HIGHER ORDER MATCHING POLYNOMIALS AND d-ORTHOGONALITY

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ABSTRACT. We show combinatorially that the higher-order matching polynomials of several families of graphs are d-orthogonal polynomials. The matching polynomial of a graph is a generating function for coverings of a graph by disjoint edges; the higher-order matching polynomial corresponds to coverings by paths. Several families of classical orthogonal polynomials—the Chebyshev, Hermite, and Laguerre polynomials—can be interpreted as matching polynomials of paths, cycles, complete graphs, and complete bipartite graphs. The notion of d-orthogonality is a generalization of the usual idea of orthogonality for polynomials and we use sign-reversing involutions to show that the higher-order Chebyshev (first and second kinds), Hermite, and Laguerre polynomials are d-orthogonal. We also investigate the moments and find generating functions of those polynomials.

1. INTRODUCTION AND BACKGROUND

A *matching* of a graph is a subset of mutually disjoint edges in the graph. Given a matching, we can assign a weight to each matching by giving a weight of -1 to each edge in the matching and weight x to each vertex not adjacent to an edge in the matching, then multiplying those weights together. We define the *matching polynomial* of a graph as the sum of weights of all matchings of the graph.

Matching polynomials have long been an object of interest in graph theory, and it is well-known that the matching polynomials for some classes of graphs—namely paths, cycles, complete graphs, and complete bipartite graphs—are in fact classical orthogonal polynomials: respectively, the Chebyshev polynomials of the second and first kinds, Hermite polynomials, and Laguerre polynomials. See [6, 13, 14, 29] and also [28, §4], which all treat the links between classical orthogonal polynomials and matching polynomials.

The number of matchings of a graph was used by Hosoya to develop his "topological index" Z, which relates chemical properties of hydrocarbons with their molecular structure. Later, Randić, Morales, and Araujo [22] generalized the Z index to the so-called higher-order Hosoya numbers by considering coverings of graphs not by disjoint edges (which can be thought of as paths of length one), but by paths of length two, three, and so on. Araujo, Estrada, Morales, and Rada, starting from the higher-order Hosoya numbers and working with Farrell's F-coverings [7, 8, 9], described the higher-order matching polynomial of a graph, derived recurrence relations, exact formulas, and also found expressions for those polynomials as hypergeometric series [2].

Apart from combinatorics and graph theory, Van Iseghem [30] and Maroni [21] introduced a generalization of orthogonality for polynomials. A set of polynomials $\{P_n\}_{n\geq 0}$ that is orthogonal in the usual sense has an associated positive measure μ such that

$$\int P_n P_m \, \mathrm{d}\mu = 0 \quad \text{if } n > m, \text{ and} \quad \int P_n^2 \, \mathrm{d}\mu \neq 0.$$

(We also demand that the degree of P_n be n.) We call the integral of P_n^2 the L^2 norm of P_n. For our purposes, instead of providing a measure, it is equivalent to give a sequence of moments $\{\mu_n\}_{n \geq 0}$ and define a linear functional $\mathcal{L}$ on the space of polynomials by declaring $\mathcal{L}(x^n) = \mu_n$. A set of orthogonal polynomials must satisfy a recurrence relation of the form

(1) $$P_{n+1} = (x - b_n)P_n - \lambda_n P_{n-1}.$$

Van Iseghem and Maroni defined the concept of d-orthogonality, or orthogonality of dimension d. Here d is a positive integer, and we say that a monic set of polynomials $\{P_n\}_{n \geq 0}$ is d-orthogonal if there is a measure μ (or, for us, a sequence of moments) such that

(2) $$\int P_n P_m \, d\mu = 0 \quad \text{if } n > dm, \text{ and} \quad \int P_{dn} P_n \, d\mu = 1.$$

Observe that usual orthogonal polynomials correspond to $d = 1$. We will commit a minor abuse of language and call the integral of $P_{dn} P_n$ the L^2 norm of P_n. Sets of d-orthogonal polynomials satisfy a recurrence relation of order $d + 2$ analogous to the one above.

In this paper, we establish the t-orthogonality for higher-order matching polynomials corresponding to coverings of paths, cycles, complete graphs, and complete bipartite graphs by paths with t edges. We will find formulas and combinatorial descriptions for the moments, the recurrence relation, a sign-reversing involution that proves the orthogonality and L^2 norms, and generating functions for the moments and polynomials.

1.1. **Notation and terminology.** A path with t edges and $t + 1$ vertices will be called a "t-path". Vertices of a graph not adjacent to an edge in the matching will often be called "fixed points"; the term comes from thinking of a matching of a graph as giving an involution on the vertices. The set of integers from 1 to n, inclusive, will be written as "$[n]$". Finally, we use $A \sqcup B$ for the disjoint union of two sets (usually graphs).

The vertices of our graphs are all labeled 1 to n, and we often draw matchings by arranging the vertices horizontally and drawing arcs for edges or paths in the matching; we say that a matching is noncrossing when such a diagram has no crossings. We will draw set partitions in a similar manner, as Kasraoui and Zeng do in [18].

2. Warmup: Chebyshev polynomials of the second kind

We begin with the Chebyshev polynomials of the second kind, which is the simplest example. The basic combinatorics of Chebyshev polynomials of the second kind were described by de Sainte-Catherine and Viennot in [6, 31, 32] and we review the theory to familiarize the reader with our basic strategy and aims.

The Chebyshev polynomial of the second kind $U_n(x)$ is defined here as the matching polynomial of a path with n vertices. With this normalization, they are also called Fibonacci polynomials, since matchings of a path with n vertices corresponds in a natural way to "pavings" of length n composed of dominos and monominos, and such pavings are counted by Fibonacci numbers. The last vertex of such a matching must be a fixed point or in an edge in the matching, so the recurrence relation corresponding to (1) is clear:

(3) $$U_{n+1}(x) = xU_n(x) - U_{n-1}(x),$$

so these polynomials have $b_n = 0$ and $\lambda_n = 1$ for all n. Viennot established that the nth moment of a set of orthogonal polynomials with recurrence coefficients b_n and λ_n as in (1) equals the total weight of all weighted Motzkin paths of length n; a Motzkin path is a lattice path that never goes below the x-axis and takes upsteps, horizontal steps, and downsteps (that is, steps of the form $(1, 1)$, $(1, 0)$, and $(1, -1)$), with upsteps of weight 1, horizontal steps at height n of weight b_n, and downsteps leaving from height n of weight λ_n. Knowing that, we see that the nth

moment of the Chebyshev polynomials of the second kind is the number of Dyck paths—lattice paths with only up- and down-steps—of length n.

The number of Dyck paths of length $2m$ is the Catalan number $\binom{2m}{m}/(m+1)$, which here we interpret as the number of noncrossing complete matchings of K_{2m}, which are the same as noncrossing set partitions of $[2m]$ in which all blocks have size two.

The orthogonality of these Chebyshev polynomials can be proved with a sign-reversing involution: $U_n(x)U_m(x)$ is the generating function for pairs of matchings of an n-vertex path and and m-vertex path. Integrating that product can be interpreted as the generating function for complete noncrossing matchings on $[n] \sqcup [m]$ with:

- black edges of weight -1 that connect adjacent vertices and are homogeneous—that is, they stay within $[n]$ or $[m]$, and
- dashed edges of weight 1 between any two vertices.

The weight of such a configuration is the product of the weights of the edges. See Figure 1 for an example. Given such a configuration, we can produce another configuration by finding the leftmost edge that connects adjacent vertices and changing it to black if it is dashed, or vice versa. This process is a sign-reversing involution that cancels all configurations with a homogeneous edge, so $\mathcal{L}(U_n(x)U_m(x))$ equals the number of complete noncrossing inhomogeneous matchings of $[n] \sqcup [m]$. If $n \neq m$, there are obviously zero such matchings, and if $n = m$, there's exactly one: a "rainbow" configuration in which vertex $n - k$ on the left is connected to vertex k on the right ($1 \leq k \leq n$).
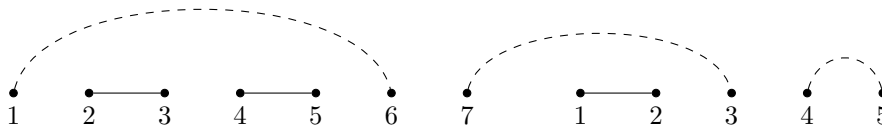

FIGURE 1. A configuration that contributes weight -1 to the "integral" $\mathcal{L}(U_7(x)U_5(x))$. The orthogonality involution would change the edge connecting vertices 2 and 3 on the left from solid to dashed. There must always be at least one homogeneous adjacent edge, so all configurations are canceled and the integral is zero.

