A Combinatorial Survey of Identities for the Double Factorial

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Abstract

We survey combinatorial interpretations of some dozen identities for the double factorial such as, for instance, $(2n-2)!! + \sum_{k=2}^{n} \frac{(2n-1)!!(2k-4)!!}{(2k-1)!!} = (2n-1)!!$. Our methods are mostly bijective.

1 Introduction

There are a surprisingly large number of identities for the odd double factorial $(2n-1)!! = (2n-1)\cdot(2n-3)\cdots 3\cdot 1 = \frac{(2n)!}{2^n n!}$ that involve round numbers (small prime factors), as well as several that don't. The purpose of this paper is to present (and in some cases, review) combinatorial interpretations of these identities. Section 2 reviews combinatorial constructs counted by $(2n-1)!!$. Section 3 uses Hafnians to establish one of these manifestations. The subsequent sections contain the main results and treat individual identities, presenting one or more combinatorial interpretations for each; Section 4 is devoted to round-number identities, Section 5 to non-round identities, Section 6 to refinements involving double summations interpreted by two statistics in addition to size.

The even double factorial is $(2n)!! = 2n\cdot(2n-2)\cdots 2 = 2^n n!$. The recurrence $k!! = k(k-2)!!$ allows the definition of the double factorial to be extended to odd negative arguments. In particular, the values $(-1)!! = 1$ and $(-3)!! = -1$ will arise in some of the identities. We use the notations $[n]$ for the set $\{1,2,\ldots,n\}$ and $[a,b]$ for the closed interval of integers from a to b. For negative k, $\binom{n}{k} = 0$ as usual, except that identities (2) and (10) below require $\binom{-1}{-1} := 1$. It has become customary to draw trees down but when a construction involves growing a tree, it seems more natural to draw it up.

Also, to visualize ordered trees as Dyck paths, they must go up. So we will combine arborological pictures conjuring roots and leaves with the usual genealogical terminology of parents, children, and siblings. We sometimes refer to (clockwise) *walkaround* order of the edges/vertices in a tree; more formally, it is the order edges are visited in depth-first search and the preorder of the vertices.

2 Combinatorial manifestations of $(2n-1)!!$

We begin with a review of some combinatorial manifestations of $(2n-1)!!$, each illustrated for the case $n=2$. In all cases, the parameter n is the *size* of the object.

2.1 Trapezoidal words

The Cartesian product $[1] \times [3] \times \ldots \times [2n-1]$.

$$11, \quad 12, \quad 13.$$

The elements of this Cartesian product, the most obvious manifestation of $(2n-1)!!$, were called *trapezoidal* words by Riordan [1]. A minor variation is *symmetric trapezoidal* words: the Cartesian product $[0,0] \times [-1,1] \times [-2,2] \times \ldots \times [-(n-1), n-1)]$.

2.2 Perfect matchings

Perfect matchings of $[2n]$.

$$12/34, \quad 13/24, \quad 14/23.$$

A *perfect matching* of $[2n] = \{1, 2, \ldots, 2n\}$ is a partition of $[2n]$ into 2-element subsets $a(1)\,b(1)\,/\,a(2)\,b(2)\,/\,\ldots\,/\,a(n)\,b(n)$ written, in standard form, so that $a(i) < b(i)$ for all i, and $a(1) < a(2) < \ldots < a(n)$. Erasing the virgules (slashes) then gives a bijection to the *perfect matching permutations* of $[2n]$, denoted $\mathcal{P}_n$: the permutations $a(1)\,b(1)\,a(2)\,b(2)\,\ldots\,a(n)\,b(n)$ (denoted $(\mathbf{a}, \mathbf{b})$) of $[2n]$ satisfying $a(i) < b(i)$ for all i, and $a(1) < a(2) < \ldots < a(n)$. Given a perfect matching permutation $(\mathbf{a}, \mathbf{b})$, form a list whose $a(i)$-th and $b(i)$-th entries are both i. This is a bijection to the permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ in which the first occurrences of $1, 2, \ldots, n$ occur in that order.

A perfect matching of $[2n]$ can be regarded as a fixed-point-free involution on $[2n]$ and also as a 1-regular graph on $[2n]$, whose pictorial representation is sometimes called a Brauer diagram.

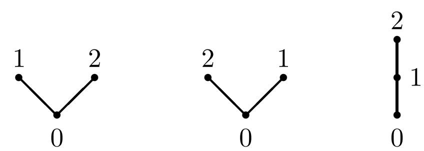
2.3 Stirling permutations

Permutations of the multiset $\{1, 1, 2, 2, \ldots, n, n\}$ in which, for each i, all entries between the two occurrences of i exceed i [2].

$$1122, \quad 1221, \quad 2211.$$

2.4 Increasing ordered trees

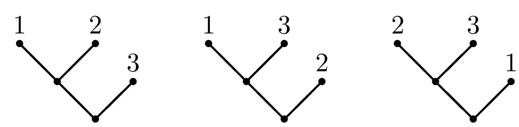
Increasing vertex-labeled ordered trees of n edges, label set $[0, n]$ [3, 4].



Given such a tree, delete the root label and transfer the remaining labels from vertices to parent edges. Walk clockwise around the tree thereby traversing each edge twice and record labels in the order encountered. This is a bijection to Stirling permutations due to Svante Janson [5]. Label sets other than $[0, n]$ may arise and the term *standard* then emphasizes that a tree's label set is an initial segment of the nonnegative integers.

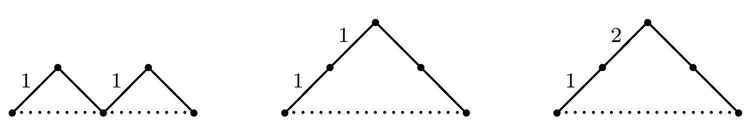
2.5 Leaf-labeled 0-2 trees

0-2 trees (rooted, unordered, each vertex has 0 or 2 children) with $n + 1$ labeled leaves, label set $[1, n + 1]$ [6, Chapter 5.2.6].

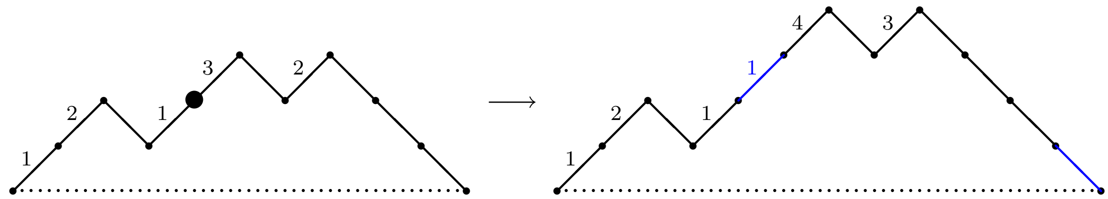


2.6 Height-labeled Dyck paths

Dyck paths [6, Exercise 6.19] of n upsteps and n downsteps in which each upstep is labeled with a positive integer $\leq$ the height of its top vertex.



This item is due to Jean Françon and Gérard Viennot [7, 8]; they observe that it is a consequence of their bijection from permutations to certain marked-up lattice paths. Here is perhaps the simplest proof and several further bijections that prove the result will appear in the course of the paper. A height-labeled (HL) Dyck path P of size $n-1$ has $2n-1$ vertices, each of which can be used to construct a height-labeled Dyck path of size n: split P at the specified vertex into subpaths P_1 and P_2, insert an upstep between P_1 and P_2, increment by 1 the labels on P_2, and append a downstep, as illustrated below.

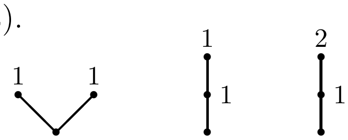


HL Dyck 5-path with marked vertex HL Dyck 6-path

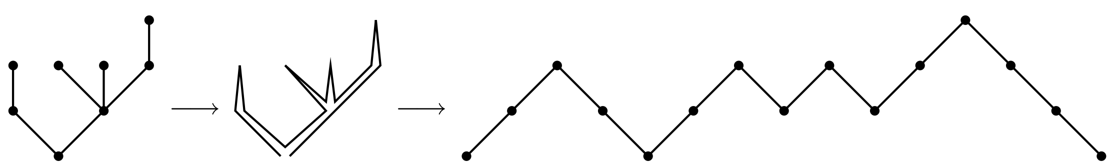
The process can be reversed by locating the last upstep with label 1 in a size-n path. Thus there is a multiplying factor of $2n-1$ from size $n-1$ to size n and the number of height-labeled Dyck n-paths is indeed $(2n-1)!!$.

2.7 Height-labeled ordered trees

Ordered trees of n edges in which each non-root vertex is labeled with a positive integer $\leq$ its height (distance from root).



The "accordion" bijection from ordered trees to Dyck paths—burrow up the branches from the root and open out the tree as illustrated—sends non-root vertices to tops of upsteps and preserves height.

