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Editor of Journal of Number Theory

Dear Professor Zagier

Please accept our paper titled "Platonic solids in Z^3" for publication in the *Journal of Number Theory*. This short communication is extending previous results on a characterization of all equilateral triangle in space having vertices with integer coordinates ("in Z^3"), we look at the problem of characterizing all regular polyhedra (Platonic Solids) with the same property. To summarize, we show first that there is no regular icosahedron/ dodecahedron in Z^3. On the other hand, as shown before there is a finite (6 or 12) class of regular tetrahedra in Z^3, associated naturally to each nontrivial solution (a, b, c, d) of the Diophantine equation $a^2 + b^2 + c^2 = 3d^2$ and for every nontrivial integer solution (m, n, k) of the equation $m^2 - mn + n^2 = k^2$. Every regular tetrahedron in Z^3 belongs, up to an integer translation and/or rotation, to one of these classes. We then show that each such tetrahedron can be completed to a cube with integer coordinates. The study of regular octahedra is reduced to the cube case via the duality between the two. This work allows one to basically give a description the orthogonal group $O(3, Q)$ in terms of the seven integer parameters satisfying the two relations mentioned above. Also, this allows one to calculate easily the sequences A098928, A103158 (already introduced in the Online Encyclopedia of Integer Sequences) and the sequence (that we are planning to submit to OEIS) of the number of regular octahedrons in $\{0, 1, 2, ..., n\}^3$.

Thank you for your consideration and time spent with our submittal.

Sincerely yours,

Professor Eugen J. Ionascu

PLATONIC SOLIDS IN $\mathbb{Z}^3$

EUGEN J. IONASCU AND ANDREI MARKOV

Abstract. Extending previous results on a characterization of all equilateral triangle in space having vertices with integer coordinates ("in $\mathbb{Z}^3$"), we look at the problem of characterizing all regular polyhedra (Platonic Solids) with the same property. To summarize, we show first that there is no regular icosahedron/ dodecahedron in $\mathbb{Z}^3$. On the other hand, there is a finite (6 or 12) class of regular tetrahedra in $\mathbb{Z}^3$, associated naturally to each nontrivial solution (a, b, c, d) of the Diophantine equation $a^2 + b^2 + c^2 = 3d^2$ and for every nontrivial integer solution (m, n, k) of the equation $m^2 - mn + n^2 = k^2$. Every regular tetrahedron in $\mathbb{Z}^3$ belongs, up to an integer translation and/or rotation, to one of these classes. We then show that each such tetrahedron can be completed to a cube with integer coordinates. The study of regular octahedra is reduced to the cube case via the duality between the two. This work allows one to basically give a description the orthogonal group $O(3, \mathbb{Q})$ in terms of the seven integer parameters satisfying the two relations mentioned above.

1. INTRODUCTION

The set of equilateral triangles in the three dimensional space with integers coordinates for its vertices is very rich. If one counts all of these triangles with the coordinates in $\{0, 1, 2, 3, ..., 100\}$, finds that there are $10, 588, 506, 416$ of them ([10]). A constructive way to find these triangles is described in [1] and [4], and this method was implemented in Maple (see [5]). In [6] a characterization of regular tetrahedra in $\mathbb{Z}^3$ is given. One may ask naturally if other Platonic solids may happen to have integer coordinates. Perhaps the simplest different examples are those of cubes in space whose vertices have integer coordinates can be found by dilating the so called unit cube by an integer factor: $(0, 0, 0)$, $(n, 0, 0)$, $(0, n, 0)$, $(0, 0, n)$, $(n, n, 0)$, $(n, 0, n)$, $(0, n, n)$, and (n, n, n), $n \in \mathbb{Z}$. The figure below shows a less obvious cube which has vertices: $(1, 4, 3)$, $(3, 3, 1)$, $(1, 1, 0)$, $(-1, 2, 2)$, $(2, 2, 5)$, $(4, 1, 3)$, $(2, -1, 2)$ and $(0, 0, 4)$.

