EQUATIONS OF THE FORM $t(x+a) = t(x)$ AND $t(x+a) = 1 - t(x)$ FOR THUE-MORSE SEQUENCE

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Abstract. For every $a \geq 1$ we give a recursion algorithm of building of set of solutions of equations of the form $t(x+a) = t(x)$ and $t(x+a) = 1-t(x)$, where $\{t(n)\}$ is Thue-Morse sequence. We pose an open problem and two conjectures.

1. Introduction and main results

The Thue-Morse (or Prouhet-Thue-Morse [1]) sequence $\{t_n\}_{n\geq 0}$ is one of the most known and useful $(0,1)$-sequences. By the definition, $t_n = 0$, if the binary expansion of n contains an even number of $1's$, and $t_n = 1$ otherwise. It is sequence A010060 in OEIS [8]. Numerous applications of this sequence and a large bibliography one can find in [1] (see also the author's articles [6]-[7] and especially the applied papers [4]-[5] in combinatorics and [3] and [11] in informative theory, in which the Thue-Morse sequence plays a key role in their constructions). Let $\mathbb{N}_0$ be the set of nonnegative integers. For $a \in \mathbb{N}$, consider on $\mathbb{N}_0$ equations

$$t(x+a) = t(x), \tag{1}$$

$$t(x+a) = 1 - t(x). \tag{2}$$

Denote C_a and B_a the sets of solutions of equations (1) and (2) correspondingly. Evidently we have

$$B_a \cup C_a = \mathbb{N}_0, \;\; B_a \cap C_a = \oslash. \tag{3}$$

The following lemma is proved straightforward (cf.[8], A079523, A121539).

Lemma 1. *B_1 (C_1) consists of nonnegative integers the binary expansion of which ends in an even (odd) number of 1's.*

For a set of integers $A = \{a_1, a_2, ...\}$ let us introduce a translation operator

$$E_h(A) = \{a_1 - h, a_2 - h, ...\}. \tag{4}$$

One of our main result is the following.

Theorem 1. *B_a and C_a are obtained by a finite set of operations of translation, union and intersection over B_1 and C_1.*

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It is well known that the Thue-Morse sequence is not periodic (a very attractive proof of this fact is given in [9]). Nevertheless, it is trivial to note that for every $n \in \mathbb{N}_0$ there exists $x = 1, 2$ *or* 3 such that $t(n+x) = t(n)$. Indeed, as is well-known, the Thue-Morse sequence does not contain configurations of the form 000 or 111. Therefore, if to suppose that the equalities $t(n+x) = 1 - t(n),\ \ x = 1, 2, 3,$ are valid simultaneously, then we have a contradiction. In connection with this, it is natural to pose the following problem.

Question 1. *For which numbers* a, b, c *one can state that for every* $n \in \mathbb{N}_0$ *there exists* $x = a, b$ *or* c *such that* $t(n+x) = t(n)$?

In conclusion of introduction, we pose a quite another conjecture. Recall that $n \in \mathbb{N}_0$ is called evil (odious) if the number of 1's in its binary expansion is even (odd). Thus, by the above definition of Thue-Morse sequence, for evil (odious) n we have $t_n = 0$ $(t_n = 1)$. Denote $\{B_a(n)\}(\{C_a(n)\})$ the sequence of elements of $B_a(C_a)$ in the increasing order. Denote, furthermore, $\{\beta_a(n)\}(\{\gamma_a(n)\})$ $(0,1)$-sequence, which is obtained from $\{B_a(n)\}(\{C_a(n)\})$ by replacing the odious terms by 1's and the evil terms by 0's.

Conjecture 1. *1)Sequence* $\{\gamma_a(n)\}$ *is periodic; 2) if* $2^m || a$, *then the minimal period has* 2^{m+1} *terms, moreover, 3) if* a *is evil, then the minimal period contains the first* 2^{m+1} *terms of Thue-Morse sequence* $\{t_n\}$, *otherwise, it contains the first* 2^{m+1} *terms of sequence* $\{1 - t_n\}$; *4)* $\beta_a(n) + \gamma_a(n) = 1$.

