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SUPER CONGRUENCES AND EULER NUMBERS

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Abstract. Let $p>3$ be a prime. A p-adic congruence is called a super congruence if it happens to hold modulo some higher power of p. The topic of super congruences is related to many fields including Gauss and Jacobi sums and hypergeometric series. We prove that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv(-1)^{(p-1)/2}-p^2E_{p-3}\pmod{p^3},$$
$$\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\equiv(-1)^{(p+1)/2}\frac{8}{3}pE_{p-3}\pmod{p^2},$$
$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^{(p-1)/2}+p^2E_{p-3}\pmod{p^3},$$

where $E_0,E_1,E_2,\ldots$ are Euler numbers. Our new approach is of combinatorial nature. We also formulate many conjectures concerning super congruences and relate most of them to Euler numbers or Bernoulli numbers. Motivated by our investigation of super congruences, we also raise a conjecture on 7 new series for π^2, π^{-2} and the constant $K:=\sum_{k=1}^{\infty}(\frac{k}{3})/k^2$ (with $(-)$ the Jacobi symbol), two of which are

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2}\quad\text{and}\quad\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^{k-1}}{k^3\binom{2k}{k}^2\binom{3k}{k}}=K.$$

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1. Introduction

Let p be an odd prime. Clearly

$$\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\equiv 0 \pmod p \quad \text{for every } k=\frac{p+1}{2},\ldots,p-1.$$

After a series of work on combinatorial congruences involving central binomial coefficients (cf. [PS], [ST1] and [ST2]), Z. W. Sun [S10b] determined $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k$ modulo p^2 for any integer $m\not\equiv 0 \pmod p$. In particular, he showed that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}\equiv(-1)^{(p-1)/2} \pmod{p^2}$$

and conjectured that there are no odd composite numbers $n>1$ satisfying the congruence $\sum_{k=0}^{n-1}\binom{2k}{k}/2^k\equiv(-1)^{(n-1)/2} \pmod{n^2}$. He also searched those *exceptional primes* p such that $\sum_{k=0}^{p-1}\binom{2k}{k}/2^k\equiv(-1)^{(p-1)/2} \pmod{p^3}$ and only found two such primes: 149 and 241.

Let p be an odd prime. A p-adic congruence is said to be a *super congruence* if it happens to hold modulo some higher power of p. In 2003 Rodriguez-Villegas [RV] conjectured 22 super congruences via his analysis of the p-adic analogues of Gaussian hypergeometric series and the Calabi-Yau manifolds. The most elegant one of the 22 super congruences is as follows:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^{(p-1)/2} \pmod{p^2},$$

which was later proved by E. Mortenson [M03a] via the p-adic Γ-function and the Gross-Koblitz formula. See also K. Ono's book [O] and the papers [AO], [K], [MO], [M03b], [M05], [M08], [Mc] and [OS] for such advanced approach to super congruences. Recently the author's twin brother Z. H. Sun, as well as R. Tauraso [T], gave a simple proof of the last super congruence, and Z. H. Sun also proved the author's conjectural congruence

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{16^k}\equiv\frac{(-1)^{(p+1)/2}}{4} \pmod{p^2}$$

via a combinatorial identity. Note that Stirling's formula $n!\sim\sqrt{2\pi n}(n/e)^n$ implies that

$$\lim_{k\to+\infty}\frac{k\binom{2k}{k}^2}{16^k}=\frac{1}{\pi}.$$

Surprisingly, we find that the above topics are related to Euler numbers.

Recall that Euler numbers E_n ($n \in \mathbb{N} = \{0, 1, 2, \ldots\}$) are integers defined by

$$E_0 = 1, \text{ and } \sum_{\substack{k=0\\ 2\mid k}}^{n} \binom{n}{k} E_{n-k} = 0 \quad \text{for } n \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}.$$

It is well known that $E_{2n+1} = 0$ for all $n \in \mathbb{N}$ and

$$\sec x = \sum_{n=0}^{\infty} (-1)^n E_{2n} \frac{x^{2n}}{(2n)!} \ \text{ for } |x| < \frac{\pi}{2}.$$

Now we state the main results and key conjectures in this paper.

Theorem 1.1. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}}{2^k} \equiv (-1)^{(p-1)/2} - p^2 E_{p-3} \pmod{p^3}. \tag{1.1}$$

If $p > 3$, then

$$\sum_{k=1}^{(p-1)/2} \frac{\binom{2k}{k}}{k} \equiv (-1)^{(p+1)/2} \frac{8}{3} p E_{p-3} \pmod{p^2} \tag{1.2}$$

and

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2} \frac{4}{3} E_{p-3} \pmod{p}. \tag{1.3}$$

We also have

$$\sum_{k=1}^{(p-1)/2} \frac{4^k}{k^2 \binom{2k}{k}} \equiv (-1)^{(p-1)/2}\, 4E_{p-3} \pmod{p} \tag{1.4}$$

and

$$\sum_{p/2<k<p} \frac{\binom{2k}{k}}{k4^k} \equiv (-1)^{(p-1)/2}\, 2pE_{p-3} \pmod{p^2}. \tag{1.5}$$

Remark 1.1. By (1.1) those exceptional primes are just those odd primes p with $p \mid E_{p-3}$; the two exceptional primes 149 and 241 offer the main clue to our discovery of (1.2)-(1.5). Also, Sun and Tauraso [ST1] showed that

$$\sum_{k=1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv \frac{8}{9} p^2 B_{p-3} \pmod{p^3}$$

for any prime $p>3$, where $B_0,B_1,B_2,\ldots$ are Bernoulli numbers. It is remarkable that

$$\sum_{k=1}^\infty\frac{1}{k^2\binom{2k}{k}}=\frac{\pi^2}{18}\ \ \text{and}\ \ \sum_{k=1}^\infty\frac{4^k}{k^2\binom{2k}{k}}=\frac{\pi^2}{2}$$

(see [Po, (3)] and problem 44b of [St, Chapter 1] for the first series, and [Ma] and [Sp] for the second series), which were even known in the nineteenth century. Tauraso ([T1],[T2]) showed that

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k4^k}\equiv-\sum_{k=1}^{(p-1)/2}\frac{1}{k}\ (\mathrm{mod}\ p^3)\ \ \text{and}\ \ \sum_{k=1}^{p-1}\frac{1}{k^2\binom{2k}{k}}\equiv\frac{1}{3p}\sum_{k=1}^{p-1}\frac{1}{k}\ (\mathrm{mod}\ p^3)$$

for any prime $p>5$.

Recall that harmonic numbers are those integers

$$H_n:=\sum_{0<k\leqslant n}\frac{1}{k}\quad(n=0,1,2,\ldots).$$

It is known (cf. [S1]) that

$$\frac{H_{p-1}}{p^2}\equiv-\frac{B_{p-3}}{3}\ (\mathrm{mod}\ p)\ \ \text{and}\ \ \frac{5}{p^2}\sum_{k=1}^{p-1}\frac{1}{k^3}\equiv-6B_{p-5}\ (\mathrm{mod}\ p)$$

for any prime $p>3$.

Now we present our first conjecture.