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Key words and phrases. Platonic solids, equilateral triangles, integers, orthogonal matrices, integer parametrization, tetrahedrons, cubes, octahedrons, dodecahedrons, icosahedrons .

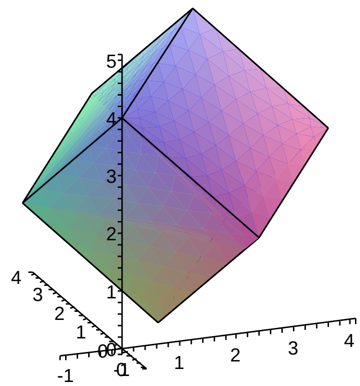

Figure 1(a): Non−trivial cube

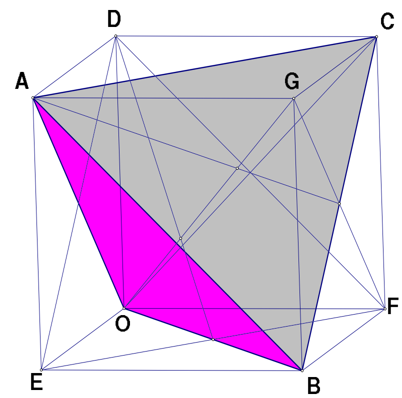

Figure 1(b): Regular tetrahedron inscribed: OABC

On the other side of the question asked earlier, we will show that dodecahedrons or icosahedrons do not exist "in $\mathbb{Z}^3$".

Given a cube in general, there is a natural way to imbed a regular tetrahedron in it like in Figure 1(b). In fact, one can find two such regular tetrahedra that have vertices a subset of the cube's vertices but if we require that both the cube and the tetrahedron have one of the vertices the origin, the correspondence is one to one. If the cube has integer coordinates then the regular tetrahedra considered must have the same property. Conversely, if we start with a regular tetrahedra having integer coordinates this can be completed uniquely by adding four more vertices, to a cube as in Figure 1(b). In general, the resulting cube, as we will see, may have coordinates only in $\frac{1}{2}\mathbb{Z}$, but it works out that the coordinates are always in $\mathbb{Z}$.

Finally, we study the regular octahedrons with integers coordinates which are in duality with the cubes.

2. Previous Results

Our starting point is the theorem below about how to obtain all equilateral triangles in "in $\mathbb{Z}^3$". Each such triangle is contained in a lattice of points of the form

$$\text{(1)} \qquad \mathcal{P}_{a,b,c} := \{(\alpha,\beta,\gamma) \in \mathbb{Z}^3 | \ \ a\alpha + b\beta + c\gamma = 0, \ \ a^2+b^2+c^2 = 3d^2, \ \ a,b,c,d \in \mathbb{Z}\}.$$

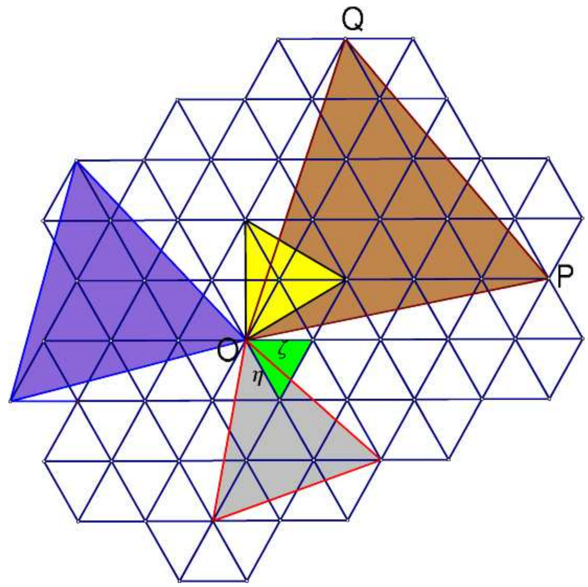

Figure 2: The lattice $\mathcal{P}_{a,b,c}$

In general, the vertices of the equilateral triangles that dwell in $\mathcal{P}_{a,b,c}$ form a strict sub-lattice of $\mathcal{P}_{a,b,c}$ which is generated by only two vectors, $\overrightarrow{\zeta}$ and $\overrightarrow{\eta}$ (see Figure 2). These two vectors are described by the Theorem 2.1 proved in [1].