Below we prove the conjecture in case of $a = 2^m$.

2. Some formulas for B_a and C_a

Theorem 2.

$$B_{a+1} = (C_a \cap E_a(B_1)) \cup (B_a \cap E_a(C_1)), \tag{5}$$

$$C_{a+1} = (C_a \cap E_a(C_1)) \cup (B_a \cap E_a(B_1)). \tag{6}$$

Proof. Denote the right hand sides of (5) and (6) via B^*_{a+1} and C^*_{a+1} correspondingly. Show that $B^*_{a+1} \cup C^*_{a+1} = \mathbb{N}_0$. Indeed, using (3)-(6), we have

$$B^*_{a+1} \cup C^*_{a+1} = (C_a \cap (E_a(B_1) \cup E_a(C_1))) \cup (B_a \cap (E_a(C_1) \cup E_a(B_1))) =$$

$$(C_a \cap E_a(\mathbb{N}_0)) \cup (B_a \cap E_a(\mathbb{N}_0)) = C_a \cup B_a = \mathbb{N}_0.$$

Now it is sufficient to show that B^*_{a+1} contains only solutions of (2) for $a := a+1$, while C^*_{a+1} contains only solutions of (1) for $a := a+1$. Indeed, let $x \in B^*_{a+1}$. Distinguish two cases: 1) $x \in C_a \cap E_a(B_1)$ and 2) $x \in B_a \cap E_a(C_1)$. In case 1) (1) is valid and $x + a \in B_1$. Thus

$$t(x + a + 1) + t(x + a) = 1,$$

or, taking into account (1), we have

$$t(x + a + 1) = 1 - t(x).$$

In case 2) (2) is valid and $x + a \in C_1$. Thus

$$t(x + a + 1) = t(x + a),$$

or, taking into account (2), we have

$$t(x + a + 1) = 1 - t(x).$$

Let now $x \in C^*_{a+1}$. Again distinguish two cases: 1) $x \in C_a \cap E_a(C_1)$ and 2) $x \in B_a \cap E_a(B_1)$. In case 1) (1) is valid and $x + a \in C_1$. Thus

$$t(x + a + 1) = t(x + a),$$

or, taking into account (1), we have

$$t(x + a + 1) = t(x).$$

In case 2) (2) is valid and $x + a \in B_1$. Thus

$$t(x + a + 1) = 1 - t(x + a),$$

or, taking into account (2), we have

$$t(x + a + 1) = t(x).$$

Consequently, $B^*_{a+1} \cap C^*_{a+1} = \oslash$ and $B^*_{a+1} = B_{a+1}, \ \ C^*_{a+1} = C_{a+1}$.■
From Theorem 3, evidently follows Theorem 1. ■

Example 1. *(cf.A081706[8]; this sequence is closely connected with sequence of Allouche et al [2], A003159[8])*

According to Theorem 2, we have

$$C_2 = (C_1 \cap E_1(C_1)) \cup (B_1 \cap E_1(B_1)).$$

Since, evidently, $C_1 \cap E_1(C_1) = \oslash$, then we obtain a representation

$$C_2 = B_1 \cap E_1(B_1). \tag{7}$$

Example 2. *(cf. our sequences A161916, A161974 in [8]) Denote* $C_3^{(0)}$ *the subset of* C_3 *such that for* $n \in C_3^{(0)}$ *we have:* $\min\{x : t(n+x) = t(x)\} = 3.$ *The following simple formula is valid:*

$$C_3^{(0)} = E_1(C_1).$$

Proof. Using (7), consider the following partition of $\mathbb{N}_0$:

$$\mathbb{N}_0 = C_1 \cup B_1 = C_1 \cup (B_1 \cap E_1(B_1)) \cup (B_1 \cap \overline{E_1(B_1)}) = C_1 \cup C_2 \cup D,$$

where

$$D = B_1 \cap \overline{E_1(B_1)}$$

Evidently,

$$D \cap C_1 = \oslash, \;\; D \cap C_2 = D \cap (B_1 \cap E_1(B_1)) = \oslash.$$

Thus $D = C_3^{(0)}$. On the other hand, we have

$$D = B_1 \cap \overline{E_1(B_1)} = B_1 \cap E_1(C_1) = E_1(C_1).\blacksquare$$

By the same way one can prove the following more general results.