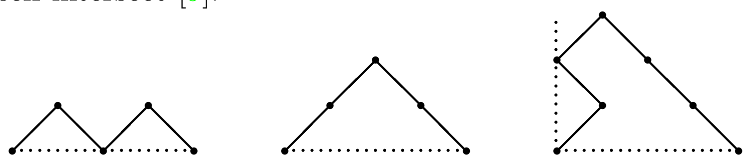


The "accordion" bijection

Thus height-labeled ordered trees transparently correspond to height-labeled Dyck paths.

2.8 Overhang paths

Lattice paths of steps $(1,1),(1,-1),(-1,1)$ from (0,0) to $(2n,0)$ that lie in the first quadrant and do not self-intersect [9].



Listing the ordinates of the upstep tops is a bijection to trapezoidal words.

3 Pfaffians, Hafnians and Dyck paths

The Pfaffian is usually defined for a skew-symmetric matrix but it can just as well be defined for the upper triangular array $T=(x_{ij})_{1\le i<j\le 2n}$:

$$\mathrm{Pf}\,(T)=\sum_{(\mathbf{a},\mathbf{b})\in\mathcal{P}_n}\mathrm{sgn}((\mathbf{a},\mathbf{b}))x_{a(1)b(1)}x_{a(2)b(2)}\cdots x_{a(n)b(n)},$$

a sum over all $(2n-1)!!$ perfect matching permutations $(\mathbf{a},\mathbf{b})$ in $\mathcal{P}_n$ (where $\mathrm{sgn}((\mathbf{a},\mathbf{b}))$ is the sign of the permutation). The Hafnian of T is given by the same sum but without the signs:

$$\mathrm{Hf}\,(T)=\sum_{(\mathbf{a},\mathbf{b})\in\mathcal{P}_n}x_{a(1)b(1)}x_{a(2)b(2)}\cdots x_{a(n)b(n)}$$

Obviously, the Hafnian of the all 1s array is $(2n-1)!!$.

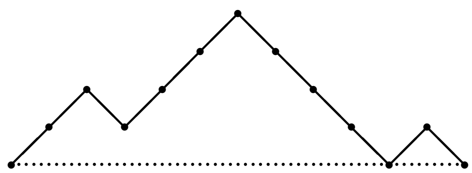
Proposition 1. *For an array $T=(x_{ij})_{1\le i<j\le 2n}$ with constant rows $x_{ij}:=x_i$,*

$$\mathrm{Hf}\,(T)=\sum_{\mathbf{a}\in\mathcal{A}_n}\prod_{i=1}^{n}(2i-a(i))x_{a(i)}$$

where $\mathcal{A}_n$ denotes the set of increasing sequences $\mathbf{a}=(a(1),a(2),\dots,a(n))$ satisfying $1\le a(i)\le 2i-1$.

Proof. The map $(\mathbf{a},\mathbf{b})\to\mathbf{a}$ maps perfect matching permutations onto $\mathcal{A}_n$. Thus every term in $\mathrm{Hf}\,(T)$ has the form $\prod_{i=1}^{n}x_{a(i)}$ for some $\mathbf{a}\in\mathcal{A}_n$ and the question is, how many of each form? Given $\mathbf{a}\in\mathcal{A}_n$, set $R=\{a(1),a(2),\dots,a(n)\}$ and $C=[2n]\setminus R$, and let $C_i=\{c\in C\,:\,c>a(i)\}$ and $c_i=|\,C_i\,|$. Clearly, $C_n\subseteq C_{n-1}\subseteq\ldots\subseteq C_1$. The b_i's in a perfect matching permutation $(\mathbf{a},\mathbf{b})$ are subject only to the two restrictions: $b_i\in C_i$ and all b_i's distinct. Thus there are c_n choices for b_n, $c_{n-1}-1$ choices for b_{n-1}, $c_{n-2}-2$ choices for b_{n-2} and so on. Hence the coefficient of $\prod_{i=1}^{n} x_{a(i)}$ in $\mathrm{Hf}\,(T)$ is $\prod_{i=1}^{n} \big(c_i - (n-i)\big)$ and a simple check shows that $c_i - (n-i) = 2i - a(i)$. ☐

A Dyck path can be coded by the positions in the path of its upsteps and this coding is a bijection from Dyck n-paths onto $\mathcal{A}_n$. A Dyck path can also be coded by the heights of the tops of its upsteps, giving a bijection from Dyck n-paths to the sequences $\mathcal{B}_n = \{\big(b(i)\big)_{i=1}^{n}\}$ satisfying $b(1) = 1$ and $1 \le b(i+1) \le b(i) + 1$ for $1 \le i \le n-1$. The two codes are related by the equality $b(i) = 2i - a(i)$ for all i.



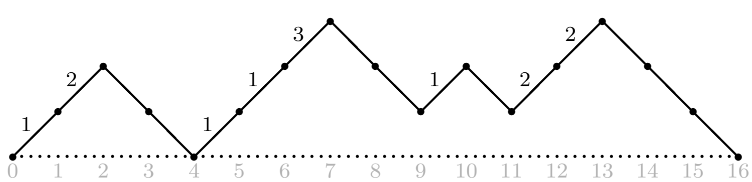
a Dyck 6-path

positions of upsteps $\mathbf{a} = (1, 2, 4, 5, 6, 11)$

heights of upstep tops $\mathbf{b} = (1, 2, 2, 3, 4, 1)$

The correspondences, Dyck n-paths $\to \mathcal{A}_n \to \mathcal{B}_n$, along with the assertion of Prop. 1 and the fact that the Hafnian of the all 1s array is $(2n-1)!!$, now establish item 2.6.

This proof can be worked up into a bijection from height-labeled Dyck paths to perfect matchings $(\mathbf{a}, \mathbf{b})$. The positions of the upsteps in the Dyck path give $\mathbf{a}$, and $\mathbf{b}$ is formed from the positions of the downsteps, using the sequence of height labels $\mathbf{h} = \big(h(i)\big)_{i=1}^{n}$, as follows. First, write the downstep position list $\mathbf{d}$ in decreasing order. Then $b(n)$ is the $h(n)$-th entry in $\mathbf{d}$, $b(n-1)$ is the $h(n-1)$-th entry in the remaining $n-1$ elements of $\mathbf{d}$, $b(n-2)$ is the $h(n-2)$-th entry in the remaining $n-2$ elements and so on. An example is illustrated.



height-labeled Dyck 8-path

$\mathbf{a} = (1, 2, 5, 6, 7, 10, 12, 13)$,

$\mathbf{d} = (16, 15, 14, 11, 9, 8, 4, 3)$,

$\mathbf{h} = (1, 2, 1, 1, 3, 1, 2, 2)$

upsteps, downsteps, labels

So $b(8)$ is the second entry of $\mathbf{d}$, namely 15; $b(7)$ is the second entry of $(16, 14, 11, \ldots)$, namely 14; $b(3) = 16$ and so on. The result is $\mathbf{b} = (4, 3, 9, 11, 8, 16, 14, 15)$ and the perfect matching is

$$1\ 4\ /\ 2\ 3\ /\ 5\ 9\ /\ 6\ 11\ /\ 7\ 8\ /\ 10\ 16\ /\ 12\ 14\ /\ 13\ 15.$$

There is an analogue of Prop. 1 for the Pfaffian.

Proposition 2. *For an array $T = (x_{ij})_{1\le i<j\le 2n}$ with constant rows $x_{ij} := x_i$,*

$$\mathrm{Pf}\,(T) = x_1x_3x_5\dots x_{2n-1}.$$

Proof. For a perfect matching permutation $(\mathbf{a},\mathbf{b})$, take the smallest i for which $a(i)$ and $b(i)$ are not consecutive integers (if there is one). Then the entries in $(\mathbf{a},\mathbf{b})$ up through $a(i)$ necessarily form an initial segment of the positive integers ending at $a(i) = 2i-1$, and $a(i+1) = 2i$. So interchanging $b(i)$ and $b(i+1)$ gives another perfect matching permutation. Both contribute the same product to the Pfaffian but with opposite signs and hence they cancel out. The only surviving permutation under this involution is the identity, which contributes $x_1x_3x_5\dots x_{2n-1}$. □

Corollary 3. *For the array $T = (x_{ij})_{1\le i<j\le 2n}$ with $x_{ij} := i$,*

$$\mathrm{Pf}\,(T) = (2n-1)!!.$$

Proof. Put $x_i = i$ in Prop. 2.

