CORRIGENDA TO "INTERESTING SERIES INVOLVING THE CENTRAL BINOMIAL COEFFICIENT" [AM. MATH. MONTHLY VOL 92 (1985)]

RICHARD J. MATHAR

Abstract. These are seven corrigenda to equations in the Lehmer article in American Mathematical Monthly 92 (1985), pp 449–457, partially reproduced in the Apelblat tables of integrals and series.

1. Paper by D. H. Lehmer

The corrigenda to equations in an article by the late Derrick Lehmer [5] are:

(1) Multiply the right hand side of the first equation on page 451 by two:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\binom{2n}{n}}{n4^n} = 2\log\left[\frac{\sqrt{2}+1}{2}\right].$$

The numerical value is sequence A157699 in the Online Encyclopedia of Integer Sequences (OEIS) [6].

(2) Switch the sign of the first but last term in equation (7):

$$x\sum_{n=1}^{\infty} \frac{\binom{2n}{n}}{n(n+1)}x^n = 2x\log\frac{1-\sqrt{1-4x}}{x} + \frac{\sqrt{1-4x}}{2} - x(\log 4 - 1) - \frac{1}{2}.$$

(3) Multiply the right hand side of an equation on page 453 by two:

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m\binom{2m}{m}} = 2\log\left(\frac{1+\sqrt{5}}{2}\right)/\sqrt{5}.$$

The numerical value is sequence A086466 in the OEIS [6].

(4) Flip a sign on one side of another equation on page 453:

$$\sum_{m=1}^{\infty} \frac{(-1)^{m-1}}{m^2\binom{2m}{m}} = 2\left\{\log\left(\frac{\sqrt{5}+1}{2}\right)\right\}^2.$$

The numerical value is sequence A086467 in the OEIS [6].

(5) Multiply the right hand side of the last line of another formula on page 453 by four:

$$\sum_{m=1}^{\infty} \frac{1}{m^3\binom{2m}{m}} = -\frac{4\zeta(3)}{3} - \frac{\pi\sqrt{3}}{18}\left\{\psi\left(\frac{1}{3}\right) - \psi\left(\frac{2}{3}\right)\right\}.$$

Date: May 2, 2009.
2000 *Mathematics Subject Classification.* Primary 40-01, 40H05; Secondary 00A20.
Key words and phrases. Errata, Central Binomial Coefficient, series.

Variants in notation have appeared since then, see theorem 3.3 of a paper by Borwein et al. [2], which is equation (47) in a later work [3], or equation (2.67) by Davydychev and Kalmykov [4]. The numerical value is sequence A145438 in the OEIS [6].

(6) Drop a factor two on the right hand side of a formula on page 455 and change sign on either side:

$$\sum \frac{(-1)^m m^3}{\binom{2m}{m}} = \frac{2}{625}[14\sigma + 5].$$

The numerical value is sequence A157701 [6].

(7) The last equation on page 457 cannot be reproduced; the value of the sum is 0.280851790115... whereas $\pi^2/36 = \zeta(2)/6 \approx 0.27415567\ldots$ Equation (13) of page 453 yields the substitute

$$\sum \frac{2^m(2-\sqrt{3})^m}{m^2\binom{2m}{m}} = 2\left(\arcsin\tau\right)^2, \quad \tau \equiv \frac{\sqrt{3}-1}{2} = \sqrt{2}\sin\frac{\pi}{12} = \sin\frac{\pi}{3} - \sin\frac{\pi}{6}.$$

2. Apelblat Tables

Equivalent corrections concern section 4.1 of the Apelblat tables [1]:

(1) Insert an exponent 2 on the left hand side of formula 40:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1} n^2 (n!)^2}{(2n)!} = \frac{4}{125}\left[5 - \sqrt{5}\ln\left(\frac{\sqrt{5}+1}{2}\right)\right].$$

The numerical value is sequence A145434 [6].

(2) Change two numbers on the right hand side of formula 42:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n!)^2}{n(2n)!} = \frac{2}{\sqrt{5}}\ln\left(\frac{1+\sqrt{5}}{2}\right).$$

(3) Multiply the right hand side of formula 47 by four and show that the ψ are trigamma functions:

$$\sum_{n=1}^{\infty} \frac{(n!)^2}{n^3(2n)!} = 4\left\{\frac{\pi\sqrt{3}}{72}\left[\psi^{(1)}\left(\frac{2}{3}\right) - \psi^{(1)}\left(\frac{1}{3}\right)\right] - \frac{1}{3}\zeta(3)\right\}.$$

(4) Replace the right hand side of formula 81:

$$\sum_{n=1}^{\infty} \frac{2^n(2-\sqrt{3})^n(n!)^2}{n^2(2n)!} = 2\arcsin^2\left(\frac{\sqrt{3}-1}{2}\right).$$

References

1. Alexander Apelblat, *Tables of integrals and series*, 1 ed., Harri Deutsch, Thun, 1996.
2. Jonathan Michael Borwein, David J. Broadhurst, and Joel Kamnitzer, *Central binomial sums, multiple Clausen values and Zeta values*, Exp. Math. **10** (2001), no. 1, 25–34. MR 1821569 (2002k:11105)
3. Jonathan Michael Borwein and Roland Girgensohn, *Evaluations of binomial series*, Aequat. Math. **70** (2005), no. 1–2, 25–36. MR 2167981
4. A. I. Davydychev and M. Yu. Kalmykov, *Massive Feynman diagrams and inverse binomial sums*, Nucl. Phys. B **699** (2004), no. 1–2, 3–64. MR 2095786
5. Derrick Henry Lehmer, *Interesting series involving the central binomial coefficient*, Amer. Math. Monthly **92** (1985), no. 7, 449–457. MR 0801217 (87c:40002)

6. Neil J. A. Sloane, *The On-Line Encyclopedia Of Integer Sequences*, Notices Am. Math. Soc. **50** (2003), no. 8, 912–915, http://www.research.att.com/~njas/sequences/. MR 1992789 (2004f:11151)

URL: `http://www.strw.leidenuniv.nl/~mathar`
E-mail address: `mathar@strw.leidenuniv.nl`

Leiden Observatory, Leiden University, P.O. Box 9513, 2300 RA Leiden, The Netherlands