \sum_{\substack{s=v' \\ (u=s-v')}}^{A/2+v'/2} - \sum_{\substack{s=0 \\ (u=s)}}^{A/2} \tag{4.13}
$$

The binomial coefficient is independent of u in the first sum and of s in the remaining two and hence the innermost sum can be done using the identity

$$\sum_{s=v}^{n/2} C_{n-2s}^{B+2s} = \binom{B+n-1}{n-2v}. \tag{4.14}$$

This reduces the sum (after a change of variables) to

$$\begin{aligned} S_2 = &\sum_{k=0}^{n-3}\sum_{v=0}^{k}\binom{2k+3}{k-v}\binom{2n-2k-4}{n-k-v-2} \\ &-\sum_{k=0}^{n-3}\binom{2k+1}{k+1}\binom{2n-2k-3}{n-k-3} \end{aligned} \tag{4.15}$$

These sums are handled as special cases of the Chu-Vandermonde identity to yield

$$S_2 = \frac{n+2}{4}\binom{2n}{n} - 3\cdot 2^{2n-3}, \tag{4.16}$$

which appears as sequence A045720 [4] because it is the threefold convolution of the sequence $a_n = \binom{2n+1}{n+1}$.

4.2. The sum S_3.

$$S_3 = \sum_{i=1}^{n-2}\sum_{j=i+1}^{n-1}\sum_{r=0}^{i-1}\sum_{s=0}^{n-j-1} C_r^{2i-r-1} C_s^{2n-2j-s}\binom{2j-2i-1}{n-s-r-1}, \tag{4.17}$$

which after replacing $i \to i-r-1$ and subsequently $j \to j-r-i-2$ and rearranging becomes

$$S_3 = \sum_{j=0}^{n-3}\sum_{i=0}^{n-j-3}\sum_{s=0}^{n-j-i-3}\sum_{r=0}^{n-j-i-s-3} C_r^{2i+r+1} C_{n-j-i-r-s-3}^{n-j-r-i+s-1}\binom{2j+1}{s+j+i+2}. \tag{4.18}$$

We now use (4.11) to do the r sum and get

$$S_3 = \sum_{j=0}^{n-3}\sum_{i=0}^{n-j-3}\sum_{s=0}^{n-j-i-3} C_{n-j-i-s-3}^{n-j+i+s+1}\binom{2j+1}{s+j+i+2}. \tag{4.19}$$

Now, replacing s by $k = i + s$, we get

$$\begin{aligned} S_3 &= \sum_{j=0}^{n-3} \sum_{k=0}^{n-j-3} \sum_{i=0}^{k} C_{n-j-k-3}^{n-j+k+1} \binom{2j+1}{k+j+2} \\ &= \sum_{j=0}^{n-3} \sum_{k=0}^{n-j-3} (k+1) C_{n-j-k-3}^{n-j+k+1} \binom{2j+1}{k+j+2} \\ &= \sum_{k=0}^{n-3} \sum_{j=0}^{n-k-3} (k+1) C_{n-j-k-3}^{n-j+k+1} \binom{2j+1}{k+j+2}. \end{aligned} \tag{4.20}$$

We now use the identity

$$\sum_{j=C}^{B} C_{B-j}^{A-j} \binom{2j+1}{j-C} = \binom{A+B+2}{B-C}, \tag{4.21}$$

for the j sum to get

$$\begin{aligned} S_3 &= \sum_{k=0}^{n-3} (k+1) \binom{2n}{n-4-2k}, \\ &= \sum_{k=0}^{(n-4)/2} (k+1) \binom{2n}{n-4-2k}, \\ &= \frac{1}{2} \sum_{k=0}^{(n-4)/2} (2k+4) \binom{2n}{n-4-2k} - \sum_{k=0}^{(n-4)/2} \binom{2n}{n-4-2k} \\ &= \frac{n}{2} \binom{2n-2}{n-4} - \sum_{k=0}^{(n-4)/2} \binom{2n}{n-4-2k} \\ &= \frac{n}{2} \binom{2n-2}{n-4} - 2^{2n-2} + \frac{(2n)!(3n^2+n+2)}{2n!(n+2)!}. \end{aligned} \tag{4.22}$$