Let's finish this section by mentioning the generating functions of the polynomials and the moments. For $U_n(x)$, any such polynomial is a sequence of fixed points, which have size 1 and weight x, and edges, which have size 2 and weight -1. When your objects are composed of sequences of smaller objects, the generating function is typically just a geometric series:

$$(4) \qquad \sum_{n \geq 0} U_n(x)z^n = \frac{1}{1 - (xz - z^2)}.$$

The moments are Catalan numbers, whose generating function is well known; for example, see Aigner [1, §3.1 and §7.3]:

$$(5) \qquad f(z) = \sum_{n \geq 0} \mu_n z^n = \frac{1 - \sqrt{1 - 4z^2}}{2z^2},$$

but for our purposes, we will focus more on the functional equation satisfied by $f(z)$ and the corresponding continued fraction. The functional equation is

$$(6) \qquad f(z) = 1 + z^2 f(z)^2$$

and is easy to explain using Dyck paths, which are counted by the Catalan numbers. Think of $f(z)$ as standing for "any possible Dyck path"; such a path is either empty (with weight 1), or is of the form "upstep-(some Dyck path)-downstep-(some Dyck path)", which has weight $z^2 f(z)^2$; see Figure 2.
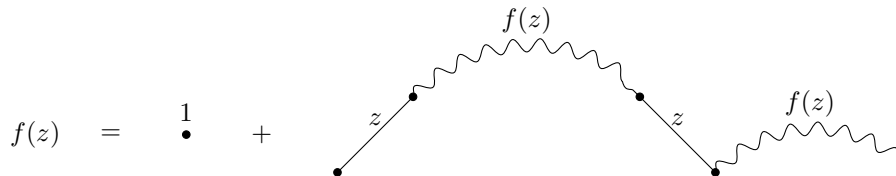

FIGURE 2. A pictorial explanation of the functional equation of (6).

By rearranging (6), one is easily led to a continued fraction expression for $f(z)$:

$$(7) \qquad f(z) = \cfrac{1}{1 - \cfrac{z^2}{1 - \cfrac{z^2}{1 - \cdots}}}$$

Now we generalize the above work to higher-order matching polynomials.

3. HIGHER-ORDER CHEBYSHEV POLYNOMIALS OF THE SECOND KIND

Following Araujo et al. in [2], let's now cover the path with n vertices by paths with t edges, and give t-paths weight -1 and fixed points weight x. We will denote the generating function for such coverings by $U_n^{(t)}(x)$ and call them *Chebyshev polynomials of the second kind and order t*.

These polynomials satisfy a recurrence relation similar to (3):

$$(8) \qquad U_{n+1}^{(t)}(x) = x U_n^{(t)}(x) - U_{n-t}^{(t)}(x);$$

the proof is effectively the same: vertex $n + 1$ is either fixed or the final vertex in a t-path, and the rest of the vertices can be covered by a smaller configuration.

Let $\mu_n^{(t)}$ be the number of noncrossing set partitions of $[n]$ in which all blocks have size $t + 1$, and let $\mathcal{L}^{(t)}$ be the linear functional on the space of polynomials defined by $\mathcal{L}^{(t)}(x^n) = \mu_n^{(t)}$. Then we have our first theorem, which generalizes a result of de Sainte-Catherine and Viennot [6, Theorem 7].

Theorem 3.1. *Let $n_1, n_2, \ldots, n_k$ be nonnegative integers. The integral*

$$\mathcal{L}^{(t)} \left(\prod_{i=1}^{k} U_{n_i}^{(t)}(x) \right)$$

equals the number of inhomogeneous noncrossing coverings of $[n_1] \sqcup \cdots \sqcup [n_k]$ by t-paths, or equivalently, the number of noncrossing set partitions of $[n_1] \sqcup \cdots \sqcup [n_k]$ in which all blocks have size $t + 1$ and no block is a subset of any $[n_i]$.

Proof. The proof uses a sign-reversing involution analogous to the above involution we used for the usual Chebyshev polynomials of the second kind. The product of the polynomials is the generating function for k-tuples of coverings of n_i-vertex paths by fixed points and t-paths. Integrating the product yields the generating function for complete noncrossing coverings of $[n_1] \sqcup \cdots \sqcup [n_k]$ by t-paths with

- homogeneous black t-paths of weight -1 that connect a sequence of adjacent vertices in the underlying path, and
- dashed t-paths of weight 1 that can go anywhere.

Call a path (black or dashed) that connects $t+1$ adjacent vertices in the underlying path "flat". The sign-reversing involution is simple: find the leftmost homogeneous flat t-path and change its "color" from black to dashed, or vice versa. Any configuration that has at least one homogeneous t-path must have a flat t-path, so this involution will cancel any configuration with a homogeneous edge. Uncanceled configurations have only edges of weight 1, so $\mathcal{L}^{(t)}\big(U_n^{(t)}(x)U_m^{(t)}(x)\big)$ equals the number of configurations with only inhomogeneous edges. □

The above theorem immediately implies that the polynomials $U_n^{(t)}$ are, in fact, t-orthogonal with respect to those moments.

Corollary 3.2. *The polynomials $U_n^{(t)}$ are t-orthogonal with respect to the above moments: if $m > nt$, then*

$$(9) \qquad \mathcal{L}\big(U_m^{(t)}(x)U_n^{(t)}(x)\big) = 0 \quad and \quad \mathcal{L}\big(U_{nt}^{(t)}(x)U_n^{(t)}(x)\big) = 1.$$

Proof. If $m > tn$, then any configuration of black and dashed edges must have at least one homogeneous adjacent t-path in $[m]$, so the integral is zero. The integral of $U_n^{(t)}U_{nt}^{(t)}$ is 1 because there is exactly one inhomogeneous configuration, an example of which is pictured in Figure 3. □
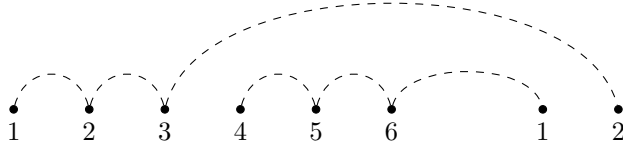

FIGURE 3. The sole uncanceled configuration in $\mathcal{L}\big(U_6^{(3)}(x)U_2^{(3)}(x)\big)$.

The usual Chebyshev polynomials of the second kind are the generating functions for matchings of a path; the number of such matchings is a Fibonacci number, and can be obtained with the appropriate substitution: $F_n = U_n(i)/i^n$. The higher-order polynomials lead to the higher-order Fibonacci numbers of Randić et al. [22], and can be obtained by

$$(10) \qquad {}^t F_n = \left(\frac{1}{w}\right)^n U_n^{(t)}(w),$$

where w is any $(t+1)$th root of -1 and ${}^t F_n$ is the notation of Randić et al. for the higher-order Fibonacci numbers. The explanation for this is simple: we want each t-edge path to have weight $+1$, but $U_n^{(t)}(x)$ gives them weight -1. By multiplying by $1/w^n$, we effectively give each vertex weight $1/w$, and hence the total weight of each t-path is $+1$. Then we plug in w to give each fixed point weight $+1$ as well.

Before we start investigating the moments of these polynomials, note that the generating function for the polynomials is a straightforward generalization of (4).

Proposition 3.3. *The ordinary generating function of the Chebyshev polynomials of the second kind and order t is*

$$(11) \qquad UP(t,x,z) = \sum_{n \geq 0} U_n^{(t)}(x)z^n = \frac{1}{1 - (xz - z^{t+1})}.$$

The "UP" is intended to be mnemonic: the U is for $U_n^{(t)}(x)$, and the P is for "polynomials". We'll meet UM, the generating function for the moments, shortly.