Theorem 3. *(A generalization) Let* $l + m = a + 1$. *Then we have*

$$B_{a+1} = (C_l \cap E_l(B_m)) \cup (B_l \cap E_l(C_m)), \tag{8}$$

$$C_{a+1} = (C_l \cap E_l(C_m)) \cup (B_l \cap E_l(B_m)). \tag{9}$$

In particular together with (5)-(6) we have

$$B_{a+1} = (C_1 \cap E_1(B_a)) \cup (B_1 \cap E_1(C_a)), \tag{10}$$

$$C_{a+1} = (C_1 \cap E_1(C_a)) \cup (B_1 \cap E_1(B_a)). \tag{11}$$

Further, for a set of integers $A = \{a_1, a_2, ...\}$, denote hA the set $A = \{ha_1, ha_2, ...\}$.

Theorem 4. *For* $m \in \mathbb{N}$ *we have*

$$B_{2^m} = \bigcup_{k=0}^{2^m-1} E_{-k}(2^m B_1), \tag{12}$$

$$C_{2^m} = \bigcup_{k=0}^{2^m-1} E_{-k}(2^m C_1). \tag{13}$$

Proof. It is sufficient to consider numbers of the form

$$n = ...011...1 \times \times ... \times, \tag{14}$$

where the m last digits are arbitrary. The theorem follows from a simple observation that the indicated in (14) series of 1's contains an odd (even) number of 1's if and only if $n \in C_{2^m}$ $(n \in B_{2^m})$.$\blacksquare$

Example 3.

$$C_2 = (2C_1) \cup E_{-1}(2C_1). \tag{15}$$

Comparison with (7) leads to an identity

$$(2C_1) \cup E_{-1}(2C_1) = B_1 \cap E_1(B_1). \tag{16}$$

On the other hand, the calculating B_2 by Theorems 3,5 leads to another identity

$$(2B_1) \cup E_{-1}(2B_1) = C_1 \cup E_1(C_1). \tag{17}$$

Corollary 1. *For $a = 2^m$, Conjecture 1 is true.*

Proof. In view of the structure of formulas (12)-(13), it is sufficient to prove that in sequences $\{B_1(n)\}, \{C_1(n)\}$ odious and evil terms alternate. Indeed, in the mapping $\{B_{2^m}(n)\}(\{C_{2^m}(n)\})$ on $\{\beta_{2^m}(n)\}(\{\gamma_{2^m}(n)\})$ correspondingly, for any $x \in B_1(n)$ the ordered subset

$$\bigcup_{k=0}^{2^m-1} E_{-k}(2^m x)$$

of B_{2^m} (12) maps on the first 2^m terms of sequence $\{t_n\}$ or $\{1-t_n\}$ depending on the number x is evil or odious. Therefore, if odious and evil terms of $B_1(n)$ alternate, then we obtain the minimal period 2^{m+1} for $\{\beta_{2^m}(n)\}$. By the same way we prove that if odious and evil terms of $C_1(n)$ alternate, then we obtain the minimal period 2^{m+1} for $\{\gamma_{2^m}(n)\}$. Now we prove that odious and evil terms of, e.g., $C_1(n)$, indeed, alternate. If the binary expansion of n ends in more than 1 odd 1's, then the nearest following number from $\{C_1(n)\}$ is $n+2$, and it is easy to see that the relation $t(n+2) = 1 - t(n)$ satisfies; if the binary expansion of n ends in one isolated 1, and before it we have a series of more than 1 0's, then the nearest following number from $\{C_1(n)\}$ is $n+4$, and it is easy to see that the relation $t(n+4) = 1 - t(n)$ again satisfies; at last, if the binary expansion of n ends in one isolated 1, and before it we have one isolated 0, i.e. n has the form ...011...101, then we distinguish two cases: the series of 1's before two last digits 01 contains a)odd and b)even 1's. In case a) the nearest following number from $\{C_1(n)\}$ is $n+2$, with the relation $t(n+2) = 1 - t(n)$, while in case b) it is $n+4$ with the relation $t(n+4) = 1 - t(n)$. Thus odious and evil terms of $\{C_1(n)\}$, indeed, alternate. For $\{B_1(n)\}$ the statement is proved quite analogously.■