Theorem 2.1. *Let a, b, c, d be odd integers such that $a^2+b^2+c^2=3d^2$ and $\gcd(a,b,c)=1$. Then for every $m,n\in\mathbb{Z}$ (not both zero) the triangle OPQ, determined by*

$$\overrightarrow{OP}=m\overrightarrow{\zeta}+n\overrightarrow{\eta},\quad \overrightarrow{OQ}=m(\overrightarrow{\zeta}-\overrightarrow{\eta})+n\overrightarrow{\zeta},\ \text{with}\ \overrightarrow{\zeta}=(\zeta_1,\zeta_1,\zeta_2),\ \overrightarrow{\eta}=(\eta_1,\eta_2,\eta_3), \tag{2}$$

$$\begin{cases}\zeta_1=-\dfrac{rac+dbs}{q},\\[2ex] \zeta_2=\dfrac{das-bcr}{q},\\[2ex] \zeta_3=r,\end{cases}\quad,\quad \begin{cases}\eta_1=-\dfrac{db(s-3r)+ac(r+s)}{2q},\\[2ex] \eta_2=\dfrac{da(s-3r)-bc(r+s)}{2q},\\[2ex] \eta_3=\dfrac{r+s}{2},\end{cases} \tag{3}$$

where $q=a^2+b^2$ and (r,s) is a suitable solution of $2q=s^2+3r^2$ that makes all the numbers in (3) *integers, forms an equilateral triangle in $\mathbb{Z}^3$ contained in the lattice* (1) *and having sides-lengths equal to $d\sqrt{2(m^2-mn+n^2)}$.*

Conversely, there exists a choice of the integers r and s such that given an arbitrary equilateral triangle in $\mathbb{R}^3$ whose vertices, one at the origin and the other two in the lattice (1)*, then there also exist integers m and n such that the two vertices not at the origin are given by* (2) *and* (3)*.*

Out of all the equilateral triangles in $\mathbb{Z}^3$ only those for which $m^2 - mn + n^2 = \lambda^2$, $\lambda \in \mathbb{Z}$, may give rise to regular tetrahedra in $\mathbb{Z}^3$ according to the following theorem. As a notation, we let for every $k \in \mathbb{Z}$, $\Omega(k) = \{(m,n) \in \mathbb{Z}^2 | m^2 - mn + n^2 = k^2\}$.

Theorem 2.2. *Every tetrahedron whose side lengths are $\lambda\sqrt{2}$, $\lambda \in \mathbb{N}$, which has a vertex at the origin, can be obtained by taking as one of its faces an equilateral triangle having the origin as a vertex and the other two vertices given by* (2) *and* (3) *with a, b, c and d odd integers satisfying $a^2 + b^2 + c^2 = 3d^2$ with d a divisor of λ, and then completing it with the fourth vertex R with coordinates*

$$\left(\frac{\begin{array}{c}(2\zeta_1 - \eta_1)m \\ -(\zeta_1 + \eta_1)n \\ \pm 2ak\end{array}}{3}, \frac{\begin{array}{c}(2\zeta_2 - \eta_2)m \\ -(\zeta_2 + \eta_2)n \\ \pm 2bk\end{array}}{3}, \frac{\begin{array}{c}(2\zeta_3 - \eta_3)m \\ -(\zeta_3 + \eta_3)n \\ \pm 2ck\end{array}}{3} \right), \ for\ some\ (m,n) \in \Omega(k),\ k := \frac{\lambda}{d}. \tag{4}$$

Conversely, if we let a, b, c and d be a primitive solution of $a^2 + b^2 + c^2 = 3d^2$, let $k \in \mathbf{N}$ and $(m,n) \in \Omega(k)$, then the coordinates of the point R in (4)*, which completes the equilateral triangle OPQ given as in* (2) *and* (3)*, are*

(a) all integers, if $k \equiv 0 \pmod 3$ regardless of the choice of signs or

(b) integers, precisely for only one choice of the signs if $k \not\equiv 0 \pmod 3$.