Conjecture 1.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=1}^{p-1}\frac{4^k}{k^2\binom{2k}{k}}+\frac{4q_p(2)}{p}\equiv-2q_p^2(2)+pB_{p-3}\ (\mathrm{mod}\ p^2)$$

and

$$p\sum_{k=1}^{p-1}\frac{2^k}{k\binom{2k}{k}}\equiv\left(\frac{-1}{p}\right)-1-p\,q_p(2)+p^2E_{p-3}\ (\mathrm{mod}\ p^3),$$

where $(-)$ denotes the Jacobi symbol and $q_p(2)$ stands for the Fermat quotient $(2^{p-1}-1)/p$. Also,

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^3}\equiv\frac{2}{3}B_{p-3}\ (\mathrm{mod}\ p),$$

and furthermore

$$\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k^3}\equiv-\frac{2}{p^2}H_{p-1}-\frac{13}{27}\sum_{k=1}^{p-1}\frac{1}{k^3}\pmod{p^4}\quad \text{if } p>7.$$

When $p>5$ we have

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^3\binom{2k}{k}}\equiv-2B_{p-3}\pmod{p},$$

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\binom{2k}{k}\equiv\frac{56}{15}pB_{p-3}\pmod{p^2},$$

$$\sum_{k=1}^{p-1}\frac{1}{k^4\binom{2k}{k}}-\frac{H_{p-1}}{p^3}\equiv-\frac{7}{45}pB_{p-5}\pmod{p^2}.$$

Remark 1.2. It is known that

$$\sum_{k=1}^{\infty}\frac{2^k}{k\binom{2k}{k}}=\frac{\pi}{2},\quad \sum_{k=1}^{\infty}\frac{(-1)^k}{k^3\binom{2k}{k}}=-\frac{2}{5}\zeta(3)\quad\text{and}\quad \sum_{k=1}^{\infty}\frac{1}{k^4\binom{2k}{k}}=\frac{17}{36}\zeta(4).$$

Tauraso [T2] determined $\sum_{k=1}^{p-1}(-1)^k/(k^3\binom{2k}{k})$ and $\sum_{k=1}^{p-1}(-1)^k\binom{2k}{k}/k^2$ modulo p^2 (for any prime $p>5$) in terms of H_{p-1}.

Theorem 1.2. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{8^k}\equiv\left(\frac{2}{p}\right)+\left(\frac{-2}{p}\right)\frac{p^2}{4}E_{p-3}\pmod{p^3};\tag{1.6}$$

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv(-1)^{(p-1)/2}+p^2E_{p-3}\pmod{p^3},\tag{1.7}$$

$$\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{16^k}\equiv\frac{(-1)^{(p+1)/2}}{4}+\frac{p^2}{4}(1-E_{p-3})\pmod{p^3};\tag{1.8}$$

$$\sum_{p/2<k<p}\frac{\binom{2k}{k}^2}{16^k}\equiv-2p^2E_{p-3}\pmod{p^3},\tag{1.9}$$

$$\sum_{p/2<k<p}\frac{k\binom{2k}{k}^2}{16^k}\equiv\frac{p^2}{2}E_{p-3}\pmod{p^3}.\tag{1.10}$$

Furthermore,

$$(-1)^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv 1-\frac{3}{8}p\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\pmod{p^4} \tag{1.11}$$

and

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{16^k}\equiv&\frac{(-1)^{(p+1)/2}}{4}+\frac{p^2}{4}(2^p-1)\\&+(-1)^{(p-1)/2}\frac{3}{32}p\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\pmod{p^4}.\end{aligned} \tag{1.12}$$

Remark 1.3. The reason why we don't include (1.6) in Theorem 1.1 is that its proof is similar to that of (1.11) and (1.12). For any prime $p>3$, R. Osburn and C. Schneider [OS] used Jacobi sums and the p-adic Γ-function to prove that

$$(-1)^{(p-1)/2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{16^k}\equiv 1-\frac{3}{8}p\sum_{k=1}^{(p-1)/2}\frac{\binom{2k}{k}}{k}\pmod{p^3}.$$

The author observed that a combination of (1.11) and (1.12) yields that

$$\sum_{k=0}^{(p-1)/2}\frac{4k+1}{16^k}\binom{2k}{k}^2\equiv p^2(2^p-1)\pmod{p^4}$$

for any prime $p>3$. After reading this, Tauraso noted the following identity

$$\sum_{k=0}^{n}\frac{4k+1}{16^k}\binom{2k}{k}^2=\frac{(n+1)^2}{16^n}\binom{2n+1}{n}^2=\frac{(2n+1)^2}{16^n}\binom{2n}{n}^2,$$

which can be easily proved by induction. This identity implies the author's following observation:

$$\sum_{p/2<k<p}\frac{4k+1}{16^k}\binom{2k}{k}^2\equiv 6p^2(1-2^{p-1})\pmod{p^4}$$

for each odd prime p.

Conjecture 1.2. *Let p be an odd prime and let $a \in \mathbb{Z}^+$. If $p \equiv 1 \pmod 4$ or $a > 1$, then*

$$\sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}}{(-4)^k} \equiv \left(\frac{2}{p^a}\right) \pmod{p^2} \quad \text{and} \quad \sum_{k=0}^{\lfloor \frac{3}{4}p^a \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p^a}\right) \pmod{p^3}. \tag{1.13}$$

If $p > 3$, and $p \equiv 1, 3 \pmod 8$ or $a > 1$, then

$$\sum_{k=0}^{\lfloor \frac{r}{8}p^a \rfloor} \frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p^a}\right) \pmod{p^3} \quad \text{for } r = 5, 7. \tag{1.14}$$

Here is our third theorem.

Theorem 1.3. *Let p be a prime and let $a \in \mathbb{Z}^+$. Then*

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} (21k+8)\binom{2k}{k}^3 \equiv 8 + 16p^3 B_{p-3} \pmod{p^4}, \tag{1.15}$$

where B_{-1} is regarded as zero.

Remark 1.4. In [S11a] the author conjectured that for any odd prime p we have

$$\sum_{k=0}^{p-1} \binom{2k}{k}^3 \equiv \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } (\frac{p}{7}) = 1 \ \& \ p = x^2 + 7y^2 \ (x, y \in \mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{p}{7}) = -1, \text{ i.e., } p \equiv 3, 5, 6 \pmod 7. \end{cases}$$

Let p be a prime with $(\frac{p}{7}) = -1$. We also conjecture that $\frac{1}{n}\sum_{k=0}^{n-1}\binom{2k}{k}^3$ is a p-adic integer for any $n \in \mathbb{Z}^+$, and that

$$\sum_{k=0}^{p^a-1} \binom{2k}{k}^3 \equiv \begin{cases} 0 \pmod{p^{a+1}} & \text{if } a \in \{1, 3, 5, \dots\}, \\ p^a \pmod{p^{a+3-\delta_{p,3}}} & \text{if } a \in \{2, 4, 6, \dots\}, \end{cases}$$

where the Kronecker symbol $\delta_{p,3}$ takes 1 or 0 according as $p = 3$ or not.

In a previous version of this paper, the author conjectured that for any positive integer n the arithmetic mean

$$s_n := \frac{1}{n} \sum_{k=0}^{n-1} (21k+8)\binom{2k}{k}^3 \tag{1.16}$$

is always an integer divisible by $4\binom{2n}{n}$, and observed the recursion

$$n^3(n+1)s_{n+1} = n^4 s_n + 8(2n-1)^3(21n+8)\binom{2(n-1)}{n-1}^3 \quad (n = 1, 2, 3, \dots).$$

On Feb. 11, 2010, Kasper Andersen noted that this recurrence relation yields the following recursion for $t_n := s_n/(4\binom{2n}{n})$:

$$(4n+2)t_{n+1} - nt_n = (21n+8)\binom{2n-1}{n}^2 \ (n=1,2,3,\ldots).$$

Then Andersen used Zeilberger's algorithm (cf. [PWZ]) to find that

$$r_n := \sum_{k=0}^{n-1}\binom{n+k-1}{k}^2 \ (n=1,2,3,\ldots) \tag{1.17}$$

satisfies the same recursion and hence he obtained that $t_n = r_n \in \mathbb{Z}$ since $t_1 = r_1$. Thanks to Andersen's discovery, we are now able to prove Theorem 1.3 which was an earlier conjecture of the author.

