New pattern matching conditions for wreath products of the cyclic groups with symmetric groups.

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Abstract

We present several multi-variable generating functions for a new pattern matching condition on the wreath product $C_k \wr S_n$ of the cyclic group C_k and the symmetric group S_n. Our new pattern matching condition requires that the underlying permutations match in the usual sense of pattern matching for S_n and that the corresponding sequence of signs match in the sense of words, rather than the exact equality of signs which has been previously studied. We produce the generating functions for the number of matches that occur in elements of $C_k \wr S_n$ for any pattern of length 2 by applying appropriate homomorphisms from the ring of symmetric functions over an infinite number of variables to simple symmetric function identities. We also provide multi-variable generating functions for the distribution of nonoverlapping matches and for the number of elements of $C_k \wr S_n$ which have exactly 2 matches which do not overlap for several patterns of length 2.

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1 Introduction

The goal of this paper is to study pattern matching conditions on the wreath product $C_k \wr S_n$ of the cyclic group C_k and the symmetric group S_n. $C_k \wr S_n$ is the group of $k^n n!$ signed permutations where we allow k signs of the form $1 = \omega^0$, ω, ω^2, ..., ω^{k-1} for some primitive k-th root of unity ω. We can think of the elements $C_k \wr S_n$ as pairs $\gamma = (\sigma, \epsilon)$ where $\sigma = \sigma_1 \dots \sigma_n \in S_n$ and $\epsilon = \epsilon_1 \dots \epsilon_n \in \{1, \omega, \dots, \omega^{k-1}\}^n$. For ease of notation, if $\epsilon = (\omega^{w_1}, \omega^{w_2}, \dots, \omega^{w_n})$ where $w_i \in \{0, \dots, k-1\}$ for $i = 1, \dots, n$, then we simply write $\gamma = (\sigma, w)$ where $w = w_1 w_2 \dots w_n$.

Given a sequence $\sigma = \sigma_1 \cdots \sigma_n$ of distinct integers, let $\text{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\text{red}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation τ in the symmetric group S_j, define a permutation $\sigma = \sigma_1 \cdots \sigma_n \in S_n$ to have a *τ-match at place i* provided $\text{red}(\sigma_i \cdots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ. Similarly, we say that τ *occurs* in σ if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. We say that σ *avoids* τ if there are no occurrences of τ in σ.

We can define similar notions for words over a finite alphabet $[k] = \{0, 1, \dots, k-1\}$. Given a word $w = w_1 \cdots w_n \in [k]^n$, let $\text{red}(w)$ be the word found by replacing the i^{th} largest integer that appears in w by $i - 1$. For example, if $w = 2\ 7\ 2\ 4\ 7$, then $\text{red}(w) = 0\ 2\ 0\ 1\ 2$. Given a word $u \in [k]^j$ such that $\text{red}(u) = u$, define a word $w \in [k]^n$ to have a *u-match at place i* provided $\text{red}(w_i \cdots w_{i+j-1}) = u$. Let $u\text{-mch}(w)$ be the number of u-matches in the word w. Similarly, we say that u *occurs* in a word w if there exist $1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(w_{i_1} \cdots w_{i_j}) = u$. We say that w *avoids* u if there are no occurrences of u in w.

There are a number of papers on pattern matching and pattern avoidance in $C_k \wr S_n$ [7, 13, 14, 15]. For example, the following pattern matching condition was studied in [13, 14, 15].

Definition 1.
1. *We say that an element $(\tau, u) \in C_k \wr S_j$* **occurs** *in an element $(\sigma, w) \in C_k \wr S_n$ if there are $1 \leq i_1 < i_2 < \cdots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \dots \sigma_{i_j}) = \tau$ and $w_{i_p} = u_p$ for $p = 1, \dots, j$.*
2. *We say that an element $(\sigma, w) \in C_k \wr S_n$* **avoids** *$(\tau, u) \in C_k \wr S_j$ if there are no occurrences of (τ, u) in (σ, w).*
3. *If $(\sigma, w) \in C_k \wr S_n$ and $(\tau, u) \in C_k \wr S_j$, then we say that there is a (τ, u)-***match in** *(σ, w)* **starting at position** *i if $\text{red}(\sigma_i \sigma_{i+1} \dots \sigma_{i+j-1}) = \tau$ and $w_{i+p-1} = u_p$ for $p = 1, \dots, j$.*

That is, an occurrence or match of $(\tau, u) \in C_k \wr S_j$ in an element $(\sigma, w) \in C_k \wr S_n$ is just an ordinary occurrence or match of τ in σ where the corresponding signs agree exactly. For example, Mansour [14] proved via recursion that for any $(\tau, u) \in C_k \wr S_2$, the number of (τ, u)-avoiding elements in $C_k \wr S_n$ is $\sum_{j=0}^{n} j!(k-1)^j \binom{n}{j}^2$. This generalized a result of Simion [22] who proved the same result for the hyperoctrahedral group $C_2 \wr S_n$. Similarly, Mansour and West [15] determined the number of permutations in $C_2 \wr$

S_n that avoid all possible 2 and 3 element set of patterns of elements of $C_2 \wr S_2$. For example, let K_n^1, the number of $(\sigma, \epsilon) \in C_2 \wr S_n$ that avoid all the patterns in the set $\{(1\ 2, 0\ 0), (1\ 2, 0\ 1), (2\ 1, 1\ 0)\}$, K_n^2, the number of $(\sigma, \epsilon) \in C_2 \wr S_n$ that avoid all the patterns in the set $\{(1\ 2, 0\ 1), (1\ 2, 1\ 0), (2\ 1, 0\ 1)\}$, and K_n^3, the number of $(\sigma, \epsilon) \in C_2 \wr S_n$ that avoid all the patterns in the set $\{(1\ 2, 0\ 0), (1\ 2, 0\ 1), (2\ 1, 0\ 0)\}$. They proved that

$$\begin{aligned} K_n^1 &= F_{2n+1}, \\ K_n^2 &= n! \sum_{j=0}^{n} \binom{n}{j}^{-1}, \text{ and} \\ K_n^3 &= n! + n! \sum_{j=1}^{n} \frac{1}{j} \end{aligned}$$

where F_n is n-th Fibonacci number.

In this paper, we shall drop the requirement of the exact matching of signs and replace it by the condition that the two sequences of signs match in the sense of words described above. That is, we shall consider the following pattern matching conditions:

Definition 2. *Suppose that $(\tau, u) \in C_k \wr S_j$ and $\mathrm{red}(u) = u$.*

1. *We say that (τ, u)* **bi-occurs** *in $(\sigma, w) \in C_k \wr S_n$ if there are $1 \le i_1 < i_2 < \cdots < i_j \le n$ such that $\mathrm{red}(\sigma_{i_1} \ldots \sigma_{i_j}) = \tau$ and $\mathrm{red}(w_{i_1} \ldots w_{i_j}) = u$.*

2. *We say that an element $(\sigma, w) \in C_k \wr S_n$* **bi-avoids** *$(\tau, u)$ if there are no bi-occurrences of (τ, u) in (σ, w).*

3. *We say that there is a (τ, u)***-bi-match in** *$(\sigma, w) \in C_k \wr S_n$* **starting at position** *i if $\mathrm{red}(\sigma_i \sigma_{i+1} \ldots \sigma_{i+j-1}) = \tau$ and $\mathrm{red}(w_i w_{i+1} \ldots w_{i+j-1}) = u$.*

For example, suppose that $(\tau, u) = (1\ 2, 0\ 0)$ and $(\sigma, w) = (1\ 3\ 2\ 4, 1\ 2\ 2\ 2)$. Then there are no occurrences or matches of (τ, u) in (σ, w) according to Definition 1. However, there is a (τ, u)-bi-match in (σ, w) starting at position 3; additionally, 34 and 24 in σ are bi-occurrences of (τ, u) in (σ, w). Let (τ, u)-mch$((\sigma, w))$ be the number of (τ, u)-bi-matches in $(\sigma, w) \in C_k \wr S_n$. Let (τ, u)-nlap$((\sigma, w))$ be the maximum number of non-overlapping (τ, u)-bi-matches in (σ, w) where two (τ, u)-bi-matches are said to overlap if there are positions in (σ, w) that are involved in both (τ, u)-bi-matches.

One can easily extend these notions to sets of elements of $C_k \wr S_j$. That is, suppose that $\Upsilon \subseteq C_k \wr S_j$ is such that every $(\tau, u) \in \Upsilon$ has the property that $\mathrm{red}(u) = u$. Then (σ, w) has a Υ-bi-match at place i provided $(\mathrm{red}(\sigma_i \ldots \sigma_{i+j-1}), \mathrm{red}(w_i \ldots w_{i+j-1})) \in \Upsilon$. Let Υ-mch$((\sigma, w))$ and Υ-nlap$((\sigma, w))$ be the number of Υ-bi-matches and non-overlapping Υ-bi-matches in (σ, w), respectively.

In this paper, we shall mainly study the distribution of bi-matches for patterns of length 2, i.e. where $(\tau, u) \in C_k \wr S_2$. This is closely related to the analogue of rises and descents in $C_k \wr S_n$ where we compare pairs using the product order. That is, instead of thinking of an element of $C_k \wr S_n$ as a pair $(\sigma_1 \ldots \sigma_n, w_1 \ldots w_n)$, we can think of it as a sequence of pairs $(\sigma_1, w_1)(\sigma_2, w_2)\ldots(\sigma_n, w_n)$. We then define a partial order on such pairs by the usual product order. That is, $(i_1, j_1) \leq (i_2, j_2)$ if and only if $i_1 \leq i_2$ and $j_1 \leq j_2$. Then we define the following statistics for elements $(\sigma, w) \in C_k \wr S_n$.

$$
\begin{aligned}
Des((\sigma,w)) &= \{i : \sigma_i > \sigma_{i+1} \ \&\ w_i \geq w_{i+1}\}, \mathrm{des}((\sigma,w)) = |Des((\sigma,w))|,\\
Ris((\sigma,w)) &= \{i : \sigma_i < \sigma_{i+1} \ \&\ w_i \leq w_{i+1}\}, \mathrm{ris}((\sigma,w)) = |Ris((\sigma,w))|,\\
WDes((\sigma,w)) &= \{i : \sigma_i > \sigma_{i+1} \ \&\ w_i = w_{i+1}\}, \mathrm{wdes}((\sigma,w)) = |WDes((\sigma,w))|,\\
WRis((\sigma,w)) &= \{i : \sigma_i < \sigma_{i+1} \ \&\ w_i = w_{i+1}\}, \mathrm{wris}((\sigma,w)) = |WRis((\sigma,w))|,\\
SDes((\sigma,w)) &= \{i : \sigma_i > \sigma_{i+1} \ \&\ w_i > w_{i+1}\}, \mathrm{sdes}((\sigma,w)) = |SDes((\sigma,w))|,\\
SRis((\sigma,w)) &= \{i : \sigma_i < \sigma_{i+1} \ \&\ w_i < w_{i+1}\}, \mathrm{sris}((\sigma,w)) = |SRis((\sigma,w))|.
\end{aligned}
$$

We shall refer to $Des((\sigma,w))$ as the *descent set* of (σ,w), $WDes((\sigma,w))$ as the *weak descent set* of (σ,w), and $SDes((\sigma,w))$ as the *strict descent set* of (σ,w). Similarly, we shall refer to $Ris((\sigma,w))$ as the *rise set* of (σ,w), $WRis((\sigma,w))$ as the *weak rise set* of (σ,w), and $SRis((\sigma,w))$ as the *strict rise set* of (σ,w). It is easy to see that

$i \in WDes((\sigma,w))$ if and only if there is a $(2\ 1, 0\ 0)$-bi-match starting at position i,

$i \in SDes((\sigma,w))$ if and only if there is a $(2\ 1, 1\ 0)$-bi-match starting at position i, and

$i \in Des((\sigma,w))$ if and only if there is a Υ-bi-match starting at position i where $\Upsilon = \{(2\ 1, 0\ 0), (2\ 1, 1\ 0)\}$.

Similarly,

$i \in WRis((\sigma,w))$ if and only if there is a $(1\ 2, 0\ 0)$-bi-match starting at position i,

$i \in SRis((\sigma,w))$ if and only if there is a $(1\ 2, 0\ 1)$-bi-match starting at position i, and

$i \in Ris((\sigma,w))$ if and only if there is a Υ-bi-match starting at position i where $\Upsilon = \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\}$.

If $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, then we define the reverse of σ, σ^r, by $\sigma^r = \sigma_n \ldots \sigma_1$. Similarly, if $w = w_1 \ldots w_n \in [k]^n$, then we define $w^r = w_n \ldots w_1$. It is easy to see that

$$
\begin{aligned}
\mathrm{ris}((\sigma,w)) &= \mathrm{des}((\sigma^r,w^r)),\\
\mathrm{wris}((\sigma,w)) &= \mathrm{wdes}((\sigma^r,w^r)),\ \text{and}\\
\mathrm{sris}((\sigma,w)) &= \mathrm{sdes}((\sigma^r,w^r)).
\end{aligned}
$$

Thus we need to find the distributions for only one of the corresponding pairs. We shall prove the following generating functions.

$$
\sum_{n\geq 0} \frac{t^n}{n!} \sum_{(\sigma,w)\in C_k\wr S_n} x^{\mathrm{ris}((\sigma,w))} = \frac{1-x}{1-x+\sum_{n\geq 1} \frac{((x-1)t)^n}{n!}\binom{n+k-1}{n}}. \tag{1}
$$

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{\mathrm{wris}((\sigma,w))} = \frac{1-x}{1-x+k(e^{(x-1)t}-1)}. \tag{2}$$

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{\mathrm{sris}((\sigma,w))} = \frac{1-x}{1-x+\sum_{n\geq 1}\frac{((x-1)t)^n}{n!}\binom{k}{n}}. \tag{3}$$

Other distributions results for (τ,u)-bi-matches follow from these results. For example, if $\sigma = \sigma_1 \ldots \sigma_n \in S_n$, then we define the complement of σ, σ^c, by

$$\sigma^c = (n+1-\sigma_1)\ldots(n+1-\sigma_n).$$

If $w = w_1 \ldots w_n \in [k]^n$, then we define the complement of w, w^c, by

$$w^c = (k-1-w_1)\ldots(k-1-w_n).$$

We can then consider maps $\phi_{a,b} : C_k \wr S_n \to C_k \wr S_n$ where $\phi_{a,b}((\sigma,w)) = (\sigma^a, w^b)$ for $a,b \in \{r,c\}$. Such maps will easily allow us to establish that the distribution of (τ,u)-bi-matches is the same for various classes of (τ,u)'s. For example, one can use such maps to show that the distributions of $(1\ 2, 0\ 1)$-bi-matches, $(2\ 1, 0\ 1)$-bi-matches, $(1\ 2, 1\ 0)$-bi-matches, and $(2\ 1, 1\ 0)$-bi-matches are all the same.