4.3. **The sum S_1.**

$$S_1 = \sum_{i=1}^{n-1} \sum_{r=1}^{i} \sum_{s=1}^{i} C_{i-r}^{i-1+r} C_{n-i-s}^{n-i+s-1} \left[\binom{0}{r-s} - \binom{0}{r+s-1} \right]. \tag{4.23}$$

The first term forces $r = s$ and the second term is identically zero because $r + s \geq 2$. This means we are left with

$$\begin{aligned} S_1 &= \sum_{i=1}^{n-1} \sum_{r=1}^{i} C_{i-r}^{i-1+r} C_{n-i-r}^{n-i+r-1} \\ &= (n-1) C_{n-1}. \end{aligned} \tag{4.24}$$

Thus the total number of Gessel words where an $\bar{1}$ occurs before an 1 defined in (4.5) is given, using (4.16),(4.22) and (4.24), by

$$4(S_2 - S_3) + S_1 = \frac{(n^3 + 4n^2 + 5n + 2)(2n)!}{2n!(n+2)!} - 2^{(2n-1)}, \tag{4.25}$$

and therefore, the total number of complete Gessel words is

$$G_1(n) = 4(S_2 - S_3) + S_1 + (2n-1)\binom{2n-2}{n-1} = \frac{(2n+1)}{2}\binom{2n}{n} - 2^{2n-1}, \tag{4.26}$$

which is exactly the same expression as (2.5). ■

5. Remarks

This section is intended to be speculative in nature and consequently, the statements are unproven as far as we know, though not necessarily very deep. In 2001, Simon Norton made the following conjecture in A000531 [4].

A conjectured definition: Let $0 < a_1 < a_2 < ... < a_{2n} < 1$. Then how many ways are there in which one can add or subtract all the a_i to get an odd number. For example, take $n = 2$. Then the options are $a_1 + a_2 + a_3 + a_4 = 1$ or 3; one can change ths sign of any of the a_i's and get 1; or $-a_1 - a_2 + a_3 + a_4 = 1$. That's a total of 7, which is the 2nd number of this sequence.

We want to connect this conjecture to Theorem 1. Before that, we need some preliminaries. One can represent every equation of the form $\pm a_1 \cdots \pm a_{2n} = 1$ as a $2n$-tuple of $+, -$ symbols. Let us replace every $-$ by a 0 and every $+$ by a 1. Then, one can represent all possible ways of ordering the $+$'s and $-$'s by binary words of length $2n$.

Let w be such a binary word. Then define $n_1(w)$ to be number of 1's in w. Also define $n_{10}(w)$ to be the number of occurences of distinct 10 subwords in w. For example, $n_{10}(1110) = 1$ and $n_{10}(0110000) = 2$. We now form the multiset S, where each word w occurs

$$m(w) = \left\lfloor \frac{n_1(w) - n_{10}(w)}{2} \right\rfloor \tag{5.1}$$

times. Note that if $m(w)$ is zero or negative, it never appears. Then, it seems that the cardinality of S is the same as the conjecture in the sequence. Moreover there is a bijection from the $\pm$ notation to the binary notation. This means that the number of times a binary word appears in S seems to be the same as the number of positive odd integers in the right hand side of the equation corresponding to the same binary word which admit solutions. We give a concrete example in Table 1.

$\pm$ word	Odd integer sums	Binary word	$n_1(w)$	$n_{10}(w)$	$m(w)$
$++++$	1,3	1111	4	0	2
$+++-$	1	1110	3	1	1
$++-+$	1	1101	3	1	1
$+-++$	1	1011	3	1	1
$-+++$	1	0111	3	0	1
$--++$	1	0011	2	0	1

TABLE 1. All allowed possibilities for $n = 2$.