Proof. The rational function in (11) equals

$$\sum_{k \geq 0} \left(xz - z^{t+1} \right)^k,$$

which can be interpreted as the generating function for finite sequences of objects with either weight xz or weight $-z^{t+1}$ grouped by number of objects; the sum in (11) is the generating function for the same thing, just grouped by coefficient of z. $\qquad\square$

3.1. **Moments of $U_n^{(t)}(x)$.** The moments for the usual Chebyshev polynomials of the second kind (order $t = 1$) are "aerated" Catalan numbers: $\mu_{2n+1} = 0$ and $\mu_{2n} = \binom{2n}{n}/(n+1)$ for all nonnegative n. In what follows, we will work with the following interpretations of Catalan numbers: as Dyck paths with $2n$ steps, as noncrossing matchings of $[2n]$, as binary trees, and as triangulations of an $(n+2)$-gon.

To understand the moments for Chebyshev polynomials of the second kind and order t, we need to understand the *Fuss-Catalan numbers*, also called *k-Catalan* or *generalized Catalan numbers*. The Fuss-Catalan numbers $C_n^{(k)}$ are defined by

(12) $$C_n^{(k)} = \frac{1}{kn+1} \binom{(k+1)n}{n},$$

and have been extensively studied; see Hilton and Pedersen [15] for an introduction to these numbers, which were likely first described by Fuss [12] (see the table in that paper on page 249), nearly 50 years before Catalan [5]. The above interpretations of the usual Catalan numbers generalize to the following interpretations for the Fuss-Catalan numbers:

- Dissections of an $(nk + 2)$-gon into $(k + 2)$-gons.
- Rooted plane trees in which all non-leaf vertices have $k + 1$ children; i.e., $(k+1)$-ary trees.
- Lattice paths of length n composed of $(1, 1)$ and $(1, -k)$ steps that start and end on the x-axis, and never go below the x-axis.
- Noncrossing set partitions with all blocks of size $k + 1$.

The first item was studied by Fuss; the second and third are connected by the Łukasiewicz language; see [20, chap. 11]. The last item is not as ubiquitous as the others and we will use it to prove the following.

Theorem 3.4. *The moments $\mu_n^{(t)}$ for $U_n^{(t)}(x)$ are "aerated" Fuss-Catalan numbers: $\mu_{(t+1)n}^{(t)} = C_n^{(t)}$ for all $n \geq 0$, and are zero otherwise.*

Proof. The moments for a d-orthogonal set of polynomials are unique (this follows from an inductive argument, or by the "spanning argument" of [24]), so from Theorem 3.2, we know that noncrossing set partitions with all blocks of size $t+1$ are the correct moments for $U_n^{(t)}$. Therefore we need nothing more than a bijection to one of the above families of objects counted by the aerated Fuss-Catalan numbers. One easy bijection is to $(t+1)$-ary trees. Given a $(t+1)$-ary tree on $(t + 1)n + 1$ vertices, number the vertices of the tree according to a depth-first, left-to-right search. The labels on each set of $t + 1$ siblings in the tree describe the corresponding block in the set partition—see Figure 4. Injectivity is obvious, and surjectivity follows from the nesting structure of such a set partition. Consider the usual diagram of the set partition, and find all blocks of the set partition that contain $t + 1$ consecutive numbers; those blocks will be sets of siblings in the tree that have no subtree. The root of the subtree corresponding to a block P gets connected to vertex i corresponding to block Q when the smallest number in P is one larger

than the ith smallest number in Q. (For example, in Figure 4, the block $\{4,5,6\}$ is immediately nested by block $\{2,3,7\}$, and we connect the root of the $\{4,5,6\}$ subtree to 3 because 4 is one larger than 3.) This produces a $(t+1)$-ary tree from a noncrossing set partition with all blocks of size $t+1$, so surjectivity—hence bijectivity—holds. $\square$
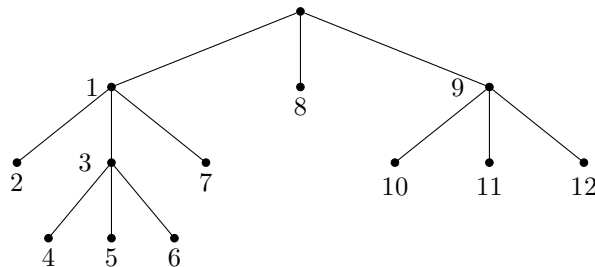

FIGURE 4. An example illustrating the bijection from ternary trees to noncrossing set partitions in which all blocks have size three; the above tree corresponds to the set partition $\{\{1,8,9\},\{2,3,7\},\{4,5,6\},\{10,11,12\}\}$.

There is a closely related bijection, analogous to Flajolet's path diagrammes [10] and Viennot's Laguerre histories [31, 32].

A Łukasiewicz path is a generalization of a Motzkin path in which, in addition to upsteps and horizontal steps, downsteps of the form $(1,-k)$ are allowed. The paths begin and end on the x-axis and never go below the x-axis. For a general set of d-orthogonal polynomials, the nth moment is the generating function for weighted Łukasiewicz paths of length n that have downsteps $(1,-1),(1,-2),\ldots,(1,-d)$. The weight of each horizontal step and downstep is given by the recurrence coefficients for the polynomials. See [23, §4.2] for more details.

The recurrence relation (8) tells us that the moments for Chebyshev polynomials of the second kind and order t are also given by Łukasiewicz paths with upsteps $(1,1)$ and downsteps $(1,-t)$, with all steps of weight 1. The bijection between those paths and the set partitions counted by $\mu_n^{(t)}$ is the obvious generalization of the classical Motzkin path bijection to set partitions: given such a Łukasiewicz path with n steps, one produces a set partition of $[n]$ with the following procedure: begin with an empty set partition and read through the path. if the kth step is an upstep, add k to a set of "candidates". If step k is a downstep, add a block to the set partition consisting of k and the t largest elements of the candidate set. This produces a set partition in which every block has size $t+1$. Choosing the t largest candidates guarantees that the set partition is noncrossing, and the procedure is a bijection because every downstep corresponds to a unique set of t upsteps. This is in some sense a generalization of the map between set partitions and Charlier diagrams found in [18, §3.1] and [10, 31]; in the language of that bijection, we place an "opener" vertex whenever one sees an upstep, and placing a "closer" vertex and connecting the rightmost t open vertices.

For example, if U stands for an upstep and D for a $(1,-2)$ step, the tree and set partition in Figure 4 correspond to the lattice path $UUUUUDDUDUUD$.

The tree and lattice path representations for the moments of $U_n^{(t)}$ make the generalizations of (6) and (7) obvious.

Theorem 3.5. *Let $UM(t,z)$ be the ordinary generating function of the moments of the Chebyshev polynomials of the second kind and order t:*

$$UM(t,z) = \sum_{n\geq 0} \mu_n^{(t)} z^n.$$

Then $UM(z)$ satisfies

$$(13) \qquad UM(t,z) = 1 + z^{t+1}UM(t,z)^{t+1}$$

and has the continued fraction expansion

$$(14) \qquad UM(t,z) = \cfrac{1}{1 - \cfrac{z^{t+1}}{\left(1 - \cfrac{z^{t+1}}{\left(1 - \cfrac{z^{t+1}}{(1 - \cdots)^t}\right)^t}\right)^t}}.$$

Viennot called the above continued fraction an L-fraction when he derived a generalization of the above theorem in [31, chapter V, §6]. Note that the continued fraction expansion only requires the recursion coefficients of the polynomials, which Araujo et al. found [2, eq. (2.7)], and are clear from the combinatorial description.

For these polynomials, and every class of higher-ordering matching polynomials we consider here, we note that Araujo et al. [2] give exact formulas and expressions for these polynomials as generalized hypergeometric functions. They give an exact formula for $U_n^{(t)}(x)$ in equation (2.11) and express that polynomial as a $_{t+1}F_t$ in (3.6).

4. CHEBYSHEV POLYNOMIALS OF THE FIRST KIND

Now let's move on to the Chebyshev polynomials of the first kind and order t, which are the higher-order matching polynomials for a cycle with, as usual, weight x for fixed points and weight -1 for a path with t edges. We denote them $T_n^{(t)}(x)$, and use $\mathrm{Cyc}(n)$ for the underlying labeled n-cycle. Here a 1-cycle is a single vertex[1] and a 2-cycle, of course, is two vertices with two edges between them. The interpretation of Chebyshev polynomials as the matching polynomial of a cycle has been studied by several authors; see Benjamin and Walton [3], Bergeron [4], and Hosoya [16, 17], whose $Q(Y)$ polynomials for cycloparaffins in the first reference are essentially Chebyshev polynomials of the first kind. The Chebyshev polynomials of the first kind are often defined by $P_n(\cos\theta) = \cos(n\theta)$, and those polynomials are related to ours by $T_n^{(1)}(x) = 2P_n(x/2)$.