Another interesting case is when we let $\Upsilon = \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}$. In this case we have a Υ-bi-match in (σ,w) starting at i if and only if $\sigma_i < \sigma_{i+1}$ and $w_i \neq w_{i+1}$. In that case, we shall show that

$$\sum_{n\geq 0}\frac{t^n}{n!}\sum_{(\sigma,w)\in C_k\wr S_n} x^{\Upsilon\text{-mch}((\sigma,w))} = \frac{(k-1)(1-x)}{(k-1)(1-x)+k(e^{(k-1)(x-1)t}-1)}. \tag{4}$$

In fact, all of the generating functions (1)–(4) are special cases of more refined generating functions for $C_k \wr S_n$ where we keep track of more statistics. For $\Upsilon \subseteq C_k \wr S_j$, we shall consider generating functions of the form

$$D_k^{\Upsilon}(x,p,q,r,t) = \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}x^{\Upsilon\text{-mch}((\sigma,w))} \tag{5}$$

where $||w|| = ||w_1 \ldots w_n|| = w_1 + \cdots + w_n$, and $\mathrm{inv}(\sigma)$ (resp. $\mathrm{coinv}(\sigma)$) is the number of inversions (resp. coinversions) of σ defined for $\sigma = \sigma_1 \ldots \sigma_n$ as the number of pairs $i < j$ such that $\sigma_i > \sigma_j$ (resp. $\sigma_i < \sigma_j$). Let

$$\begin{aligned}
\Upsilon_{\mathbf{r}} &= \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\},\\
\Upsilon_{\mathbf{w}} &= \{(1\ 2, 0\ 0)\},\\
\Upsilon_{\mathbf{s}} &= \{(1\ 2, 0\ 1)\}, \text{ and}\\
\Upsilon_{\mathbf{d}} &= \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}.
\end{aligned}$$

Thus $\Upsilon_{\mathbf{r}}$-matches correspond to rises, $\Upsilon_{\mathbf{w}}$-matches correspond to weak rises, and $\Upsilon_{\mathbf{s}}$-matches correspond to strict rises. We shall find $D_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t)$ for $\mathbf{a} \in \{\mathbf{r},\mathbf{w},\mathbf{s}\}$ and find $D_k^{\Upsilon_{\mathbf{d}}}(x,p,q,1,t)$. For example, define the p,q-analogues of n, $n!$, $\binom{n}{k}$, and $\binom{n}{a_1,\ldots,a_m}$ by

$$\begin{aligned}
[n]_{p,q} &= \frac{p^n-q^n}{p-q} = p^{n-1}+p^{n-2}q+\cdots+pq^{n-2}+q^{n-1},\\
[n]_{p,q}! &= [n]_{p,q}[n-1]_{p,q}\cdots[2]_{p,q}[1]_{p,q},\\
\begin{bmatrix} n\\ k\end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}, \text{ and}\\
\begin{bmatrix} n\\ a_1,\ldots,a_m\end{bmatrix}_{p,q} &= \frac{[n]_{p,q}!}{[a_1]_{p,q}!\cdots[a_m]_{p,q}!},
\end{aligned}$$

respectively. We define the q-analogues of n, $n!$, $\binom{n}{k}$, and $\binom{n}{a_1,\ldots,a_m}$ by $[n]_{1,q}$, $[n]_{1,q}!$, $\begin{bmatrix} n\\ k\end{bmatrix}_{1,q}$, and $\begin{bmatrix} n\\ a_1,\ldots,a_m\end{bmatrix}_{1,q}$, respectively. Then we will prove that

$$D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t) = \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}x^{\mathrm{ris}((\sigma,w))} = \frac{1-x}{1-x+\sum_{n\geq 1}\frac{p^{\binom{n}{2}}((x-1)t)^n}{[n]_{p,q}!}\begin{bmatrix} n+k-1\\ n\end{bmatrix}_r} \tag{6}$$

which reduces to (1) when we set $p=q=r=1$.

We shall prove our formulas for the generating functions $D_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t)$ for $\mathbf{a}\in\{\mathbf{r},\mathbf{w},\mathbf{s}\}$ and $D_k^{\Upsilon_{\mathbf{d}}}(x,p,q,1,t)$ by applying a ring homomorphism, defined on the ring Λ of symmetric functions over infinitely many variables $x_1,x_2,\ldots$, to a simple symmetric function identity. There has been a long line of research, [2], [3], [1], [10], [11], [17], [19], [21], [25], [16], which shows that a large number of generating functions for permutation statistics can be obtained by applying homomorphisms defined on the ring of symmetric functions Λ over infinitely many variables $x_1,x_2,\ldots$ to simple symmetric function identities. For example, the n-th elementary symmetric function, e_n, and the n-th homogeneous symmetric function, h_n, are defined by the generating functions

$$E(t)=\sum_{n\geq 0}e_nt^n=\prod_i(1+x_it) \tag{7}$$

and

$$H(t)=\sum_{n\geq 0}h_nt^n=\prod_i\frac{1}{1-x_it}. \tag{8}$$

We let $P(t)=\sum_{n\geq 0}p_nt^n$ where $p_n=\sum_i x_i^n$ is the n-th power symmetric function. A partition of n is a sequence $\mu=(\mu_1,\ldots,\mu_k)$ such that $0<\mu_1\leq\cdots\leq\mu_k$ and $\mu_1+\cdots+\mu_k=n$. We write $\mu\vdash n$ if μ is partition of n and we let $\ell(\mu)$ denote the number of parts of μ. If $\mu\vdash n$, we set $h_\mu=\prod_{i=1}^{\ell(\mu)}h_{\mu_i}$, $e_\mu=\prod_{i=1}^{\ell(\mu)}e_{\mu_i}$, and $p_\mu=\prod_{i=1}^{\ell(\mu)}p_{\mu_i}$. Let Λ_n denote the space of homogeneous symmetric functions of degree n over infinitely many variables $x_1,x_2,\ldots$ so that $\Lambda=\oplus_{n\geq 0}\Lambda_n$. It is well know that $\{e_\lambda:\lambda\vdash n\}$,

$\{h_\lambda : \lambda \vdash n\}$, and $\{p_\lambda : \lambda \vdash n\}$ are all bases of Λ_n. It follows that $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ and hence we can define a ring homomorphism $\xi : \Lambda \to R$ where R is a ring by simply specifying $\xi(e_n)$ for all $n \geq 0$.

Now it is well-known that

$$H(t) = 1/E(-t) \tag{9}$$

and

$$P(t) = \frac{\sum_{n\geq 1}(-1)^{n-1} n e_n t^n}{E(-t)}. \tag{10}$$

A surprisingly large number of results on generating functions for various permutation statistics in the literature and large number of new generating functions can be derived by applying homomorphisms on Λ to simple identities such as (9) and (10). We shall show that all our generating functions arise by applying appropriate ring homomorphisms to identity (9). For example, we shall show that (6) arises by applying the ring homomorphism ξ to identity (9) where $\xi(e_0) = 1$ and

$$\xi(e_n) = \frac{(-1)^{n-1}(x-1)^{n-1} p^{\binom{n}{2}}}{[n]_{p,q}!} \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r$$

for all $n \geq 0$.

We can use our formulas for the generating functions $D_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t)$ for $\mathbf{a} \in \{\mathbf{r},\mathbf{w},\mathbf{s}\}$ and $D_k^{\Upsilon_{\mathbf{d}}}(x,p,q,1,t)$, to derive a number of other generating functions. For example, for any $\Upsilon \subseteq C_k \wr S_j$ such that $red(u) = u$ for all $(\tau, u) \in \Upsilon$, let

$$A_k^{\Upsilon}(p,q,r,t) = \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} \chi(\Upsilon\text{-mch}((\sigma,w)) = 0) \tag{11}$$

where for any statement B, we let $\chi(B)$ equal 1 if B is true and equal 0 if B is false. Thus $A_k^{\Upsilon}(p,q,r,t)$ is the generating function counting elements of $C_k \wr S_n$ with no Υ-matches. We shall prove that if

$$N_k^{\Upsilon}(x,p,q,r,t) = \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} x^{\Upsilon\text{-nlap}((\sigma,w))}, \tag{12}$$

then

$$N_k^{\Upsilon}(x,p,q,r,t) = \frac{A_k^{\Upsilon}(p,q,r,t)}{1 - x(([k]_r t - 1)A_k^{\Upsilon}(p,q,r,t) + 1)}. \tag{13}$$

This result is an analogue of a result by Kitaev [9] for permutations. Since our generating functions will allow us to derive expressions for $A_k^{\Upsilon_{\mathbf{a}}}(p,q,r,t)$ for $\mathbf{a} \in \{\mathbf{r},\mathbf{w},\mathbf{s}\}$, we will automatically be able to find the generating functions for the distribution of non-overlapping $\Upsilon_{\mathbf{a}}$-matches for $\mathbf{a} \in \{\mathbf{r},\mathbf{w},\mathbf{s}\}$. There are two additional generating functions that we can obtain in each of our examples. For example, it is easy to see that since $\Upsilon_{\mathbf{r}}$-matches correspond to rises, then the coefficient of x in $N_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t) - D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)$, written

$$(N_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t) - D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t))|_x,$$

is the generating function for $(\sigma, w) \in C_k \wr S_n$ such that (σ, w) has exactly 2 rises which overlap, i.e. there is exactly one pattern match of

$$\Upsilon = \{(0\ 1\ 2, 0\ 0\ 0), (0\ 1\ 2, 0\ 0\ 1), (0\ 1\ 2, 0\ 1\ 1), (0\ 1\ 2, 0\ 1\ 2)\}$$

and no other rises. Moreover,

$$D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)|_{x^2} - [N_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t) - D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)|_x]$$

is the generating function for $(\sigma, w) \in C_k \wr S_n$ such that (σ, w) has exactly 2 rises which do not overlap. Our results will allow us to find explicit formulas for these two additional types of generating functions for rises, weak rises, and strict rises.

The outline of this paper is as follows. In section 2, we shall provide the necessary background in symmetric functions that we shall need to derive our generating functions. In section 3, we shall give our proofs of the generating functions $D_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t)$ for $\mathbf{a} \in \{\mathbf{r}, \mathbf{w}, \mathbf{s}\}$ and $D_k^{\Upsilon_{\mathbf{d}}}(x,p,q,1,t)$. Finally, in sections 4 and 5, we shall find explicit expressions for

$$(N_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t) - D_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t))|_x$$

and

$$D_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t)|_{x^2} - [N_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t) - D_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t)|_x]$$

for $\mathbf{a} \in \{\mathbf{r}, \mathbf{w}, \mathbf{s}\}$. In section 6, we shall give tables of the number of various types of permutations $(\sigma, w) \in C_k \wr S_n$ that can be computed from our generating functions for small values of k and n. We shall see that various sequences associated with our sets of permutations appear in OEIS [23] and hence our sequences count other combinatorial objects. Moreover, we shall see that for fixed n, some of the sequences are generated by certain natural polynomials in k. For example, we let $A_{n,k}^{\Upsilon}$ denote the number of $(\sigma, w) \in C_k \wr S_n$ for which Υ-mch$((\sigma, w)) = 0$. We shall show that if $\Upsilon = \{(1\ 2, 0\ 0)\}$, then $A_{n,k}^{\Upsilon} = \sum_{j=1}^{n} (-1)^{n-j} j! S_{n,j} k^j$ for all $k \geq 2$, where $S_{n,k}$ is the Stirling number of the second kind. Similarly, if $\Upsilon = \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}$, then $A_{n,k}^{\Upsilon}$ is an Eulerian polynomial. That is, for all $k \geq 2$, $A_{n,k}^{\Upsilon} = \sum_{\sigma \in S_n} k^{\mathrm{des}(\sigma)+1}$ for where $\mathrm{des}(\sigma)$ is the number of descents of σ. The connections to the Stirling numbers of the second kind and to the Eulerian polynomials were observed by Einar Steingrímsson and we prove this in this paper. Finally, in section 7, we shall state a few problems for further research.

2 Symmetric Functions

In this section we give the necessary background on symmetric functions needed for our proofs of the generating functions.

Let Λ denote the ring of symmetric functions over infinitely many variables $x_1, x_2, \ldots$ with coefficients in the field of complex numbers $\mathbb{C}$. The n^{th} elementary symmetric function e_n in the variables $x_1, x_2, \ldots$ is given by

$$E(t) = \sum_{n \geq 0} e_n t^n = \prod_i (1 + x_i t)$$

and the n^{th} homogeneous symmetric function h_n in the variables $x_1, x_2, \ldots$ is given by

$$H(t) = \sum_{n \geq 0} h_n t^n = \prod_i \frac{1}{1 - x_i t}.$$

Thus

$$H(t) = 1/E(-t). \tag{14}$$

Let $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ be an integer partition, that is, λ is a finite sequence of weakly increasing positive integers. Let $\ell(\lambda) = l$ denote the number of parts of λ. If the sum of these integers is n, we say that λ is a partition of n and write $\lambda \vdash n$. For any partition $\lambda = (\lambda_1, \ldots, \lambda_\ell)$, let $e_\lambda = e_{\lambda_1} \cdots e_{\lambda_\ell}$. The well-known fundamental theorem of symmetric functions says that $\{e_\lambda : \lambda \text{ is a partition}\}$ is a basis for Λ or that $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ. Similarly, if we define $h_\lambda = h_{\lambda_1} \cdots h_{\lambda_\ell}$, then $\{h_\lambda : \lambda \text{ is a partition}\}$ is also a basis for Λ. Since $\{e_0, e_1, \ldots\}$ is an algebraically independent set of generators for Λ, we can specify a ring homomorphism θ on Λ by simply defining $\theta(e_n)$ for all $n \geq 0$.

Since the elementary symmetric functions e_λ and the homogeneous symmetric functions h_λ are both bases for Λ, it makes sense to talk about the coefficient of the homogeneous symmetric functions when written in terms of the elementary symmetric function basis. These coefficients has been shown to equal the sizes of a certain sets of combinatorial objects up to a sign. A *brick tabloid* of shape (n) and type $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a filling of a row of n squares of cells with bricks of lengths $\lambda_1, \ldots, \lambda_k$ such that bricks do not overlap. One brick tabloid of shape (12) and type $(1, 1, 2, 3, 5)$ is displayed below.

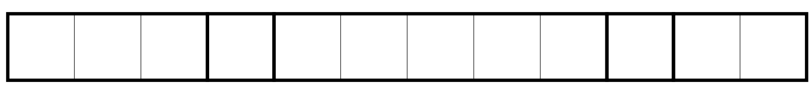

Figure 1: A brick tabloid of shape (12) and type $(1, 1, 2, 3, 5)$.

Let $\mathcal{B}_{\lambda,n}$ denote the set of all λ-brick tabloids of shape (n) and let $B_{\lambda,n} = |\mathcal{B}_{\lambda,n}|$. Through simple recursions stemming from (14), Eğecioğlu and Remmel proved in [6] that

$$h_n = \sum_{\lambda \vdash n} (-1)^{n - \ell(\lambda)} B_{\lambda,n} e_\lambda. \tag{15}$$

We end this section with two lemmas that will be needed in later sections. Both of the lemmas follow from simple codings of a basic result of Carlitz [4] that

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \sum_{r \in \mathcal{R}(1^k 0^{n-k})} q^{inv(r)},$$

where $\mathcal{R}(1^k 0^{n-k})$ is the number of rearrangements of k 1's and $n-k$ 0's. We start with a lemma from [20]. Fix a brick tabloid $T = (b_1, \ldots, b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}$. Let $IF(T)$ denote the set of all fillings of the cells of $T = (b_1, \ldots, b_{\ell(\mu)})$ with the numbers $1, \ldots, n$ so that the numbers increase within each brick reading from left to right. We then think of each such filling as a permutation of S_n by reading the numbers from left to right in each row. For example, Figure 2 pictures an element of $IF(3,6,3)$ whose corresponding permutation is 4 6 12 1 5 7 8 10 11 2 3 9.

4	**6**	**12**	**1**	**5**	**7**	**8**	**10**	**11**	**2**	**3**	**9**

Figure 2: An element of $IF(3,6,3)$.

Then the following lemma which is proved in [20] gives a combinatorial interpretation to $p^{\sum_{i=1}^{\ell(\mu)} \binom{b_i}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q}$.

Lemma 3. *If $T = (b_1, \ldots, b_{\ell(\mu)})$ is a brick tabloid in $\mathcal{B}_{\mu,n}$, then*

$$p^{\sum_{i=1}^{\ell(\mu)} \binom{b_i}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q} = \sum_{\sigma \in IF(T)} q^{inv(\sigma)} p^{coinv(\sigma)}.$$

Another well-known combinatorial interpretation for $\begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q$ is that it is equal to the sum of the sizes of the partitions that are contained in an $n \times (k-1)$ rectangle. Thus we have the following lemma.

Lemma 4.

$$\sum_{0 \le a_1 \le \cdots \le a_n \le k-1} q^{a_1+\cdots+a_n} = \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_q.$$

3 Generating Functions

The main goal of this section is to prove generating functions that specialize to the generating functions (1)–(4) given in the introduction.

We start by proving a generating function which specializes to (1).