The connection between the two problems is as follows. For each fixed number n_1 of $+$ signs from 2 to $2n$, count only those sums in which all possible $\binom{2n}{n_1}$ combinations give rise to that sum and add them up. This number is precisely the same as the number of Gessel words stated in Theorem 1 in which the 1 precedes the $\bar{1}$. The formula for the number of such Gessel words is given by (4.1). If one considers the set of only those $\pm$ words for fixed n_1 such that a number strictly smaller than $\binom{2n}{n_1}$ contribute, then this set is equinumerous with the Gessel words stated above in which the $\bar{1}$ precedes the 1 and is given by (4.25). This leads us to conjecture the presence of a bijection between the multiset S and the number of complete Gessel words with exactly one 1 and $\bar{1}$.

Number of $+$ and and $-$ signs	Sum=1	Sum=3	Sum=5	Sum=7
$8+$	1	1	1	1
$7+, 1-$	8	8	8	
$6+, 2-$	28	28	1	
$5+, 3-$	56	8		
$4+, 4-$	28	1		
$3+, 5-$	8			
$2+, 6-$	1			

TABLE 2. The number of words for a fixed number of $+$ and $-$ signs and fixed sum in the case $n = 4$.

For example, there are 6 complete Gessel words for $n = 2$ where the 1 precedes the $\bar{1}$. From Table 1, one sees that all possible terms contribute when we have either $4+$ or $3+, 1-$ signs. There are two possibilities for the former (when the sums are 1 and 3) and four for the latter (when the sum is 1). Similarly, there is only one complete Gessel word for $n = 2$ where $\bar{1}$ precedes 1, which is given by $2\bar{1}21\bar{2}\bar{2}$ and for $2+, 2-$ signs, there are 6 possible words, but only one contributes.

For any fixed n_1 and any fixed odd integer sum, the number of words which allow this seem to be of the form $\binom{2n}{k}$ where k varies from 0 to $n-1$! We illustrate this via another concrete example in Table 2. Notice that the only integers appearing in the table are the binomial coefficients $\binom{8}{k}$ with $k = 0, 1, 2$ or 3. Another observation is that if one draws lines of 45° starting from the first column in Table 2 and looks at the diagonal columns, one finds the pattern,

$$\left|\begin{array}{c|c|c|c|c|c|c} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & 8 & 8 & 8 & 8 & 8 & \\ & & 28 & 28 & 28 & & \\ & & & 56 & & & \end{array}\right| \tag{5.2}$$

from which it is clear that each of these diagonal columns in (5.2) starts with $\binom{2n}{0}$ with subsequent values of the lower index increasing by 1. The first four columns above correspond exactly to the Gessel words where 1 precedes $\bar{1}$ is the sum of the entries is precisely $(2n-1)\binom{2n-2}{n-1}$ with $n = 4$. This pattern persists up until $n = 6$.

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References

[1] Marko Petkovsek and Herbert S. Wilf, On a conjecture of Ira Gessel, **preprint**, arXiv:0807.3202.

[2] Mireille Bousquet-Mélou and Marni Mishna, Walks with small steps in the quarter plane, **preprint**, arXiv:0810.4387.

[3] Manuel Kauers, Christoph Koutschan, and Doron Zeilberger, Proof of Ira Gessel's Lattice Path Conjecture, **preprint**, arXiv:0806.4300.

[4] N. J. A. Sloane, *The On-Line Encyclopedia of Integer Sequences*, `http://www.research.att.com/~njas/sequences`

[5] D. P. Robbins, Areas of polygons inscribed in a circle, Amer. Math. Monthly, **102** (1995), 523–530.

[6] M. Fedorchuk and I. Pak, Rigidity and polynomial invariants of convex polytopes, *Duke Math. J.* **129** (2005), 371–404.

[7] F. M. Malay, D. P. Robbins and J. Roskies, On the areas of cyclic and semicyclic polyons, **preprint**, arXiv:math/0407300.

[8] Igor Pak, The area of cyclic polygons: recent progress on Robbins' conjectures, *Advances in Applied Mathematics* **34** no. 4 (2005), 690–696.

[9] L. Comtet, *Advanced Combinatorics*, D. Reidel, Dordrecht, Holland, 1974.

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