4 Round-Number Identities

4.1

$$\sum_{k=0}^{n-1}\binom{n}{k+1}(2k-1)!!(2n-2k-3)!! = (2n-1)!! \tag{1}$$

This identity counts increasing ordered trees of size n by size k of the leftmost subtree of the root. To see this, simply condition on the vertex set of the leftmost subtree. The bivariate generating function $\sum_{n\ge1,k\ge0}\binom{n}{k+1}(2k-1)!!(2n-2k-3)!!\frac{x^n}{n!}y^k$ is

$$\frac{1-\sqrt{1-2xy}}{y\sqrt{1-2x}},$$

and the first few values are

$n \setminus k$	0	1	2	3	4
1	1				
2	2	1			
3	9	3	3		
4	60	18	12	15	
5	525	150	90	75	105

4.2

$$\sum_{k=0}^{n} \binom{2n-k-1}{k-1} \frac{(2n-2k-1)(2n-k+1)}{k+1} (2n-2k-3)!! = (2n-1)!! \qquad (2)$$

This identity counts increasing ordered trees of size n by length k of the rightmost path from the root—the path that starts at the root and successively goes to the rightmost child until it reaches a leaf. To see this, let $u(n,k)$ be the number of such trees. Clearly, $u(0,0) = 1$ and, for $n \geq 1$, $u(n,1) = n(2n-3)!!$ since there are n choices for the rightmost child of the root and this child is a leaf. Now suppose $n \geq k \geq 2$ and consider trees of size $n-1$. If the rightmost path has length $\geq k-1$ then there is just one way to insert n to get a size-n tree with rightmost path of length k, and n then ends the rightmost path. On the other hand, if the rightmost path has length k, then adding n anywhere except as the rightmost child of one of the $k+1$ vertices on the rightmost path gives a size-n tree with rightmost path of length k, and in this case n does not end the rightmost path. Thus we have the recurrence

$$\begin{aligned} u(n,1) &= n(2n-3)!! \\ u(n,k) &= \sum_{j=k-1}^{n-1} u(n-1,j) + (2n-k-2)u(n-1,k) \qquad n \geq k \geq 2, \end{aligned}$$

and the summand in (2) satisfies this recurrence. The recurrence leads to a partial differential equation for the generating function $F(x,y) = \sum_{n \geq k \geq 0} u(n,k) \frac{x^n}{n!} y^k$:

$$(1-y)(1-2x)F_x(x,y) + y(1-y)F_y(x,y) + y^2 F(x,y) = y/\sqrt{1-2x},$$

with solution

$$F(x,y) = \frac{1-(1-y)e^{y(1-\sqrt{1-2x})}}{y\sqrt{1-2x}}.$$

The first few values of $u(n,k)$ are

$n \backslash k$	0	1	2	3	4	5
0	1					
1	0	1				
2	0	2	1			
3	0	9	5	1		
4	0	60	35	9	1	
5	0	525	315	90	14	1

(The top left entry for $n = k = 0$ in the array wants to be included in order to get the nice generating function.)

Remark The summand in (2) can be written somewhat more compactly by distinguishing the cases k even or odd: the summand is

$$\binom{n-j-1}{j-1}\frac{(2n-2j+1)!!}{(2j+1)!!} \qquad \text{if } k=2j \text{ is even, and}$$

$$\binom{n-j+1}{j}\frac{(2n-2j-1)!!}{(2j-3)!!} \qquad \text{if } k=2j-1 \text{ is odd.}$$

Our interpretation of (2) is equivalent to the assertion that the number of increasing ordered trees of size n whose rightmost path from the root has length $\ge k$ is $\binom{2n-k}{k}(2n-2k-1)!!$. We now prove this assertion bijectively. Form a "vertex" set $V=\{1,2,\ldots,n\}$ and an "edge" set $E=\{1,2,\ldots,n-k\}$ and make them disjoint by using subscripts V and E on their entries. Thus $|\,V\cup E\,|=2n-k$. It suffices to exhibit a bijection from the trees being counted to pairs consisting of a k-element subset X of $V\cup E$ and a standard increasing ordered tree T_0 of size $n-k$, since these pairs are clearly counted by $\binom{2n-k}{k}(2n-2k-1)!!$. We use the tree illustrated in Fig. 1a) as a working example with $n=14$ and $k=5$.

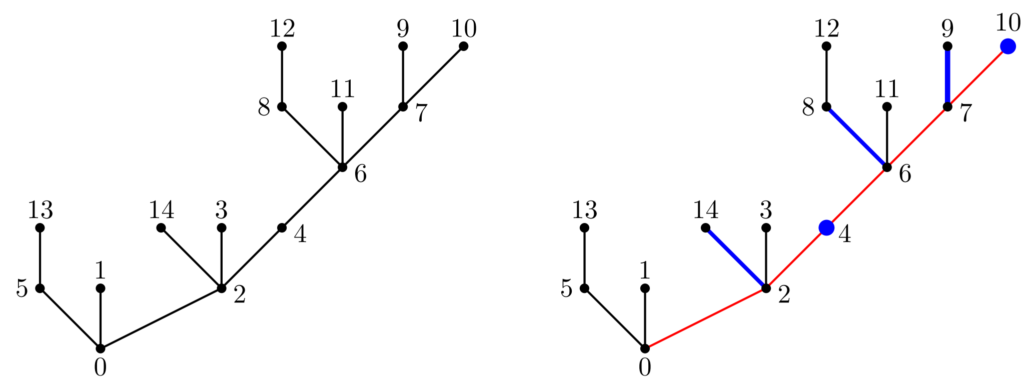


Fig. 1a) Fig. 1b)

Call the first k edges of the rightmost path from the root the *base* edges and their child vertices the *base* vertices $(2,4,6,7,10)$. Say a base vertex is *fertile* if it has a child that is not a base vertex and *barren* otherwise. Highlight each barren vertex and the leftmost edge from each fertile vertex, and color the base edges red as in Fig 1b). Thus there are k highlighted vertices/edges. The labels on the highlighted vertices are themselves the contribution from V to the k-element set X. The red edges will be deleted by an iterative cut-and-paste procedure to get the required $(n-k)$-edge tree T_0 and then the positions of the highlighted edges in T_0 will determine the contribution from E to X.

First, erase the barren vertices and standardize the vertex labeling—replace smallest by 0, next smallest by 1, and so on, to get the first tree in Fig. 2.

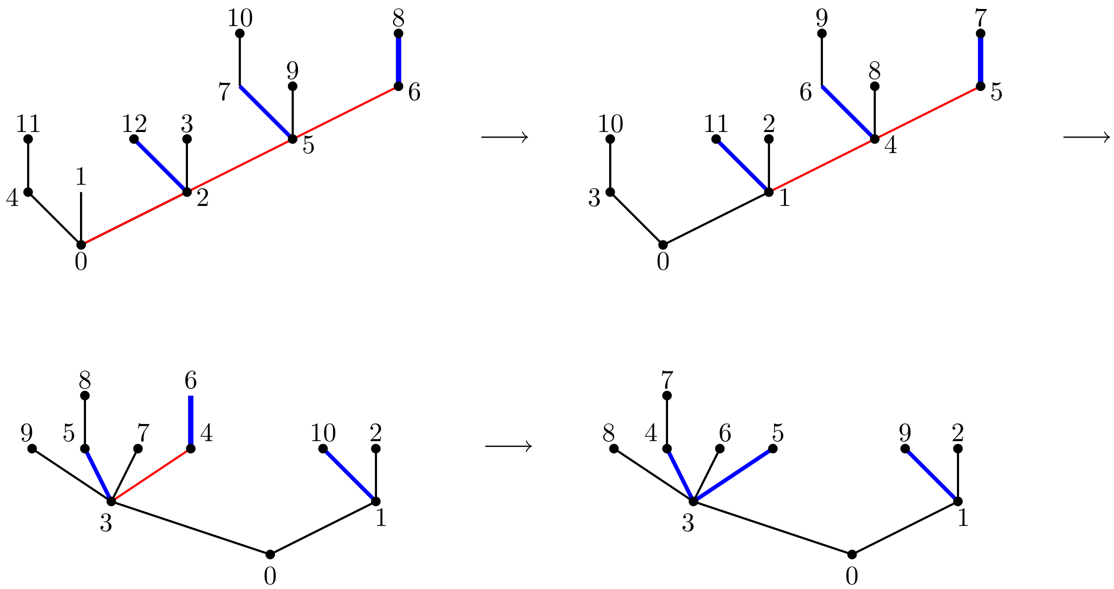


Fig. 2

Now eliminate the remaining red edges, one at a time, in increasing order of their child vertices. Let b denote the current smallest base vertex (initially $b = 2$). Cut out the subtree rooted at b, discarding the label b and its parent edge, and re-root this subtree at $b - 1$, placed so that it lies to the right of the existing subtree rooted at $b - 1$, and standardize vertex labels. Repeat this process on the second, third, ..., red edge in turn to obtain the desired $(n - k)$-edge tree T_0 (the last tree in Fig. 2) with some edges highlighted. Observe that the progress of individual highlighted edges in the successive trees of Fig. 2 can be discerned even though the labels on their endpoints will change.