The combinatorial model for these polynomials is well known, but it is much harder to find models for the moments and an involution proof like that of Theorem 3.1 to prove orthogonality. The Chebyshevs of the first kind satisfy the same recurrence relation as those of the second kind, but they have different initial conditions. Before we describe the moments and orthogonality involution, let's show directly that $T_n^{(t)}(x)$ satisfies the same recurrence as $U_n^{(t)}(x)$.

4.1. A weight-preserving bijection for the recurrence relation. In this section, we find a weight-preserving bijection that shows

$$(15) \qquad T_{n+1}^{(t)}(x) = xT_n^{(t)}(x) - T_{n-t}^{(t)}(x),$$

with $T_i^{(t)}(x) = x^i$ for $0 \le i \le t$ and $T_{t+1}^{(t)}(x) = x^{t+1} - (t+1)$. Araujo et al. [2, §2.2] and Farrell [9, §6] both find this recurrence, but here we present a direct bijection. The initial conditions are clear; either there are not enough edges to have a t-path (so every vertex is fixed), or there are $t+1$ edges, and we can choose any one of them to be the single edge not in a t-path.

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[1]One can also have a 1-cycle be a vertex with a loop, with the convention that such a graph has no one-edge matching since the edge would be incident with itself—but that's effectively the same as just saying it's a single vertex with no edge at all.
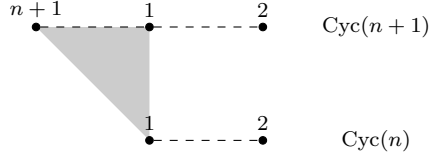
FIGURE 5. Case (a) for the $T_{n+1}^{(t)}(x)$ recurrence. When the edge $(1, 2)$ in $\mathrm{Cyc}(n)$ is not covered by a t-path, we simply insert a new edge behind 1, label the new vertex $n + 1$, and attach the previous $(n, 1)$ edge from n to $n + 1$. The weight of the new configuration is x times the weight of the old.
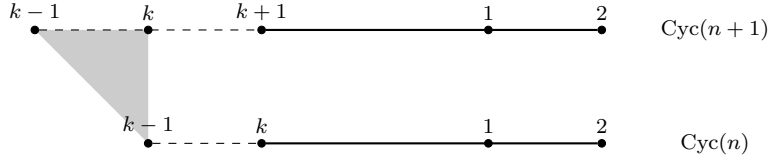

FIGURE 6. Case (b) for the $T_{n+1}^{(t)}(x)$ recurrence. If the edge $(1, 2)$ in $\mathrm{Cyc}(n)$ is covered by a t-path, say the path extends back to vertex k. The "long edges" from k to 1, and from $k + 1$ to 1, represent $n + 1 - k$ edges in a t-path. Expand vertex $k - 1$ into vertices $k - 1$ and k, with an edge not in a t-path between them, and relabel the vertices between the old vertex k and vertex 1. Note that k could be 1, in which case $k - 1$ is n and no relabeling is necessary. The weight of the new configuration is x times the weight of the old.

Assume $n > t$. There are four cases for the bijection between coverings of $T_{n+1}^{(t)}(x)$ and the union of coverings of $T_n^{(t)}(x)$ with extra weight x and coverings of $T_{n-t}^{(t)}(x)$ with extra weight -1.

Given a configuration for $T_n^{(t)}(x)$, the edge from 1 to 2 is either in a t-path or not. Case (a) is when the edge is not in a t-path and is illustrated in Figure 5. In all four figures, we have a section of $\mathrm{Cyc}(n + 1)$ on top, and a section of $\mathrm{Cyc}(n)$ or $\mathrm{Cyc}(n - t)$ on the bottom; the rest of the vertices are omitted for clarity. Black edges are in a t-path, and dashed edges are not. The gray triangles indicate expanding (or contracting) edges in the map from one configuration to another. In case (a), we simply insert a new edge "behind" vertex 1 that is not in a t-path.

In case (b), the edge from 1 to 2 in $\mathrm{Cyc}(n)$ is in a t-path, which extends back to vertex k. We expand vertex $k - 1$ into an edge not in a t-path and relabel the relevant vertices in the t-path. Figure 6 shows the process. Observe that this always yields a configuration in which $(1, 2)$ is in a t-path, and that path is preceded by at least two edges not in a path.

Both operations multiply the weight by x and yield a configuration for $T_{n+1}^{(t)}(x)$.

Now consider $T_{n-t}^{(t)}(x)$. Case (c) is when the $(1, 2)$ edge is not in a t-path. Insert $t + 1$ new edges immediately behind 1: an edge not in a t-path, followed by a t-path. See Figure 7.

Case (d) is shown in Figure 8: if the $(1, 2)$ edge of $\mathrm{Cyc}(n - t)$ is in a t-path, there are k edges in the path preceding vertex 1 for some k with $0 \le k < t$. Immediately behind vertex 1, add $t + 1$ new edges: k edges in a t-path, an edge not in a t-path, and $t - k$ edges in a t-path. This splits a single t-path into two t-paths.

Both operations yield a configuration with one more t-path than we started with, so we've multiplied the weight by -1 and have a configuration for $T_{n+1}^{(t)}(x)$.
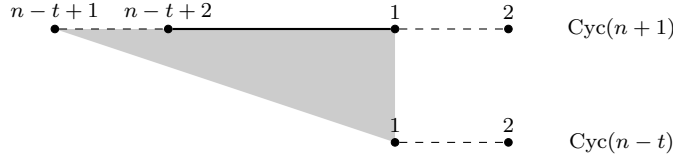
FIGURE 7. Case (c) of the $T_{n+1}^{(t)}(x)$ recurrence. If the $(1,2)$ edge of $\mathrm{Cyc}(n-t)$ is not in a t-path, insert a t-path preceded by an edge immediately behind vertex 1. The old $(n-t,1)$ edge is now an $(n-t,n-t+1)$ edge. This adds a t-path and multiplies the weight by -1.
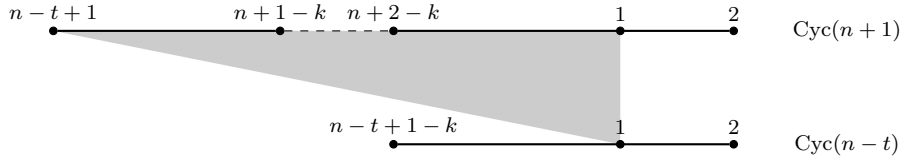

FIGURE 8. Case (d) of the $T_{n+1}^{(t)}(x)$ recurrence. The bottom is $\mathrm{Cyc}(n-t)$, and edge $(1,2)$ is covered by a t-path with k edges preceding 1, with $0 \le k \le t-1$. Expand vertex 1 into a sequence of $t+1$ edges of the form "$t-k$ edges in a t-path, one edge not in a t-path, k edges in a t-path".

This is a bijection: in $T_{n+1}^{(t)}(x)$, consider if $(1,2)$ is in a t-path. If it isn't, is $(n+1,1)$ in a t-path? If no, then the configuration came from $T_n^{(t)}(x)$ in case (a); if yes, it came from case (c) and $T_{n-t}^{(t)}(x)$. If $(1,2)$ is indeed in a t-path: is the path preceded by one, or more than one, edges not in a t-path? If exactly one, the configuration came from case (d) and $T_{n-t}^{(t)}(x)$, and if more than one, it came from $T_n^{(t)}(x)$ in case (b). Every possible configuration in $T_{n+1}^{(t)}(x)$ is accounted for, so the map is surjective. Since we always simply expand a vertex into new edges, and because the above argument shows that the cases are distinguishable, the map is injective.

4.2. **Orthogonality involution.** Now we address the moments and orthogonality for higher order Chebyshev polynomials of the first kind. Instead of the Fuss-Catalan numbers, the moments of $T_n^{(t)}(x)$ are what we will call $(t+1)$-*reciprocal binomial coefficients*:

$$(16) \qquad B_n^{(t)} = \binom{(t+1)n}{n}.$$

Note the similarity to (12). We say "$(t+1)$-reciprocal" because these coefficients give the number of ways to choose exactly $1/2$, $1/3$, $1/4$, etc., of the elements in a set.