3. New Facts

As observed in [6] the dihedral angle between the faces of a regular tetrahedron has a cosine of $\frac{1}{3}$. This is compatible with the normals to equilateral triangles given implicitly by (1). This is not the case for regular icosahedra and regular dodecahedrons in $\mathbb{Z}^3$. The next result may come to no surprise to our reader since there are so many Diophantine equations that need to be satisfied.

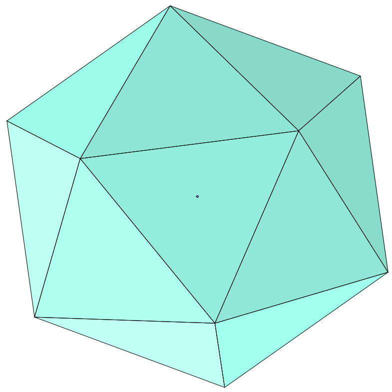

Figure 3 (*a*): *Regular Icosahedron*

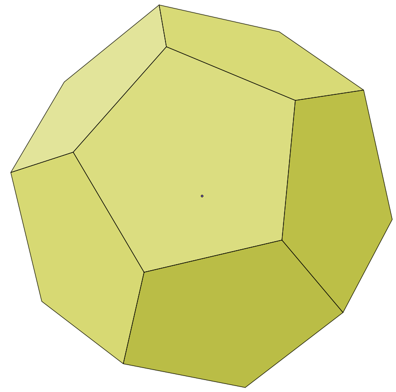

Figure 3 (*b*): *Regular Dodecahedron*

Theorem 3.1. *There is no regular icosahedron and no regular dodecahedron in* $\mathbb{Z}^3$*.*

Proof. We are beginning with the case of an icosahedron. We are arguing by way of contradiction. It is public knowledge that the cosine of the dihedral angle between the two adjacent faces of a regular icosahedron is $-\frac{\sqrt{5}}{3}$ (which gives an angle of approximately 138.1896851°). We consider two such adjacent faces which are equilateral triangles (see Figure 3 (a)) and their normals: $\overrightarrow{n} = \frac{(a,b,c)}{d\sqrt{3}}$ and $\overrightarrow{n'} = \frac{(a',b',c')}{d'\sqrt{3}}$ for some odd integers a, b, c, d, a', b', c', and d'. But the cosine of the two vectors above is equal to their scalar product $\overrightarrow{n}\overrightarrow{n'} = \frac{aa'+bb'+cc'}{3dd'} \in \mathbb{Q}$. This contradicts the fact that $\sqrt{5}$ is irrational.

For dodecahedrons, if there is one that is regular in $\mathbb{Z}^3$, the dual polyhedron (the one obtained by taking the centers of mass of each face), is a regular icosahedron which must have coordinates that are rational numbers with denominators of 1 or 5. Translating it so that a vertex becomes the origin and expanding everything by a factor of five it becomes an regular icosahedron in $\mathbb{Z}^3$. By what we have just shown this last object cannot exist. Hence, there cannot be any regular dodecahedron in $\mathbb{Z}^3$. ■

Next let us use the Theorem 2.2 to give a characterization of all cubes in $\mathbb{Z}^3$.