Finally, list the edges of T_0 in *standard* order, that is, in increasing order of their parent vertices (preserving, of course, the order of edges with a common parent vertex), and take the positions of the highlighted edges in this list as the contribution from E to the k-element set X. In the example, T_0 has defining edge list

$$\big((0,3), (0,1), (1,9), (1,2), (3,8), (3,4), (3,6), (3,5), (4,7)\big)$$

and the highlighted edges—$(1,9), (3,4), (3,5)$—are in positions 3,6,8. The net result is the pair (X, T_0) with $X = \{4_V, 10_V, 3_E, 6_E, 8_E\}$ and T_0 as just given.

Is this process reversible? The entries in X with subscript E determine the highlighted edges in T_0, and the entire process can now be reversed step-by-step provided we know the order in which the highlighted edges were originally processed. But this order is precisely their left-to-right order in the standard listing of the edges of T_0. $\square$

Remark $\binom{2n-k}{k}(2n-2k-1)!!$ is also the number of increasing ordered trees of size $n+1$ whose root has $k+1$ children, increasing from left to right: given an increasing ordered tree of size n whose rightmost path from the root has length $\geq k$, delete the base edges, increment all labels by 1, and then attach the old root and the k base vertices in increasing order to a new root 0. This is a bijection to the trees in question.

4.3

$$\sum_{k=1}^{n} \frac{(n-1)!}{(k-1)!}\, k\,(2k-3)!! = (2n-1)!! \tag{3}$$

This identity counts increasing ordered trees of size n by the maximum k of the young leaves where a *young* leaf is a leaf with no left sibling (leaf or otherwise). Every nonempty increasing ordered tree has at least one young leaf. To establish this count, consider the $2n-1$ ways to produce an increasing ordered tree of n edges by inserting a leaf n into an increasing ordered tree of $n-1$ edges: either as the leftmost child of one of the n vertices or so that the new edge is the immediate right neighbor edge of one of the $n-1$ existing edges. In the first case, the maximum young leaf becomes n; in the second, it is preserved. In particular, the maximum young leaf becomes n in n ways from *any* increasing ordered tree of $n-1$ edges. Hence, if $u(n,k)$ denotes the number of increasing ordered trees with n edges whose maximum young leaf is k, we have $u(n,n) = n(2n-3)!!$ and the recurrence $u(n,k) = (n-1)u(n-1,k)$ for $1 \leq k < n$. Iterating the recurrence yields $u(n,k) = (n-1)^{\underline{n-k}}\, k\,(2k-3)!!$ for all $1 \leq k \leq n$.

The bivariate generating function $\sum_{n,k\geq 1} \frac{(n-1)!}{(k-1)!}\, k\,(2k-3)!! \frac{x^{n-1}}{(n-1)!} y^{k-1}$ is

$$\frac{1-xy}{(1-x)(1-2xy)^{3/2}},$$

and the first few values are

$n \setminus k$	1	2	3	4	5
1	1				
2	1	2			
3	2	4	9		
4	6	12	27	60	
5	24	48	108	240	525

4.4

$$\sum_{k=0}^{n/2} \binom{n}{2k}\binom{2k}{k}\frac{n!}{2^{2k}} = (2n-1)!! \tag{4}$$

This identity counts perfect matchings of $[2n]$ by number k of matches in which both entries are $\le n$. In fact, (4) is the special case $r = 0$ of a family of identities indexed by a nonnegative integer r:

$$\sum_{k=r}^{(n+r)/2} \binom{n}{2k-r}\binom{2k-r}{k}(n+r)^{\underline{r}}\,(n-r)!\frac{1}{2^{2k-r}} = (2n-1)!! \tag{5}$$

where $(n+r)^{\underline{r}} = (n+r)(n+r-1)\ldots(n+1)$ is the falling factorial. For given r, as a straightforward direct count shows, (5) counts perfect matchings of $[2n]$ by number k of matches in which both entries are $\le n+r$ (also by number k of matches in which both entries are $\le n-r$). The bivariate generating function $\sum_{n,k\ge r}\binom{2k-r}{k}(n+r)^{\underline{r}}(n-r)!\frac{1}{2^{2k-r}}\,\frac{x^{n-r}}{(n-r)!}\,y^{k-r}$ is

$$\frac{(2r-1)!!}{\big((1-x)^2 - x^2y\big)^{(2r+1)/2}}.$$

For $r = 0$, the first few values are

$n\setminus k$	0	1	2	3
0	1			
1	1			
2	2	1		
3	6	9		
4	24	72	9	
5	120	600	225	
6	720	5400	4050	225

and for $r = 1$, the first few values are

$n\setminus k$	1	2	3
1	1		
2	3		
3	12	3	
4	60	45	
5	360	540	45
6	2520	6300	1575

By transposing factors, (5) can be written in the alternative form

$$\sum_{k=r}^{(n+r)/2} \binom{n}{2k-r}\binom{2k-r}{k} 2^{n-2k+r} = \binom{2n}{n-r}, \tag{6}$$

an identity that counts bicolored *UDF* paths of length n ending at height r by number k of upsteps. A *UDF* path is a lattice path of upsteps $(1,1)$, downsteps $(1,-1)$, and flatsteps $(1,0)$; bicolored means each flatstep is colored red or blue; height is relative to the horizontal line through the initial vertex.

The case $r = 1$ of (5) also counts increasing ordered trees of n edges by number k of young leaves (as defined in Section 4.3). To see this, note that inserting a leaf n into an increasing ordered tree of $n-1$ edges ($2n-1$ possible ways) always either preserves or increments (by 1) the number of young leaves, and the number of ways to increment is $n-2k$ where k is the number of young leaves. This observation leads to the recurrence

$$u(n,k) = (n+2k-1)u(n-1,k) + (n-2k+2)u(n-1,k-1) \qquad 1 \le k \le (n+1)/2$$

for the number $u(n,k)$ of increasing ordered trees of n edges and k young leaves, and the recurrence is satisfied by the summand.

4.5

$$\sum_{k=1}^{n} \frac{(2n-2)!!(2k-3)!!}{(2k-2)!!} = (2n-1)!! \tag{7}$$

This identity counts

(1) Stirling permutations of size n by first entry k,

(2) Stirling permutations of size n by position $2k-1$ of the first 1 (the position of the first 1 is necessarily odd),

(3) increasing ordered trees of size n by the parent $k-1$ of n,

(4) increasing ordered trees of size n by the label k on the leaf in the minimal path from the root. The *minimal path* starts at the root and successively travels to the smallest child vertex until it arrives at a leaf.

Proofs

(1) and (4) The number of choices in building up the object, inserting a pair ii in the permutation or a leaf i in the tree for $i = 1$ to n, is successively $1, 3, \ldots, 2k-3, 1, 2k, 2k+2, \ldots, 2n-2$ and their product is the summand.

(2) If a letter occurs to the left of the first 1, then both occurrences do so, and so the number of such permutations is $\binom{n-1}{k-1}$ [choose support set for the first $2k-2$ entries] $\times(2k-3)!!$ [form a Stirling permutation on this support] $\times(2n-2k)!!$ [form a Stirling permutation of size $n-k+1$ that starts with a 1], and $\binom{n-1}{k-1}(2k-3)!!(2n-2k)!!$ is an equivalent expression for the summand.

(3) Here, build up the tree by successively inserting leaves $1, 2, \ldots, k-1$, then n must be inserted as a child of $k-1$, then proceed to insert $k, k+1, \ldots, n-1$. The number of choices is the same as in (1) and (4) above.

The bivariate generating function $\sum_{n\geq 1, k\geq 1} \frac{(2n-2)!!(2k-3)!!}{(2k-2)!!} \frac{x^{n-1}}{(n-1)!} y^{k-1}$ is

$$\frac{1}{(1-2x)\sqrt{1-2xy}},$$

and the first few values are

$n \setminus k$	1	2	3	4	5
1	1				
2	2	1			
3	8	4	3		
4	48	24	18	15	
5	384	192	144	120	105

With rows reversed, this array is entry A122774 in OEIS [10], and the reversed array counts (i) height-labeled Dyck paths by the position among the upsteps of the last upstep with label 1, and (ii) increasing ordered trees by the maximum child of 1.

