THE ALTERNATIVE OPERAD IS NOT KOSZUL

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In the very interesting online compendium [Lod], the alternative operad is marked conjecturally as Koszul. The purpose of this note is to demonstrate that this, unfortunately, is not true. In doing so, we are helped with the programs Albert [A1] and PARI/GP [P].

1. The alternative operad and its dual

Recall that an algebra is called *right-alternative* if it satisfies the identity

$$(xy)y = x(yy),$$

and *left-alternative* if it satisfies the identity

$$(xx)y = x(xy).$$

An algebra which is both right-alternative and left-alternative is called *alternative*. Linearizing these identities, we get

$$(xy)z + (xz)y - x(yz) - x(zy) = 0 \qquad \text{(RA)}$$

and

$$(xy)z + (yx)z - x(yz) - y(xz) = 0 \qquad \text{(LA)}$$

respectively.

If characteristic of the ground field is different from 2, these identities are equivalent to the initial ones, so the corresponding operads $\mathcal{RA}lt$, $\mathcal{LA}lt$ and $\mathcal{A}lt$ (dubbed *right-alternative*, *left-alternative* and *alternative operads*) are quadratic. In characteristic 2 these operads are not quadratic, and, generally, things are going berserk, so we will exclude this case from our considerations.

Right- and left-alternative algebras are opposed to each other, i.e. if A is a right-alternative algebra, then the algebra defined on the same vector space A with multiplication $x \circ y = yx$ is a left-alternative algebra, and vice versa. Hence all the statements below for left-alternative algebras automatically follow from the corresponding statements for right-alternative ones, and in the proofs we will consider the right-alternative case only.

Note that free alternative algebras are much more difficult objects than, for example, their associative or Lie counterparts, and are still not understood sufficiently well.

Proposition. *Each of the operads dual to the right-alternative, left-alternative and alternative operad is defined by two identities: associativity and the identity*

$$xyz + xzy = 0 \qquad (\mathrm{RA}^!)$$

in the right-alternative case,

$$xyz + yxz = 0$$

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in the left-alternative case, and

$$xyz + yxz + zxy + xzy + yzx + zyx = 0. \tag{A$^!$}$$

in the alternative case.

In the alternative case, this is stated in [Lod] without proof, so we will provide a simple (and pretty much standard for such situations) proof for completeness. Following [Lod], we will call algebras over the corresponding dual operads *dual right-alternative*, *dual left-alternative* and *dual alternative*, respectively.

Proof. Let L be an alternative algebra, and A be a dual alternative algebra. Then the space $L \otimes A$ equipped with the bracket

$$[x \otimes a, y \otimes b] = xy \otimes ab - yx \otimes ba$$

for $x, y \in L$, $a, b \in A$, is a Lie algebra.

Identities (RA) and (LA) imply that we may take 7 monomials $(xy)z$, $(yx)z$, $(xz)y$, $(zx)y$, $(yz)x$, $(zy)x$, $z(xy)$ as the basis of $\mathcal{A}lt(3)$, with the rest of monomials expressed through them as follows:

$$\begin{aligned}
z(yx) &= (zx)y + (zy)x - z(xy)\\
y(zx) &= -(zx)y + (yz)x + z(xy)\\
y(xz) &= (yx)z + (zx)y - z(xy)\\
x(yz) &= (xy)z - (zx)y + z(xy)\\
x(zy) &= (xz)y + (zx)y - z(xy).
\end{aligned} \tag{1}$$

In particular, $\dim \mathcal{A}lt(3) = 7$.

Writing the Jacobi identity for triple $x \otimes a$, $y \otimes b$, $z \otimes c$ for $x, y, z \in L$, $a, b, c \in A$, substituting in it all equalities (1), and collecting similar terms, we get:

$$\begin{aligned}
&\ (xy)z \otimes ((ab)c - a(bc))\\
&+ (yx)z \otimes (b(ac) - (ba)c)\\
&+ (xz)y \otimes (a(cb) - (ac)b)\\
&+ (zx)y \otimes (a(bc) + a(cb) + b(ac) + b(ca) + (ca)b + c(ba))\\
&+ (yz)x \otimes ((bc)a - b(ca))\\
&+ (zy)x \otimes (c(ba) - (cb)a)\\
&- z(xy) \otimes (a(bc) + a(cb) + b(ac) + b(ca) + c(ab) + c(ba))\\
&= 0,
\end{aligned}$$

and the claimed statement obviously follows.

In the right-alternative case $\dim \mathcal{RA}lt(3) = 9$ and the computations are similar. □

Corollary.