We guess that any integer $n > 1$ satisfying $s_n \equiv 8 \pmod{n^3}$ must be a prime; this has been verified for $n \leqslant 10^4$. It seems that $t_n \not\equiv 3 \pmod 4$, and t_n is composite for all $n = 3, 4, \ldots$. It is interesting to compare s_n and t_n with Apéry numbers (cf. [Po]).

Conjecture 1.3. *If p is a prime and a is a positive integer with $p^a \equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{\lfloor\frac{2}{3}p^a\rfloor}(21k+8)\binom{2k}{k}^3 \equiv 8p^a \pmod{p^{a+5+(-1)^p}}. \tag{1.18}$$

Also, for each prime $p > 5$ we have

$$\sum_{k=1}^{p-1}\frac{21k-8}{k^3\binom{2k}{k}^3} + \frac{p-1}{p^3} \equiv \frac{H_{p-1}}{p^2}(15p-6) + \frac{12}{5}p^2B_{p-5} \pmod{p^3}. \tag{1.19}$$

It is interesting to compare Theorem 1.3 and Conjecture 1.3 with the following elegant identity

$$\sum_{k=1}^{\infty}\frac{21k-8}{k^3\binom{2k}{k}^3} = \zeta(2) = \frac{\pi^2}{6}$$

obtained by D. Zeilberger [Z] via the WZ method. In the same spirit, we formulate the following conjecture inspired by our observations of some congruences (see Conjectures 5.3-5.6, Remark 5.2 and Conj. 5.15(i) in Section 5).

Conjecture 1.4. *We have*

$$\sum_{k=1}^{\infty}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=\frac{\pi^2}{2},\ \sum_{k=1}^{\infty}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=8\pi^2,\ \sum_{k=1}^{\infty}\frac{(35k-8)81^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=12\pi^2. \tag{1.20}$$

Also,

$$\sum_{k=1}^{\infty}\frac{(15k-4)(-27)^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}=-27K \quad \text{and} \quad \sum_{k=1}^{\infty}\frac{(5k-1)(-144)^k}{k^3\binom{2k}{k}^2\binom{4k}{2k}}=-\frac{45}{2}K, \tag{1.21}$$

where

$$K:=L\left(2,\left(\frac{\cdot}{3}\right)\right)=\sum_{k=1}^{\infty}\frac{(\frac{k}{3})}{k^2}=0.78130241289648629686718742962 4\ldots.$$

Moreover,

$$\sum_{n=0}^{\infty}\frac{18n^2+7n+1}{(-128)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2=\frac{4\sqrt{2}}{\pi^2} \tag{1.22}$$

and

$$\sum_{n=0}^{\infty}\frac{40n^2+26n+5}{(-256)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}=\frac{24}{\pi^2}. \tag{1.23}$$

One can easily check the identities in Conjecture 1.4 numerically. Let us take the first identity in (1.20) as an example. The series converges rapidly since

$$\binom{2k}{k}^2\binom{3k}{k}\sim\frac{\sqrt{3}}{2}\cdot\frac{108^k}{(k\pi)^{1.5}}\quad(k\to+\infty)$$

by Stirling's formula. Via `Mathematica` we find that

$$\left|\frac{2}{\pi^2}\sum_{k=1}^{200}\frac{(10k-3)8^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}-1\right|<\frac{1}{10^{227}}.$$

This provides a powerful evidence to support the first identity in (1.20).

We will show Theorems 1.1–1.3 in Sections 2-4 respectively; our new approach to super congruences is of combinatorial nature. In Section 5 we will raise many new conjectures for further research.

2. Proof of Theorem 1.1

Proof of (1.1). By [ST1, (2.1)], we have

$$\sum_{k=0}^{p-1}\binom{2k}{k}2^{p-1-k}=\sum_{k=0}^{p-1}\binom{2p}{k}u_{p-k},$$

where $u_0=0,\ u_1=1$ and $u_{n+1}=-u_{n-1}$ for $n=1,2,3,\ldots$. Clearly $u_{2n}=0$ and $u_{2n+1}=(-1)^n$ for all $n\in\mathbb{N}$. Thus

$$\sum_{k=0}^{p-1}\binom{2k}{k}2^{p-1-k}=\sum_{k=0}^{(p-1)/2}\binom{2p}{2k}(-1)^{(p-2k-1)/2}. \tag{2.1}$$

For $k=1,\ldots,(p-1)/2$, we have

$$\begin{aligned}\binom{2p}{2k}=&\frac{2p}{2k}\binom{2p-1}{2k-1}=\frac{p}{k}\prod_{j=1}^{2k-1}\frac{2p-j}{j}=-\frac{p}{k}\prod_{j=1}^{2k-1}\left(1-\frac{2p}{j}\right)\\ \equiv&-\frac{p}{k}(1-2pH_{2k-1})=\frac{p}{k}(1-2(1-pH_{2k-1}))\\ \equiv&\frac{p}{k}\left(1+2\binom{p-1}{2k-1}\right)=4\binom{p}{2k}+\frac{p}{k}\pmod{p^3}.\end{aligned}$$

Thus

$$\begin{aligned}&(-1)^{(p-1)/2}\sum_{k=0}^{p-1}\binom{2k}{k}2^{p-1-k}-1\\ \equiv&\sum_{k=1}^{(p-1)/2}(-1)^k\left(4\binom{p}{2k}+\frac{p}{k}\right)=p\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k}+4\sum_{\substack{k=1\\ 2\mid k}}^{p-1}\binom{p}{k}(-1)^{k/2}\pmod{p^3}.\end{aligned}$$

By Lehmer [L],

$$H_{(p-1)/2}\equiv-2q_p(2)+p\,q_p^2(2)\pmod{p^2}. \tag{2.2}$$

In view of [S2, Corollary 3.3] we also have

$$H_{\lfloor p/4\rfloor}\equiv-3q_p(2)+\frac{3}{2}p\,q_p^2(2)-(-1)^{(p-1)/2}pE_{p-3}\pmod{p^2}. \tag{2.3}$$

Therefore

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k}=&\sum_{k=1}^{(p-1)/2}\frac{1+(-1)^k}{k}-\sum_{k=1}^{(p-1)/2}\frac{1}{k}=H_{\lfloor p/4\rfloor}-H_{(p-1)/2}\\ \equiv&-q_p(2)+\frac{p}{2}q_p^2(2)+(-1)^{(p+1)/2}pE_{p-3}\pmod{p^2}.\end{aligned}$$

Note also that

$$\sum_{\substack{k=1\\2|k}}^{p-1}\binom{p}{k}(-1)^{k/2}=\left(\frac{2}{p}\right)2^{(p-1)/2}-1$$

by [S02, (3.2)]. Combining the above we obtain

$$\begin{aligned}&\left(\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}}{2^k}-1\right)2^{p-1}+2^{p-1}-1\\ \equiv&4\left(\left(\frac{2}{p}\right)2^{(p-1)/2}-1\right)-p\,q_p(2)+\frac{p^2}{2}q_p^2(2)+(-1)^{(p+1)/2}p^2E_{p-3}\pmod{p^3}.\end{aligned}$$