4.6

$$(2n-2)!! + \sum_{k=2}^{n} \frac{(2n-1)!!(2k-4)!!}{(2k-1)!!} = (2n-1)!! \tag{8}$$

This identity counts increasing ordered trees of size n by smallest child k of 1 ($k = 1$ if vertex 1 has no children). To see this, let $u(n, k)$ be the number of such trees. Consideration of the effect of inserting a leaf n into an increasing ordered tree of size $n-1$ on the smallest child of 1 yields the recurrence

$$\begin{aligned} u(n,1) &= (2n-2)!! \\ u(n,k) &= (2n-1)u(n-1,k) \qquad 2 \leq k \leq n-1 \\ u(n,n) &= (2n-4)!!, \end{aligned}$$

satisfied by the summands. The recurrence leads to a first-order ordinary differential equation for the generating function $F(x,y) = \sum_{n\geq k\geq 1} u(n,k)\ \frac{x^{n-1}}{(n-1)!}y^{k-1}$:

$$(1-2x)F_x - 3F + 1/(1-2x) = \frac{y}{1-2xy},$$

with solution

$$F(x,y) = \frac{1}{(1-2x)^{3/2}} + \frac{\sqrt{y-y^2}}{(1-2x)^{3/2}y}\ \tan^{-1}\left(\frac{\sqrt{y-y^2}\,(-1+2xy+\sqrt{1-2x})}{1-2y+2xy^2}\right).$$

The first few values of $u(n,k)$ are

$n \backslash k$	1	2	3	4	5
1	1				
2	2	1			
3	8	5	2		
4	48	35	14	8	
5	384	315	126	72	48

With rows reversed, this array has an interpretation for which the recurrence relation is not immediately obvious: let $v(n,k)$ denote the number of Stirling permutations of size n for which the maximum M of the entries preceding the first 1 (taken as $n+1$ if the permutation starts with a 1) is k ($2 \leq k \leq n+1$). Then $v(n,k) = u(n,n+2-k)$. This follows from the next Proposition by comparing recurrences.

Proposition 4.

$$\begin{array}{rrcl} (i) & v(n,n+1) & = & (2n-2)!!, \\ (ii) & v(n,2) & = & (2n-4)!!, \quad \textit{and} \\ (iii) & v(n,k) & = & (2n-1)v(n-1,k-1) \quad \textit{for } 3 \leq k \leq n. \end{array}$$

Proof

(i) Consider permutations that start with a 1. Inserting two adjacent n's immediately after any one of the $2n-2$ entries in such a permutation of size $n-1$ gives one of size n. Thus $v(n,n+1) = (2n-2)v(n-1,n)$, and (i) follows.

(ii) A Stirling permutation with $M=2$ necessarily begins 221.... Deleting the initial 2s is a bijection to Stirling permutations of size $n-1$ that start with a 1, counted by $v(n-1,n) = (2n-4)!!$.

(iii) Now suppose $3 \le k \le n$. A Stirling permutation of size $n-1$ with $M = k-1$ yields $2n-1$ Stirling permutations of size n with $M = k$ (all distinct) as follows. Increment each entry ≥ 2 by 1 and tentatively insert a pair of adjacent 2s anywhere in the resulting permutation ($2n-1$ choices) to obtain a permutation σ. Let I denote the initial segment of σ terminating at the first 2. If no entry exceeding 2 occurs exactly once in I, then σ is already a Stirling permutation and we may leave the tentative 2s in place. Otherwise, choose the smallest $i > 2$ that occurs exactly once in I, say i_1, and interchange the 2s and i_1s in σ to obtain a perm σ_1. Let I_1 denote the initial segment of σ_1 terminating at the first i_1. If no entry exceeding i_1 occurs exactly once in I_1, then σ_1 is a Stirling permutation, and stop. Otherwise, proceed similarly to get $i_2 > i_1$, interchange the i_1s and i_2s to get σ_2 and continue until you arrive at a permutation σ_k that does have the Stirling property. The original Stirling permutation of size $n-1$ and the location of the tentative 2s can be recovered from σ_k and so this process is a bijection $\mathcal{V}(n-1, k-1) \times [2n-1] \to \mathcal{V}(n, k)$ where $\mathcal{V}(n, k)$ is the set counted by $v(n, k)$. □

Also, $v(n, k)$ is the number of increasing ordered trees of size n with k the maximal descendant of 1 (taken as $n+1$ if 1 is a leaf since "descendant" here means "proper descendant"). The generating function $\sum_{n\ge1}\sum_{k=2}^{n+1} v(n,k)\frac{x^{n-1}}{(n-1)!}y^{k-2}$ for $v(n, k)$ is marginally more concise than that for $u(n, k)$:

$$\frac{1}{(1-2xy)^{3/2}} + \frac{\sqrt{y-1}}{(1-2xy)^{3/2}} \tan^{-1}\left(\frac{\sqrt{y-1}\,(1-2x-\sqrt{1-2xy})}{2-2x-y}\right).$$

4.7

$$(2n-3)!! + \sum_{k=1}^{n-1} \frac{2\,(2n-1)!!}{(2k+1)(2k-1)} = (2n-1)!! \tag{9}$$

This identity counts Stirling permutations of size n by smallest entry k following the last n ($k = 0$ if the last entry is n). The recurrence relation for these permutations is

$$\begin{aligned} u(n,0) &= (2n-3)!! \\ u(n,k) &= (2n-1)u(n-1,k) \qquad 1 \le k \le n-2 \\ u(n,n-1) &= 2(2n-5)!!, \end{aligned}$$

satisfied by the summand.

Remark The identity itself is trivial to prove, since the sum is telescoping, but the generating function $\sum_{n\geq 1,\, 0\leq k\leq n-1} u(n,k)\frac{x^{n-1}}{(n-1)!}y^k$ is cumbersome:

$$\frac{1}{(1-2x)^{3/2}} + \frac{1}{\sqrt{1-2x}} - \frac{\sqrt{1-2xy}}{1-2x} - \frac{1-y}{2\sqrt{y}(1-2x)^{3/2}} \log\left(\frac{1+y-4xy-2\sqrt{(1-2x)y(1-2xy)}}{(1-\sqrt{y})^2}\right).$$

The first few values of $u(n,k)$ are

$n \setminus k$	0	1	2	3	4
1	1				
2	1	2			
3	3	10	2		
4	15	70	14	6	
5	105	630	126	54	30

4.8

$$\sum_{k=1}^{n} k!\binom{2n-k-1}{k-1}(2n-2k-1)!! = (2n-1)!! \tag{10}$$

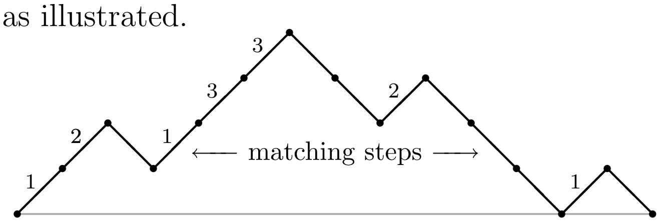
This identity counts increasing ordered trees of n edges by outdegree k of the root. The $k!$ factor allows us to assume that the k children of the root increase from left to right, and the number of such trees is given by the Remark at the end of Section 4.2.

The identity also counts height-labeled Dyck n-paths by length k of first ascent. To show this, we exhibit a bijection from height-labeled Dyck n-paths to increasing ordered n-trees that sends "length of first ascent" to "number of children of the root". The tree is built up using trees with some of their leaves unlabeled. The construction involves a pair of sequences $\big(a(i)\big)_{i=1}^{n}$ and $\big(b(i)\big)_{i=1}^{n}$ that characterizes the path:

$a(i) = \#$ upsteps immediately preceding the ith downstep, and

$b(i) =$ label on the upstep matching the ith downstep.

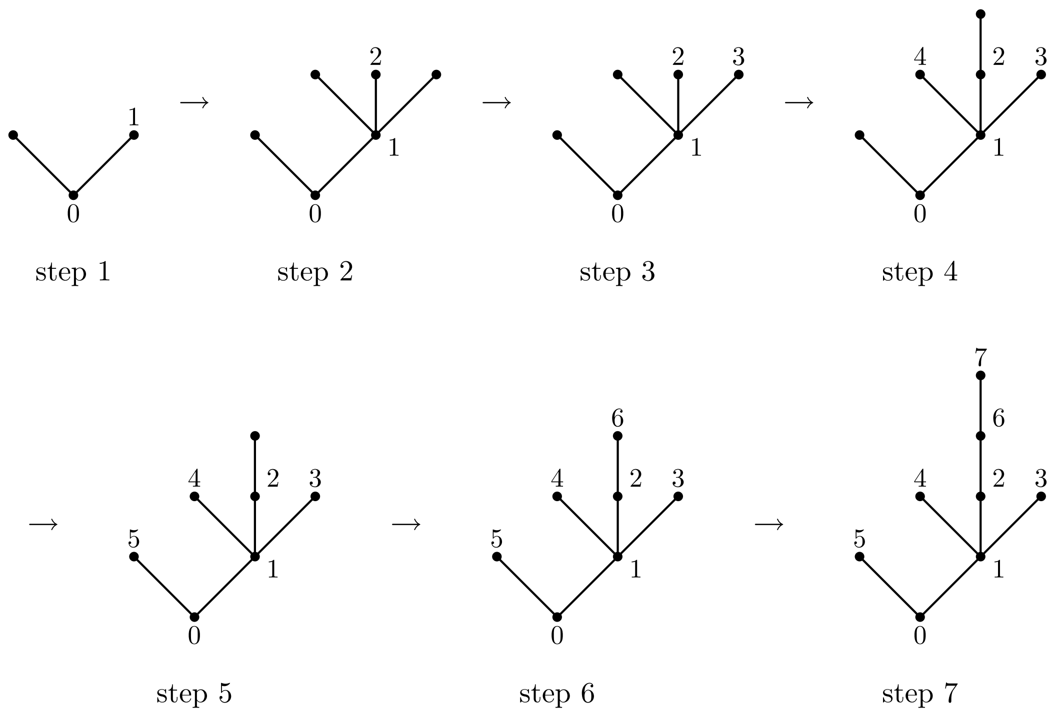
Thus $0 \leq a(i) \leq n$ and $\sum_{i=1}^{n} a(i) = n$. At step i ($1 \leq i \leq n$), attach $a(i)$ unlabeled leaves to (leaf) vertex $i-1$ and apply label i to the $b(i)$-th unlabeled leaf in walkaround order as illustrated.