Theorem 5. *Let $\Upsilon_{\mathbf{r}} = \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\}$. For all $k \ge 2$,*

$$\begin{aligned} D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t) &= \sum_{n\ge 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} x^{\mathrm{ris}((\sigma,w))} \\ &= \frac{1-x}{1-x+\sum_{n\ge 1} \frac{p^{\binom{n}{2}}((x-1)t)^n}{[n]_{p,q}!} \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r}. \end{aligned} \tag{16}$$

Proof. Define a ring homomorphism $\Gamma : \Lambda \to \mathbb{Q}(p,q,r,x)$ by setting $\Gamma(e_0) = 1$ and

$$\Gamma(e_n) = (-1)^{n-1}(x-1)^{n-1} \frac{\begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r}{[n]_{p,q}!} p^{\binom{n}{2}} \tag{17}$$

for $n \geq 1$. Then we claim that

$$[n]_{p,q}!\Gamma(h_n) = \sum_{(\sigma,w) \in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} x^{\mathrm{ris}((\sigma,w))} \tag{18}$$

for all $n \geq 1$. That is,

$$\begin{aligned}
&[n]_{p,q}!\Gamma(h_n) = \\
&[n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,(n)} \Gamma(e_\mu) = \\
&[n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \prod_{j=1}^{\ell(\mu)} (-1)^{b_j-1}(x-1)^{b_j-1} \frac{\begin{bmatrix} b_j+k-1 \\ b_j \end{bmatrix}_r}{[b_j]_{p,q}!} p^{\binom{b_j}{2}} = \\
&\sum_{\mu \vdash n} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} p^{\sum_{j=1}^{\ell(\mu)} \binom{b_j}{2}} \begin{bmatrix} n \\ b_1, \ldots, b_{\ell(\mu)} \end{bmatrix}_{p,q} \prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1} \begin{bmatrix} b_j+k-1 \\ b_j \end{bmatrix}_r .
\end{aligned} \tag{19}$$

Next we want to give a combinatorial interpretation to (19). By Lemma 3, for each brick tabloid $T = (b_1, \ldots, b_{\ell(\mu)})$, we can interpret $p^{\sum_{j=1}^{\ell(\mu)} \binom{b_j}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q}$ as the sum of the weights of all fillings of T with a permutation $\sigma \in S_n$ such that σ is increasing in each brick and we weight σ with $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}$. By Lemma 4, we can interpret the term $\prod_{j=1}^{\ell(\mu)} \begin{bmatrix} b_j+k-1 \\ b_j \end{bmatrix}_r$ as the sum of the weights of fillings $w = w_1 \ldots w_n$ where the elements of w are between 0 and $k-1$ and are weakly increasing in each brick and where we weight w by $r^{w_1+\cdots+w_n}$. Finally, we interpret $\prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1}$ as all ways of picking a label of the cells of each brick except the final cell with either an x or a -1. For completeness, we label the final cell of each brick with 1. We shall call all such objects created in this way filled labelled brick tabloids and let $\mathcal{F}_n$ denote the set of all filled labelled brick tabloids that arise in this way. Thus a $C \in \mathcal{F}_n$ consists of a brick tabloid T, a permutation $\sigma \in S_n$, a sequence $w \in \{0, \ldots, k-1\}^n$, and a labelling L of the cells of T with elements from $\{x, 1, -1\}$ such that

1. σ is strictly increasing in each brick,
2. w is weakly increasing in each brick,
3. the final cell of each brick is labelled with 1, and
4. each cell which is not a final cell of a brick is labelled with x or -1.

We then define the weight $w(C)$ of C to be $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}$ times the product of all the x labels in L and the sign $sgn(C)$ of C to be the product of all the -1 labels in L. For example, if $n = 12$, $k = 4$, and $T = (4, 3, 3, 2)$, then Figure 3 pictures such a composite object $C \in \mathcal{F}_{12}$ where $w(C) = q^{24}p^{32}r^{17}x^5$ and $sgn(C) = -1$.

Thus

$$[n]_{p,q}!\Gamma(h_n) = \sum_{C \in \mathcal{F}_n} sgn(C)w(C). \tag{20}$$

L	x	x	−1	1	x	−1	1	x	x	1	−1	1
w	0	1	1	3	1	1	3	0	2	3	0	2
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 3: A composite object $C \in \mathcal{F}_{12}$.

Next we define a weight-preserving sign-reversing involution $I : \mathcal{F}_n \to \mathcal{F}_n$. To define $I(C)$, we scan the cells of $C = (T, \sigma, w, L)$ from left to right looking for the leftmost cell t such that either (i) t is labelled with -1 or (ii) t is at the end of a brick b_j and the brick b_{j+1} immediately following b_j has the property that σ is strictly increasing in all the cells corresponding to b_j and b_{j+1} and w is weakly increasing in all the cells corresponding to b_j and b_{j+1}. In case (i), $I(C) = (T', \sigma', w', L')$ where T' is the result of replacing the brick b in T containing t by two bricks b^* and b^{**} where b^* contains the cell t plus all the cells in b to the left of t and b^{**} contains all the cells of b to the right of t, $\sigma = \sigma'$, $w = w'$, and L' is the labelling that results from L by changing the label of cell t from -1 to 1. In case (ii), $I(C) = (T', \sigma', r', L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma = \sigma'$, $w = w'$, and L' is the labelling that results from L by changing the label of cell t from 1 to -1. If neither case (i) or case (ii) applies, then we let $I(C) = C$. For example, if C is the element of $\mathcal{F}_{12}$ pictured in Figure 3, then $I(C)$ is pictured in Figure 4.

L	x	x	1	1	x	−1	1	x	x	1	−1	1
w	0	1	1	3	1	1	3	0	2	3	0	2
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 4: $I(C)$ for C in Figure 3.

It is easy to see that I is a weight-preserving sign-reversing involution and hence I shows that

$$[n]_{p,q}!\Gamma(h_n) = \sum_{C \in \mathcal{F}_n, I(C)=C} sgn(C)w(C). \tag{21}$$

Thus we must examine the fixed points $C = (T, \sigma, w, L)$ of I. First there can be no -1 labels in L so that $sgn(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is the last cell of b_j, then it can not be the case that $\sigma_t < \sigma_{t+1}$ and $w_t \leq w_{t+1}$ since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we associate an element $(\sigma, w) \in C_k \wr S_n$. For example, a fixed point of I is pictured in Figure 5 where

$$\begin{aligned}\sigma &= 2\ 3\ 4\ 6\ 9\ 10\ 11\ 1\ 8\ 12\ 5\ 7 \text{ and}\\ w &= 0\ 1\ 1\ 3\ 1\ 1\ 3\ 0\ 2\ 3\ 3\ 3.\end{aligned}$$

It follows that if cell t is at the end of a brick, then $t \notin Ris((\sigma, \epsilon))$. However if v is a cell which is not at the end of a brick, then our definitions force $\sigma_v < \sigma_{v+1}$ and $w_v \leq w_{v+1}$ so that $v \in Ris((\sigma, \epsilon))$. Since each such cell v must be labelled with an x, it follows that

$sgn(C)w(C) = q^{inv(\sigma)}p^{coinv(\sigma)}r^{||w||}x^{\mathrm{ris}((\sigma,\epsilon))}$. Vice versa, if $(\sigma, w) \in C_k \wr S_n$, then we can create a fixed point $C = (T, \sigma, w, L)$ by having the bricks in T end at cells of the form t where $t \notin Ris((\sigma,\epsilon))$, and labelling each cell $t \in Ris((\sigma,\epsilon))$ with x and labelling the remaining cells with 1. Thus we have shown that

$$[n]_{p,q}!\Gamma(h_n) = \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}x^{\mathrm{ris}((\sigma,w))}$$

as desired.

L	x	x	x	1	x	x	1	x	x	1	x	1
w	0	1	1	3	1	1	3	0	2	3	3	3
σ	2	3	4	6	9	10	11	1	8	12	5	7

Figure 5: A fixed point of I.

Applying Γ to the identity $H(t) = (E(-t))^{-1}$, we get

$$\begin{aligned}
\sum_{n\geq 0}\Gamma(h_n)t^n &= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}x^{\mathrm{ris}((\sigma,w))} \\
&= \frac{1}{1+\sum_{n\geq 1}(-t)^n\Gamma(e_n)} \\
&= \frac{1}{1+\sum_{n\geq 1}(-1)^n t^n \frac{(-1)^{n-1}(x-1)^{n-1}p^{\binom{n}{2}}}{[n]_{p,q}!}\begin{bmatrix} n+k-1 \\ k-1\end{bmatrix}_r} \\
&= \frac{1-x}{1-x+\sum_{n\geq 1}\frac{p^{\binom{n}{2}}(x-1)^n t^n}{[n]_{p,q}!}\begin{bmatrix} n+k-1 \\ k-1\end{bmatrix}_r}
\end{aligned}$$

which proves (16). $\square$

In essentially the same way, we can prove a result which specializes to (2).

Theorem 6. *Let $\Upsilon_{\mathbf{w}} = \{(1\ 2, 0\ 0)\}$. Then for all $k \geq 2$,*

$$\begin{aligned}
D_k^{\Upsilon_{\mathbf{w}}}(x,p,q,r,t) &= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}x^{\mathrm{wris}((\sigma,w))} \\
&= \frac{1-x}{1-x+\sum_{n\geq 1}\frac{p^{\binom{n}{2}}((x-1)t)^n}{[n]_{p,q}!}[k]_{r^n}}. \qquad (22)
\end{aligned}$$

Proof. Define a ring homomorphism $\Gamma_w : \Lambda \to \mathbb{Q}(p,q,r,x)$ by setting $\Gamma_w(e_0) = 1$ and

$$\Gamma_w(e_n) = (-1)^{n-1}(x-1)^{n-1}\frac{[k]_{r^n}}{[n]_{p,q}!}p^{\binom{n}{2}} \qquad (23)$$

for $n \geq 1$. Then we claim that

$$[n]_{p,q}!\Gamma_w(h_n) = \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} x^{\mathrm{wris}((\sigma,w))} \tag{24}$$

for all $n \geq 1$. That is,

$$\begin{aligned}
&[n]_{p,q}!\Gamma_w(h_n) = \\
&[n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,(n)} \Gamma_w(e_\mu) = \\
&[n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)})\in \mathcal{B}_{\mu,n}} \prod_{j=1}^{\ell(\mu)} (-1)^{b_j-1}(x-1)^{b_j-1} \frac{[k]_{r^{b_j}}}{[b_j]_{p,q}!} p^{\binom{b_j}{2}} = \\
&\sum_{\mu \vdash n} \sum_{(b_1,\ldots,b_{\ell(\mu)})\in \mathcal{B}_{\mu,n}} p^{\sum_{j=1}^{\ell(\mu)} \binom{b_j}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q} \prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1} [k]_{r^{b_j}}.
\end{aligned} \tag{25}$$

Next we want to give a combinatorial interpretation to (25). By Lemma 3 for each brick tabloid $T = (b_1,\ldots,b_{\ell(\mu)})$, we can interpret $p^{\sum_{j=1}^{\ell(\mu)} \binom{b_j}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q}$ as the sum of the weights of all fillings of T with a permutation $\sigma \in S_n$ such that σ is increasing in each brick and we weight σ by $q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}$. For each j, we have a factor

$$[k]_{r^{b_j}} = r^{0\cdot b_j} + r^{1\cdot b_j} + \cdots + r^{(k-1)\cdot b_j}.$$

We shall interpret the term r^{sb_j} as indicating that we will fill the top of each cell of a brick b_j with s. Thus we can interpret $\prod_{j=1}^{\ell(\mu)} [k]_{r^{b_j}}$ as filling of the brick with a sequence $w_1 \ldots w_n \in [k]^n$ such that w is constant in each brick and where we weight w by $r^{||w||}$. Finally, we interpret $\prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1}$ as all ways of picking a label of the cells of each brick except the final cell with either an x or a -1. For completeness, we label the final cell of each brick with 1. We shall call all such objects created in this way filled labelled brick tabloids and let $\mathcal{G}_n$ denote the set of all filled labelled brick tabloids that arise in this way. Thus a $C \in \mathcal{G}_n$ consists of a brick tabloid T, a permutation $\sigma \in S_n$, a sequence $w \in \{0,\ldots,k-1\}^n$, and a labelling L of the cells of T with elements from $\{x, 1, -1\}$ such that

1. σ is strictly increasing in each brick,
2. w is constant in each brick,
3. the final cell of each brick is labelled with 1, and
4. each cell which is not a final cell of a brick is labelled with x or -1.

We then define the weight $w(C)$ of C to be $q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||}$ times the product of all the x labels in L and the sign $sgn(C)$ of C to be the product of all the -1 labels in L. For example, if $n = 12$, $k = 4$, and $T = (4, 3, 3, 2)$, then Figure 6 pictures such a composite object $C \in \mathcal{G}_{12}$ where $w(C) = q^{24}p^{42}r^{19}x^5$ and $sgn(C) = -1$.

Thus

$$[n]_{p,q}!\Gamma_w(h_n) = \sum_{C \in \mathcal{G}_n} sgn(C)w(C). \tag{26}$$

L	x	x	−1	1	x	−1	1	x	x	1	−1	1
w	3	3	3	3	1	1	1	0	0	0	2	2
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 6: A composite object $C \in \mathcal{G}_{12}$.

Next we define a weight-preserving sign-reversing involution $I_w : \mathcal{G}_n \to \mathcal{G}_n$. To define $I_w(C)$, we scan the cells of $C = (T, \sigma, w, L)$ from left to right looking for the leftmost cell t such that either (i) t is labelled with -1 or (ii) t is at the end a brick b_j and the brick b_{j+1} immediately following b_j has the property that σ is strictly increasing in all the cells corresponding to b_j and b_{j+1} and w is constant in all the cells corresponding to b_j and b_{j+1}. In case (i), $I_w(C) = (T', \sigma', w', L')$ where T' is the result of replacing the brick b in T containing t by two bricks b^* and b^{**} where b^* contains the cell t plus all the cells in b to the left of t and b^{**} contains all the cells of b to the right of t, $\sigma = \sigma'$, $w = w'$, and L' is the labelling that results from L by changing the label of cell t from -1 to 1. In case (ii), $I_w(C) = (T', \sigma', r', L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma = \sigma'$, $w = w'$, and L' is the labelling that results from L by changing the label of cell t from 1 to -1. If neither case (i) or case (ii) applies, then we let $I_w(C) = C$. For example, if C is the element of $\mathcal{G}_{12}$ pictured in Figure 6, then $I_w(C)$ is pictured in Figure 7.

L	x	x	1	1	x	−1	1	x	x	1	−1	1
w	3	3	3	3	1	1	1	0	0	0	2	2
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 7: $I_w(C)$ for C in Figure 6.

It is easy to see that I_w is a weight-preserving sign-reversing involution and hence I_w shows that

$$[n]_{p,q}!\Gamma_w(h_n) = \sum_{C \in \mathcal{G}_n, I_w(C) = C} sgn(C)w(C). \tag{27}$$

Thus we must examine the fixed points $C = (T, \sigma, w, L)$ of I_w. First there can be no -1 labels in L so that $sgn(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is that last cell of b_j, then it can not be the case that $\sigma_t < \sigma_{t+1}$ and $w_t = w_{t+1}$ since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we associate an element $(\sigma, w) \in C_k \wr S_n$. For example, a fixed point of I_w is pictured in Figure 8 where

$$\begin{aligned} \sigma &= 2\ 3\ 4\ 6\ 9\ 10\ 11\ 1\ 8\ 12\ 5\ 7 \text{ and} \\ w &= 3\ 3\ 3\ 3\ 1\ 1\ 1\ 2\ 2\ 2\ 3\ 3. \end{aligned}$$

It follows that if cell t is at the end of a brick, then $t \notin WRis((\sigma, \epsilon))$. However if v is a cell which is not at the end of a brick, then our definitions force $\sigma_v < \sigma_{v+1}$ and $w_v = w_{v+1}$ so that $v \in WRis((\sigma, \epsilon))$. Since each such cell v must be labelled with an x, it follows that $sgn(C)w(C) = q^{inv(\sigma)}p^{coinv(\sigma)}r^{||w||}x^{\text{wris}((\sigma,\epsilon))}$. Vice versa, if $(\sigma, w) \in C_k \wr S_n$, then we can create a fixed point $C = (T, \sigma, w, L)$ by having the bricks in T end at cells of the form t where $t \notin WRis((\sigma, \epsilon))$, and labelling each cell $t \in WRis((\sigma, \epsilon))$ with x and labelling the remaining cells with 1. Thus we have shown that

$$[n]_{p,q}!\Gamma_w(h_n) = \sum_{(\sigma,w)\in C_k \wr S_n} q^{\text{inv}(\sigma)}p^{\text{coinv}(\sigma)}r^{||w||}x^{\text{wris}((\sigma,w))}$$

as desired.

L	x	x	x	1	x	x	1	x	x	1	x	1
w	3	3	3	3	1	1	1	2	2	2	3	3
σ	2	3	4	6	9	10	11	1	8	12	5	7

Figure 8: A fixed point of I_w.