Observe that

$$\begin{aligned}&2\left(\left(\frac{2}{p}\right)2^{(p-1)/2}-1\right)-p\,q_p(2)\\ =&2\left(\frac{2}{p}\right)2^{(p-1)/2}-2^{p-1}-1=-\left(\left(\frac{2}{p}\right)2^{(p-1)/2}-1\right)^2.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{(\frac{-1}{p})\sum_{k=0}^{p-1}\binom{2k}{k}/2^k-1}{p^2}\\ \equiv&-2\left(\frac{(\frac{2}{p})2^{(p-1)/2}-1}{p}\right)^2+\frac{q_p^2(2)}{2}+(-1)^{(p+1)/2}E_{p-3}\pmod{p}.\end{aligned}$$

Since

$$q_p(2)=\frac{(\frac{2}{p})2^{(p-1)/2}-1}{p}\left(\left(\frac{2}{p}\right)2^{(p-1)/2}+1\right)\equiv 2\times\frac{(\frac{2}{p})2^{(p-1)/2}-1}{p}\pmod{p},$$

we finally obtain (1.1). □

Lemma 2.1. *Let p be an odd prime. Then, for any $k=1,\ldots,p-1$ we have*

$$k\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv(-1)^{\lfloor 2k/p\rfloor-1}2p\pmod{p^2}.\tag{2.4}$$

Proof. For $k=1,\ldots,(p-1)/2$, if

$$(p-k)\binom{2(p-k)}{p-k}\binom{2k}{k}\equiv(-1)^{\lfloor 2(p-k)/p\rfloor-1}2p=2p\pmod{p^2}$$

then
$$k\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv -2p=(-1)^{\lfloor 2k/p\rfloor-1}2p \pmod{p^2}$$
since $\binom{2(p-k)}{p-k}\equiv 0 \pmod p$. So it suffices to show (2.4) for any $k=(p+1)/2,\dots,p-1$.

Let $k\in\{(p+1)/2,\dots,p-1\}$. Then

$$\begin{aligned}\frac{1}{p}\binom{2k}{k}=&\frac{1}{p}\times\frac{(2k)!}{(k!)^2}=\frac{1}{p}\times\frac{p!(p+1)\cdots(p+(2k-p))}{((p-1)!/\prod_{j=1}^{p-1-k}(p-j))^2}\\ =&\frac{1}{(p-1)!}\prod_{i=1}^{2k-p}(p+i)\times\prod_{j=1}^{p-1-k}(p-j)^2\\ \equiv&\frac{(2k-p)!}{(p-1)!}((p-1-k)!)^2=\frac{((p-1-k)!)^2}{\prod_{j=1}^{2(p-k)-1}(p-j)}\\ \equiv&-\frac{((p-1-k)!)^2}{(2(p-k)-1)!}=-\frac{2}{(p-k)\binom{2(p-k)}{p-k}}\equiv\frac{2}{k\binom{2(p-k)}{p-k}} \pmod p\end{aligned}$$

and hence
$$k\binom{2k}{k}\binom{2(p-k)}{p-k}\equiv 2p \pmod{p^2}$$
as desired. □

Remark 2.1. [T2] contains certain technique similar to Lemma 2.1.

Lemma 2.2. *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}=\frac{n+1}{3}\binom{2n+1}{n}\sum_{k=1}^{n}\frac{1}{k^2\binom{n}{k}^2} \tag{2.5}$$

and

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^2\binom{n}{k}\binom{n+k}{k}}=(-1)^{n-1}\left(3\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}}+2\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right). \tag{2.6}$$

Remark 2.2. These two identities are known results. (2.5) is due to T. B. Staver [Sta] (see also (5.2) of [Go, p. 50]), and (2.6) was discovered by Apéry (see [Ap] and [Po]) during his study of the irrationality of $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$.

Lemma 2.3. *We have the new combinatorial identity*

$$\sum_{k=1}^{n}\frac{1}{k^2\binom{n+k}{k}}=3\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}}-\sum_{k=1}^{n}\frac{1}{k^2}. \tag{2.7}$$

Proof. Observe that

$$\begin{aligned}&\sum_{k=1}^{n}\frac{1}{k^2\binom{n+k}{k}}-\sum_{k=1}^{n}\frac{1}{k^2\binom{n+1+k}{k}}\\ =&\sum_{k=1}^{n}\frac{\binom{n+k+1}{k}-\binom{n+k}{k}}{k^2\binom{n+k}{k}\binom{n+k+1}{k}}=\sum_{k=1}^{n}\frac{\binom{n+k}{k-1}}{k^2\binom{n+k}{k}\binom{n+k+1}{k}}\\ =&\sum_{k=1}^{n}\frac{n!(k-1)!}{(n+k+1)!}=\sum_{j=0}^{n-1}\frac{n!j!}{(n+2+j)!}=\frac{1}{(n+1)(n+2)}\sum_{k=0}^{n-1}\frac{1}{\binom{n+2+k}{n+2}}.\end{aligned}$$

By (2.26) of [Go, p. 21],

$$\sum_{k=0}^{m}\frac{1}{\binom{x+k}{l}}=\frac{l}{l-1}\left(\frac{1}{\binom{x-1}{l-1}}-\frac{1}{\binom{x+m}{l-1}}\right)$$

for any $l\in\mathbb{Z}^+$. So we have

$$\sum_{k=1}^{n}\frac{1}{k^2\binom{n+k}{k}}-\sum_{k=1}^{n}\frac{1}{k^2\binom{n+1+k}{k}}=\frac{1}{(n+1)(n+2)}\times\frac{n+2}{n+1}\left(1-\frac{1}{\binom{(n+2)+n-1}{n+1}}\right)$$

and hence

$$\begin{aligned}\sum_{k=1}^{n+1}\frac{1}{k^2\binom{n+1+k}{k}}-\sum_{k=1}^{n}\frac{1}{k^2\binom{n+k}{k}}=&\frac{1}{(n+1)^2\binom{2n+2}{n+1}}-\frac{1}{(n+1)^2}\left(1-\frac{1}{\binom{2n+1}{n}}\right)\\ =&\frac{3}{(n+1)^2\binom{2n+2}{n+1}}-\frac{1}{(n+1)^2}.\end{aligned}$$

Therefore (2.7) follows by induction. □

The following lemma is essentially known, but we will include a simple proof.

Lemma 2.4. *For any prime $p>3$ we have*

$$\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv 0 \pmod p \ \textit{and}\ \sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\equiv(-1)^{(p-1)/2}\,2E_{p-3} \pmod p.$$

Proof. Since $\sum_{j=1}^{p-1} 1/(2j)^2 \equiv \sum_{k=1}^{p-1} 1/k^2 \pmod p$, we have the well-known congruence $\sum_{k=1}^{p-1} 1/k^2 \equiv 0 \pmod p$. Thus

$$\sum_{k=1}^{(p-1)/2} \frac{1}{k^2} \equiv \frac{1}{2} \sum_{k=1}^{(p-1)/2} \left(\frac{1}{k^2} + \frac{1}{(p-k)^2} \right) = \frac{1}{2} \sum_{k=1}^{p-1} \frac{1}{k^2} \equiv 0 \pmod p.$$

By Lehmer [L, (20)],

$$\sum_{k=1}^{\lfloor p/4 \rfloor} \frac{1}{k^2} \equiv (-1)^{(p-1)/2}\, 4E_{p-3} \pmod p.$$

Therefore

$$\sum_{k=1}^{(p-1)/2} \frac{1+(-1)^k}{k^2} = \sum_{j=1}^{\lfloor p/4 \rfloor} \frac{2}{(2j)^2} \equiv (-1)^{(p-1)/2}\, 2E_{p-3} \pmod p$$

and hence the second congruence in Lemma 2.4 also holds. □

Proof of (1.2)-(1.5). Note that (1.4) and (1.5) hold trivially when $p = 3$. Below we assume that $p = 2n + 1 > 3$.