Let $\mathcal{L}^{(t)}$ be the linear functional whose moments are defined by aerated $(t+1)$-reciprocal binomial coefficients: $\mu_{(t+1)n}^{(t)}$ equals $B_n^{(t)}$ and is zero otherwise. We can easily prove an analogue of Theorem 3.1:

Theorem 4.1. *Let* $n_1, n_2, \ldots, n_k$ *be nonnegative integers. The integral*

$$\mathcal{L}^{(t)} \left(\prod_{i=1}^{k} T_{n_i}^{(t)}(x) \right)$$

equals the number of ways to mark exactly $1/(t+1)$ *of the vertices in* $\mathrm{Cyc}(n_1) \sqcup \cdots \sqcup \mathrm{Cyc}(n_k)$ *such that no marked vertex is followed by* t *unmarked vertices.*

Proof. The proof follows the now-familiar mantra: the product is the generating function for coverings of $\mathrm{Cyc}(n_1) \sqcup \cdots \sqcup \mathrm{Cyc}(n_k)$ by t-edge paths with weight -1 and with fixed points of weight x. Applying $\mathcal{L}^{(t)}$ can be interpreted as changing all fixed points to have weight 1 and marking exactly $1/(t+1)$ of them. Now we apply a sign-reversing involution to the set of those configurations: scan through the cycles, and find the first occurrence of a t-edge path and turn it into a marked vertex followed by t unmarked vertices, or vice versa. $\square$

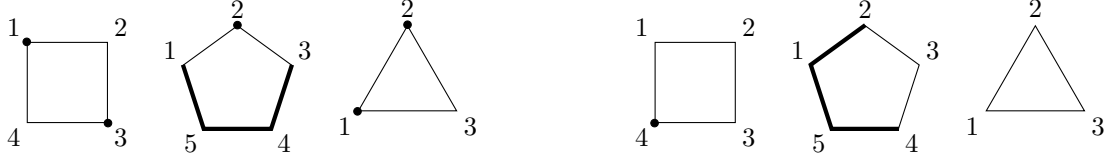See Figure 9 for two examples of the involution.

FIGURE 9. Two (unrelated) configurations in $\mathcal{L}_3\big(T_4^{(3)}(x)T_5^{(3)}(x)T_3^{(3)}(x)\big)$. Marked vertices are indicated by a black circle, unmarked vertices are regular corners of the shapes, and the thick edges represent paths with $t = 3$ edges. In the left configuration, the sign-reversing involution would remove the edge in the 5-cycle and leave the 5-cycle with vertices 2 and 3 marked and the others unmarked. In the right configuration, vertex 4 in the 4-cycle is followed by three unmarked vertices, so the involution would replace the marked vertex by a path on vertices 4–1–2–3. Observe that it is perfectly acceptable to have no (or all) marked vertices in a cycle.

The above theorem immediately gives us the orthogonality relation and L^2 norm for $T_n^{(t)}(x)$.

Corollary 4.2. *The Chebyshev polynomials of the first kind and order t are t-orthogonal with respect to the moments given by aerated $(t + 1)$-reciprocal binomial coefficients. That is, with $\mathcal{L}^{(t)}$ defined as above, whenever $m > nt$, then*

$$(17) \qquad \mathcal{L}^{(t)}\big(U_m^{(t)}(x)U_n^{(t)}(x)\big) = 0 \quad and \quad \mathcal{L}^{(t)}\big(U_{nt}^{(t)}(x)U_n^{(t)}(x)\big) = t + 1.$$

Proof. Assume $m > nt$. The above theorem tells us that, to find $\mathcal{L}^{(t)}\big(U_m^{(t)}(x)U_n^{(t)}(x)\big)$, we need to count the number of ways to mark exactly $(n + m)/(t + 1)$ vertices in $\mathrm{Cyc}(m) \sqcup \mathrm{Cyc}(n)$ such that no marked vertex is followed by t unmarked vertices. Since $(n + m)/(t + 1)$ is greater than n, we must mark at least one vertex in $\mathrm{Cyc}(m)$, and to insure that we leave no marked vertex in $\mathrm{Cyc}(m)$ followed by t unmarked vertices, we must mark more than m/t vertices there—but $(n+m)/(t+1)$ is strictly smaller than m/t, so there are zero configurations meeting the criteria.

When we consider $\mathcal{L}^{(t)}\big(U_{nt}^{(t)}(x)U_n^{(t)}(x)\big)$, we must mark exactly n vertices. We can do that and meet the marked-unmarked condition by either marking all vertices of $\mathrm{Cyc}(n)$, or by marking those vertices whose label is congruent to $0, 1, 2, \ldots$ modulo t in $\mathrm{Cyc}(nt)$, for a total of $t + 1$ different configurations. $\square$

4.3. **Generating functions.** In analogy with UP and UM, define

$$TP(t, x, z) = \sum_{n \geq 0} T_n^{(t)}(x)z^n$$

and

$$TM(t, z) = \sum_{n \geq 0} \mu_n^{(t)} z^n.$$

These generating functions are not difficult to derive. For the polynomials, we decompose $T_n^{(t)}(x)$ by considering vertex 1. That vertex is either a fixed point of weight x, or is one of the $t+1$ vertices in a t-path. If one removes the "component" that vertex 1 is in, the result is a covering of a path. Therefore for $n \geq 1$,

$$T_n^{(t)}(x) = xU_{n-1}^{(t)}(x) - (t+1)U_{n-(t+1)}^{(t)}(x),$$

where we take polynomials with negative indices to equal zero. By multiplying the above equation by z^n and summing over n, the above equation immediately yields the generating function for $T_n^{(t)}(x)$:

$$TP(t, x, z) = 1 + xzUP(t, x, z) - (t+1)z^{t+1}UP(t, x, z)$$

(18)
$$= \frac{1 - tz^{t+1}}{1 - (xz - z^{t+1})}.$$

The generating function for the moments is also similar to that for the Chebyshevs of the second kind. The recurrence coefficients of (15) tell us the L-fraction expression [31, chapter V, §6]:

(19)
$$TM(t, z) = \cfrac{1}{1 - \cfrac{(t+1)z^{t+1}}{\left(1 - \cfrac{z^{t+1}}{\left(1 - \cfrac{z^{t+1}}{(1 - \cdots)^t}\right)^t}\right)^t}}.$$

The only recurrence coefficient here that is different from those for the Chebyshev polynomials of the second kind is the very first one, so reasoning as in Figure 2, we have the following:

(20)
$$TM(t, z) = 1 + (t+1)z^{t+1}(UM(t, z))^t TM(t, z).$$

That equation also follows from (19) and the definition of UM. The recurrence coefficients also directly tell us that the moments are the generating function for weighted Łukasiewicz paths with upsteps and steps $(1, -t)$ in which all steps have weight 1 except for a downstep leaving from height t—such a step has weight $t+1$.

5. Hermite polynomials

Having addressed paths and cycles, we turn now to complete graphs. The higher-order matching polynomial for coverings of K_n by t-paths is $H_n^{(t)}(x)$, the Hermite polynomial of order t. These polynomials are very similar to Chebyshev polynomials of the second kind, but now, since the underlying graph is the complete graph instead of just a path, we may have crossings. We will draw coverings for $H_n^{(t)}(x)$ similar to how we drew coverings for $U_n^{(t)}(x)$ but now edges may be between any two vertices. Figure 10 shows an example configuration.

The recurrence relation is hardly any more difficult than that for higher-order Chebyshev polynomials of the second kind (3). First, observe that there are $(t+1)!/2$ possible t-paths on a set of $t+1$ vertices. Given $H_{n+1}^{(t)}(x)$, consider vertex $n+1$: it is either fixed, or is in a t-path with t other vertices, and we have

(21)
$$H_{n+1}^{(t)}(x) = xH_n^{(t)}(x) - \binom{n}{t}\frac{(t+1)!}{2}H_{n-t}^{(t)}(x)$$

for $n > 0$, with the usual convention that polynomials with negative indices are zero and the zeroth polynomial equals one. See Araujo et al. [2, §2.3] for an explicit formula.