Theorem 3.2. *Every cube in* $\mathbb{Z}^3$ *can be obtained by a translation along a vector with integer coordinates from a cube with a vertex the origin containing a regular tetrahedron with a vertex at the origin and all integer coordinates (see Figure 1(b)) and as a result it must have side lengths equal to* n *for some* $n \in \mathbb{N}$*. Conversely, given a regular tetrahedron in* $\mathbb{Z}^3$*, this can be completed to a cube which is going to be automatically in* $\mathbb{Z}^3$*.*

Proof. The first part of this statement follows from Theorem 2.2 which prescribes the sides of a regular tetrahedron to be of the form $n\sqrt{2}$, $n \in \mathbb{N}$. The converse requires other parts of Theorem 2.2. We may assume that vertices of the given regular tetrahedron are given by the origin and three other points, say $A(a_1, a_2, a_3)$, $B(b_1, b_2, b_3)$ and $C(c_1, c_2, c_3)$. By Theorem 2.2, we may assume that A, B are given by (2), (3) and C is given by (4) (Figure 2(b)). We observe that the planes defined by ABC and DEF are parallel and cut the diagonal $\overline{OG}$ into three equal parts. Hence the coordinates of G must be $(\frac{a_1+b_1+c_1}{2}, \frac{a_2+b_2+c_2}{2}, \frac{a_3+b_3+c_3}{2})$. Similarly, the coordinates of D, E and F are respectively $(\frac{a_1-b_1+c_1}{2}, \frac{a_2-b_2+c_2}{2}, \frac{a_3-b_3+c_3}{2})$, $(\frac{a_1+b_1-c_1}{2}, \frac{a_2+b_2-c_2}{2}, \frac{a_3+b_3-c_3}{2})$, and $(\frac{-a_1+b_1+c_1}{2}, \frac{-a_2+b_2+c_2}{2}, \frac{-a_3+b_3+c_3}{2})$. It suffices to show that the coordinates of G are integers. Let us look at $a_1 + b_1 + c_1 \pmod 2$. We have

$$a_1 + b_1 + c_1 = m\zeta_1 + n\eta_1 + m(\zeta_1 - \eta_1) + n\zeta_1 + (2\zeta_1 - \eta_1)m - (\zeta_1 + \eta_1)n \mp 2ak \equiv 0 \ (mod\ 2).$$

By symmetry the other two components must satisfy similar relations. ■

We will refer to cubes or other Platonic solids in $\mathbb{Z}^3$ as being *irreducible* if it cannot be obtained by an integer vector translation and multiplication by an integer factor, greater than one in absolute value, from another analog object in $\mathbb{Z}^3$.

Corollary 3.3. *Every irreducible cube must have side lengths which are only odd natural numbers.*

Proof. If a given cube is irreducible, then it must arrive from an irreducible tetrahedron $\mathcal{T}$. Indeed, if the formulae for the vertices of a tetrahedron $\mathcal{T}$ can be simplified to a smaller tetrahedron $\mathcal{T}'$ then we can use the construction above and obtain that the given cube is obtained from the cube corresponding to $\mathcal{T}'$. Hence, we may assume that the tetrahedron $\mathcal{T}$ is irreducible, and by Theorem 2.1 and Theorem 2.2, its side lengths are $d\sqrt{2(m^2 - mn + n^2)}$ with m and n relative prime numbers such that $m^2 - mn + n^2 = k^2$, $k \in \mathbb{N}$ and d an odd number. This implies that that the sides of the given cube are equal to dk. Here, k must be also odd, otherwise m and n must be divisible by 2. ■

The problem of finding the number of cubes in space with coordinates in $\{0, 1, 2, ..., n\}$ has been studied in [8]. In [7], we implemented the method of obtaining cubes from tetrahedra and the former from equilateral triangles. So we extended the sequences A098928 and A103158. We list here a few more terms in the sequence A098928.

n	1	2	3	4	5	6	7	8	9	10	11
A098928	1	9	36	100	229	473	910	1648	2795	4469	6818

n	12	13	14	15	16	17	18
A098928	10032	14315	19907	27190	36502	48233	62803

It is clear that $A098928 \leq A103158$. For $n \geq 4$ we have actually a strict inequality, $A098928 < A103158$, and this is due to the fact that some of the tetrahedrons inside of grid $\{0, 1, .., n\}^3$ extend beyond of its boundaries.