With the help of Lemma 2.1, we have

$$\begin{aligned} \sum_{k=n+1}^{p-1} \frac{\binom{2k}{k}}{k} &= \sum_{k=n+1}^{p-1} \frac{k\binom{2k}{k}}{k^2} \\ &\equiv \sum_{k=n+1}^{p-1} \frac{2p}{k^2 \binom{2(p-k)}{p-k}} = \sum_{j=1}^{n} \frac{2p}{(p-j)^2 \binom{2j}{j}} \equiv \sum_{k=1}^{n} \frac{2p}{k^2 \binom{2k}{k}} \pmod{p^2}. \end{aligned}$$

As $\sum_{k=0}^{p-1} \binom{2k}{k}/k \equiv 0 \pmod{p^2}$ by [ST1], we obtain that

$$\sum_{k=1}^{n} \frac{\binom{2k}{k}}{k} \equiv - \sum_{k=n+1}^{p-1} \frac{\binom{2k}{k}}{k} \equiv -2p \sum_{k=1}^{n} \frac{1}{k^2 \binom{2k}{k}} \pmod{p^2}. \tag{2.8}$$

In view of (2.5),

$$\sum_{k=1}^{n} \frac{\binom{2k}{k}}{k} = \frac{2n+1}{3} \binom{2n}{n} \sum_{k=1}^{n} \frac{1}{k^2 \binom{n}{k}^2} \equiv \frac{p}{3}(-1)^n \sum_{k=1}^{n} \frac{1}{k^2 \binom{n}{k}^2} \pmod{p^2}. \tag{2.9}$$

Since

$$\binom{n+k}{k}(-1)^k = \binom{-n-1}{k} \equiv \binom{p-n-1}{k} = \binom{n}{k} \pmod p$$

for every $k = 1, \dots, n$, (2.6) yields that

$$\sum_{k=1}^{n} \frac{1}{k^2\binom{n}{k}^2} \equiv (-1)^{n-1}\left(3\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}} + 2\sum_{k=1}^{n}\frac{(-1)^k}{k^2}\right) \pmod p. \tag{2.10}$$

Combining (2.8)–(2.10) we get

$$3\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k} + 2p\sum_{k=1}^{n}\frac{(-1)^k}{k^2} \equiv -3p\sum_{k=1}^{n}\frac{1}{k^2\binom{2k}{k}} \equiv \frac{3}{2}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k} \pmod{p^2}. \tag{2.11}$$

In view of Lemma 2.4,

$$\sum_{k=1}^{n}\frac{(-1)^k}{k^2} \equiv (-1)^n 2E_{p-3} \pmod p.$$

So, we have (1.2) and (1.3) by (2.11) and (2.8).

By Lemma 2.1, (1.4) and (1.5) are equivalent. Since $\sum_{k=1}^{n} 1/k^2 \equiv 0 \pmod p$ (by Lemma 2.4) and

$$\binom{n+k}{k} \equiv \binom{k-1/2}{k} = \frac{\binom{2k}{k}}{4^k} \qquad \text{for every } k = 1, \dots, n,$$

we obtain (1.4) from (2.7) and (1.3). □

3. Proof of Theorem 1.2

Lemma 3.1. *Let $p = 2n+1$ be an odd prime. For $k = 0, \dots, n$ we have*

$$\binom{n+k}{2k} - \frac{p\binom{n}{k}}{4^{k+1}}(H_{n+k} - H_{n-k}) \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^3} \tag{3.1}$$

and

$$\binom{n}{k}\binom{n+k}{k}(-1)^k\left(1 - \frac{p}{4}(H_{n+k} - H_{n-k})\right) \equiv \frac{\binom{2k}{k}^2}{16^k} \pmod{p^4}. \tag{3.2}$$

Proof. Both (3.1) and (3.2) hold trivially when $k = 0$. Below we fix $k \in \{1, \dots, n\}$.

As noted by the author's brother Z. H. Sun,

$$\begin{aligned}\binom{n+k}{2k} =& \frac{\prod_{j=1}^{k}(p^2 - (2j-1)^2)}{4^k \times (2k)!}\\ =& \frac{\prod_{j=1}^{k}(-(2j-1)^2)}{4^k \times (2k)!}\prod_{j=1}^{k}\left(1 - \frac{p^2}{(2j-1)^2}\right)\\ \equiv& \frac{\binom{2k}{k}}{(-16)^k}\left(1 - \sum_{j=1}^{k}\frac{p^2}{(2j-1)^2}\right) \pmod{p^4}.\end{aligned}$$

Observe that

$$\begin{aligned}H_{n+k}-H_{n-k}=&H_n+\sum_{j=1}^k\frac{1}{n+j}-H_n+\sum_{j=1}^k\frac{1}{n+1-j}\\=&\sum_{j=1}^k\frac{2n+1}{(n+j)(n+1-j)}=\sum_{j=1}^k\frac{p}{p^2/4-(j-1/2)^2}\\\equiv&-4\sum_{j=1}^k\frac{p}{(2j-1)^2}\pmod{p^3}.\end{aligned}$$

Therefore

$$\binom{n+k}{2k}\equiv\frac{\binom{2k}{k}}{(-16)^k}\left(1+\frac{p}{4}(H_{n+k}-H_{n-k})\right)\pmod{p^4}. \tag{3.3}$$

Note that $p(H_{n+k}-H_{n-k})\equiv 0\pmod{p^2}$ and

$$\binom{n}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod{p}.$$

Thus (3.1) follows from (3.3) immediately.

In light of (3.3), we have

$$\begin{aligned}&\binom{n}{k}\binom{n+k}{k}(-1)^k=\binom{n+k}{2k}\binom{2k}{k}(-1)^k\\\equiv&\frac{\binom{2k}{k}^2}{16^k}\left(1+\frac{p}{4}(H_{n+k}-H_{n-k})\right)\\\equiv&\frac{\binom{2k}{k}^2}{16^k}+\binom{n}{k}\binom{n+k}{k}(-1)^k\frac{p}{4}(H_{n+k}-H_{n-k})\pmod{p^4}.\end{aligned}$$

(Recall that $p(H_{n+k}-H_{n-k})\equiv 0\pmod{p^2}$.) So (3.2) also holds. □

Lemma 3.2. *For any $n\in\mathbb{Z}^+$ we have*

$$(-1)^n\sum_{k=0}^n\binom{n}{k}(-2)^{n-k}(H_{n+k}-H_{n-k})=\sum_{k=1}^n\frac{(-1)^k}{k}-\frac{H_n}{2}, \tag{3.4}$$

$$(-1)^n\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}(-1)^k(H_{n+k}-H_{n-k})=\frac{3}{2}\sum_{k=1}^n\frac{\binom{2k}{k}}{k} \tag{3.5}$$

and

$$\begin{aligned}(-1)^n\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}(-1)^k k(H_{n+k}-H_{n-k})\\ =&(2n+1)\left(1-\binom{2n}{n}\right)+\frac{3}{2}n(n+1)\sum_{k=1}^n\frac{\binom{2k}{k}}{k}.\end{aligned}\tag{3.6}$$