For the higher-order Chebyshev U polynomials, the moments were noncrossing set partitions with all blocks of size $t+1$; now, the moments involve the same sort of set partitions, but with
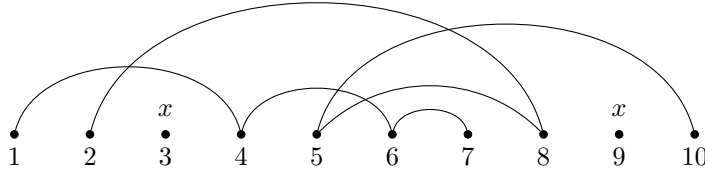
FIGURE 10. A covering of K_{10} by 3-paths that contributes weight $(-1)^2 x^2$ to $H_{10}^{(3)}(x)$, which equals $x^{10} - 2520x^6 + 226800x^2$. For clarity, only the edges in covering paths are drawn; the remaining 39 edges of the underlying graph are considered understood.

crossings allowed. In both cases, the moments are "complete" configurations. Let $\mu_n^{(t)}$ equal the number of ways to completely cover K_n by t-paths and $\mathcal{L}^{(t)}$ the corresponding linear functional. Clearly $\mu_n^{(t)}$ is zero if n is not a multiple of $t + 1$, and if it is a multiple of $t + 1$, then we can count the number of such coverings by first finding the number of set partitions of $[n]$ with all blocks of size t (which is just a multinomial coefficient) and then multiplying by an appropriate power of $(t + 1)!/2$, the number of t-paths through a given set of vertices. That is,

$$(22) \qquad \mu_{(t+1)n}^{(t)} = \binom{(t+1)n}{t+1, t+1, \ldots, t+1} \left(\frac{(t+1)!}{2}\right)^n$$

with n copies of $t + 1$ in the "denominator" of the multinomial coefficient. Corollary 5.2 will show that these numbers really are the moments for the higher-order Hermite polynomials.

For $t = 1$, the moments have the very well-known integral representation as the moments of a positive measure on the real axis, namely

$$\mu_n^{(1)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} x^n \exp\left(-x^2/2\right) \, \mathrm{d}x.$$

For $t = 2$ and $t = 3$, integral representations are also known:

$$(23) \qquad \mu_{3n}^{(2)} = \frac{3^n}{\pi} \int_0^\infty x^n \sqrt{\frac{2}{3x}} K_{1/3}\left(\frac{2\sqrt{2x}}{3}\right) \, \mathrm{d}x,$$

where $K_{1/3}$ is the modified Bessel function of the second kind. This expression follows from the formula given in sequence A025035 of the OEIS [27]. For $t = 3$, a formula in sequence A025036 gives a representation for $\mu_{4n}^{(3)}$:

$$(24) \quad 12^n \int_0^\infty x^{n-1/4} \frac{3^{1/4}}{2^{1/2}} \left(\frac{3^{1/2}F(5/4, 3/2)\Gamma(3/4)}{2^{3/4}\pi} - \frac{3^{1/4}F(5/4, 3/4)}{\pi^{1/2}x^{1/4}} + \frac{F(1/2, 3/4)}{2^{3/4}x^{1/2}\Gamma(3/4)}\right) \, \mathrm{d}x,$$

where $F(a, b) = {}_0F_2\left(\begin{smallmatrix} - \\ a \quad b \end{smallmatrix}; -3x/32\right)$, a generalized hypergeometric function.

With $\mathcal{L}^{(t)}$ in hand, our next task is to show what happens when one integrates a product of higher-order Hermite polynomials; like Theorem 3.1, this is a generalization of a theorem of de Sainte-Catherine and Viennot [6, Theorem 2].

Theorem 5.1. *Let $n_1, n_2, \ldots, n_k$ be nonnegative integers. The integral*

$$\mathcal{L}^{(t)}\left(\prod_{i=1}^k H_{n_i}^{(t)}(x)\right)$$

equals the number of inhomogeneous coverings of $[n_1] \sqcup \cdots \sqcup [n_k]$ by t-paths.
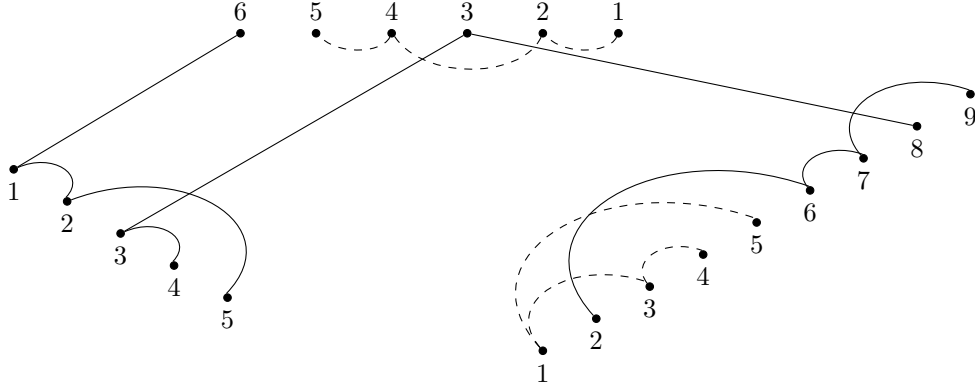
FIGURE 11. A configuration in $\mathcal{L}^{(3)}\big(H_5^{(3)}(x)H_9^{(3)}(x)H_6^{(3)}(x)\big)$. The sign-reversing involution of Theorem 5.1 would change the path on vertices 1, 3, 4 and 5 in [9] to black.

Proof. Just as in Theorem 3.1, the integral of the product is the generating function for complete coverings of $[n_1] \sqcup \cdots \sqcup [n_k]$ by two kinds of t-paths: black paths, which have weight -1 and must stay within on of the $[n_i]$, and dashed paths, which have weight 1 and can go anywhere. We can cancel all the black paths with a simple sign-reversing involution: given a homogeneous path in $[n_i]$, label the path with (i, j), where j is the smallest index among the $t + 1$ vertices of the path. Order those labels lexicographically, and switch the first path in that ordering from dashed to black, or vice versa. This cancels every configuration with a homogeneous t-path. $\square$

Figure 11 shows an example of such a configuration and the action of the above sign-reversing involution on it.

The next result, along with uniqueness of moments, implies that the $\mu_n^{(t)}$ we defined above are the moments for the higher-order Hermite polynomials.

Corollary 5.2. *The Hermite polynomials of order t are t-orthogonal with respect to the moments given by $\mu_n^{(t)}$: whenever $m > nt$, then*

$$(25) \qquad \mathcal{L}^{(t)}\big(H_m^{(t)}(x)H_n^{(t)}(x)\big) = 0 \quad and \quad \mathcal{L}^{(t)}\big(H_{nt}^{(t)}(x)H_n^{(t)}(x)\big) = \left(\frac{t+1}{2}\right)^n (nt)!$$

for $n \geq 0$.

Proof. The first relation is clear from Theorem 5.1, since if $m > nt$, there must be a homogeneous t-path in $[m]$ and so all such configurations are canceled.

The L^2 norm requires a bit of work. The involution implies that the only configurations we need consider are those in which every t-path has one vertex in $[n]$, and the other t vertices in $[nt]$. To construct such a covering, take a permutation of $[nt]$ and insert a bar after every t vertices to form n groups. The first group of vertices will be in a path with vertex 1 from $[n]$, the second group in a path with vertex 2 from $[n]$, and so on. For each of the groups, there are $t + 1$ ways to insert the vertex from $[n]$ into the path given by the ordering of the group. However, since the paths have no orientation, we have counted every possibility twice, so we divide by 2 for each of the n paths, and obtain the claimed L^2 norm. $\square$

5.1. **Generating functions.** Generating functions for the higher-order Hermite polynomials and their moments are quite easy to find, since moving from a path to a complete graph gives

"more symmetry". First, the polynomials: we define

$$HP(t, x, z) = \sum_{n \geq 0} H_n^{(t)}(x) \frac{z^n}{n!}.$$

Any configuration contributing to $H_n^{(t)}(x)$ has two kinds of connected components: t-paths and fixed points. The former has weight -1 and there are $(t+1)!/2$ of them on a labeled set of $t+1$ points, the latter has weight x and there's obviously just one on a point. The exponential formula [1, §3.3] immediately gives us

$$(26) \qquad HP(t, x, z) = \exp\left(xz - \frac{z^{t+1}}{2} \right).$$

The above generating function is a specialization of one found by Farrell [7, Theorem 2]. The same reasoning gives us the exponential generating function for the moments:

$$(27) \qquad HM(t, z) = \sum_{n \geq 0} \mu_n^{(t)} \frac{z^n}{n!} = \exp\left(\frac{z^{t+1}}{2} \right).$$