(i) *A dual right- or left-alternative algebra over the field of characteristic different from 2 is nilpotent of degree 4.*
(ii) *A dual alternative algebra over the field of characteristic different from 2 and 3 is nilpotent of degree 6.*

Proof. (i) We have, by subsequent application of associativity and ($RA^!$):

$$(xyz)t = -(xzy)t = -x(zyt) = x(zty) = x(zt)y = -xy(zt).$$

(ii) Substituting in ($A^!$) $x = y = z$, we get $6x^3 = 0$. The claim then follows from the results centered around the classical Dubnov–Ivanov–Nagata–Higman theorem about nilpotency of associative nil algebras (see, for example, [Dr, §8.3]).

These claims also could be proved with the help of Albert. □

2. Dimension sequence

We are going to establish non-Koszulity using the well-known Ginzburg–Kapranov criterion [GiK, Proposition 4.14(b)] which tells that if a quadratic operad $\mathcal{P}$ over the field of characteristic zero is Koszul, then

$$g_{\mathcal{P}}(g_{\mathcal{P}^!}(x)) = x, \tag{2}$$

where

$$g_{\mathcal{P}}(x) = \sum_{n=1}^{\infty} (-1)^n \frac{\dim \mathcal{P}(n)}{n!} x^n$$

is the Poincaré series of operad $\mathcal{P}$, and $\mathcal{P}^!$ is the dual of $\mathcal{P}$. For this, we need to know the first few terms of the sequence $\dim \mathcal{P}(n)$ for the corresponding operads and their duals. This is achieved with the help of Albert.

Albert computes over a fixed prime field, and we are going to explain now how these computations imply results valid in characteristic zero.

For an operad $\mathcal{P}$, the space $\mathcal{P}(n)$ coincides with all multilinear monomials in n variables, modulo corresponding relations. Thus $\dim \mathcal{P}(n)$ is equal to the number of all nonassociative multilinear monomials in n variables (equal to $n!$ multiplied by the nth Catalan number) minus the rank of matrix M which is obtained from coefficients of all possible consequences of the corresponding identities valid for elements of $\mathcal{P}(n)$. As coefficients of identities defining our operads are integers, this is an integer matrix, and it is possible to consider its reduction M_p modulo a given prime p.

It is clear that $rk\, M \ge rk\, M_p$. The question is how to ensure equality of these values. What follows is a variation on the standard theme in numerical linear algebra – how to substitute rational or integer arithmetic by modular one.

Let represent the matrix M in the Smith normal form, i.e. as a product

$$M = A\, diag(d_1, \dots, d_r, 0, \dots, 0)\, B,$$

where A and B are integer quadratic matrices with determinant equal to ± 1, $r = rk\, M$, and $d_1, \dots, d_r$ are nonzero integers such that d_i is divided by d_{i+1}. Reduction of this product modulo p will produce the Smith normal form of M_p, i.e. A_p, B_p are still matrices with determinant ± 1 over $\mathbb{Z}_p$, and the number of nonzero elements in the diagonal matrix

$$diag(d_1 (mod\, p), \dots, d_r (mod\, p), 0, \dots, 0)$$

is equal to $rk\, M_p$.

If we pick primes $p_1, \dots, p_n$ in such a way that

$$p_1 \dots p_n > |d_1 \dots d_r|, \tag{3}$$

then

$$d_1 \dots d_n \not\equiv 0 \,(mod\, p_1 \dots p_n),$$

and by the Chinese Remainder Theorem,

$$d_1 \dots d_n \not\equiv 0 \,(mod\, p_i),$$

and hence $rk\, M_{p_i} = rk\, M$ for some p_i. Consequently, if $rk\, M_{p_i} = r$ for *all* $i = 1, \dots, n$, then $rk\, M = r$.

It remains to estimate $p_1 \dots p_n$ to ensure inequality (3). The product $d_1 \dots d_r$ is equal, up to sign, to the determinant of a certain minor T of M of size $r \times r$. As the identities defining our operads have coefficients 1 or -1, all nonzero elements of the matrix M could be chosen to be equal to 1 or -1, so the usual estimate in such situations is provided by the Hadamard inequality: $|\det(T)| \le r^{\frac{r}{2}}$ (see, for example, [HJ, §7.8.2]).

To summarize: if there are primes $p_1, \dots, p_n$ such that Albert produces the same value

$$\dim \mathcal{P}(n) = r \tag{4}$$

modulo these primes, and

$$p_1 \dots p_n > r^{\frac{r}{2}}, \tag{5}$$

then (4) holds over integers, and, consequently, over any field of characteristic zero.

Albert allows to compute over a prime field $\mathbb{Z}_p$ with $p \le 251$. We have modified Albert [A2] to allow primes up to the largest possible value of the largest signed integer type, which is $2^{63} - 1$ on the standard modern computer architectures, both 32-bit and 64-bit. We also have modified it to facilitate batch processing.