Applying Γ_w to the identity $H(t) = (E(-t))^{-1}$, we get

$$\begin{aligned} \sum_{n\geq 0} \Gamma_w(h_n)t^n &= \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k \wr S_n} q^{\text{inv}(\sigma)}p^{\text{coinv}(\sigma)}r^{||w||}x^{\text{wris}((\sigma,w))} \\ &= \frac{1}{1 + \sum_{n\geq 1}(-t)^n \Gamma_w(e_n)} \\ &= \frac{1}{1 + \sum_{m\geq 1}(-1)^m t^m \frac{(-1)^{m-1}(x-1)^{m-1}p^{\binom{m}{2}}}{[m]_{p,q}!}[k]_{r^m}} \\ &= \frac{1-x}{1 - x + \sum_{m\geq 1} \frac{p^{\binom{m}{2}}(x-1)^m t^m}{[m]_{p,q}!}[k]_{r^m}} \end{aligned}$$

which proves (22). □

Next we prove a result which specializes to (3).

Theorem 7. *Let $\Upsilon_{\mathbf{s}} = \{(1\ 2, 0\ 1)\}$. For all $k \geq 2$,*

$$\begin{aligned} D_k^{\Upsilon_{\mathbf{s}}}(x,p,q,r,t) &= \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k \wr S_n} q^{\text{inv}(\sigma)}p^{\text{coinv}(\sigma)}r^{||w||}x^{\text{sris}((\sigma,w))} \\ &= \frac{1-x}{1 - x + \sum_{n\geq 1} \frac{p^{\binom{n}{2}}((x-1)t)^n}{[n]_{p,q}!} r^{\binom{n}{2}} \begin{bmatrix} k \\ n \end{bmatrix}_r}. \end{aligned} \tag{28}$$

Proof. Define a ring homomorphism $\Gamma_s : \Lambda \to \mathbb{Q}(p,q,r,x)$ by setting $\Gamma_s(e_0) = 1$ and

$$\Gamma_s(e_n) = (-1)^{n-1}(x-1)^{n-1}\frac{r^{\binom{n}{2}}\begin{bmatrix}k\\n\end{bmatrix}_r}{[n]_{p,q}!}p^{\binom{n}{2}} \tag{29}$$

for $n \geq 1$. Then we claim that

$$[n]_{p,q}!\Gamma_s(h_n) = \sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}x^{\mathrm{sris}((\sigma,w))} \tag{30}$$

for all $n \geq 1$. That is,

$$\begin{aligned}
&[n]_{p,q}!\Gamma_s(h_n) = \\
&[n]_{p,q}!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}B_{\mu,(n)}\Gamma_s(e_\mu) = \\
&[n]_{p,q}!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,n}}\prod_{j=1}^{\ell(\mu)}(-1)^{b_j-1}(x-1)^{b_j-1}\frac{r^{\binom{b_j}{2}}\begin{bmatrix}k\\b_j\end{bmatrix}_r}{[b_j]_{p,q}!}p^{\binom{b_j}{2}} = \\
&\sum_{\mu\vdash n}\sum_{(b_1,\ldots,b_{\ell(\mu)})\in\mathcal{B}_{\mu,n}} p^{\sum_{j=1}^{\ell(\mu)}\binom{b_j}{2}}\begin{bmatrix}n\\b_1,\ldots,b_{\ell(\mu)}\end{bmatrix}_{p,q}\prod_{j=1}^{\ell(\mu)}(x-1)^{b_j-1}r^{\binom{b_j}{2}}\begin{bmatrix}k\\b_j\end{bmatrix}_r.
\end{aligned} \tag{31}$$

Next we want to give a combinatorial interpretation to (31). By Lemma 3 for each brick tabloid $T = (b_1,\ldots,b_{\ell(\mu)})$, we can interpret $p^{\sum_{j=1}^{\ell(\mu)}\binom{b_j}{2}}\begin{bmatrix}n\\b_1,\ldots,b_{\ell(\mu)}\end{bmatrix}_{p,q}$ as the sum of the weights of all fillings of T with a permutation $\sigma \in S_n$ such that σ is increasing in each brick and we weight σ by $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}$. By Lemma 4,

$$\sum_{0\leq j_1\leq\cdots\leq j_n\leq k-n} q^{j_1+j_2+\cdots+j_n} = \begin{bmatrix}k\\n\end{bmatrix}_q.$$

If we replace each j_s in the sum above by $i_s = j_s + s - 1$, then we see that

$$\sum_{0\leq i_1<\cdots<i_n\leq k-1} q^{i_1+i_2+\cdots+i_n} = q^{\binom{n}{2}}\begin{bmatrix}k\\n\end{bmatrix}_q. \tag{32}$$

It follows from (32) that we can interpret the term $\prod_{j=1}^{\ell(\mu)} r^{\binom{b_j}{2}}\begin{bmatrix}k\\b_j\end{bmatrix}_r$ as the sum of the weights of fillings $w = w_1 \ldots w_n$ where the elements of w are between 0 and $k-1$ and are strictly increasing in each brick and where we weight w by $r^{w_1+\cdots+w_n}$. Finally, we interpret $\prod_{j=1}^{\ell(\mu)}(x-1)^{b_j-1}$ as all ways of picking a label x or -1 for each of the cells of each brick except the final cell. For completeness, we label the final cell of each brick with 1. We shall call all such objects created in this way filled labelled brick tabloids and let $\mathcal{H}_n$ denote the set of all filled labelled brick tabloids that arise in this way. Thus a $C \in \mathcal{H}_n$ consists of a brick tabloid T, a permutation $\sigma \in S_n$, a sequence $w \in \{0,\ldots,k-1\}^n$, and a labelling L of the cells of T with elements from $\{x, 1, -1\}$ such that

1. σ is strictly increasing in each brick,
2. w is strictly increasing in each brick,
3. the final cell of each brick is labelled with 1, and
4. each cell which is not a final cell of a brick is labelled with x or -1.

We then define the weight $w(C)$ of C to be $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}$ times the product of all the x labels in L and the sign $sgn(C)$ of C to be the product of all the -1 labels in L. For example, if $n = 12$, $k = 5$, and $T = (4, 3, 3, 2)$, then Figure 9 pictures such a composite object $C \in \mathcal{H}_{12}$ where $w(C) = q^{24}p^{42}r^{20}x^5$ and $sgn(C) = -1$.

Thus

$$[n]_{p,q}!\Gamma_s(h_n) = \sum_{C \in \mathcal{H}_n} sgn(C)w(C). \tag{33}$$

L	x	x	−1	1	x	−1	1	x	x	1	−1	1
w	0	1	2	3	0	1	4	0	2	3	0	4
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 9: A composite object $C \in \mathcal{H}_{12}$.

Next we define a weight-preserving sign-reversing involution $I_s : \mathcal{H}_n \to \mathcal{H}_n$. To define $I_s(C)$, we scan the cells of $C = (T, \sigma, w, L)$ from left to right looking for the leftmost cell t such that either (i) t is labelled with -1 or (ii) t is at the end a brick b_j and the brick b_{j+1} immediately following b_j has the property that σ is strictly increasing in all the cells corresponding to b_j and b_{j+1} and w is strictly in all the cells corresponding to b_j and b_{j+1}. In case (i), $I_s(C) = (T', \sigma', w', L')$ where T' is the result of replacing the brick b in T containing t by two bricks b^* and b^{**} where b^* contains the cell t plus all the cells in b to the left of t and b^{**} contains all the cells of b to the right of t, $\sigma = \sigma'$, $w = w'$, and L' is the labelling that results from L by changing the label of cell t from -1 to 1. In case (ii), $I_s(C) = (T', \sigma', r', L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma = \sigma'$, $w = w'$, and L' is the labelling that results from L by changing the label of cell t from 1 to -1. If neither case (i) or case (ii) applies, then we let $I_s(C) = C$. For example, if C is the element of $\mathcal{H}_{12}$ pictured in Figure 9, then $I_s(C)$ is pictured in Figure 10.

L	x	x	1	1	x	−1	1	x	x	1	−1	1
w	0	1	2	3	0	1	4	0	2	3	0	4
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 10: $I_s(C)$ for C in Figure 9.

It is easy to see that I_s is a weight-preserving sign-reversing involution and hence I_s shows that

$$[n]_{p,q}!\Gamma_s(h_n) = \sum_{C \in \mathcal{H}_n, I_s(C)=C} sgn(C)w(C). \tag{34}$$

Thus we must examine the fixed points $C = (T, \sigma, w, L)$ of I_s. First there can be no -1 labels in L so that $sgn(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is that last cell of b_j, then it can not be the case that $\sigma_t < \sigma_{t+1}$ and $w_t < w_{t+1}$ since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we associate an element $(\sigma, w) \in C_k \wr S_n$. For example, a fixed point of I_s is pictured in Figure 11 where

$$\begin{aligned} \sigma &= 2\ 3\ 4\ 6\ 9\ 10\ 11\ 1\ 8\ 12\ 5\ 7 \text{ and} \\ w &= 0\ 1\ 2\ 3\ 0\ 1\ 4\ 0\ 1\ 3\ 3\ 4. \end{aligned}$$

It follows that if cell t is at the end of a brick, then $t \notin SRis((\sigma, \epsilon))$. However if v is a cell which is not at the end of a brick, then our definitions force $\sigma_v < \sigma_{v+1}$ and $w_v < w_{v+1}$ so that $v \in SRis((\sigma, \epsilon))$. Since each such cell v must be labelled with an x, it follows that $sgn(C)w(C) = q^{inv(\sigma)}p^{coinv(\sigma)}r^{||w||}x^{\mathrm{sris}((\sigma,\epsilon))}$. Vice versa, if $(\sigma, w) \in C_k \wr S_n$, then we can create a fixed point $C = (T, \sigma, w, L)$ by having the bricks in T end at cells of the form t where $t \notin SRis((\sigma, \epsilon))$, and labelling each cell $t \in SRis((\sigma, \epsilon))$ with x and labelling the remaining cells with 1. Thus we have shown that

$$[n]_{p,q}!\Gamma_s(h_n) = \sum_{(\sigma,w) \in C_k \wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}x^{\mathrm{sris}((\sigma,w))}$$

as desired.

L	x	x	x	1	x	x	1	x	x	1	x	1
w	0	1	2	3	0	1	4	0	1	3	3	4
σ	2	3	4	6	9	10	11	1	8	12	5	7

Figure 11: A fixed point of I_s.

Applying Γ_s to the identity $H(t) = (E(-t))^{-1}$, we get

$$\begin{aligned} \sum_{n\geq 0} \Gamma_s(h_n)t^n &= \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||}x^{\mathrm{sris}((\sigma,w))} \\ &= \frac{1}{1+\sum_{n\geq 1}(-t)^n\Gamma_s(e_n)} \\ &= \frac{1}{1+\sum_{m\geq 1}(-1)^m t^m \frac{(-1)^{m-1}(x-1)^{m-1}p^{\binom{m}{2}}}{[m]_{p,q}!} r^{\binom{m}{2}} \begin{bmatrix} k \\ m \end{bmatrix}_r} \\ &= \frac{1-x}{1-x+\sum_{m\geq 1} \frac{p^{\binom{m}{2}}(x-1)^m t^m}{[m]_{p,q}!} r^{\binom{m}{2}} \begin{bmatrix} k \\ m \end{bmatrix}_r} \end{aligned}$$

which proves (28). □

We end this section by proving a generating function which specializes to (4).

Theorem 8. *Let* $\Upsilon_{\mathbf{d}} = \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}$ *For all* $k \geq 2$*,*

$$\begin{aligned} D_k^{\Upsilon_{\mathbf{d}}}(x,p,q,r,t) &= \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} x^{\Upsilon\text{-mch}((\sigma,w))} \\ &= \frac{(k-1)(1-x)}{(k-1)(1-x) + k\sum_{n\geq 1} \frac{p^{\binom{n}{2}}((k-1)(x-1)t)^n}{[n]_{p,q}!}}. \end{aligned} \tag{35}$$

Proof. Define a ring homomorphism $\Gamma_U : \Lambda \rightarrow \mathbb{Q}(p,q,r,x)$ by setting $\Gamma_U(e_0) = 1$ and

$$\Gamma_U(e_n) = (-1)^{n-1}(x-1)^{n-1} \frac{k(k-1)^{n-1}}{[n]_{p,q}!} p^{\binom{n}{2}} \tag{36}$$

for $n \geq 1$. Then we claim that

$$[n]_{p,q}!\Gamma_U(h_n) = \sum_{(\sigma,w)\in C_k \wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} x^{\Upsilon\text{-mch}((\sigma,w))} \tag{37}$$

for all $n \geq 1$. That is,

$$\begin{aligned} &[n]_{p,q}!\Gamma_U(h_n) = \\ &[n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} B_{\mu,(n)} \Gamma_U(e_\mu) = \\ &[n]_{p,q}! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)})\in \mathcal{B}_{\mu,n}} \prod_{j=1}^{\ell(\mu)} (-1)^{b_j-1}(x-1)^{b_j-1} \frac{k(k-1)^{b_j-1}}{[b_j]_{p,q}!} p^{\binom{b_j}{2}} = \\ &\sum_{\mu \vdash n} \sum_{(b_1,\ldots,b_{\ell(\mu)})\in \mathcal{B}_{\mu,n}} p^{\sum_{j=1}^{\ell(\mu)} \binom{b_j}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q} \prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1} k(k-1)^{b_j-1}. \end{aligned} \tag{38}$$

Next we want to give a combinatorial interpretation to (38). By Lemma 3 for each brick tabloid $T = (b_1,\ldots,b_{\ell(\mu)})$, we can interpret $p^{\sum_{j=1}^{\ell(\mu)} \binom{b_j}{2}} \begin{bmatrix} n \\ b_1,\ldots,b_{\ell(\mu)} \end{bmatrix}_{p,q}$ as the sum of the weights of all fillings of T with a permutation $\sigma \in S_n$ such that σ is increasing in each brick and we weight σ by $q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)}$. For any n, there are $k(k-1)^{n-1}$ words $w = w_1 w_2 \ldots w_n \in [k]^n$ such that for $1 \leq i < n$, $w_i \neq w_{i+1}$. That is, we have k choices for the first letter w_1, but then, for any given i, we have only $k-1$ choices for w_{i+1} since w_{i+1} cannot equal w_i. Thus we can interpret $\prod_{j=1}^{\ell(\mu)} k(k-1)^{b_j-1}$ as the number of words $w_1 \ldots w_n$ so that within any brick, there are never two consecutive letters of w which are equal. Finally, we interpret $\prod_{j=1}^{\ell(\mu)} (x-1)^{b_j-1}$ as all ways of picking a label of the cells of each brick except the final cell with either an x or a -1. For completeness, we label the final cell of each brick with 1. We shall call all such objects created in this way filled labelled brick tabloids and let $\mathcal{K}_n$ denote the set of all filled labelled brick tabloids that arise in this way. Thus a $C \in \mathcal{K}_n$ consists of a brick tabloid T, a permutation $\sigma \in S_n$, a sequence $w \in \{0,\ldots,k-1\}^n$, and a labelling L of the cells of T with elements from $\{x, 1, -1\}$ such that

1. σ is strictly increasing in each brick,
2. w is such that there are never two consecutive letters that lie in the same brick which are equal,
3. the final cell of each brick is labelled with 1, and
4. each cell which is not a final cell of a brick is labelled with x or -1.

We then define the weight $w(C)$ of C to be $q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}$ times the product of all the x labels in L and the sign $sgn(C)$ of C to be the product of all the -1 labels in L. For example, if $n=12$, $k=5$, and $T=(4,3,3,2)$, then Figure 12 pictures such a composite object $C \in \mathcal{K}_{12}$ where $w(C)=q^{24}p^{42}x^5$ and $sgn(C)=-1$.

Thus

$$[n]_{p,q}!\Gamma_U(h_n) = \sum_{C \in \mathcal{K}_n} sgn(C)w(C). \tag{39}$$

L	x	x	−1	1	x	−1	1	x	x	1	−1	1
w	0	1	3	1	1	3	1	3	2	0	0	2
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 12: A composite object $C \in \mathcal{K}_{12}$.