Proof. Via the software `Sigma` we find the identities

$$\sum_{k=0}^n\binom{n}{k}(-2)^{n-k}H_{n+k}=(-1)^n\frac{H_n}{2},\tag{3.7}$$

$$\sum_{k=0}^n\binom{n}{k}(-2)^{n-k}H_{n-k}=(-1)^nH_n-(-1)^n\sum_{k=1}^n\frac{(-1)^k}{k},\tag{3.8}$$

$$\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}(-1)^kH_{n-k}=2(-1)^nH_n-\frac{3}{2}(-1)^n\sum_{k=1}^n\frac{\binom{2k}{k}}{k}.\tag{3.9}$$

These identities can be easily proved by the WZ method (see, e.g., [PWZ]). (The reader may consult [OS] to see how to produce such identities.) Also, it is known that (cf. [OS] and [Pr])

$$\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}(-1)^kH_{n+k}=(-1)^n2H_n.$$

By [OS, (36) and (37)], we have

$$(-1)^n\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}(-1)^kkH_{n+k}=2n(n+1)H_n-n^2$$

and

$$\begin{aligned}(-1)^n\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}(-1)^kkH_{n-k}\\ =&(2n+1)\binom{2n}{n}-(n+1)^2+2n(n+1)H_n-\frac{3}{2}n(n+1)\sum_{k=1}^n\frac{\binom{2k}{k}}{k}.\end{aligned}$$

In view of the above six identities we immediately obtain the desired (3.4)-(3.6). □

Remark 3.1. S. Ahlgren and Ono [AO] employed the identity

$$\sum_{k=1}^n\binom{n}{k}^2\binom{n+k}{k}^2(1+2k(H_{n+k}+H_{n-k})-4kH_k)=0$$

to prove a super congruence conjectured by F. Beukers [Be].

Proof of Theorem 1.2. Set $n=(p-1)/2$. In light of (3.1) and (3.4), we have

$$\begin{aligned}&\sum_{k=0}^{n}\frac{\binom{2k}{k}}{8^k}-\sum_{k=0}^{n}(-2)^k\binom{n+k}{2k}\\ \equiv&-\frac{p}{4}\sum_{k=0}^{n}\frac{\binom{n}{k}}{(-2)^k}(H_{n+k}-H_{n-k})\\ =&\frac{p}{4\times 2^n}\left(\frac{H_n}{2}-\sum_{k=1}^{n}\frac{(-1)^k}{k}\right)\pmod{p^3}.\end{aligned}$$

By a known identity (cf. (1.62) of [Go, p.8]),

$$\sum_{k=0}^{n}(-1)^k\binom{2n-k}{k}(2\cos x)^{2(n-k)}=\frac{\sin((2n+1)x)}{\sin x}$$

and hence

$$\begin{aligned}\sum_{k=0}^{n}(-2)^k\binom{n+k}{2k}=&\sum_{k=0}^{n}(-1)^{n-k}\binom{2n-k}{k}\left(2\cos\frac{\pi}{4}\right)^{2(n-k)}\\ =&(-1)^n\frac{\sin((2n+1)\pi/4)}{\sin(\pi/4)}=\left(\frac{2}{2n+1}\right)=\left(\frac{2}{p}\right).\end{aligned}$$

In view of (2.2) and (2.3), we also have

$$\begin{aligned}\frac{H_n}{2}-\sum_{k=1}^{n}\frac{(-1)^k}{k}=&\frac{3}{2}H_n-\sum_{k=1}^{n}\frac{1+(-1)^k}{k}\\ =&\frac{3}{2}H_{(p-1)/2}-H_{\lfloor p/4\rfloor}\\ \equiv&(-1)^{(p-1)/2}pE_{p-3}\pmod{p^2}.\end{aligned}$$

Therefore

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}}{8^k}-\left(\frac{2}{p}\right)\equiv\frac{p}{4\times 2^n}\left(\frac{-1}{p}\right)pE_{p-3}\equiv\frac{p^2}{4}\left(\frac{-2}{p}\right)E_{p-3}\pmod{p^3}.$$

This proves (1.6).

By (3.2) and (3.5),

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{16^k}-\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k\equiv-\frac{p}{4}(-1)^n\frac{3}{2}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\pmod{p^4}.$$

With the help of the Chu-Vandermonde identity (cf. [GKP, p. 169]),

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^k=\sum_{k=1}^{n}\binom{n}{n-k}\binom{-n-1}{k}=\binom{-1}{n}=(-1)^n.$$

Thus

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{16^k}-(-1)^n\equiv-(-1)^n\frac{3}{8}p\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\pmod{p^4}$$

which gives (1.11).

By the Chu-Vandermonde identity we also have

$$\begin{aligned}&\sum_{k=0}^{n}k\binom{n}{k}\binom{n+k}{k}(-1)^k\\=&n\sum_{k=1}^{n}\binom{n-1}{k-1}\binom{-n-1}{k}=n\sum_{k=0}^{n}\binom{n-1}{n-k}\binom{-n-1}{k}\\=&n\binom{-2}{n}=(-1)^n n(n+1).\end{aligned}$$

Recall Morley's congruence (cf. [Mo] and [P])

$$\binom{2n}{n}\equiv(-1)^n4^{p-1}\pmod{p^3}.$$

Note also that $n(n+1)=(p^2-1)/4$ and

$$\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\equiv\sum_{k=1}^{p-1}\frac{\binom{2k}{k}}{k}\equiv 0\pmod{p}.$$

Therefore, by (3.2) and (3.6) we have

$$\begin{aligned}&\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2}{16^k}-(-1)^n\frac{p^2-1}{4}\\\equiv&-\frac{p}{4}\left(p((-1)^n-4^{p-1})+(-1)^n\frac{3}{2}\cdot\frac{p^2-1}{4}\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\right)\\\equiv&\frac{p^2}{4}(4^{p-1}-(-1)^n)+\frac{3}{32}\left(\frac{-1}{p}\right)p\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\pmod{p^4}\end{aligned}$$

and hence

$$\begin{aligned}&\sum_{k=0}^{n}\frac{k\binom{2k}{k}^2}{16^k}+\frac{(-1)^n}{4}\\ \equiv&\frac{p^2}{4}(1+(2^{p-1}-1))^2+\frac{3}{32}\left(\frac{-1}{p}\right)p\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\\ \equiv&\frac{p^2}{4}(2^p-1)+\frac{3}{32}\left(\frac{-1}{p}\right)p\sum_{k=1}^{n}\frac{\binom{2k}{k}}{k}\ (\mathrm{mod}\ p^4).\end{aligned}$$

This proves (1.12).

In light of (1.2), clearly (1.7) and (1.8) follow from (1.11) and (1.12) respectively.

Now we prove (1.9). By Lemma 2.1,

$$\begin{aligned}\frac{1}{p^2}\sum_{p/2<k<p}\frac{\binom{2k}{k}^2}{16^k}\equiv&\sum_{p/2<k<p}\frac{4/\binom{2(p-k)}{p-k}^2}{k^2 16^k}=\sum_{k=1}^{n}\frac{4}{(p-k)^2\binom{2k}{k}^2 16^{p-k}}\\ \equiv&\frac{1}{4}\sum_{k=1}^{n}\frac{16^k}{k^2\binom{2k}{k}^2}\equiv\frac{1}{4}\sum_{k=1}^{n}\frac{1}{k^2\binom{n}{k}^2}\ (\mathrm{mod}\ p).\end{aligned}$$

This, together with (2.9) and (1.2), yields (1.9).