height-labeled Dyck 7-path

i	1	2	3	4	5	6	7
$a(i)$	2	3	0	1	0	0	1
$b(i)$	2	3	3	2	1	1	1

defining sequences



construction of corresponding tree

We remark without proof that the identity also counts leaf-labeled 0-2 trees of size n by length of the path from 1 to the root.

The bivariate generating function $\sum_{n\geq 0,k\geq 0} k!\binom{2n-k-1}{k-1}\ (2n-2k-1)!!\frac{x^n}{n!}y^k$ is

$$\frac{1-y-y\sqrt{1-2x}}{1-2y+2xy^2}.$$

Omitting the empty tree ($n=k=0$), the generating function $\sum_{n\geq 1,k\geq 1} k!\binom{2n-k-1}{k-1}\ (2n-2k-1)!!\frac{x^{n-1}}{(n-1)!}y^{k-1}$ looks rather different:

$$\frac{1}{\sqrt{1-2x}\left(1-y+y\sqrt{1-2x}\right)^2},$$

and the first few values are

$n \setminus k$	1	2	3	4	5
1	1				
2	1	2			
3	3	6	6		
4	15	30	36	24	
5	105	210	270	240	120

This array is entry A102625 in OEIS.

The result of Section 4.8 can be refined to give the joint distribution of first ascent length and first peak upstep label: let $u(n, j, k)$ denote the number of height-labeled Dyck n-paths whose first peak upstep has label j and whose initial ascent has length k ($n \ge k \ge j \ge 1$). Then, since j is uniformly distributed over $[k]$, $u(n, j, k) = (k-1)!\binom{2n-k-1}{k-1}(2n-2k-1)!!$ for $1 \le j \le k$, leading to the generating function $\sum_{n\ge1,j\ge1,k\ge1} u(n, j, k)\frac{x^{n-1}}{(n-1)!}y^{j-1}z^{k-1} =$

$$\frac{1}{\sqrt{1-2x}\left(1-yz+yz\sqrt{1-2x}\right)\left(1-z+z\sqrt{1-2x}\right)}.$$

The bijection of this section shows that $u(n, j, k)$ also counts increasing ordered trees whose root has k children among which vertex 1 is the j-th.

5 Non-Round Identities

5.1

$$n! + \sum_{k=1}^{n-1}\big((k-1)!!(2n-k)!! - k!!(2n-k-1)!!\big) = (2n-1)!! \tag{11}$$

This identity is trivial to prove—the sum is telescoping and collapses to $(2n-1)!! - (n-1)!!n!!$—but it has an interesting interpretation: it counts height-labeled Dyck n-paths by length k of the first descent where a *descent* is a maximal sequence of contiguous downsteps and the term $n!$ corresponds to $k = n$. To see this, let $u(n, k)$ denote the number of height-labeled Dyck n-paths whose first descent has length $\ge k$ ($1 \le k \le n$). The next-size-up construction described in (2.6) yields one of these paths precisely when the $(n-1)$-path has first descent of length $\ge k$ and the specified vertex is not the terminal vertex of one of the first $k-1$ downsteps in the first descent. Thus $u(n, k) = \big((2n-1)-(k-1)\big)u(n-1, k) = (2n-k)u(n-k)$ for $n \ge k$. Together with the obvious initial case $u(k, k) = k!$, this recurrence yields that $u(n, k) = k!(k+2)^{\overline{n-k,2}}$, where $k^{\overline{n,2}} = k(k+2)(k+4)\cdots$ to n factors is the rising double factorial. Equivalently, $u(n, k) = (k-1)!!(2n-k)!!$. Thus the number of height-labeled Dyck n-paths with first descent of length k is $u(n, n) = n!$ for $k = n$, and $u(n, k) - u(n, k+1) = (k-1)!!(2n-k)!! - k!!(2n-k-1)!!$ for $1 \le k \le n-1$, and the first few values are

$n \setminus k$	1	2	3	4	5
1	1				
2	1	2			
3	7	2	6		
4	57	18	6	24	
5	561	174	66	24	120

5.2

$$\sum_{k=1}^{n} \left\langle\!\!\left\langle {n \atop k} \right\rangle\!\!\right\rangle = (2n-1)!! \tag{12}$$

Here $\left\langle\!\left\langle {n \atop k} \right\rangle\!\right\rangle$ is the second-order Eulerian number (indexed so that $1 \le k \le n$) A008517. This identity counts

(1) Stirling permutations of size n by number k of descents (including a conventional descent at the end),

(2) Stirling permutations of size n by number k of plateaus, that is, pairs of adjacent equal entries,

(3) increasing ordered trees of size n by number k of leaves,

(4) height-labeled Dyck n-paths by number k of upstep-free vertices, and

(5) trapezoidal words of length n by number k of distinct entries.

Furthermore, the second-order Eulerian triangle *with reversed rows* counts

(6) increasing ordered trees by number of descents where a descent in a tree is a pair of adjacent sibling vertices with the first larger than the second (no conventional descents), and

(7) height-labeled Dyck n-paths by number of peaks.

Proofs

(1) Several proofs are given in [2].

(2) Stirling permutations are usually defined on support set $[n]$ but of course can be similarly defined on an arbitrary set of positive integers. We present a bijection ϕ, actually an involution, on Stirling permutations of arbitrary support set that preserves size and interchanges "# descents" and "# plateaus". First, ϕ is the identity on the empty permu-

tation. A nonempty Stirling permutation can be written as the concatenation $A\,m\,B\,m\,C$ where m is the smallest integer in its support set and A, B, C are perforce themselves Stirling permutations. Now define ϕ recursively by

$$\phi(A\,m\,B\,m\,C) = \phi(A)\,m\,\phi(C)\,m\,\phi(B).$$

One checks, using induction, that ϕ has the required properties.

(3) This result follows from Janson's bijection (see Section 2.4) since it clearly sends leaves to plateaus.

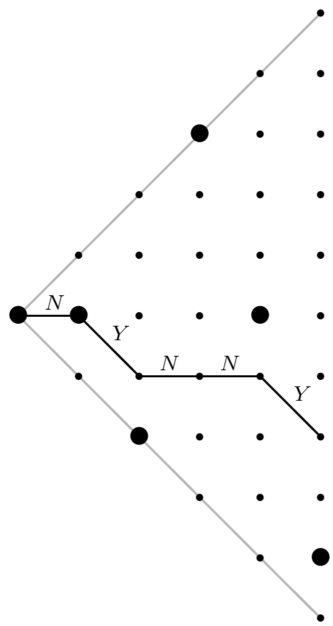
(4) There are $2n+1$ vertices in a Dyck n-path and a vertex is *upstep-free* if it is not incident with an upstep. The recurrence of (2.6) for height-labeled Dyck paths can easily be refined to a recurrence for $h(n,k)$, the number of height-labeled Dyck n-paths with k upstep-free vertices, which turns out to be $h(n,k) = k\,h(n-1,k) + (2n-k)h(n-1,k-1)$—the defining recurrence for second-order Eulerian numbers [2, p. 27].