Next we define a weight-preserving sign-reversing involution $I_U : \mathcal{K}_n \to \mathcal{K}_n$. To define $I_U(C)$, we scan the cells of $C=(T,\sigma,w,L)$ from left to right looking for the leftmost cell t such that either (i) t is labelled with -1 or (ii) t is at the end a brick b_j and the brick b_{j+1} immediately following b_j has the property that σ is strictly increasing in all the cells corresponding to b_j and b_{j+1} and there are never two consecutive elements of w that are equal in all the cells corresponding to b_j and b_{j+1}. In case (i), $I_U(C)=(T',\sigma',w',L')$ where T' is the result of replacing the brick b in T containing t by two bricks b^* and b^{**} where b^* contains the cell t plus all the cells in b to the left of t and b^{**} contains all the cells of b to the right of t, $\sigma=\sigma'$, $w=w'$, and L' is the labelling that results from L by changing the label of cell t from -1 to 1. In case (ii), $I_U(C)=(T',\sigma',r',L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma=\sigma'$, $w=w'$, and L' is the labelling that results from L by changing the label of cell t from 1 to -1. If neither case (i) or case (ii) applies, then we let $I_U(C)=C$. For example, if C is the element of $\mathcal{K}_{12}$ pictured in Figure 12, then $I_U(C)$ is pictured in Figure 13.

L	x	x	1	1	x	−1	1	x	x	1	−1	1
w	0	1	3	1	1	3	1	3	2	0	0	2
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 13: $I_U(C)$ for C in Figure 12.

It is easy to see that I_U is a weight-preserving sign-reversing involution and hence I_U shows that

$$[n]_{p,q}!\Gamma_U(h_n) = \sum_{C\in\mathcal{K}_n, I_U(C)=C} sgn(C)w(C). \tag{40}$$

Thus we must examine the fixed points $C = (T, \sigma, w, L)$ of I_U. First there can be no -1 labels in L so that $sgn(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is the last cell of b_j, then it can not be the case that $\sigma_t < \sigma_{t+1}$ and $w_t \neq w_{t+1}$ since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we associate an element $(\sigma, w) \in C_k \wr S_n$. For example, a fixed point of I_U is pictured in Figure 14 where

$$\begin{aligned} \sigma &= 2\ 3\ 4\ 6\ 9\ 10\ 11\ 1\ 8\ 12\ 5\ 7 \text{ and} \\ w &= 0\ 1\ 3\ 1\ 1\ 0\ 3\ 3\ 2\ 3\ 3\ 0. \end{aligned}$$

It follows that if cell t is at the end of a brick, then there is no Υ-match in (σ, w) starting at position t. However if v is a cell which is not at the end of a brick, then our definitions force $\sigma_v < \sigma_{v+1}$ and $w_v \neq w_{v+1}$ so that there is Υ-match in (σ, w) starting at position v. Since each such cell v must be labelled with an x, it follows that $sgn(C)w(C) = q^{inv(\sigma)}p^{coinv(\sigma)}x^{\Upsilon\text{-mch}((\sigma,\epsilon))}$. Vice versa, if $(\sigma, w) \in C_k \wr S_n$, then we can create a fixed point $C = (T, \sigma, w, L)$ by having the bricks in T end at cells of the form t where there is no Υ-match in (σ, w) starting at position t, and labelling each cell t where there is an Υ-match in (σ, w) starting at position t with x and labelling the remaining cells with 1. Thus we have shown that

$$[n]_{p,q}!\Gamma_U(h_n) = \sum_{(\sigma,w)\in C_k\wr S_n} q^{\text{inv}(\sigma)}p^{\text{coinv}(\sigma)}x^{\Upsilon\text{-mch}((\sigma,w))}$$

as desired.

L	x	x	x	1	x	x	1	x	x	1	x	1
w	0	1	3	1	1	0	3	3	2	3	3	0
σ	2	3	4	6	9	10	11	1	8	12	5	7

Figure 14: A fixed point of I_U.

Applying Γ_U to the identity $H(t) = (E(-t))^{-1}$, we get

$$\begin{aligned} \sum_{n\geq 0}\Gamma_U(h_n)t^n &= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in C_k\wr S_n} q^{\text{inv}(\sigma)}p^{\text{coinv}(\sigma)}x^{\Upsilon\text{-mch}((\sigma,w))} \\ &= \frac{1}{1+\sum_{n\geq 1}(-t)^n\Gamma_U(e_n)} \\ &= \frac{1}{1+\sum_{m\geq 1}(-1)^m t^m \frac{(-1)^{m-1}(x-1)^{m-1}p^{\binom{m}{2}}}{[m]_{p,q}!}k(k-1)^{m-1}} \\ &= \frac{(k-1)(1-x)}{(k-1)(1-x)+k\sum_{m\geq 1}\frac{p^{\binom{m}{2}}((k-1)(x-1)t)^m}{[m]_{p,q}!}} \end{aligned}$$

which proves (35). ☐

4 Distribution of non-overlapping Υ-bi-matches

In this section we provide arguments similar to those in [9, Sect. 4] to determine the generating function for Υ-nlap((σ, w)), the maximum number of non-overlapping Υ-bi-matches. That is, suppose that $\Upsilon \subseteq C_k \wr S_j$. Recall that

$$N_k^{\Upsilon}(x,p,q,r,t) = \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} x^{\Upsilon\text{-nlap}((\sigma,w))}, \tag{41}$$

and

$$A_k^{\Upsilon}(p,q,r,t) = \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in C_k\wr S_n} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} \chi(\Upsilon\text{-mch}((\sigma,w))=0). \tag{42}$$

Let $(C_k \wr S_n)_{\Upsilon\text{-mch}(end)}$ denote the set of all (σ, w) such that (σ, w) has exactly one Υ-match which occurs at the end of (σ, w), i.e. the unique Υ-match in (σ, w) starts at position $n - j + 1$. We then let

$$B_k^{\Upsilon}(p,q,r,t) = \sum_{n\geq 1} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in (C_k\wr S_n)_{\Upsilon\text{-mch}(end)}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||}. \tag{43}$$

Lemma 9. *For all $k \geq 2$, we have $B_k^{\Upsilon}(p,q,r,t) = ([k]_r t - 1)A_k^{\Upsilon}(p,q,r,t) + 1$.*

Proof. Suppose that $(\sigma, w) \in C_k \wr S_{n-1}$, let σ^j be the result of replacing $j,\ldots,n-1$ in σ by $j+1,\ldots,n$ respectively and then adding j at the end. For example, if $\sigma = 1\ 3\ 4\ 2$, then $\sigma^2 = 1\ 4\ 5\ 3\ 2$. Clearly,

$$\sum_{i=0}^{k-1}\sum_{j=1}^{n} q^{\mathrm{inv}((\sigma^j,wi))} p^{\mathrm{coinv}((\sigma^j,wi))} r^{||wi||} =$$
$$(1+r+\cdots+r^{k-1})(p^{n-1}+qp^{n-2}+\cdots+pq^{n-2}+q^{n-1}) q^{\mathrm{inv}((\sigma,w))} p^{\mathrm{coinv}((\sigma,w))} r^{||w||} =$$
$$[k]_r [n]_{p,q} q^{\mathrm{inv}((\sigma,w))} p^{\mathrm{coinv}((\sigma,w))} r^{||w||}.$$

Now if $(\sigma, w) \in C_k \wr S_{n-1}$ and $\Upsilon\text{-mch}((\sigma, w)) = 0$, then for any $0 \leq i \leq k-1$ and $1 \leq j \leq n-1$, the pair (σ^j, wi) either has no Υ-match or has exactly one Υ-match which occurs at the end. It follows that

$$[k]_r[n]_{p,q}! A_k^{\Upsilon}(p,q,r,t)|_{\frac{t^{n-1}}{[n-1]_{p,q}!}} = A_k^{\Upsilon}(p,q,r,t)|_{\frac{t^n}{[n]_{p,q}!}} + B_k^{\Upsilon}(p,q,r,t)|_{\frac{t^n}{[n]_{p,q}!}}. \tag{44}$$

If we multiply both sides of (44) by $\frac{t^n}{[n]_{p,q}!}$ and sum for $n \geq 1$, we get that

$$[k]_r t A_k^{\Upsilon}(p,q,r,t) = A_k^{\Upsilon}(p,q,r,t) - 1 + B_k^{\Upsilon}(p,q,r,t)$$

or that

$$B_k^{\Upsilon}(p,q,r,t) = 1 + ([k]_r t - 1)A_k^{\Upsilon}(p,q,r,t).$$

☐

Theorem 10. *For all* $\Upsilon \subseteq C_k \wr S_j$ *and* $k \geq 2$,

$$N_k^{\Upsilon}(x,p,q,r,t) = \frac{A_k^{\Upsilon}(p,q,r,t)}{1 - x(1 + ([k]_r t - 1)A_k^{\Upsilon}(p,q,r,t))}. \tag{45}$$

Proof. Suppose that Υ-nlap$((\sigma, w)) = i \geq 0$. One can read any such (σ, w) from left to right making a cut right after a Υ-bi-occurrence counted by Υ-nlap$((\sigma, w))$. As the result, one obtains i signed words which have exactly one Υ-match and that Υ-match occurs at the end of the word that is followed by a possibly empty word that has no Υ-matches. In terms of generating functions, this says that

$$\begin{aligned}
&N_k^{\Upsilon}(x,p,q,r,t) = \\
&A_k^{\Upsilon}(p,q,r,t) + xB_k^{\Upsilon}(p,q,r,t)A_k^{\Upsilon}(p,q,r,t) + (xB_k^{\Upsilon}(p,q,r,t))^2 A_k^{\Upsilon}(p,q,r,t) + \cdots = \\
&\frac{A_k^{\Upsilon}(p,q,r,t)}{1 - xB_k^{\Upsilon}(p,q,r,t)}.
\end{aligned}$$

The result then follows from Lemma 9. □

Using our results in Section 3, we immediately have the following corollaries setting $x = 0$ in our formulas for $D_k^{\Upsilon}(x,p,q,r,t)$.

Corollary 11. *Let* $\Upsilon_{\mathbf{r}} = \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\}$. *Then for all* $k \geq 2$,

$$A_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t) = \frac{1}{1 + \sum_{n\geq 1} \frac{p^{\binom{n}{2}}(-t)^n}{[n]_{p,q}!} \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r}. \tag{46}$$

Corollary 12. *Let* $\Upsilon_{\mathbf{w}} = \{(1\ 2, 0\ 0)\}$. *Then for all* $k \geq 2$,

$$A_k^{\Upsilon_{\mathbf{w}}}(p,q,r,t) = \frac{1}{1 + \sum_{n\geq 1} \frac{p^{\binom{n}{2}}(-t)^n}{[n]_{p,q}!} [k]_{r^n}}. \tag{47}$$

Corollary 13. *Let* $\Upsilon_{\mathbf{s}} = \{(1\ 2, 0\ 1)\}$. *Then for all* $k \geq 2$,

$$A_k^{\Upsilon_{\mathbf{s}}}(p,q,r,t) = \frac{1}{1 + \sum_{n\geq 1} \frac{p^{\binom{n}{2}}(-t)^n}{[n]_{p,q}!} r^{\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_r}. \tag{48}$$

Thus it follows that we can obtain the generating functions for $N_k^{\Upsilon_{\mathbf{a}}}(x,p,q,r,t)$ for $\mathbf{a} \in \{\mathbf{r}, \mathbf{s}, \mathbf{w}\}$ immediately from Theorem 10.

Similarly, it follows from Theorem 8 that we have the following corollary.

Corollary 14. *Let* $\Upsilon_{\mathbf{d}} = \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}$. *Then for all* $k \geq 2$,

$$A_k^{\Upsilon_{\mathbf{d}}}(p,q,1,t) = \frac{k-1}{k - 1 + k\sum_{n\geq 1} \frac{p^{\binom{n}{2}}(-(k-1)t)^n}{[n]_{p,q}!}}. \tag{49}$$

Thus we can obtain $N_k^{\Upsilon_{\mathbf{d}}}(x,p,q,1,t)$ from Theorem 10.

5 More generating functions

For $\mathbf{a} \in \{\mathbf{r}, \mathbf{w}, \mathbf{s}, \mathbf{d}\}$, let $onenlap^{\Upsilon_{\mathbf{a}}}(C_k \wr S_n)$ denote the set of permutations $(\sigma, w) \in C_k \wr S_n$ such that $\Upsilon_{\mathbf{a}}$-nlap$(\sigma, w) = 1$, $onemch^{\Upsilon_{\mathbf{a}}}(C_k \wr S_n)$ denote the set of permutations $(\sigma, w) \in C_k \wr S_n$ such that $\Upsilon_{\mathbf{a}}$-mch$(\sigma, w) = 1$, and $twomch^{\Upsilon_{\mathbf{a}}}(C_k \wr S_n)$ denote the set of permutations $(\sigma, w) \in C_k \wr S_n$ such that $\Upsilon_{\mathbf{a}}$-mch$(\sigma, w) = 2$. It is easy to see that

$$onemch^{\Upsilon_{\mathbf{a}}}(C_k \wr S_n) \subseteq onenlap^{\Upsilon_{\mathbf{a}}}(C_k \wr S_n).$$

Now define

$$\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{a}}} = onenlap^{\Upsilon_{\mathbf{a}}}(C_k \wr S_n) - onemch^{\Upsilon_{\mathbf{a}}}(C_k \wr S_n). \tag{50}$$

Thus $\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{a}}}$ consists of those permutations (σ, w) such that there is an s with $1 \leq s < n-1$ such that (σ, w) has a $\Upsilon_{\mathbf{a}}$-match starting at positions s and $s+1$ and these are the only $\Upsilon_{\mathbf{a}}$-matches in (σ, w). For example, $\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{r}}}$ consists of those permutations $(\sigma, w) \in C_k \wr S_n$ such that $Ris(\sigma, w) = \{s, s+1\}$ for some s. Similarly $\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{w}}}$ consists of those permutations $(\sigma, w) \in C_k \wr S_n$ such that $WRis(\sigma, w) = \{s, s+1\}$ for some s and $\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{s}}}$ consists of those permutations $(\sigma, w) \in C_k \wr S_n$ such that $SRis(\sigma, w) = \{s, s+1\}$ for some s. It is also the case that

$$\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{a}}} \subseteq twomch^{\Upsilon_{\mathbf{a}}}(C_k \wr S_n).$$

Now define

$$\mathcal{V}_{n,k}^{\Upsilon_{\mathbf{a}}} = twomch^{\Upsilon_{\mathbf{a}}}(C_k \wr S_n) - \mathcal{U}_{n,k}^{\Upsilon_{\mathbf{a}}}. \tag{51}$$

Then $\mathcal{V}_{n,k}^{\Upsilon_{\mathbf{a}}}$ consists of those permutations $(\sigma, w) \in C_k \wr S_n$ such that (σ, w) has exactly two $\Upsilon_{\mathbf{a}}$-matches and those $\Upsilon_{\mathbf{a}}$-matches do not overlap. Thus, $\mathcal{V}_{n,k}^{\Upsilon_{\mathbf{r}}}$ consists of those permutations $(\sigma, w) \in C_k \wr S_n$ such that $WRis(\sigma, w) = \{i, j\}$ where $i + 2 \leq j$. Similarly, $\mathcal{V}_{n,k}^{\Upsilon_{\mathbf{w}}}$ consists of those permutations $(\sigma, w) \in C_k \wr S_n$ such that $Ris(\sigma, w) = \{i, j\}$ where $i+2 \leq j$ and $\mathcal{V}_{n,k}^{\Upsilon_{\mathbf{s}}}$ consists of those permutations $(\sigma, w) \in C_k \wr S_n$ such that $SRis(\sigma, w) = \{i, j\}$ where $i + 2 \leq j$.