By Tauraso's identity mentioned in Remark 1.3,

$$\sum_{p/2<k<p}\frac{4k+1}{16^k}\binom{2k}{k}^2=\frac{p^2}{16^{p-1}}\binom{2p-1}{p-1}^2-\frac{p^2}{4^{p-1}}\binom{p-1}{(p-1)/2}^2\equiv 0\ (\mathrm{mod}\ p^3).$$

So (1.10) follows from (1.9).

The proof of Theorem 1.2 is now complete. □

4. Proof of Theorem 1.3

As we mentioned in the paragraph after Remark 1.4, based on the author's conjecture that $t_n\in\mathbb{Z}$ for all $n\in\mathbb{Z}^+$, Kasper Andersen obtained the following lemma.

Lemma 4.1 (Kasper Andersen)**.** *For any $n\in\mathbb{Z}^+$ the number*

$$t_n:=\frac{1}{4n\binom{2n}{n}}\sum_{k=0}^{n-1}(21k+8)\binom{2k}{k}^3$$

is indeed an integer as conjectured by Z. W. Sun; in fact,

$$t_n=\sum_{k=0}^{n-1}\binom{n+k-1}{k}^2.$$

We also need the following result.

Lemma 4.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=1}^{p-1} \frac{1+2pH_{k-1}}{k^2} \equiv \frac{8}{3} pB_{p-3} \pmod{p^2}.$$

Proof. Note that

$$\sum_{k=1}^{p-1} \frac{1}{(-2k)^3} \equiv \sum_{j=1}^{p-1} \frac{1}{j^3} \pmod{p}$$

and hence $\sum_{k=1}^{p-1} 1/k^3 \equiv 0 \pmod{p}$ since $1-(-2)^3 \not\equiv 0 \pmod{p}$. By [ST1, (5.3)], we have

$$\sum_{k=1}^{p-1} \frac{1+2pH_k}{k^2} \equiv \frac{8}{3} pB_{p-3} \pmod{p^2}.$$

As $H_k = H_{k-1} + 1/k$ for $k \in \mathbb{Z}^+$, the desired result follows. $\square$

Proof of Theorem 1.3. In light of Lemma 4.1,

$$\frac{1}{p^a} \sum_{k=0}^{p^a-1} (21k+8) \binom{2k}{k}^3 = 4 \binom{2p^a}{p^a} \sum_{k=0}^{p^a-1} \binom{p^a+k-1}{k}^2.$$

So we turn to determining $\frac{1}{2}\binom{2p^a}{p^a}$ and $\sum_{k=1}^{p^a-1} \binom{p^a+k-1}{k}^2$ modulo p^4.

By a result of Glaisher [G1, G2], if $p > 3$ then

$$\frac{1}{2}\binom{2p}{p} = \binom{2p-1}{p-1} \equiv 1 - \frac{2}{3} p^3 B_{p-3} \pmod{p^4}.$$

In view of [SD, Lemma 3.2],

$$\frac{1}{2}\binom{2p^{i+1}}{p^{i+1}} \equiv \frac{1}{2}\binom{2p^i}{p^i} \pmod{p^{2i+2}} \quad \text{for every } i = 1, 2, 3, \ldots.$$

Thus

$$\frac{1}{2}\binom{2p^a}{p^a} \equiv \frac{1}{2}\binom{2p}{p} \equiv \begin{cases} p^2 + (-1)^{p-1} \pmod{p^4} & \text{if } p \in \{2,3\}, \\ 1 - \frac{2}{3} p^3 B_{p-3} \pmod{p^4} & \text{if } p > 3. \end{cases}$$

Observe that

$$\sum_{k=1}^{p^a-1} \binom{p^a+k-1}{k}^2 = \sum_{k=1}^{p^a-1} \left(\frac{p^a}{k} \prod_{0<j<k} \left(1 + \frac{p^a}{j}\right) \right)^2$$

$$\equiv \sum_{k=1}^{p^a-1} \left(\frac{p^a}{k}\right)^2 \prod_{0<j<k} \left(1 + 2\frac{p^a}{j}\right) \equiv \sum_{k=1}^{p-1} \left(\frac{p^a}{p^{a-1}k}\right)^2 \prod_{0<j<k} \left(1 + 2\frac{p^a}{p^{a-1}j}\right)$$

$$\equiv p^2 \sum_{k=1}^{p-1} \frac{1+2pH_{k-1}}{k^2} \equiv \begin{cases} p^2/(p-1) \pmod{p^4} & \text{if } p \in \{2,3\}, \\ 8p^3 B_{p-3}/3 \pmod{p^4} & \text{if } p > 3. \end{cases}$$

(In the last step we apply Lemma 4.2.)

Combining the above we see that

$$\begin{aligned}&\frac{1}{p^a}\sum_{k=0}^{p^a-1}(21k+8)\binom{2k}{k}^3\\ \equiv&\begin{cases}8(p^2+(-1)^{p-1})(p^2/(p-1)+1)\pmod{p^4}&\text{if }p\in\{2,3\},\\ 8(1-\frac{2}{3}p^3B_{p-3})(1+\frac{8}{3}p^3B_{p-3})\pmod{p^4}&\text{if }p>3,\end{cases}\\ \equiv&8(1+2p^3B_{p-3})\pmod{p^4}.\end{aligned}$$

This proves (1.15). □

5. More conjectures

In 1914 S. Ramanujan [R] found the following curious identities (see [BB], [B, pp. 353-354] and [BBC] for more such series):

$$\sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{256^k}=\frac{4}{\pi},\quad \sum_{k=0}^{\infty}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}=\frac{2\sqrt{2}}{\pi},$$

and

$$\sum_{k=0}^{\infty}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}=\frac{16}{\pi}.$$

(They are usually stated in terms of Gaussian hypergeometric series.) For an odd prime p, L. van Hamme [vH] conjectured the following p-adic analogues of the above three identities of Ramanujan:

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{256^k}\equiv&p\left(\frac{-1}{p}\right)\pmod{p^4}\ \text{ if }p>3,\\ \sum_{k=0}^{(p-1)/2}(6k+1)\frac{\binom{2k}{k}^3}{(-512)^k}\equiv&p\left(\frac{-2}{p}\right)\pmod{p^3},\\ \sum_{k=0}^{(p-1)/2}(42k+5)\frac{\binom{2k}{k}^3}{4096^k}\equiv&5p\left(\frac{-1}{p}\right)\pmod{p^4}.\end{aligned}$$

The first of these was recently shown by L. Long [Lo]; the second and the third remain open.

Motivated by Theorem 1.3 and Lemma 4.1, we propose the following conjecture.