The recurrence leads to a bijection from height-labeled Dyck n-paths to Stirling permutations that sends # upstep-free vertices to # conventional descents. The bijection uses identical "codings" for the two classes of objects. Consider the alphabet consisting of two copies of the positive integers, distinguished by subscripts Y and N. Let $\mathcal{C}_n$ denote the set of words $\big(w(i)\big)_{i=1}^{n}$ over this alphabet satisfying the following condition for each i. If $w(i)$ has subscript Y, then its value is $\le 1 + \#$ subscripts N preceding $w(i)$, while if $w(i)$ has subscript N, then its value is $\le 2i-2-\#$ subscripts N preceding $w(i)$. Clearly, there are $2i-1$ choices for $w(i)$ regardless of the preceding entries, and so $|\,\mathcal{C}_n\,| = (2n-1)!!$. Indeed, a word w in $\mathcal{C}_n$ can be represented as a lattice path along with a set of lattice points: discard $w(1)$—necessarily 1_Y—and subtract 1 from each remaining value to get a sequence of $n-1$ nonnegative integers $\big(b(i)\big)$ and a sequence of $n-1$ subscripts. The subscripts give the path letting Y denote a downstep $(1,-1)$ and N a flatstep $(1,0)$, and the ith lattice point is $b(i)$ units above (resp. below) the terminal point of the ith step if the corresponding subscript is N (resp. Y), as illustrated.

$$(1_Y\ 1_N\ 2_Y\ 5_N\ 2_N\ 3_Y)$$

$$\downarrow$$

$$\begin{pmatrix} 0 & 1 & 4 & 1 & 2 \\ N & Y & N & N & Y \end{pmatrix} \longrightarrow$$



word in $\mathcal{C}_n \longrightarrow$ path and set of lattice points

The ordinates of the lattice points with a prepended 0 form a symmetric trapezoidal word (Section 2.1). The path is redundant since it can be recovered from the lattice points and so we have an explicit bijection from $\mathcal{C}_n$ to symmetric trapezoidal words.

To code a height-labeled Dyck n-path by a word in $\mathcal{C}_n$, delete the last upstep U with label 1 along with the last downstep and decrement labels after U to get an $(n-1)$-path P_1 with a distinguished vertex v (where the upstep was deleted). If v is upstep-free (resp. upstep-incident) in P_1, the subscript on $w(n)$ is Y (resp. N) and its value is the number of upstep-free (resp. upstep-incident) vertices weakly preceding v. Repeat on P_1 to get $w(n-1)$ and so on, ending with $w(1) := 1_Y$. This gives a bijection that sends # upstep-free vertices to # subscripts Y.

Likewise, to code a Stirling n-permutation by a word in $\mathcal{C}_n$, delete the two ns (necessarily adjacent) to get an $(n-1)$-permutation with a distinguished gap (possibly at either end). Recalling that the last gap is considered a descent, if the distinguished gap is a descent (resp. non-descent), the subscript on $w(n)$ is Y (resp. N) and its value is the number of descent (resp. non-descent) gaps weakly preceding the distinguished gap. Repeat to get $w(n-1)$ and so on, again ending with $w(1) := 1_Y$. This gives a bijection that sends # descents to # subscripts Y.

(5) This result is stated without proof in [1]. A Stirling permutation can be built up in a unique way by starting with a plateau of two 1s, inserting a plateau 22 in one of the 3 gaps

in –1–1–, then inserting a plateau 33 in one of 5 gaps and so on. Define a mapping from Stirling permutations of size n to trapezoidal words $\boldsymbol{w} = (w_i)_{i=1}^{n}$ as follows. Set $w_1 = 1$. If 22 is placed between the two 1s, set $w_2 = 1$; if 22 is placed to the left of the ones, set $w_2 = 2$ (the smallest number not yet appearing in $\boldsymbol{w}$), else $w_2 = 3$. In general, if kk is placed inside a plateau, say the ith plateau (left to right), w_k is the ith smallest number already appearing in $\boldsymbol{w}$; otherwise kk is placed in one of the remaining gaps, say in the jth of the remaining gaps, and w_k is the jth smallest positive integer not yet appearing in $\boldsymbol{w}$. For example, 5512234431 → 11442 as follows.

$$\begin{array}{ccccccccc} 11 & \rightarrow & 1221 & \rightarrow & 122331 & \rightarrow & 12234431 & \rightarrow & 5512234431 \\ w_1 = 1 & & w_2 = 1 & & w_3 = 4^* & & w_4 = 4^\dagger & & w_5 = 2 \end{array}$$

* 4 is the third smallest number not yet appearing in $\boldsymbol{w}$

† 4 is the second smallest number already appearing in $\boldsymbol{w}$

It is easy to check that this algorithm defines a bijection from Stirling permutations of size n to trapezoidal words of length n and that it sends "# plateaus" in the permutation to "# distinct entries" in the word.

(6) Translated to Stirling permutations using Janson's bijection described above, descents in a tree become strong descents in the permutation where a *strong descent* is a descent $a > b$ involving the *first* of the two occurrences of b in the permutation. Note that the analogous notion of strong ascent is superfluous because all ascents in a Stirling permutation are strong. Consideration of the effect of inserting nn into a Stirling permutation of size $n-1$ on the number of strong descents leads to the defining recurrence for the reversed second-order Eulerian triangle using the following fact, proved by induction: in a Stirling permutation of size n, # strong descents + # ascents = $n-1$.

(7) The number of upstep-free vertices in a Dyck n-path is related to the number of peaks: their sum is $n+1$. It follows that the number of height-labeled Dyck n-paths with k peaks is $\left\langle\!\left\langle {n \atop n+1-k} \right\rangle\!\right\rangle$, the second-order Eulerian triangle with reversed rows.

5.3

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} 2^{n-k} = (2n-1)!! \tag{13}$$

Here $\begin{bmatrix} n \\ k \end{bmatrix}$ is the Stirling cycle number (and $\begin{bmatrix} 0 \\ 0 \end{bmatrix} = 1$). This identity counts

(1) Stirling permutations of size n by number k of left-to-right (LR) minima. (For example, 5 5 2 3 4 4 3 2 1 1 has 3 LR minima, namely 5,2,1),

(2) increasing ordered trees of size n by number k of edges in the minimal path from the root as defined in Section 4.5, and

(3) height-labeled Dyck n-paths by number k of upsteps in the first ascent with label 1.

Proofs

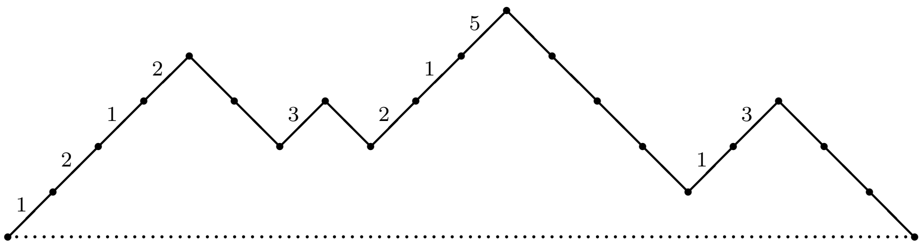
(1) It suffices to exhibit a bijection from Stirling permutations σ of size n with k left-to-right minima to pairs (A, τ) where A is a subset of $[n-k]$ and τ is a permutation of $[n]$ with k left-to-right minima (recall the number of such τ is $\left[{n \atop k}\right]$). Split σ just before each LR minimum to write σ as a concatenation $\sigma_1, \sigma_2, \ldots, \sigma_k$ of lists. Each σ_i is itself a Stirling permutation and their support sets, taken in order, form a partition of $[n]$ into blocks whose smallest elements are decreasing left to right. In view of these observations, the general case follows from the special case $k = 1$ by amalgamating subsets and concatenating permutations, both taken on the appropriate support sets. So suppose $k = 1$, implying that σ starts with a 1. For $j \in [2, n]$, let i_j denote the last entry preceding j in σ that is $< j$. Thus, for $\sigma = 1\,2\,5\,5\,2\,1\,4\,4\,3\,3$, i_2, i_3, i_4 all $= 1$ and $i_5 = 2$. Now take $A = \{j \in [2, n]$: both occurrences of i_j in σ precede the first occurrence of $j\}$, a subset of the $(n-1)$-element support set $[2, n]$. As for τ, observe that $(i_j)_{j=2}^{n} \in [1] \times [2] \times \cdots \times [n-1]$ and so corresponds to a permutation of the $(n-1)$-element set $[2, n]$. Prepend a 1 to this permutation to get the permutation τ of $[n]$ with $k = 1$ LR minima. We leave the reader to verify that σ can be recovered from the pair (A, τ).

(2) Adding a leaf n to a tree of size $n-1$ preserves the length of the minimal path except when n is added as a child of the leaf that terminates the minimal path. We thus have the recurrence

$$u(n, k) = (2n-2)u(n-1, k) + u(n-1, k-1),$$

satisfied by the summand because it reduces to the basic recurrence for the Stirling cycle numbers: $\left[{n \atop k}\right] = (n-1)\left[{n-1 \atop k}\right] + \left[{n-1 \atop k-1}\right]$.