We can also find an expression for the ordinary generating function, since the recurrence coefficients of the polynomials tell us the continued fraction expansion for that function. We know that the moments of order t are also the generating functions for Łukasiewicz paths that consist of upsteps and t-downsteps, where upsteps all have weight 1 and a t-downstep leaving from height n has weight $\binom{n}{t}(t+1)!/2$ (the recurrence coefficient for $H_{n-t}^{(t)}(x)$ in (21)). For clarity, let $\lambda_n^{(t)} = \binom{n}{t}(t+1)!/2$, and write $HM'(t, z)$ for the ordinary generating function for $\mu_n^{(t)}$; then, decomposing Łukasiewicz paths makes the following expression clear:

$$(28) \qquad HM'(t, z) = \cfrac{1}{1 - \cfrac{\lambda_t^{(t)} z^{t+1}}{\displaystyle\prod_{k=1}^{t}\left(1 - \cfrac{\lambda_{t+k}^{(t)} z^{t+1}}{\displaystyle\prod_{j=1}^{t}\left(1 - \cfrac{\lambda_{t+k+j}^{(t)} z^{t+1}}{\prod(1 - \cdots)}\right)}\right)}}$$

This expression, like the continued fraction for $UM(t, z)$ and $TM(t, z)$, is an L-fraction [31, §6]. Figure 12, the higher-order Hermite version of Figure 2, explains the first steps of $HM'(3, z)$. Analogous to (6) and (20), HM' satisfies the functional equation

$$HM'(t, z) = 1 + \lambda_t^{(t)} z^{t+1} HM'(t, z) \prod_{k=1}^{t} \left(\delta^k HM'(t, z) \right).$$

The δ operator seen here and in Figure 12 is taken from [31, chap. V, §1]: the generating function $HM'(t, z)$ depends on t, z, and the $\lambda_n^{(t)}$'s, and we could make the dependence explicit by writing $HM'(t, z, \lambda_t^{(t)}, \lambda_{t+1}^{(t)}, \dots)$. The δ operator simply increases the subscript on all the λ's: $\delta HM' = HM'(t, z, \lambda_{t+1}^{(t)}, \lambda_{t+2}^{(t)}, \dots)$.

6. Laguerre polynomials

The final class of higher-order matching polynomials we will consider are the Laguerre polynomials. The usual Laguerre polynomial $L_n(x^2)$ is the matching polynomial for the complete bipartite graph $K_{n,n}$; often the Laguerre polynomials are defined with a parameter α, and while that parameter has combinatorial meaning—in some sense, it counts cycles; see Foata and Strehl [11], Labelle and Yeh [19], and Simion and Stanton [25, 26]—we will not use it; our polynomials correspond to $\alpha = 0$.
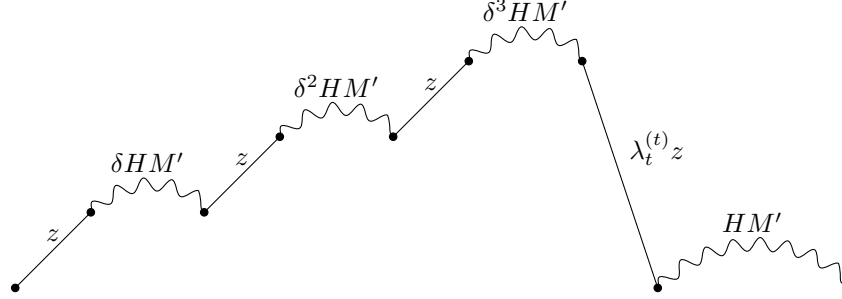
FIGURE 12. A Łukasiewicz path decomposition for $HM'(t,z)$ for $t = 3$. Any path contributing to HM' is either empty, or of the above form. The $\delta^k HM'$ notation means shift all the lower subscripts on the λ's by k.

Let $M_t(K_{n,n})$ be the higher-order matching polynomial for complete bipartite graphs with our usual weights. The degrees of the polynomials of this sequence are all even, but a d-orthogonal sequence of polynomials requires a polynomial of degree n for every nonnegative n. If t is odd (so that the number of vertices in a t-path is even) we can simply substitute $\sqrt{x}$ for x and get a proper sequence of polynomials, but if t is even, the resulting matching polynomials contain both even and odd powers of x and a simple substitution will not work. We could not find a combinatorially satisfactory way of converting the matching polynomials for even t into a proper sequence of polynomials, so in this section, we will hereafter assume that t is an odd positive integer, and define k by $t = 2k - 1$; k is the number of vertices that a t-path occupies on each "side" of $K_{n,n}$.

We therefore define the Laguerre polynomial of order t (for odd t only) by the relation

$$(29) \qquad\qquad L_n^{(t)}(x^2) = M_t(K_{n,n}).$$

Araujo et al. [2, §2.4] find explicit formulas[2] for these polynomials and derive recurrence relations for the matching polynomials—but some of the polynomials in those relations are from graphs not of the form $K_{n,n}$, so we need to derive an appropriate recurrence relation for $L_n^{(t)}(x)$ directly. Before we do that, observe that the number of ways to cover $K_{k,k}$ by a t-path is $(k!)^2$: we can orient the path by considering it to start in the left set of vertices, and then we order the vertices on the left and right.

Theorem 6.1. *With the convention that $L_n^{(t)}(x) = 0$ when $n < 0$ and $L_0^{(t)}(x) = 1$, then for $n \geq 0$, the Laguerre polynomials of order t satisfy the recurrence relation*

$$L_{n+1}^{(t)}(x) = xL_n^{(t)}(x) - (k!)^2 \left(\binom{n}{k-1}^2 + 2\binom{n}{k}\binom{n}{k-1} \right) L_{n-(k-1)}^{(t)}(x) -$$

$$\left(\binom{n}{t}\binom{t}{k}(k!)^2 \right)^2 L_{n-t}^{(t)}(x).$$

Proof. Any covering of $K_{n+1,n+1}$ by t-paths can be obtained in one or more of the following ways:

(a) Take any covering of $K_{n,n}$ and add two fixed vertices at the "bottom" of each vertex set; this corresponds to $xL_n^{(t)}$.

---

[2]Note that the terms of the sum in their equation (2.27) are missing a factor of $(-1)^k$.

(b) Take $K_{n+1,n+1}$ and choose $k-1$ vertices among the first n vertices of each set. Add in the last vertices of each set, put a t-path onto those $2k$ vertices, and then "fill in" the rest with any configuration. This corresponds to

$$-(k!)^2 \binom{n}{k-1}^2 L_{n-(k-1)}^{(t)}(x).$$

(c) Take $K_{n+1,n+1}$ and choose k vertices among the first n vertices in the left set, $k-1$ vertices among the first n vertices in the right set, put a t-path down on those $2k$ vertices, and then fill in the rest with any configuration. This, along with exchanging left and right, contributes

$$-2\binom{n}{k}\binom{n}{k-1}(k!)^2 L_{n-(k-1)}^{(t)}(x).$$

Configurations in which vertices $n+1$ on the left and right are both in a path, and are not in the same path, are counted twice by item (c) above. So we must correct for this by subtracting the total weight of those configurations. We need to choose t vertices on each side, and then choose k of those vertices to get connected to the bottom vertex on the opposite side. Finally, put down two t-paths. The total contribution of these configurations is

$$\left(\binom{n}{t}\binom{t}{k}(k!)^2\right)^2 L_{n-t}^{(t)}(x).$$

Adding together the above expressions yields the recurrence relations of the theorem. □

Figure 13 shows the four cases. Observe that if $k=1$, we indeed recover the recurrence coefficients for the classical monic Laguerre polynomials: $L_{n+1}(x) = (x - (2n+1))L_n(x) - n^2 L_{n-1}(x)$.

As usual, we define a linear functional $\mathcal{L}^{(t)}$ by $\mathcal{L}^{(t)}(x^n) = \mu_n^{(t)}$, where $\mu_n^{(t)}$ is the number of complete coverings of $K_{n,n}$ by t-paths, and can count the integral of a product of these polynomials. This is a generalization of a result of de Sainte-Catherine and Viennot [6, Theorem 5].

Theorem 6.2. *Let $n_1, n_2, \ldots, n_j$ be nonnegative integers. The integral*

$$\mathcal{L}^{(t)}\left(\prod_{i=1}^{j} L_{n_i}^{(t)}(x)\right)$$

equals the number of inhomogeneous coverings of $K_{n_1,n_1} \sqcup \cdots \sqcup K_{n_j,n_j}$ by t-paths.