We define

$$R_k^{\Upsilon_{\mathbf{a}}}(p, q, r, t) = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in \mathcal{U}_{n,k}^{\Upsilon_{\mathbf{a}}}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||}$$

and

$$S_k^{\Upsilon_{\mathbf{a}}}(p, q, r, t) = \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in \mathcal{V}_{n,k}^{\Upsilon_{\mathbf{a}}}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||}.$$

Then from our definitions

$$R_k^{\Upsilon_{\mathbf{a}}}(p, q, r, t) = [N_k^{\Upsilon_{\mathbf{a}}}(x, p, q, r, t) - D_k^{\Upsilon_{\mathbf{a}}}(x, p, q, r, t)]|_x$$

and

$$S_k^{\Upsilon_{\mathbf{a}}}(p, q, r, t) = D_k^{\Upsilon_{\mathbf{a}}}(x, p, q, r, t)|_{x^2} - R_k^{\Upsilon_{\mathbf{a}}}(p, q, r, t).$$

We shall show that we can easily find the generating functions $R_k^{\Upsilon_{\mathbf{a}}}(p,q,r,t)$ and $S_k^{\Upsilon_{\mathbf{a}}}(p,q,r,t)$ for $\mathbf{a} \in \{\mathbf{r},\mathbf{w},\mathbf{s}\}$ and the generating function $R_k^{\Upsilon_{\mathbf{d}}}(p,q,1,t)$ and $S_k^{\Upsilon_{\mathbf{d}}}(p,q,1,t)$. That is, consider the case when $\mathbf{a}=\mathbf{r}$. Then

$$N_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t) = \frac{A_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t)}{1-x(1+([k]_rt-1)A_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t))} \tag{52}$$

so that

$$N_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)|_x = A_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t)(1+([k]_rt-1)A_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t)). \tag{53}$$

In our case,

$$A_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t) = \frac{1}{P_k^{\Upsilon_{\mathbf{r}}}(t)}$$

where

$$P_k^{\Upsilon_{\mathbf{r}}}(t) = 1 + \sum_{n\geq 1} \frac{(-t)^n}{[n]_{p,q}!} \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r. \tag{54}$$

Thus

$$N_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)|_x = \frac{([k]_rt-1)+P_k^{\Upsilon_{\mathbf{r}}}(t)}{(P_k^{\Upsilon_{\mathbf{r}}}(t))^2}. \tag{55}$$

On the other hand, it follows from our results in section 3 that

$$D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t) = \frac{1}{1-\sum_{n\geq 1}(x-1)^{n-1}\frac{t^n}{[n]_{p,q}!}\begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r}.$$

Thus

$$D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)|_x = \sum_{m\geq 1}\left(\sum_{n\geq 1}(x-1)^{n-1}\frac{t^n}{[n]_{p,q}!}\begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r\right)^m |_x.$$

However,

$$\sum_{n\geq 1}(x-1)^{n-1}\frac{t^n}{[n]_{p,q}!}\begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r = \\ F_k^{\Upsilon_{\mathbf{r}}}(t) + xG_k^{\Upsilon_{\mathbf{r}}}(t) + x^2H_k^{\Upsilon_{\mathbf{r}}}(t) + O(x^3),$$

where

$$F_k^{\Upsilon_{\mathbf{r}}}(t) = \sum_{n\geq 1}(-1)^{n-1}\frac{t^n}{[n]_{p,q}!}\begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r, \tag{56}$$

$$G_k^{\Upsilon_{\mathbf{r}}}(t) = \sum_{n\geq 2}(-1)^{n-2}(n-1)\frac{t^n}{[n]_{p,q}!}\begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r, \tag{57}$$

and

$$H_k^{\Upsilon_{\mathbf{r}}}(t) = \sum_{n\geq 3}(-1)^{n-3}\binom{n-1}{2}\frac{t^n}{[n]_{p,q}!}\begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r. \tag{58}$$

Thus since

$$(F^{\Upsilon_{\mathbf{r}}}(t)+xG^{\Upsilon_{\mathbf{r}}}(t)+O(x^2))^m|_x = mG^{\Upsilon_{\mathbf{r}}}(t)(F^{\Upsilon_{\mathbf{r}}}(t))^{m-1},$$

we have

$$\begin{aligned} D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)|_x &= G^{\Upsilon_{\mathbf{r}}}(t)\sum_{m\geq 1} m(F^{\Upsilon_{\mathbf{r}}}(t))^{m-1} \\ &= \frac{G^{\Upsilon_{\mathbf{r}}}(t)}{(1-F^{\Upsilon_{\mathbf{r}}}(t))^2}. \end{aligned}$$

However

$$1-F_k^{\Upsilon_{\mathbf{r}}}(t) = 1+\sum_{n\geq 1}(-1)^n \frac{t^n}{[n]_{p,q}!}\begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r = P_k^{\Upsilon_{\mathbf{r}}}(t).$$

Thus

$$D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)|_x = \frac{G_k^{\Upsilon_{\mathbf{r}}}(t)}{(P_k^{\Upsilon_{\mathbf{r}}}(t))^2}. \tag{59}$$

It follows that

$$R_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t) = \frac{([k]_r t-1)+P_k^{\Upsilon_{\mathbf{r}}}(t)-G_k^{\Upsilon_{\mathbf{r}}}(t)}{(P_k^{\Upsilon_{\mathbf{r}}}(t))^2}. \tag{60}$$

Similarly,

$$\begin{aligned} D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)|_{x^2} &= \sum_{m\geq 1} mH_k^{\Upsilon_{\mathbf{r}}}(t)(F_k^{\Upsilon_{\mathbf{r}}}(t))^{m-1} + \binom{m}{2}(G_k^{\Upsilon_{\mathbf{r}}}(t))^2(F_k^{\Upsilon_{\mathbf{r}}}(t))^{m-2} \\ &= H_k^{\Upsilon_{\mathbf{r}}}(t)\sum_{m\geq 1} m(F_k^{\Upsilon_{\mathbf{r}}}(t))^{m-1} + (G_k^{\Upsilon_{\mathbf{r}}}(t))^2\sum_{m\geq 2}\binom{m}{2}(F_k^{\Upsilon_{\mathbf{r}}}(t))^{m-2} \\ &= \frac{H_k^{\Upsilon_{\mathbf{r}}}(t)}{(1-F_k^{\Upsilon_{\mathbf{r}}}(t))^2} + \frac{(G_k^{\Upsilon_{\mathbf{r}}}(t))^2}{(1-F_k^{\Upsilon_{\mathbf{r}}}(t))^3} \\ &= \frac{H_k^{\Upsilon_{\mathbf{r}}}(t)}{((P_k^{\Upsilon_{\mathbf{r}}}(t))^2} + \frac{(G_k^{\Upsilon_{\mathbf{r}}}(t))^2}{(P_k^{\Upsilon_{\mathbf{r}}}(t))^3} \\ &= \frac{H_k^{\Upsilon_{\mathbf{r}}}(t)P_k^{\Upsilon_{\mathbf{r}}}(t)+(G_k^{\Upsilon_{\mathbf{r}}}(t))^2}{(P_k^{\Upsilon_{\mathbf{r}}}(t))^3}. \end{aligned} \tag{61}$$

Thus

$$\begin{aligned} &S_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t) = \\ &D_k^{\Upsilon_{\mathbf{r}}}(x,p,q,r,t)|_{x^2} - R_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t) = \\ &\frac{H_k^{\Upsilon_{\mathbf{r}}}(t)P_k^{\Upsilon_{\mathbf{r}}}(t)+G_k^{\Upsilon_{\mathbf{r}}}(t))^2-([k]_r t-1)P_k^{\Upsilon_{\mathbf{r}}}(t)-(P_k^{\Upsilon_{\mathbf{r}}}(t))^2+G_k^{\Upsilon_{\mathbf{r}}}(t)P_k^{\Upsilon_{\mathbf{r}}}(t)}{(P_k^{\Upsilon_{\mathbf{r}}}(t))^3} = \\ &\frac{(G_k^{\Upsilon_{\mathbf{r}}}(t))^2+P_k^{\Upsilon_{\mathbf{r}}}(t)(H_k^{\Upsilon_{\mathbf{r}}}(t)+G_k^{\Upsilon_{\mathbf{r}}}(t)-([k]_r t-1)-P_k^{\Upsilon_{\mathbf{r}}}(t))}{(P_k^{\Upsilon_{\mathbf{r}}}(t))^3}. \end{aligned} \tag{62}$$

The exact same sequence of steps work in the other cases so that we have the following theorems.

Theorem 15. *For all $k \geq 2$,*

$$\begin{aligned} R_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t) &= \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in \mathcal{U}_{n,k}^{\Upsilon_{\mathbf{r}}}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} \\ &= \frac{([k]_r t - 1) + P_k^{\Upsilon_{\mathbf{r}}}(t) - G_k^{\Upsilon_{\mathbf{r}}}(t)}{(P_k^{\Upsilon_{\mathbf{r}}}(t))^2} \end{aligned} \tag{63}$$

and

$$\begin{aligned} S_k^{\Upsilon_{\mathbf{r}}}(p,q,r,t) &= \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in \mathcal{V}_{n,k}^{\Upsilon_{\mathbf{r}}}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} \\ &= \frac{(G_k^{\Upsilon_{\mathbf{r}}}(t))^2 + P_k^{\Upsilon_{\mathbf{r}}}(t)(H_k^{\Upsilon_{\mathbf{r}}}(t) + G_k^{\Upsilon_{\mathbf{r}}}(t) - ([k]_r t - 1) - P_k^{\Upsilon_{\mathbf{r}}}(t))}{(P_k^{\Upsilon_{\mathbf{r}}}(t))^3} \end{aligned} \tag{64}$$

where

$$P_k^{\Upsilon_{\mathbf{r}}}(t) = 1 + \sum_{n \geq 1} \frac{(-t)^n}{[n]_{p,q}!} \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r,$$

$$G_k^{\Upsilon_{\mathbf{r}}}(t) = \sum_{n \geq 2} (n-1) \frac{(-t)^n}{[n]_{p,q}!} \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r, \text{ and}$$

$$H_k^{\Upsilon_{\mathbf{r}}}(t) = -\sum_{n \geq 3} \binom{n-1}{2} \frac{(-t)^n}{[n]_{p,q}!} \begin{bmatrix} n+k-1 \\ n \end{bmatrix}_r.$$

Theorem 16. *For all $k \geq 2$,*

$$\begin{aligned} R_k^{\Upsilon_{\mathbf{w}}}(p,q,r,t) &= \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in \mathcal{U}_{n,k}^{\Upsilon_{\mathbf{w}}}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} \\ &= \frac{([k]_r t - 1) + P_k^{\Upsilon_{\mathbf{w}}}(t) - G_k^{\Upsilon_{\mathbf{w}}}(t)}{(P_k^{\Upsilon_{\mathbf{w}}}(t))^2} \end{aligned} \tag{65}$$

and

$$\begin{aligned} S_k^{\Upsilon_{\mathbf{w}}}(p,q,r,t) &= \sum_{n \geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w) \in \mathcal{V}_{n,k}^{\Upsilon_{\mathbf{w}}}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} r^{||w||} \\ &= \frac{(G_k^{\Upsilon_{\mathbf{w}}}(t))^2 + P_k^{\Upsilon_{\mathbf{w}}}(t)(H_k^{\Upsilon_{\mathbf{w}}}(t) + G_k^{\Upsilon_{\mathbf{w}}}(t) - ([k]_r t - 1) - P_k^{\Upsilon_{\mathbf{w}}}(t))}{(P_k^{\Upsilon_{\mathbf{w}}}(t))^3} \end{aligned} \tag{66}$$

where

$$P_k^{\Upsilon_{\mathbf{w}}}(t) = 1 + \sum_{n\geq 1} \frac{(-t)^n}{[n]_{p,q}!}[k]_{r^n},$$
$$G_k^{\Upsilon_{\mathbf{w}}}(t) = \sum_{n\geq 2}(n-1)\frac{(-t)^n}{[n]_{p,q}!}[k]_{r^n}, \text{ and}$$
$$H_k^{\Upsilon_{\mathbf{w}}}(t) = -\sum_{n\geq 3}\binom{n-1}{2}\frac{(-t)^n}{[n]_{p,q}!}[k]_{r^n}.$$

Theorem 17. *For all $k \geq 2$,*

$$\begin{aligned} R_k^{\Upsilon_{\mathbf{s}}}(p,q,r,t) &= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{s}}}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||} \\ &= \frac{([k]_r t - 1) + P_k^{\Upsilon_{\mathbf{s}}}(t) - G_k^{\Upsilon_{\mathbf{s}}}(t)}{(P_k^{\Upsilon_{\mathbf{s}}}(t))^2} \end{aligned} \tag{67}$$

and

$$\begin{aligned} S_k^{\Upsilon_{\mathbf{s}}}(p,q,r,t) &= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in\mathcal{V}_{n,k}^{\Upsilon_{\mathbf{s}}}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)}r^{||w||} \\ &= \frac{(G_k^{\Upsilon_{\mathbf{s}}}(t))^2 + P_k^{\Upsilon_{\mathbf{s}}}(t)(H_k^{\Upsilon_{\mathbf{s}}}(t) + G_k^{\Upsilon_{\mathbf{s}}}(t) - ([k]_r t - 1) - P_k^{\Upsilon_{\mathbf{s}}}(t))}{(P_k^{\Upsilon_{\mathbf{s}}}(t))^3} \end{aligned} \tag{68}$$

where

$$P_k^{\Upsilon_{\mathbf{s}}}(t) = 1 + \sum_{n\geq 1} \frac{(-t)^n}{[n]_{p,q}!}\begin{bmatrix} k \\ n \end{bmatrix}_r,$$
$$G_k^{\Upsilon_{\mathbf{s}}}(t) = \sum_{n\geq 2}(n-1)\frac{(-t)^n}{[n]_{p,q}!}\begin{bmatrix} k \\ n \end{bmatrix}_r, \text{ and}$$
$$H_k^{\Upsilon_{\mathbf{s}}}(t) = -\sum_{n\geq 3}\binom{n-1}{2}\frac{(-t)^n}{[n]_{p,q}!}\begin{bmatrix} k \\ n \end{bmatrix}_r.$$

Theorem 18. *For all $k \geq 2$,*

$$\begin{aligned} R_k^{\Upsilon_{\mathbf{d}}}(p,q,1,t) &= \sum_{n\geq 0}\frac{t^n}{[n]_{p,q}!}\sum_{(\sigma,w)\in\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{d}}}} q^{\mathrm{inv}(\sigma)}p^{\mathrm{coinv}(\sigma)} \\ &= \frac{(kt - 1) + P_k^{\Upsilon_{\mathbf{d}}}(t) - G_k^{\Upsilon_{\mathbf{d}}}(t)}{(P_k^{\Upsilon_{\mathbf{d}}}(t))^2} \end{aligned} \tag{69}$$

and

$$\begin{aligned} S_k^{\Upsilon_{\mathbf{d}}}(p,q,1,t) &= \sum_{n\geq 0} \frac{t^n}{[n]_{p,q}!} \sum_{(\sigma,w)\in \mathcal{V}_{n,k}^{\Upsilon_{\mathbf{d}}}} q^{\mathrm{inv}(\sigma)} p^{\mathrm{coinv}(\sigma)} \\ &= \frac{(G_k^{\Upsilon_{\mathbf{d}}}(t))^2 + P_k^{\Upsilon_{\mathbf{d}}}(t)(H_k^{\Upsilon_{\mathbf{d}}}(t) + G_k^{\Upsilon_{\mathbf{d}}}(t) - (kt-1) - P_k^{\Upsilon_{\mathbf{d}}}(t))}{(P_k^{\Upsilon_{\mathbf{d}}}(t))^3} \end{aligned} \tag{70}$$

where

$$P_k^{\Upsilon_{\mathbf{d}}}(t) = 1 + \sum_{n\geq 1} \frac{(-t)^n}{[n]_{p,q}!} k(k-1)^{n-1},$$
$$G_k^{\Upsilon_{\mathbf{d}}}(t) = \sum_{n\geq 2} (n-1) \frac{(-t)^n}{[n]_{p,q}!} k(k-1)^{n-1}, \text{ and}$$
$$H_k^{\Upsilon_{\mathbf{d}}}(t) = -\sum_{n\geq 3} \binom{n-1}{2} \frac{(-t)^n}{[n]_{p,q}!} k(k-1)^{n-1}.$$

6 Numbers involved; bijective questions

The generating functions from the previous sections allows us to easily compute the initial sequences of values for these generating functions using any computer algebra system such as Mathematica or Maple. For example, let

$$A_k^{\Upsilon}(1,1,1,t) = \sum_{n\geq 0} A_{n,k}^{\Upsilon} \frac{t^n}{n!} \tag{71}$$

so that $A_{n,k}^{\Upsilon}$ is equal to the number of $(\sigma,\epsilon) \in C_k \wr S_n$ such that $\Upsilon\text{-mch}((\sigma,\epsilon)) = 0$.

6.1 $\Upsilon_{\mathbf{r}} = \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\}$

For $\Upsilon_{\mathbf{r}} = \{(1\ 2, 0\ 0), (1\ 2, 0\ 1)\}$, $A_{n,k}^{\Upsilon_{\mathbf{r}}}$ equals the number of $(\sigma,\epsilon) \in C_k \wr S_n$ such that $\mathrm{ris}((\sigma,\epsilon)) = 0$. Table 1 gives initial values of $A_{n,k}^{\Upsilon_{\mathbf{r}}}$.

Table 1: $A_{n,k}^{\Upsilon_{\mathbf{r}}}$ for $k,n \leq 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$k=2$	1	2	5	16	65	326
$k=3$	1	3	12	64	441	3771
$k=4$	1	4	22	164	1589	19136
$k=5$	1	5	35	335	4180	64876

Several of these sequences appear in [23].