(3) We give yet another bijection from height-labeled Dyck paths to increasing ordered trees. This one sends # upsteps immediately preceding the ith downstep to the outdegree of vertex $i-1$ $(1 \le i \le n)$—in brief, full ascent sequence of path $\rightarrow$ fertility sequence of tree—and sends locations of 1s on first ascent to locations of LR minima among children of the root. In particular, # 1s = # LR minima, and Janson's bijection identifies LR minima among children of the root with left-to-right minima in a Stirling permutation. So (3) will follow from (1). The bijection is an algorithm to generate the edge list. With $n = 10$, and the path below as a working example,



height-labeled Dyck 10-path

list the total number of downsteps preceding the i-th upstep for $i = n, n-1, \ldots, 1$ (column 2 in Fig. 3) and the label on the i-th upstep (column 3).

i	$\#\, D$s $< i$-th U	label on i-th U	candidate children	selected edge
10	7	3	8 9 10	7 10
9	7	1	8 9	7 8
8	3	5	4 5 6 7 9	3 9
7	3	1	4 5 6 7	3 4
6	3	2	5 6 7	3 6
5	2	3	3 5 7	2 7
4	0	2	1 2 3 5	0 2
3	0	1	1 3 5	0 1
2	0	2	3 5	0 5
1	0	1	3	0 3

$\rightarrow$

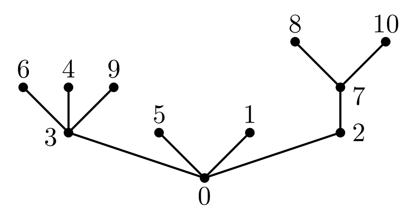


Fig. 3

Each entry j in column 2 is the parent vertex of an edge. The candidates for its child are the entries of $[j+1, n]$ not already having a parent. Initially $j = 8$, no vertex has a parent, and the candidate children are 8,9,10. The corresponding label in column 3 then determines the child, here label 3 selects the third candidate, namely 10, so 7 10 becomes an edge; 10 now has a parent and is deleted from the candidate set. The label 1 in the next row selects the first candidate, namely 8, and adds 7 8 to the edge list. Proceed similarly to get all n edges. Then the last column, read upwards, is the edge list in standard order, giving the tree shown.

The generating function $\sum_{k=1}^{n} \left[{n \atop k}\right] 2^{n-k} x^n$ for row n is $\prod_{i=0}^{n-1}(x+2i)$, and the bivariate generating function $\sum_{n,k\geq 0} \left[{n \atop k}\right] 2^{n-k} \frac{x^n}{n!} y^k$ is $(1-2x)^{-y/2}$. See A039683 in OEIS.

5.4

Consider the following modification of the minimal path from the root in increasing ordered trees: the right-then-minimal path is the path that goes from the root to its rightmost child, then follows minimal children to a leaf. Let $u(n,k)$ denote the number of increasing ordered trees of size n whose right-then-minimal path has length k, so that

$$\sum_{k=1}^{n} u(n,k) = (2n-1)!!.$$

Then $u(n,k)$ satisfies the recurrence

$$\begin{aligned} u(n,1) &= n(2n-3)!! \\ u(n,k) &= (2n-3)u(n-1,k) + u(n-1,k-1) \qquad 2 \le k \le n, \end{aligned}$$

leading to a differential equation for the generating function $F(x,y) = \sum_{n\ge k\ge 1} u(n,k)\frac{x^n}{n!}y^k$:

$$(1-2x)F_x(x,y) = \frac{y}{\sqrt{1-2x}} - (1-y)F(x,y),$$

with solution

$$F(x,y) = \frac{y\left(1-(1-2x)^{1-y/2}\right)}{(2-y)\sqrt{1-2x}}.$$

The first few values of $u(n,k)$ are

$n \backslash k$	1	2	3	4	5
1	1				
2	2	1			
3	9	5	1		
4	60	34	10	1	
5	525	298	104	17	1

6 Refinements

6.1

The interpretations of Sections 4.5 and 5.3 have a common refinement

$$\sum_{1\le j\le k\le n} 2^{n-j} \begin{bmatrix} k-1 \\ j-1 \end{bmatrix} (n-1)^{\underline{n-k}} = (2n-1)!!.$$

The summand is the number of increasing ordered trees of size n whose minimal path from the root ends at the leaf labeled k and whose length is j: the number $u(n,k,j)$ of such paths satisfies the recurrence

$$\begin{aligned}
u(n,n,1) &= (2n-2)!!, \\
u(n,k,1) &= 0 \qquad\qquad\qquad k \ge 2, \\
u(n,k,j) &= (2n-2)u(n-1,k,j) \qquad 2 \le j \le k \le n-1,
\end{aligned}$$

$$u(n,n,j) = \sum_{i=j-1}^{n-1} u(n-1,i,j-1) \qquad 2 \le j \le n.$$

The generating function $\sum_{1\le j\le k\le n} u(n,k,j)\frac{x^{n-1}}{(n-1)!}y^{j-1}z^{k-1}$ is

$$F(x,y,z) = \frac{1}{(1-2x)(1-2xy)^{z/2}}.$$

Likewise, the interpretations of Sections 4.8 and 5.3 have a common refinement:

$$\sum_{1\le j\le k\le n} \begin{bmatrix} k \\ j \end{bmatrix} \binom{2n-k-1}{k-1} (2n-2k-1)!! = (2n-1)!!.$$

The summand $v(n,k,j)$ is the number of height-labeled Dyck n-paths whose first ascent has length k and contains j 1s. The generating function $\sum_{0\le j\le k\le n} v(n,k,j)\frac{x^n}{n!}y^j z^k$ is

$$\left(\frac{1-y-y\sqrt{1-2x}}{1-2y+2xy^2}\right)^z.$$

6.2

The result of Section 4.8 can be refined to give the joint distribution of first ascent length and first peak upstep label: let $u(n,j,k)$ denote the number of height-labeled Dyck n-paths whose first peak upstep has label j and whose initial ascent has length k ($n \ge k \ge j \ge 1$). Then, since j is uniformly distributed over $[k]$, $u(n,j,k) = (k-1)!\binom{2n-k-1}{k-1}(2n-2k-1)!!$ for $1 \le j \le k$, leading to the generating function $\sum_{n\ge1,j\ge1,k\ge1} u(n,j,k)\frac{x^{n-1}}{(n-1)!}y^{j-1}z^{k-1} =$

$$\frac{1}{\sqrt{1-2x}\left(1-yz+yz\sqrt{1-2x}\right)\left(1-z+z\sqrt{1-2x}\right)}.$$

6.3

To refine the result of Section 5.1 and deduce further identities, let $u(n,k,j)$ denote the number of height-labeled Dyck n-paths whose first ascent has length $= j$ and first descent has length $\geq k$. Thus $u(n,k) = \sum_{j=k}^{n} u(n,k,j)$. Since the first peak in such a path has j possible labels, deleting this peak and its label shows that

$$u(n,k,j) = j u(n-1, k-1, j-1) \qquad \text{for } 2 \leq k \leq j \leq n \tag{14}$$

In particular, $u(n,2,j)$ is j times the number of size-$(n-1)$ height-labeled Dyck paths with first ascent of length $j-1$ and no restriction on the first descent. Hence, by the second interpretation of (10), $u(n,2,j) = j(j-1)(2n-2-j)^{\underline{j-2}}(2n-2j-1)!!$. This base case, together with (14), yields

$$u(n,k,j) = j^{\underline{k}}(2n-k-j)^{\underline{j-k}}(2n-2j-1)!!.$$

Equating $\sum_{j=k}^{n} u(n,k,j)$ and $u(n,k)$ yields the identity

$$\sum_{j=k}^{n} j^{\underline{k}}(2n-k-j)^{\underline{j-k}}(2n-2j-1)!! = (k-1)!!(2n-k)!!.$$

Two alternative forms of this identity, eliminating the double factorials, can be found by considering the cases where k is even or odd separately:

$$\sum_{j=0}^{n} \binom{j+2m}{j} 2^j \binom{2n-j}{n-j} = \binom{n+m}{n} 4^n \tag{15}$$

with $k := 2m$, n replaced by $n+2m$, and

$$\sum_{j=0}^{n} \binom{j+2m+1}{j} 2^{j+1} \binom{2n-j}{n-j} = \frac{m!(2n+2m+2)!}{n!(2m+1)!(n+m+1)!} \tag{16}$$

with $k := 2m+1$, n replaced by $n+2m+1$.

This last identity is interesting because it provides another solution to Ira Gessel's 1987 Monthly Problem E3107:

> Show that $\frac{m!(2m+2n)!}{(2m)!n!(m+n)!}$ is an integer for nonnegative integers m,n.

Replace m by $m-1$ in (16), rearrange terms, and cancel a 2 to get

$$\sum_{j=0}^{n} \binom{j+2m-1}{j} 2^j \binom{2n-j}{n-j} = \frac{m!(2n+2m)!}{n!(2m)!(n+m)!},$$

exhibiting Gessel's expression as a sum of integers, and this sum is different from that in A. A. Jagers' solution [11]. The case $m=0$ of (15) has a simple combinatorial interpretation: it counts lattice paths of upsteps and downsteps of length $2n$ by number of "returns to ground level".

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