Theorem 5. *(Formulas of complement to power of 2) Let* $2^{m-1}+1 \le a \le 2^m$*. Then we have*

$$B_a = (C_{2^m} \cap E_a(B_{2^m-a})) \cup (B_{2^m} \cap E_a(C_{2^m-a})),$$
$$C_a = (B_{2^m} \cap E_a(B_{2^m-a})) \cup (C_{2^m} \cap E_a(C_{2^m-a})).$$

Proof. Denote the right hand sides of the formulas being proved via B_a^{**} and C_a^{**} correspondingly. Show that $B_a^{**} \cup C_a^{**} = \mathbb{N}_0$. Indeed,

$$B_a^{**} \cup C_a^{**} =$$
$$(C_{2^m} \cap (E_a(B_{2^m-a}) \cup E_a(C_{2^m-a}))) \cup (B_{2^m} \cap (E_a(C_{2^m-a}) \cup E_a(B_{2^m-a}))) =$$
$$(E_a(B_{2^m-a}) \cup E_a(C_{2^m-a})) \cap (B_{2^m} \cup (C_{2^m})) = E_a(\mathbb{N}_0) \cap \mathbb{N}_0 = \mathbb{N}_0.$$

Now, by the same way as in proof of Theorem 3, it is easy to show that B_a^{**} contains only solutions of (2) , while C_a^{**} contains only solutions of (1). Then $B_a^{**} \cap C_a^{**} = \oslash$ and $B_a^{**} = B_a, \ C_a^{**} = C_a$.■

3. An approximation of Thue-Morse constant

Let $T_m \ (U_m)$ be the number which is obtained by the reading the period of $\{\beta_a(n)\}(\{\gamma_a(n)\})$ as $2^{m+1}-$bits binary number. Note that $\overline{U}_m = T_m$, i.e. U_m is obtained from T_m by replacing 0's by 1's and 1's by 0's. Therefore,

$$T_m + U_m = 2^{2^{m+1}} - 1. \tag{18}$$

Denote $U_m \vee T_m$ the concatenation of U_m and T_m. Then, using (18), we have

$$U_0 = 1, \ \ for \ \ m \ge 0,$$

$$U_{m+1} = U_m \vee T_m = 2^{2^{m+1}} U_m + 2^{2^{m+1}} - U_m - 1 = (2^{2^{m+1}} - 1)(U_m + 1). \tag{19}$$

Consider now the infinite binary fraction corresponding to sequence $\{\gamma_a(n)\}$:

$$\tau_m = .U_m \vee U_m \vee ... = U_m/(2^{2^{m+1}} - 1). \tag{20}$$

Lemma 2. *If* $F_n = 2^{2^n}+1$ *is n-th Fermat number, then we have a recursion:*

$$F_{m+1}\tau_{m+1} = 1 + (F_{m+1} - 2)\tau_m, \ \ m \ge 0 \tag{21}$$

with τ_0 *defined as the binary fraction*

$$\tau_0 = .010101... = 1/3. \tag{22}$$

Proof. Indeed, according to (19)-(20), we have

$$\tau_{m+1} = .U_{m+1} \vee U_{m+1} \vee ... =$$
$$U_{m+1}/(2^{2^{m+2}} - 1) = (2^{2^{m+1}} - 1)(U_m + 1)/(2^{2^{m+2}} - 1) =$$
$$(U_m + 1)/(2^{2^{m+1}} + 1) = (1 + \tau_m(2^{2^{m+1}} - 1))/(2^{2^{m+1}} + 1) =$$