In fact, we can easily calculate $A_{n,k}^{\Upsilon_{\mathbf{r}}}$ as a polynomial in k. For example, we have

$$\begin{aligned}
A_{0,k}^{\Upsilon_{\mathbf{r}}} &= 1 \\
A_{1,k}^{\Upsilon_{\mathbf{r}}} &= k \\
A_{2,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{2}k(3k-1) \\
A_{3,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{6}k(19k^2-15k+2) \\
A_{4,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{24}k(211k^3-270k^2+89k-6) \\
A_{5,k}^{\Upsilon_{\mathbf{r}}} &= \frac{1}{120}k(3651k^4-6490k^3+3585k^2-650k+24)
\end{aligned}$$

We point out that $A_{2,k}^{\Upsilon_{\mathbf{r}}}$ forms the familiar sequence of pentagonal numbers (A000326 in [23]). Other previously documented sequences appearing in Table 1 include the structured octagonal anti-prism numbers (A100184 in [23]) for $A_{3,k}^{\Upsilon_{\mathbf{r}}}$; as well as $A_{n,2}^{\Upsilon_{\mathbf{r}}}$ (A000522 in [23]), for which there are many known combinatorial interpretations, including the total number of arrangements of all subsets of $[n]$.

We conjecture that for $n \geq 1$ and $k \geq 2$, $A_{n,k}^{\Upsilon_{\mathbf{r}}}$ is always of the form $\frac{1}{n!}kP_n(k)$ where $P_n(k)$ is a polynomial of degree $n-1$ whose leading coefficient is positive and such that signs of the remaining coefficients alternate. Now we can prove that $A_{n,k}^{\Upsilon_{\mathbf{r}}}$ is always of the form $\frac{1}{n!}kP_n(k)$ where $P_n(k)$ is a polynomial of degree $n-1$ and the term of degree 1 in k is $(-1)^{n-1}(n-1)!$. That is, for any $k \geq 2$, if we set $p = q = r = 1$ and $x = 0$ in (18), we see that

$$n!\bar{\Gamma}(h_n) = A_{n,k}^{\Upsilon_{\mathbf{r}}} \tag{72}$$

where

$$\bar{\Gamma}(e_n) = \frac{\binom{n+k-1}{n}}{n!} = \frac{(k)\uparrow_n}{(n!)^2}. \tag{73}$$

Here we let $(q)\uparrow_0 = 1$ and $(q)\uparrow_n = q(q+1)\dots(q+n-1)$ for $n \geq 1$. But then

$$\begin{aligned}
n!\bar{\Gamma}(h_n) &= n!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}\bar{\Gamma}(e_\mu) \\
&= \frac{1}{n!}\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}\binom{n}{\mu_1,\dots,\mu_{\ell(\mu)}}^2\prod_{i=1}^{\ell(\mu)}(k)\uparrow_{\mu_i}.
\end{aligned} \tag{74}$$

It is easy to see that the right hand side of (74) is a polynomial of degree n and the lowest degree term comes from the term $(-1)^{n-1}k(k+1)\dots(k+n-1)$ corresponding to $\mu = (n)$ which is of the form $(-1)^{n-1}(n-1)!k + O(k^2)$.

6.2 $\Upsilon_{\mathbf{s}} = \{(1\ 2, 0\ 1)\}$

For $\Upsilon_{\mathbf{s}} = \{(1\ 2, 0\ 1)\}$, $A_{n,k}^{\Upsilon_{\mathbf{s}}}$ equals the number of $(\sigma,\epsilon) \in C_k \wr S_n$ such that $\mathrm{sris}((\sigma,\epsilon)) = 0$. Table 2 gives initial values of $A_{n,k}^{\Upsilon_{\mathbf{s}}}$.

Table 2: $A^{\Upsilon_{\mathbf{s}}}_{n,k}$ for $k, n \leq 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$k=2$	1	2	7	36	246	2100
$k=3$	1	3	15	109	1050	12630
$k=4$	1	4	26	244	3031	47000
$k=5$	1	5	40	460	6995	132751

In fact, we can easily calculate $A^{\Upsilon_{\mathbf{s}}}_{n,k}$ as a polynomial in k. For example, we have

$$\begin{aligned}
A^{\Upsilon_{\mathbf{s}}}_{0,k} &= 1 \\
A^{\Upsilon_{\mathbf{s}}}_{1,k} &= k \\
A^{\Upsilon_{\mathbf{s}}}_{2,k} &= \frac{1}{2}k(3k+1) \\
A^{\Upsilon_{\mathbf{s}}}_{3,k} &= \frac{1}{6}k(19k^2+15k+2) \\
A^{\Upsilon_{\mathbf{s}}}_{4,k} &= \frac{1}{24}k(211k^3+270k^2+89k+6) \\
A^{\Upsilon_{\mathbf{s}}}_{5,k} &= \frac{1}{120}k(3651k^4+6490k^3+3585k^2+650k+24)
\end{aligned}$$

We point out that $A^{\Upsilon_{\mathbf{s}}}_{2,k}$ forms the familiar sequence of the second pentagonal numbers (A005449 in [23]). None of the other rows or columns in Table 2 matched any previously known sequences in [23].

We conjecture that for $n \geq 1$ and $k \geq 2$, $A^{\Upsilon_{\mathbf{s}}}_{n,k}$ is always of the form $\frac{1}{n!}kR_n(k)$ where $R_n(k)$ is a polynomial of degree $n-1$ with positive coefficients. In fact, we see that the coefficients of $P_{n,k}$ and $R_{n,k}$ are the same up to a sign for all n. This we can prove. That is, for any $k \geq 2$, if we set $p = q = r = 1$ and $x = 0$ in (30), we see that

$$n!\bar{\Gamma}_s(h_n) = A^{\Upsilon_{\mathbf{s}}}_{n,k} \tag{75}$$

where

$$\bar{\Gamma}_s(e_n) = \frac{\binom{k}{n}}{n!} = \frac{(k)\downarrow_n}{(n!)^2}. \tag{76}$$

Here we let $(q)\downarrow_0 = 1$ and $(q)\downarrow_n = q(q-1)\dots(q-n+1)$ for $n \geq 1$. But then

$$\begin{aligned}
n!\bar{\Gamma}_s(h_n) &= n!\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}\bar{\Gamma}_s(e_\mu) \\
&= \frac{1}{n!}\sum_{\mu\vdash n}(-1)^{n-\ell(\mu)}\binom{n}{\mu_1,\dots\mu_{\ell(\mu)}}^2\prod_{i=1}^{\ell(\mu)}(k)\downarrow_{\mu_i}.
\end{aligned} \tag{77}$$

Since for any $n \geq 1$, $(k)\downarrow_n = (-1)^n (k)\uparrow_n$, it is easy to see that the right hand side of (77) is obtained from the right hand side of (74) by replacing k by $-k$ and multiplying by $(-1)^n$. Thus the conjecture that $R_n(k)$ has positive coefficients is equivalent to our conjecture that the signs of the coefficients of $P_n(k)$ alternate.

7 $\Upsilon_{\mathbf{w}} = \{(1\ 2, 0\ 0)\}$

For $\Upsilon_{\mathbf{w}} = \{(1\,2, 0\,0)\}$, $A_{n,k}^{\Upsilon_{\mathbf{w}}}$ equals the number of $(\sigma,\epsilon) \in C_k \wr S_n$ such that $\mathrm{wris}((\sigma,\epsilon)) = 0$. Table 3 gives initial values of $A_{n,k}^{\Upsilon_{\mathbf{w}}}$.

Table 3: $A_{n,k}^{\Upsilon_{\mathbf{w}}}$ for $k, n \leq 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
$k=2$	1	2	6	26	150	1082
$k=3$	1	3	15	111	1095	13503
$k=4$	1	4	28	292	4060	70564
$k=5$	1	5	45	605	10845	243005

In fact, we can easily calculate $A_{n,k}^{\Upsilon_{\mathbf{w}}}$ as a polynomial in k. For example, we have

$$\begin{aligned}
A_{0,k}^{\Upsilon_{\mathbf{w}}} &= 1 \\
A_{1,k}^{\Upsilon_{\mathbf{w}}} &= k \\
A_{2,k}^{\Upsilon_{\mathbf{w}}} &= k(2k-1) \\
A_{3,k}^{\Upsilon_{\mathbf{w}}} &= k(6k^2-6k+1) \\
A_{4,k}^{\Upsilon_{\mathbf{w}}} &= k(24k^3-36k^2+14k-1) \\
A_{5,k}^{\Upsilon_{\mathbf{w}}} &= k(120k^4-240k^3+150k^2-30k+1)
\end{aligned}$$

We point out that $A_{2,k}^{\Upsilon_{\mathbf{w}}}$ forms the familiar sequence of hexagonal numbers (A000384 in [23]). Additionally, $A_{n,2}^{\Upsilon_{\mathbf{w}}}$ matches the sequence counting the number of necklaces on set of labeled beads (A000629 in [23]). In fact, in this case we can give a completely combinatorial interpretation of $A_{n,k}^{\Upsilon_{\mathbf{w}}}$. Let $OSetpn(n)$ denote the set of ordered set partitions of $\{1,\ldots,n\}$. For any set partition $\pi \in OSetpn(n)$, let $\ell(\pi)$ denote the number of parts of π. Then we claim that

$$A_{n,k}^{\Upsilon_{\mathbf{w}}} = \sum_{\pi \in OSetpn(n)} (-1)^{n-\ell(\pi)} k^{\ell(\pi)} \tag{78}$$

so that the coefficient of k^j in $A_{n,k}^{\Upsilon_{\mathbf{w}}}$ is equal to $(-1)^{n-j} j! S_{n,j}$ where $S_{n,j}$ is the Stirling number of the second kind which is the number of set partitions of $\{1,\ldots,n\}$ into j parts. That is, for any $k \geq 2$, if we set $p=q=r=1$ and $x=0$ in (24), we see that

$$n!\bar{\Gamma}_w(h_n) = A_{n,k}^{\Upsilon_{\mathbf{w}}} \tag{79}$$

where

$$\bar{\Gamma}_w(e_n) = \frac{k}{n!}. \tag{80}$$

But then

$$\begin{aligned} n!\bar{\Gamma}_s(h_n) &= n! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \bar{\Gamma}_s(e_\mu) \\ &= n! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \prod_{i=1}^{\ell(\mu)} \frac{k}{b_i!} \\ &= \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \binom{n}{b_1,\ldots,b_{\ell(\mu)}} k^{\ell(\mu)}. \end{aligned} \tag{81}$$

Since $\binom{n}{b_1,\ldots,b_{\ell(\mu)}}$ counts the number of ordered set partitions $\pi = (\pi_1, \ldots, \pi_{\ell(\mu)})$ such that $|\pi_j| = b_j$, it is easy to see that the right hand side of (81) equals the right hand side of (78).

7.1 $\Upsilon_{\mathbf{d}} = \{(1\ 2, 0\ 1), (1\ 2, 1\ 0)\}$

Table 4 gives initial values of $A_{n,k}^{\Upsilon_{\mathbf{d}}}$.

Table 4: $A_{n,k}^{\Upsilon_{\mathbf{d}}}$ for $k, n \leq 5$.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
k=2	1	2	6	26	150	1082
k=3	1	3	12	66	480	4368
k=4	1	4	20	132	1140	12324
k=5	1	5	30	230	2280	28280

In fact, we can easily calculate $A_{n,k}^{\Upsilon_{\mathbf{d}}}$ as a polynomial in k. For example, we have

$$\begin{aligned} A_{0,k}^{\Upsilon_{\mathbf{d}}} &= 1 \\ A_{1,k}^{\Upsilon_{\mathbf{d}}} &= k \\ A_{2,k}^{\Upsilon_{\mathbf{d}}} &= k^2 + k \\ A_{3,k}^{\Upsilon_{\mathbf{d}}} &= k^3 + 4k^2 + k \\ A_{4,k}^{\Upsilon_{\mathbf{d}}} &= k^4 + 11k^3 + 11k^2 + k \\ A_{5,k}^{\Upsilon_{\mathbf{d}}} &= k^5 + 26k^4 + 66k^3 + 26k^2 + k \end{aligned}$$

In this case, we shall show that $A_{n,k}^{\Upsilon_{\mathbf{d}}}$ is just the Eulerian polynomial

$$A_{n,k}^{\Upsilon_{\mathbf{d}}} = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)+1}. \tag{82}$$

That is, for any $k \geq 2$, if we set $p = q = r = 1$ and $x = 0$ in (37), we see that

$$n!\bar{\Gamma}_U(h_n) = A_{n,k}^{\Upsilon_{\mathbf{d}}} \tag{83}$$

where

$$\bar{\Gamma}_U(e_n) = \frac{k(k-1)^{n-1}}{n!}. \tag{84}$$

But then

$$\begin{aligned} n!\bar{\Gamma}_U(h_n) &= n! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \bar{\Gamma}_U(e_\mu) \\ &= n! \sum_{\mu \vdash n} (-1)^{n-\ell(\mu)} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \prod_{i=1}^{\ell(\mu)} \frac{k(k-1)^{b_i-1}}{b_i!} \\ &= \sum_{\mu \vdash n} \sum_{(b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}} \binom{n}{b_1,\ldots,b_{\ell(\mu)}} \prod_{i=1}^{\ell(\mu)} k(1-k)^{b_i-1}. \end{aligned} \tag{85}$$

Next we want to give a combinatorial interpretation to (85). For any brick tabloid $T = (b_1,\ldots,b_{\ell(\mu)}) \in \mathcal{B}_{\mu,n}$, we can interpret $\binom{n}{b_1,\ldots,b_{\ell(\mu)}}$ as the set of all fillings of T with a permutation $\sigma \in S_n$ such that σ is increasing in each brick. We then interpret $\prod_{j=1}^{\ell(\mu)} k(1-k)^{b_j-1}$ as all ways of picking a label of the cells of each brick except the final cell with either an 1 or a $-k$ and letting the label of the last cell of each brick be k. We let $\mathcal{D}_n$ denote the set of all filled labelled brick tabloids that arise in this way. Thus a $C \in \mathcal{D}_n$ consists of a brick tabloid T, a permutation $\sigma \in S_n$ and a labelling L of the cells of T with elements from $\{k, -k, 1\}$ such that

1. σ is strictly increasing in each brick,
2. the final cell of each brick is labelled with k, and
3. each cell which is not a final cell of a brick is labelled with 1 or $-k$.

We then define the weight $w(C)$ of C to be the product of all the k labels in L and the sign $sgn(C)$ of C to be the product of all the -1 labels in L. For example, if $n = 12$, $k = 4$, and $T = (4,3,3,2)$, then Figure 15 pictures such a composite object $C \in \mathcal{D}_{12}$ where $w(C) = k^7$ and $sgn(C) = -1$.

Thus

$$n!\bar{\Gamma}_U(h_n) = \sum_{C \in \mathcal{D}_n} sgn(C) w(C). \tag{86}$$

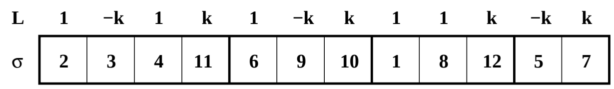

Figure 15: A composite object $C \in \mathcal{D}_{12}$.

Next we define a weight-preserving sign-reversing involution $J : \mathcal{D}_n \to \mathcal{D}_n$. To define $J(C)$, we scan the cells of $C = (T, \sigma, L)$ from left to right looking for the leftmost cell t such that either (i) t is labelled with $-k$ or (ii) t is at the end of a brick b_j and the brick b_{j+1} immediately following b_j has the property that σ is strictly increasing in all the cells corresponding to b_j and b_{j+1}. In case (i), $J(C) = (T', \sigma', L')$ where T' is the result of replacing the brick b in T containing t by two bricks b^* and b^{**} where b^* contains the cell t plus all the cells in b to the left of t and b^{**} contains all the cells of b to the right of t, $\sigma = \sigma'$, and L' is the labelling that results from L by changing the label of cell t from $-k$ to k. In case (ii), $J(C) = (T', \sigma', L')$ where T' is the result of replacing the bricks b_j and b_{j+1} in T by a single brick b, $\sigma = \sigma'$, and L' is the labelling that results from L by changing the label of cell t from k to $-k$. If neither case (i) or case (ii) applies, then we let $J(C) = C$. For example, if C is the element of $\mathcal{D}_{12}$ pictured in Figure 15, then $J(C)$ is pictured in Figure 16.