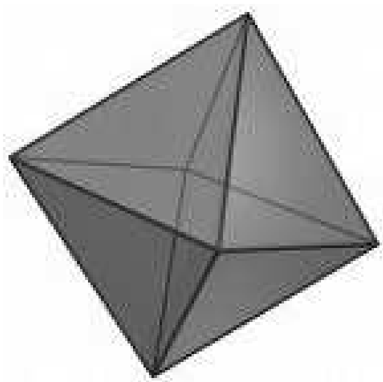

Figure 4 (a): Regular Octahedron

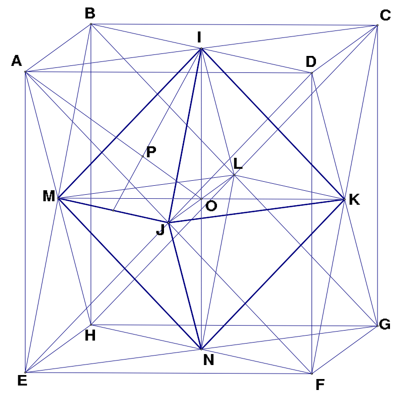

Figure 4 (b): Regular Octahedron

Regular octahedrons in $\mathbb{Z}^3$ can be obtained from cubes in $\mathbb{Z}^3$, doubling their size and then taking the centers of the faces. The converse of this statement is also true.

Theorem 3.4. *Every regular octahedrons in $\mathbb{Z}^3$ is the dual of a cube that can be obtained (up to a translation with a vector with integer coordinates) by doubling a cube in $\mathbb{Z}^3$.*

Proof. As we can see from Figure 4 (b), assuming that the octahedron IJKLMN has vertices with integer coordinates, the centers of the faces have coordinates that are rational numbers in $\frac{1}{3}\mathbb{Z}$. We may assume without loss of generality that the center of the octahedron is the origin. In the Figure 4 (b), the octahedron IJKLMN is the dual of the cube ABCDEFGH. The point P is the center of the face IJM, and from similarity of triangles we see that the coordinates are actually three times the coordinates of P. This shows that the cube ABCDEFGH is in $\mathbb{Z}^3$. Then the tetrahedron IJMA is then in $\mathbb{Z}^3$ and so by Theorem 3.2 can be completed to a cube in $\mathbb{Z}^3$. This cube if dilated by a factor of two can be shifted to get the cube ABCDEFGH. ∎

If we denote by $\mathcal{RO}(n)$, the number of regular octahedron whose vertices are in the set $\{0, 1, ..., n\}^3$, we get the following sequence:

n	1	2	3	4	5	6	7	8	9	10	11
$\mathcal{RO}(n)$	0	1	8	32	104	261	544	1000	1696	2759	4296

n	12	13	14	15	16	17	18
$\mathcal{RO}(n)$	6434	9352	13243	18304	24774	32960	43223

.

These constructions bring up the idea of a certain order on the orthogonal matrices with rational entries. For example, the first level of such matrices are the ones in which the entries are actually integers. The second tier will be formed by those matrices with denominators equal to 3, such as

$$T_3 := \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ -2 & -2 & -1 \end{pmatrix}.$$

Taking the square of T_3 we get a matrix in tier nine. There are only two representations of the type $3(9^2) = 5^2 + 7^2 + 13^2 = 1^2 + 11^2 + 11^2$, and these give the same "class" of matrices in the tier nine. The first case when we have essentially two different classes is for tier 13. An example of an orthogonal matrix with denominators equal to 2009 is given below:

$$T_{2009} := \frac{1}{2009}\begin{pmatrix} 210 & 1645 & 1134 \\ -1330 & 966 & -1155 \\ 1491 & 630 & -1190 \end{pmatrix}.$$

Since we have a structure of group on the set of orthogonal matrices, a natural question is whether one can extend this algebraic structure to all of the "equilateral triangles" not only to those for which $k^2 = m^2 - mn + n^2$.

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