Proof. The proof is the same that we've seen several times now; we start with the product of the $L_{n_i}^{(t)}$ and apply $\mathcal{L}^{(t)}$, which gives us the generating function for complete coverings of the disjoint union of the complete bipartite graphs by t-paths, in which homogeneous t-paths may be black (weight -1) or dashed (weight $+1$), and inhomogeneous paths always are black (weight $+1$). By choosing, say, the smallest i such that K_{n_i,n_i} has a homogeneous path, finding the homogeneous path with the smallest left vertex inside that subgraph, and changing the color from black to dashed or vice versa, we have a sign-reversing involution that cancels any configuration with a homogeneous t-path. □

This immediately implies the t-orthogonality of the higher order Laguerre polynomials (recall that $k = (t+1)/2$):

Corollary 6.3. *The Laguerre polynomials of order t are t-orthogonal with respect to the moments given by $\mu_n^{(t)}$: whenever $m > nt$, then*

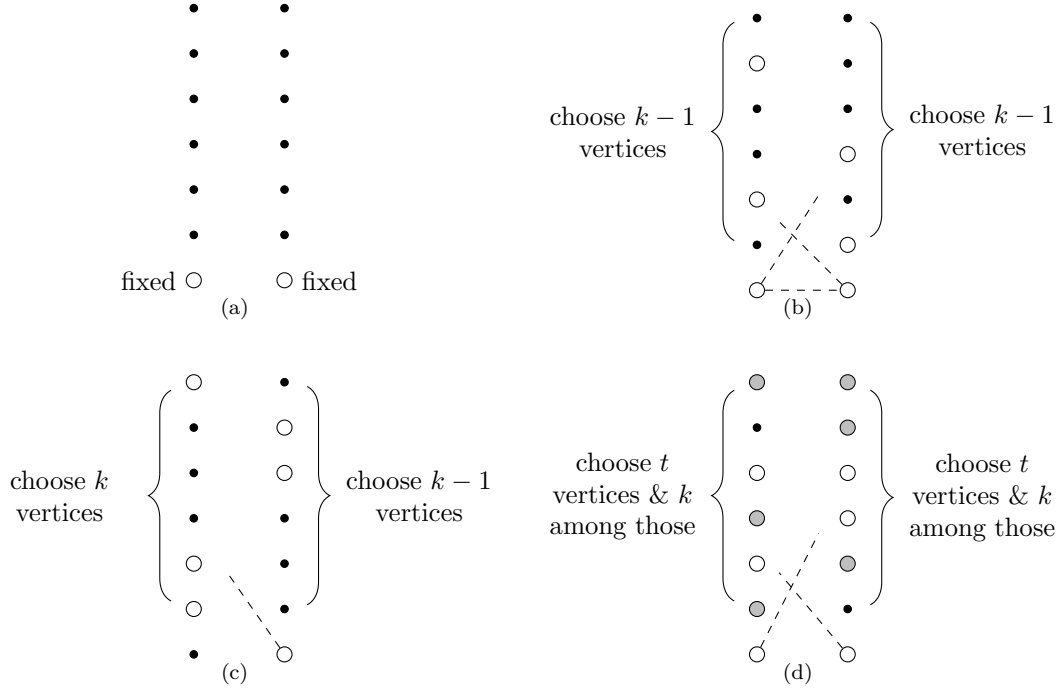$$\mathcal{L}^{(t)}\left(L_m^{(t)}(x)L_n^{(t)}(x)\right) = 0,$$

FIGURE 13. The four cases of the recurrence relation for $L_{n+1}^{(t)}(x)$; here $t = 5$ so $k = 3$. In case (a), both bottom vertices are fixed; in (b), both bottom vertices are in the same path; the bottom vertices are not necessarily adjacent in the t-path. In (c), the right vertex is in a path, and the bottom vertex on the left may or may not be in a path. Case (d) corrects the overcounting from (c) when both vertices are in a path: the gray circles are in a path together with the bottom vertex from the opposite side.

and

$$\mathcal{L}^{(t)}\left(L_{nt}^{(t)}(x)L_n^{(t)}(x)\right) = \prod_{i=0}^{n-1}\left(\binom{(n-i)t}{t}\binom{t}{k}(k!)^2\right)^2$$

for $n \geq 0$.

Proof. The orthogonality relation is clear from Theorem 6.2, since any such configuration in $L_m^{(t)}(x)L_n^{(t)}(x)$ must have a homogeneous t-path in $L_m^{(t)}(x)$. The L^2 norm can be calculated as follows: after applying the sign-reversing involution of the theorem, the only remaining configurations are those with $2n$ paths, each with one vertex in $K_{n,n}$ and the remaining vertices in $K_{nt,nt}$. Consider vertex 1 on the left and right in $K_{n,n}$: to choose a pair of paths that go through those vertices, choose t vertices among the nt vertices on the left and right in in $K_{nt,nt}$. There are $\binom{nt}{t}^2$ ways to do that. Among those t vertices on the left, choose k of them to be in the path that goes through vertex 1 on the right side of $K_{n,n}$; the same applies, *mutatis mutandis*, on the other side. There are $\binom{t}{k}^2$ ways to do this. Now take vertex 1 on the left, the $k-1$ vertices not chosen in the second step, and the k vertices chosen on the right, and put a t-path on those vertices; do the same with the remaining vertices—there are $(k!)^2$ ways to do that for each path. Altogether we've accounted for the $i = n$ factor in the above product; now repeat this procedure

with vertex 2 on the left and right in $K_{n,n}$ and the remaining $(n-1)t$ vertices in $K_{nt,nt}$, and so on; the total number of uncanceled configurations is exactly the product above. □

Using just the "left side of the above argument" for the L^2 norm, we can derive a formula for the moments. We know that the moments for the Laguerre polynomials of order t are the number of complete coverings of $K_{n,n}$ by t-paths; if n is not a multiple of k, there are zero such coverings, and otherwise if $n = mk$, the number of coverings is

$$(30) \qquad \mu_{mk}^{(t)} = \prod_{i=0}^{m-1} \binom{(m-i)k}{k}\binom{(m-i)k-1}{k-1}(k!)^2.$$

6.1. **Generating functions.** The recurrence coefficients found in Theorem 6.1 allow us to give a continued fraction expression for the moment generating function:

$$LM(t,z) = \sum_{n \geq 0} \mu_n^{(t)} z^n.$$

The recurrence relation has coefficients in front of $L_{n-(k-1)}^{(t)}(x)$ and $L_{n-t}^{(t)}(x)$, which tells us that the weighted Łukasiewicz paths whose generating function equals that of the moments have down steps of $(1, -(k-1))$ and $(1, -t)$; by decomposing the paths as in Figure 12, we can express $LM(t,z)$ as an L-fraction:

$$(31) \qquad LM(t,z) = \cfrac{1}{1 - z^{k-1}\lambda_{k-1,k-1}^{(t)}\prod_{i=1}^{k-1}\delta^i LM(t,z) - z^t\lambda_{t,t}^{(t)}\prod_{i=1}^{t}\delta^i LM(t,z)}$$

where $\lambda_{n,m}^{(t)}$ denotes the weight of a downstep leaving from height n and falling m steps and δ, as in section 5.1, acts on LM by increasing the first "coordinate" of the coefficients; it changes $\lambda_{n,m}^{(t)}$ into $\lambda_{n+1,m}^{(t)}$.

7. FURTHER WORK

In this paper, we've worked with sets of polynomials that satisfy a recurrence of order t and in each case, found a linear functional with respect to which the polynomials are t-orthogonal. However, as Van Iseghem [30] and Maroni [21] have shown, such sequences of polynomials are naturally associated to not just a single linear functional, but $t-1$ of them. These functionals are defined by

$$\mathcal{L}_k^{(t)}(P_n(x)P_m(x)) = \begin{cases} 0 & \text{if } n > tm+k, \\ \text{nonzero} & \text{if } n = tm+k, \end{cases}$$

for $k = 0, \ldots, t-2$. The nth moment of $\mathcal{L}_k^{(t)}$ is the generating function for weighted Łukasiewicz paths of length n that end at height k. In this work we have only addressed the $k = 0$ functionals and in light of the results of Van Iseghem and Maroni, the combinatorial theory of these higher-order matching polynomials is not entirely known until interpretations of the higher functionals are known.

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