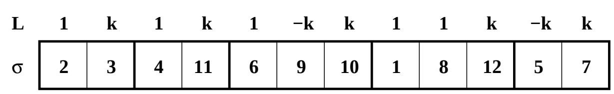

Figure 16: $J(C)$ for C in Figure 15.

It is easy to see that J is a weight-preserving sign-reversing involution and hence J shows that

$$n!\bar{\Gamma}_U(h_n) = \sum_{C \in \mathcal{D}_n, J(C)=C} sgn(C)w(C). \tag{87}$$

Thus we must examine the fixed points $C = (T, \sigma, L)$ of J. First there can be no $-k$ labels in L so that $sgn(C) = 1$. Moreover, if b_j and b_{j+1} are two consecutive bricks in T and t is the last cell of b_j, then it can not be the case that $\sigma_t < \sigma_{t+1}$ since otherwise we could combine b_j and b_{j+1}. For any such fixed point, we associate an element $(\sigma, w) \in C_k \wr S_n$. For example, a fixed point of I is pictured in Figure 17 where

$$\sigma = 2\ 3\ 4\ 11\ 6\ 9\ 10\ 1\ 8\ 12\ 5\ 7.$$

It follows that if cell t is at the end of a brick which is not the last brick, then $\sigma_t > \sigma_{t+1}$. However if v is a cell which is not at the end of a brick, then our definitions force $\sigma_v < \sigma_{v+1}$. Since each such cell v must be labelled with an 1, it follows that $sgn(C)w(C) = k^{des(\sigma)+1}$ where the $+1$ comes from the fact that the last cell of the last brick is also labeled with k. Vice versa, if $\sigma \in S_n$, then we can create a fixed point $C = (T, \sigma, L)$ by having the bricks in T end at cells of the form t where $\sigma_t > \sigma_{t+1}$ and labeling each such cell with k, labeling the last cell with k, and labelling the remaining cells with 1. Thus we have shown that

$$n!\bar{\Gamma}_U(h_n) = \sum_{\sigma \in S_n} k^{\mathrm{des}(\sigma)+1}$$

as desired.

L	1	1	1	k	1	1	k	1	1	k	1	k
σ	2	3	4	11	6	9	10	1	8	12	5	7

Figure 17: A fixed point of J.

7.2 $U_{n,k,\mathbf{a}} = |\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{a}}}|$ and $V_{n,k,\mathbf{a}} = |\mathcal{V}_{n,k}^{\Upsilon_{\mathbf{a}}}|$

We have computed similar tables for $U_{n,k,\mathbf{a}} = |\mathcal{U}_{n,k}^{\Upsilon_{\mathbf{a}}}|$ and $V_{n,k,\mathbf{a}} = |\mathcal{V}_{n,k}^{\Upsilon_{\mathbf{a}}}|$ using our formulas for the generating functions $R_k^{\Upsilon_{\mathbf{a}}}(p,q,r,t)$ and $S_k^{\Upsilon_{\mathbf{a}}}(p,q,r,t)$. Table 5 gives initial values of $U_{n,k,\mathbf{r}}$, which counts the number of $(\sigma,\epsilon) \in C_k \wr S_n$ such that $\mathrm{Ris}((\sigma,\epsilon)) = \{s, s+1\}$ for some $1 \leq s \leq n-2$.

Table 5: $U_{n,k,\mathbf{r}}$ for $n \leq 7$, $k \leq 5$.

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$k=2$	4	54	538	5064	48900
$k=3$	10	210	3363	52056	838542
$k=4$	20	570	12568	270328	6083712
$k=5$	35	1260	35328	973840	28127160

In fact, we can easily calculate $U_{n,k,\mathbf{r}}$ as a polynomial in k. For example, we have

$$\begin{aligned}
U_{3,k,\mathbf{r}} &= \frac{1}{6}k(k+1)(k+2) \\
U_{4,k,\mathbf{r}} &= \frac{1}{4}k(k+1)(k+2)(5k-1) \\
U_{5,k,\mathbf{r}} &= \frac{1}{120}k(k+1)(k+2)(903k^2-479k+36) \\
U_{6,k,\mathbf{r}} &= \frac{1}{45}k(k+1)(k+2)(2032k^3-1896k^2+419k-15) \\
U_{7,k,\mathbf{r}} &= \frac{1}{1680}k(k+1)(k+2)(482031k^4-662450k^3+268653k^2-32554k+600)
\end{aligned}$$

Thus we conjecture that for $n \geq 3$ and $k \geq 2$, $U_{n,k,\mathbf{r}}$ is always of the form $\frac{1}{n!}k(k+1)(k+2)U_{n,1}(k)$ where $U_{n,\mathbf{r}}(k)$ is a polynomial of degree $n-3$ whose leading coefficient is positive and such that signs of the remaining coefficients alternate.

Furthermore, we point out that $U_{3,k}^{\Upsilon_{\mathbf{r}}}$ forms the familiar sequence of tetrahedral numbers (A000292 in [23]). None of the other rows or columns in Table 5 matched any previously known sequence in [23].

Table 6 gives initial values of $U_{n,k,\mathbf{s}}$, which counts the number of $(\sigma,\epsilon) \in C_k \wr S_n$ such that $\mathrm{SRis}((\sigma,\epsilon)) = \{s, s+1\}$ for some $1 \leq s \leq n-2$.

Table 6: $U_{n,k,\mathbf{s}}$ for $n \le 7$, $k \le 5$.

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$k=2$	0	0	0	0	0
$k=3$	1	24	480	9760	212310
$k=4$	4	126	3280	86440	2431800
$k=5$	10	390	12503	404688	13962690

In fact, we can easily calculate $U_{n,k,\mathbf{s}}$ as a polynomial in k. For example, we have

$$\begin{aligned}
U_{3,k,\mathbf{s}} &= \frac{1}{6}k(k-1)(k-2) \\
U_{4,k,\mathbf{s}} &= \frac{1}{4}k(k-1)(k-2)(5k+1) \\
U_{5,k,\mathbf{s}} &= \frac{1}{120}k(k-1)(k-2)(903k^2+479k+36) \\
U_{6,k,\mathbf{s}} &= \frac{1}{45}k(k-1)(k-2)(2032k^3+1896k^2+419k+15) \\
U_{7,k,\mathbf{s}} &= \frac{1}{1680}k(k-1)(k-2)(482031k^4+662450k^3+268653k^2+32554k+600)
\end{aligned}$$

Thus we conjecture that for $n \ge 3$ and $k \ge 2$, $U_{n,k,\mathbf{s}}$ is always of the form $\frac{1}{n!}k(k+1)(k+2)U_{n,3}(k)$ where $U_{n,\mathbf{s}}(k)$ is a polynomial of degree $n-3$ with positive coefficients. Moreover, we conjecture that the coefficients of $U_{n,\mathbf{s}}(k)$ and $U_{n,\mathbf{r}}(k)$ are the same up to a sign for $n \ge 3$. None of the rows or columns in Table 6 matched any previously known non-trivial sequence in [23].

Table 7 gives initial values of $U_{n,k,\mathbf{w}}$, which counts the number of $(\sigma,\epsilon) \in C_k \wr S_n$ such that $\mathrm{WRis}((\sigma,\epsilon)) = \{s, s+1\}$ for some $1 \le s \le n-2$.

Table 7: $U_{n,k,\mathbf{w}}$ for $n \le 7$, $k \le 5$.

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$k=2$	2	28	326	3896	50186
$k=3$	3	66	1269	25512	556683
$k=4$	4	120	3212	90480	2773140
$k=5$	5	190	1303	235880	9303725

In fact, we can easily calculate $U_{n,k,\mathbf{w}}$ as a polynomial in k. For example, we have

$$\begin{aligned}
U_{3,k,\mathbf{w}} &= k \\
U_{4,k,\mathbf{w}} &= k(8k-2) \\
U_{5,k,\mathbf{w}} &= k(60k^2-40k+3) \\
U_{6,k,\mathbf{w}} &= 4k(120k^3-135k^2+34k-1) \\
U_{7,k,\mathbf{w}} &= k(4200k^4-6720k^3+3108k^2-392k+5)
\end{aligned}$$

Thus we conjecture that for $n \geq 3$ and $k \geq 2$, $U_{n,k,\mathbf{w}}$ is always of the form $kU_{n,\mathbf{w}}(k)$ where $U_{n,2}(k)$ is a polynomial of degree $n-3$ whose leading coefficient is positive and is such that remaining coefficients alternate in sign.

Furthermore, we point out that $U_{4,k}^{\Upsilon_{\mathbf{w}}}$ forms the sequence of alternating hexagonal numbers (A014635 in [23]). None of the other rows or columns in Table 7 matched any previously known sequence in [23].

Table 8 gives initial values of $U_{n,k,\mathbf{d}}$, which counts the number of $(\sigma,\epsilon) \in C_k \wr S_n$ such that for some $1 \leq s \leq n-2$, i is a start of $\Upsilon_{\mathbf{d}}$-match if and only if $i \in \{s, s+1\}$.

Table 8: $U_{n,k,\mathbf{d}}$ for $n \leq 7$, $k \leq 5$.

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$k=2$	2	28	326	3896	50186
$k=3$	12	240	3744	58080	958560
$k=4$	36	936	18252	345168	6860916
$k=5$	80	2560	58840	1329920	30723200

In fact, we can easily calculate $U_{n,k,\mathbf{d}}$ as a polynomial in k. For example, we have

$$\begin{aligned}
U_{3,k,\mathbf{d}} &= k(k-1)^2 \\
U_{4,k,\mathbf{d}} &= 2k(k-1)^2(3k+1) \\
U_{5,k,\mathbf{d}} &= k(k-1)^2(23k^2+34k+3) \\
U_{6,k,\mathbf{d}} &= 6k(k-1)^2(18k^3+70k^2+31k+1) \\
U_{7,k,\mathbf{d}} &= k(k-1)^2(201k^4+1660k^3+1962k^2+372k+5)
\end{aligned}$$

Thus we conjecture that for $n \geq 3$ and $k \geq 2$, $U_{n,k,\mathbf{d}}$ is always of the form $k(k-1)^2U_{n,\mathbf{d}}(k)$ where $U_{n,\mathbf{d}}(k)$ is a polynomial of degree $n-3$ with positive coefficients. None of the rows or columns in Table 8 matched any non-trivial sequence in [23].

We shall only give polynomial expressions for $V_{k,n,\mathbf{a}} = |\mathcal{V}_{n,k}^{\Upsilon_{\mathbf{a}}}|$ for $\mathbf{a} \in \{\mathbf{r}, \mathbf{w}, \mathbf{s}, \mathbf{d}\}$ and $n = 4,5,6,7$. Note that by definition $V_{n,k,\mathbf{a}} = 0$ for $n = 1,2,3$.

For $V_{k,n,\mathbf{r}}$, we have the following initial polynomials.

$$
\begin{aligned}
V_{4,k,\mathbf{r}} &= \frac{1}{24}k(k+1)(35k^2+31k-6)\\
V_{5,k,\mathbf{r}} &= \frac{1}{120}k(k+1)(2253k^3+1277k^2-1022k+72)\\
V_{6,k,\mathbf{r}} &= \frac{1}{72}k(k+1)(12781k^4+2336k^3-8911k^2+2146k-72)\\
V_{7,k,\mathbf{r}} &= \frac{1}{2520}k(k+1)(3828237k^5-943444k^4-3213331k^3\\
&\qquad +1679386k^2-207048k+3600)
\end{aligned}
$$

Thus we conjecture that for $n \geq 4$ and $k \geq 2$, $V_{n,k,\mathbf{r}}$ is always of the form $k(k+1)V_{n,\mathbf{r}}(k)$ where $V_{n,\mathbf{r}}(k)$ is a polynomial of degree $n-2$. Note that this is first example where we did not obtain polynomials whose coefficients are either positive or whose coefficients alternate in sign.

However, we still seem to have a type of reciprocity between $V_{n,k,\mathbf{r}}$ and $V_{n,k,\mathbf{s}}$. That is, we have the following initial polynomials.

$$
\begin{aligned}
V_{4,k,\mathbf{s}} &= \frac{1}{24}k(k-1)(35k^2-31k-6)\\
V_{5,k,\mathbf{s}} &= \frac{1}{120}k(k-1)(2253k^3-1277k^2-1022k-72)\\
V_{6,k,\mathbf{s}} &= \frac{1}{72}k(k-1)(12781k^4-2336k^3-8911k^2-2146k-72)\\
V_{7,k,\mathbf{s}} &= \frac{1}{2520}k(k-1)(3828237k^5+943444k^4-3213331k^3\\
&\qquad -1679386k^2-207048k-3600)
\end{aligned}
$$

Thus we conjecture that for $n \geq 4$ and $k \geq 2$, $V_{n,k,\mathbf{s}}$ is always of the form $k(k-1)V_{n,\mathbf{s}}(k)$ where $V_{n,\mathbf{s}}(k)$ is a polynomial of degree $n-2$. Moreover we conjecture that the the absolute value of the coefficients in $V_{n,\mathbf{s}}(k)$ and $V_{n,\mathbf{r}}(k)$ are the same.

For $V_{k,n,\mathbf{w}}$, we have the following initial polynomials.

$$
\begin{aligned}
V_{4,k,\mathbf{w}} &= k(6k-1)\\
V_{5,k,\mathbf{w}} &= k(90k^2-50k+3)\\
V_{6,k,\mathbf{w}} &= 2k(5050k^3-5040k^2+118k-3)\\
V_{7,k,\mathbf{w}} &= 2k(6300k^4-9240k^3+3864k^2-434k+5)
\end{aligned}
$$

Thus we conjecture that for $n \geq 4$ and $k \geq 2$, $V_{n,k,\mathbf{w}}$ is always of the form $kV_{n,\mathbf{w}}(k)$ where $V_{n,\mathbf{w}}(k)$ is a polynomial of degree $n-3$ whose leading coefficients is positive and where the signs of the remaining coefficients alternate.

For $V_{k,n,\mathbf{d}}$, we have the following initial polynomials.

$$\begin{aligned} V_{4,k,\mathbf{d}} &= k(k-1)^2(5k+1) \\ V_{5,k,\mathbf{d}} &= k(k-1)^2(43k^2+44k+3) \\ V_{6,k,\mathbf{d}} &= k(k-1)(230k^3+626k^2+218k+6) \\ V_{7,k,\mathbf{d}} &= k(k-1)^2(990k^4+5588k^3+5184k^2+838k+10) \end{aligned}$$

Thus we conjecture that for $n \geq 4$ and $k \geq 2$, $V_{n,k,\mathbf{d}}$ is always of the form $k(k-1)^2 V_{n,\mathbf{d}}(k)$ where $V_{n,\mathbf{d}}(k)$ is a polynomial of degree $n-3$ whose coefficients are positive.

Note that $U_{n,k,\mathbf{w}}$ and $V_{n,k,\mathbf{w}}$ both make sense even in the case where $k=1$. That is, $U_{n,1,\mathbf{w}}$ equals the number of $\sigma \in S_n$ such that $\text{Ris}(\sigma) = \{s, s+1\}$ for some $1 \leq s \leq n-2$ and $V_{n,1,\mathbf{w}}$ equals the number of $\sigma \in S_n$ such that $\text{Ris}(\sigma) = \{i,j\}$ where $i+2 \leq j$. Table 9 gives these values for small n.

Table 9: $U_{n,1,\mathbf{w}}$ and $V_{n,1,\mathbf{w}}$ for $n \leq 7$.

	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$U_{n,1,\mathbf{w}}$	1	6	23	72	201
$V_{n,1,\mathbf{w}}$	0	5	43	230	990

One would have expected that generating functions for $U_{n,1,\mathbf{w}}$ and $V_{n,1,\mathbf{w}}$ would have appeared before, but the sequence for $U_{n,1,\mathbf{w}}$ appears in OEIS [23] but not with our interpretation and the sequence for $V_{n,k,\mathbf{w}}$ does not even appear in OEIS [23] before our work.

8 Further research

An obvious direction of research is considering matching conditions on $C_k \wr S_n$ of length 3 or more and deriving avoidance/distribution formulas similar to those derived in this paper. Another obvious direction of research is to look at distributions of bi-occurrences of patterns in $C_k \wr S_n$. One can also consider k-tuples of words from a fixed finite alphabet with the obvious extension of our matching and occurrence conditions. All of these topics will be studied in subsequent papers.

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