As the time of Albert computations turns out not to depend significantly on the value of prime, to minimize the overall computation time, we are minimizing the number of Albert runs at the expense of larger primes, i.e. when choosing primes in the given range satisfying the condition (5), we are choosing as large primes as possible. This could be done with the help of PARI/GP.

Using all this, we establish:

Lemma 1. *Over the field of characteristic zero, the first* 5 *terms of the sequence* $\dim \mathcal{RA}lt(n)$ *are:* 1, 2, 9, 60, 530.

Proof. Over any field, the first two values are obvious, and the third could be established by hand (in fact, we already did it in the proof of Proposition in §1).

The are 3 primes $< 2^{63}$ satisfying the inequality (5) for $r = 60$:

$$2^{63} - 259, \quad 2^{63} - 165, \quad 2^{63} - 25.$$

With all these 3 primes, Albert produces $\dim \mathcal{RA}lt(4) = 60$.

Similarly, the number of largest possible primes $< 2^{63}$ satisfying the inequality (5) for $r = 530$, is 39, and Albert produces $\dim \mathcal{RA}lt(5) = 530$ for all these 39 primes. ☐

Though we do not need it for our purposes, we have also computed $\dim \mathcal{RA}lt(6) = 5820$ for a few random primes.

Lemma 2. *Over the field of characteristic zero, the first* 6 *terms of the sequence* $\dim \mathcal{A}lt(n)$ *are:* 1, 2, 7, 32, 175, 1080.

Proof. We follow the same scheme as in the proof of Lemma 1. The corresponding number of primes is 2 for $n = 4$, 11 for $n = 5$, and 87 for $n = 6$, and Albert produces the expected answers for all these primes. ☐

The first 5 terms in Lemma 2 were already specified in [Lod], but the case $n = 6$ is crucial. It requires the only time-consuming Albert computations among all computations mentioned in this note. We found that the optimal setting in this case was to add first the left-alternative identity, and then the right-alternative one, and use the static (as opposed to the sparse) matrix structure. It took about 13½ days (about 3¾ hours per prime) on a 3GHz Pentium 4 CPU.

3. Non-Koszulity

Theorem 1. *The right-alternative, left-alternative and alternative operads over the field of characteristic zero are not Koszul.*

Proof. First consider the right-alternative case. By Corollary (i) in §1, $\mathcal{RA}lt^!(n)$ vanishes for $n \geq 4$. It is easy to see that monomials $(xy)z$, $(zx)y$ and $(yz)x$ could be taken as a basis of $\mathcal{RA}lt^!(3)$, thus $\dim \mathcal{RA}lt^!(3) = 3$ and the corresponding Poincaré series is:

$$g_{\mathcal{RA}lt^!}(x) = -x + x^2 - \frac{1}{2}x^3.$$

On the other hand, by Lemma 1,

$$g_{\mathcal{RA}lt}(x) = -x + x^2 - \frac{3}{2}x^3 + \frac{5}{2}x^4 - \frac{53}{12}x^5 + O(x^6),$$

and

$$g_{\mathcal{RA}lt}(g_{\mathcal{RA}lt^!}(x)) = x + \frac{1}{6}x^5 + O(x^6),$$

what contradicts Koszulity.

Now consider the alternative case. By Corollary (ii) in §1, $\mathcal{A}lt^!(n)$ vanishes for $n \geq 6$. Either computation with Albert, or reference to [Lop, Propositions 1 and 2] provides dimensions of these spaces for small n which allows us to write down the Poincaré series of the operad $\mathcal{A}lt^!$:

$$g_{\mathcal{A}lt^!}(x) = -x + x^2 - \frac{5}{6}x^3 + \frac{1}{2}x^4 - \frac{1}{8}x^5.$$

On the other hand, by Lemma 2,

$$g_{\mathcal{A}lt}(x) = -x + x^2 - \frac{7}{6}x^3 + \frac{4}{3}x^4 - \frac{35}{24}x^5 + \frac{3}{2}x^6 + O(x^7),$$

and

$$g_{\mathcal{A}lt}(g_{\mathcal{A}lt^!}(x)) = x - \frac{11}{72}x^6 + O(x^7),$$

what contradicts Koszulity. □

4. Positive characteristic

The original Ginzburg–Kapranov operadic theory involves representation theory of the symmetric group peculiar to characteristic zero case. While extensions of the operadic theory to the case of positive characteristic exist, none of them, to our knowledge, includes an analogue of the Ginzburg–Kapranov criterion for Koszulity of a quadratic operad in terms of Poincaré series.