$$(1+\tau_m(F_{m+1}-2))/F_{m+1},$$

and the lemma follows.■

So, by (21)-(22) for $m = 0, 1, ...$ we find

$$\tau_1 = 2/5,\ \ \tau_2 = 7/17,\ \ \tau_3 = 106/257,$$

$$\tau_4 = 27031/65537,\ \ \tau_5 = 1771476586/4294967297, ...\ .$$

It follows from (21) that the numerators $\{s_n\}$ of these fractions satisfy the recursion

$$s_1 = 2,\ \ s_{n+1} = 1 + (2^{2^n} - 1)s_n,\ \ n \geq 1, \tag{23}$$

while the denominators are $\{F_n\}$. Of course, by its definition, the sequence $\{\tau_n\}$ very fast converges to the Thue-Morse constant

$$\tau = \sum_{n=1}^{\infty} \frac{t_n}{2^n} = 0.4124540336401....$$

E.g., τ_5 approximates τ up to 10^{-9}.

Conjecture 2. *For $n \geq 1$, the fraction $\tau_n = s_n/F_n$ is a convergent corresponding to the continued fraction for τ.*

Note that, the first values of indices of the corresponding convergents, according to numeration of A085394 and A085395 [8] are: 3, 5, 7, 13, 23,... Note also that the binary fraction corresponding to sequence $\{\beta_a(n)\}$:

$$\overline{\tau}_m = .T_m \vee T_m \vee ...$$

satisfies the same relation (21) but with

$$\overline{\tau}_0 = .101010... = 2/3,$$

and converges to $1 - \tau$.

4. Comparison with the Weisstein approximations

Now we want to show that Conjecture 2 is very plausible. As is well known, if the fraction $p/q,\ q > 0$, is a convergent (beginning the second one) corresponding to the continued fraction for α, then p/q is the best approximation to α between all fractions of the form $x/y,\ y > 0$, with $y \leq q$.

Weisstein [10] considered the approximations of τ of the form:

$$a_0 = 0.0_2;\ \ a_1 = 0.01_2;\ \ a_2 = 0.0110_2;\ \ a_3 = 0.01101001_2;$$

$$a_4 = 0.0110100110010110_2; ... \tag{24}$$

If to keep the "natural" denominators $F_n - 1$ (without cancelations), then, denoting w_n the numerators of these fractions, we have

$$w_{n+1} = U_n, \;\; n \geq 0. \tag{25}$$

Since, according to (19),

$$U_n = 2^{2^n} U_{n-1} + 2^{2^n} - U_{n-1} - 1, \;\; n \geq 1,$$

then

$$w = 1, \;\; w_{n+1} = 2^{2^n} - 1 + (2^{2^n} - 1)w_n, \;\; n \geq 1. \tag{26}$$

Theorem 6. *We have*

$$w_n/(F_n - 1) < s_n/F_n < \tau \tag{27}$$

Proof. It is easy to see that $s_n/F_n < \tau$. Indeed, since $t_{2^n} = 1$, then $(2^n + 1)$-th binary digit of τ after the point is 1, while the period of τ_n begins from 0. Let us now prove the left inequality. To this end, let us prove by induction that

$$s_n - w_n = 1. \tag{28}$$

Indeed, if (28) is true for some n, then, subtracting (26) from (23), we find

$$s_{n+1} - w_{n+1} = -2^{2^n} + 2 + (2^{2^n} - 1)(s_n - w_n) = 1.$$

Thus finally we have

$$w_n/(F_n - 1) = (s_n - 1)/(F_n - 1) < s_n/F_n < \tau. \blacksquare$$

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