While, therefore, checking the validity of equation (2) in positive characteristic does not make much sense, the question of computation of dimension sequence $\dim \mathcal{P}(n)$ for various operads $\mathcal{P}$ is still of interest. In this section we collect a few remarks and computational results concerning this question for the alternative and right-alternative operads and their duals.

For the dual operads, the corresponding dimension sequences terminate at low terms as indicated in the proof of Theorem 1, the same way for zero and positive characteristics, except for the case of the dual alternative operad over the field of characteristic 3.

Theorem 2. *Over the field of characteristic* 3*,* $\dim \mathcal{A}lt^!(n) = 2^n - n$*.*

For $n \le 8$ the claim could be proved with the aid of Albert. The proof in the general case is long and somewhat cumbersome, and will drive us far away from the main question considered in this note, so we will outline only the main idea. We came with this idea by inspecting the corresponding entry A000325 in [OEIS].

Sketch of the proof. For associative algebras, identity ($\mathrm{A}^!$) is equivalent to the identity

$$[[x,y],y] = 0.$$

In other words, an associative algebra over the field of characteristic 3 is dual alternative if and only if its associated Lie algebra is 2-Engel. It is well-known that 2-Engel Lie algebras are nilpotent of order 4. Free associative algebras which are Lie-nilpotent of order 4 were studied in the recent paper [EKM]. It is possible to extend some of the results of that paper to the case of characteristic 3, and, in particular, to construct a presentation of such algebras. From this, by adding more relations, one may construct a presentation of free dual alternative algebras, and using Composition (=Diamond) Lemma, to get a description of a basis of such algebras in combinatorial terms. For elements of the basis containing each free generator in the first degree, these combinatorial terms are expressed as the so-called Grassmann permutations, i.e. $\mathcal{A}lt^!(n)$ has a basis consisting of associative monomials of the form $a_{i_1} \dots a_{i_n}$ such that the permutation $(i_1 \dots i_n)$ has exactly one descent. The number of such permutations is $2^n - n$. □

The case of characteristic 3 is also exceptional for the alternative operad: in this case, the first 5 terms of $\dim \mathcal{A}lt(n)$ are the same as in Lemma 2, while the 6th term is equal, surprisingly, to 1081.

Note also that the scheme of computations presented in §2 is insufficient to deduce the validity of (4) over *all* prime fields. Either by the standard ultraproduct argument, or observing, by the same argument as in §2, that the equality (4) in characteristic zero implies the same equality in characteristic p for all $p > r^{\frac{r}{2}}$, we may deduce that it is valid for all but a finite number of characteristics. So, in principle, we could establish the validity of (4) in all characteristics by verifying it modulo all primes $\le r^{\frac{r}{2}}$ and for one prime $> r^{\frac{r}{2}}$. This is, however, computationally infeasible in almost all practical cases. Note, however, that in all characteristics we have $\dim \mathcal{P}(n) \ge r$.

To be able to establish the equality (4) in all characteristics, apparently other methods are needed. For example, one may try to utilize the capability of Albert to produce multiplication table between elements of $\mathcal{P}(n)$ up to the given degree. It seems that the scheme, based on the Chinese Reminder Theorem and similar to those presented in §2, but utilizing the multiplication table instead of just dimensions of the corresponding spaces of multilinear monomials, could be used for that, provided that all coefficients in the computed multiplication tables are rational numbers with relatively small numerators and denominators modulo the respective primes. According to a few Albert computations we have performed for $\mathcal{A}lt(6)$, the latter seems to be the case for the alternative operad.

5. Questions

There are several proofs in the literature of non-Koszulity of other operads using the Ginzburg–Kapranov criterion: in [GeK, footnote to §3.9(d)] for the so-called mock-Lie and mock commutative operads (which are dual to each other and are cyclic quadratic operads with one generator), in [GR, Proposition 2.3] for certain Lie-admissible operads dubbed G_4-Ass and G_5-Ass, in [Dz1, Theorem 10.1] for a certain skew-symmetric operad dubbed left-Alia, and in [Dz2] for the Novikov operad. In all these cases, it was enough to check Poincaré series up to the 4th or 5th term. It is interesting whether there exists a bound on the degree of Poincaré series such that the validity of the identity (2) for a binary quadratic operad $\mathcal{P}$ up to this degree guarantees its validity in all degrees.

It is also interesting to give a concrete example of a binary quadratic operad which is not Koszul but for which the equality (2) holds (such examples exist for associative quadratic algebras – see [PP, §3.5] and references therein).

Note also that it remains a challenging problem to compute the Poincaré series of $\mathcal{A}lt$.

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Software and online repositories

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[A2] Albert version 4.0M (modified); `http://justpasha.org/math/albert/` .

[OEIS] The On-line Encyclopedia of Integer Sequences.

[P] PARI/GP; `http://pari.math.u-bordeaux.fr/` .

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