Conjecture 5.1. (i) *For each $n=2,3,\ldots$ we have*

$$2n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(3k+1)\binom{2k}{k}^3(-8)^{n-1-k},$$
$$2n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(3k+1)\binom{2k}{k}^3 16^{n-1-k},$$
$$2n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(6k+1)\binom{2k}{k}^3 256^{n-1-k},$$
$$2n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(6k+1)\binom{2k}{k}^3(-512)^{n-1-k},$$
$$2n\binom{2n}{n}\ \Big|\ \sum_{k=0}^{n-1}(42k+5)\binom{2k}{k}^3 4096^{n-1-k}.$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3}\pmod{p^4},$$
$$\sum_{k=0}^{(p-1)/2}\frac{3k+1}{16^k}\binom{2k}{k}^3\equiv p+2\left(\frac{-1}{p}\right)p^3E_{p-3}\pmod{p^4},$$
$$\sum_{k=0}^{p-1}\frac{6k+1}{256^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)-p^3E_{p-3}\pmod{p^4},$$
$$\sum_{k=0}^{(p-1)/2}\frac{6k+1}{(-512)^k}\binom{2k}{k}^3\equiv p\left(\frac{-2}{p}\right)+\frac{p^3}{4}\left(\frac{2}{p}\right)E_{p-3}\pmod{p^4},$$
$$\sum_{k=0}^{p-1}\frac{42k+5}{4096^k}\binom{2k}{k}^3\equiv 5p\left(\frac{-1}{p}\right)-p^3E_{p-3}\pmod{p^4},$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv 4\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{6k+1}{(-512)^k}\binom{2k}{k}^3-3p\left(\frac{-1}{p}\right)\pmod{p^4}.$$

Also, for any $a\in\mathbb{Z}^+$ we have

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{3k+1}{16^k}\binom{2k}{k}^3\equiv 1+\frac{7}{6}p^3B_{p-3}\pmod{p^4}$$

and

$$\frac{1}{p^a}\sum_{k=0}^{(p^a-1)/2}\frac{42k+5}{4096^k}\binom{2k}{k}^3\equiv\left(\frac{-1}{p^a}\right)\left(5-\frac{3}{4}pH_{p-1}\right)\pmod{p^5}.$$

Each of Ishikawa [I], van Hamme [vH], Ahlgren [A] and Mortenson [M05] confirmed the following conjecture of Rodriguez-Villegas via certain advanced tools:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv a(p)\pmod{p^2}\ \text{ for any odd prime } p,$$

where the sequence $\{a(n)\}_{n\geqslant 1}$ is defined by

$$\sum_{n=1}^{\infty}a(n)q^n=q\prod_{n=1}^{\infty}(1-q^{4n})^6\quad(|q|<1)$$

and related to the Dedekind η-function in the theory of modular forms. In 1892 F. Klein and R. Fricke proved that (cf. [SB, Theorem 14.2])

$$a(p)=\begin{cases}4x^2-2p & \text{if } p=x^2+y^2 \text{ with } 2\nmid x \text{ and } 2\mid y,\\ 0 & \text{if } p\equiv 3 \pmod 4.\end{cases}$$

Let p be an odd prime. Since $\binom{-1/2}{k}=\binom{2k}{k}/(-4)^k$ $(k=0,1,2,\ldots)$, for any integer $x\not\equiv 0 \pmod p$ we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}x^k&\equiv\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}^3(-x)^k\\&=\sum_{j=0}^{(p-1)/2}\binom{(p-1)/2}{j}^3(-x)^{(p-1)/2-j}\\&\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\left(\frac{-x}{p}\right)x^{-k}\pmod p.\end{aligned}$$

Via computation we find that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\left(x^k-\left(\frac{-x}{p}\right)x^{-k}\right)\equiv 0\pmod{p^2}$$

for $x=1,4,-8,64$. (Note that the case $x=1$ is clear.) This leads us to propose the following conjecture.

Conjecture 5.2. *Let p be an odd prime.*

(i) *If $p\equiv 1\pmod 4$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k}\pmod{p^3};$$

if $p\equiv 3\pmod 4$ then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-512)^k}\equiv 0\pmod{p^2}.$$

(ii) *If $p\equiv 1\pmod 3$ and $p=x^2+3y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k}\pmod{p^3}$$
$$\equiv 4x^2-2p\pmod{p^2};$$

if $p\equiv 2\pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k}\equiv 0\pmod{p^2}.$$

(iii) If $p\equiv 1,2,4\pmod 7$ (i.e., $(\frac{p}{7})=1$), then

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k}\pmod{p^3};$$

if $p\equiv 3,5,6\pmod 7$ (i.e., $(\frac{p}{7})=-1$), then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{4096^k}\equiv 0\pmod{p^2}.$$

(iv) We have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{-2}{p})=1\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-2}{p})=-1,\ \text{i.e., } p\equiv 5,7\pmod 8.\end{cases}$$

Remark 5.1. Let p be an odd prime. By the theory of binary quadratic forms (cf. Cox [C]), if $p\equiv 1\pmod 3$ then there are unique $x,y\in\mathbb{Z}^+$ such that $p=x^2+3y^2$; if $p\equiv 1,3\pmod 8$ (i.e., $(\frac{-2}{p})=1$) then there are unique $x,y\in\mathbb{Z}^+$ such that $p=x^2+2y^2$.

Conjecture 5.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{81^k}\equiv\begin{cases}\sum_{k=0}^{p-1}\binom{2k}{k}^3\pmod{p^3}&\text{if }(\frac{p}{7})=1,\\ 0\pmod{p^2}&\text{if }(\frac{p}{7})=-1.\end{cases}$$

Also,

$$\frac{1}{p^a}\sum_{k=0}^{(p^a-1)/2}\frac{35k+8}{81^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 8\times 3^{p-1}\pmod{p^2}$$

and

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{35k+8}{81^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 8+\frac{416}{27}p^3B_{p-3}\pmod{p^4}$$

for all $a\in\mathbb{Z}^+$. Furthermore, for each $n=1,2,3,\ldots$ we have

$$\frac{1}{4n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(35k+8)\binom{2k}{k}^2\binom{4k}{2k}81^{n-1-k}\in 3^{-\delta(2n+1)}\mathbb{Z},$$

where $\delta(m)$ takes 1 or 0 according as m is a power of 3 or not.

The author [S11a] made a conjecture on $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/64^k \bmod p^2$ for any odd prime p. Here we give a related conjecture.

Conjecture 5.4. (i) *For any odd prime p and positive integer a, we have*

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{11k+3}{64^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 3+\frac{7}{2}p^3B_{p-3}\pmod{p^4}.$$

Moreover,

$$\frac{1}{n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(11k+3)\binom{2k}{k}^2\binom{3k}{k}64^{n-1-k}\in\mathbb{Z}$$

for all $n=2,3,\ldots$.

(ii) *If $p>3$ is a prime, then*

$$p\sum_{k=1}^{(p-1)/2}\frac{(11k-3)64^k}{k^3\binom{2k}{k}^2\binom{3k}{k}}\equiv 32q_p(2)-\frac{64}{3}p^2B_{p-3}\pmod{p^3}.$$

Conjecture 5.5. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{8^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=1\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-2}{p})=-1.\end{cases}$$

Also, for any $a\in\mathbb{Z}^+$ we have

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{10k+3}{8^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 3+\frac{49}{8}p^3B_{p-3}\pmod{p^4}.$$

Moreover, for each $n=2,3,\ldots$ we have

$$\frac{1}{n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(10k+3)\binom{2k}{k}^2\binom{3k}{k}8^{n-1-k}\in\mathbb{Z}.$$

For $n\in\mathbb{N}$ the Bernoulli polynomial of degree n is given by

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}.$$

Conjecture 5.6. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,4\pmod{15}\ \&\ p=x^2+15y^2\ (x,y\in\mathbb{Z}),\\ 2p-12x^2\pmod{p^2} & \text{if } p\equiv 2,8\pmod{15}\ \&\ p=3x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{p}{15})=-1;\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-192)^k}\equiv\begin{cases}x^2-2p\pmod{p^2} & \text{if } p\equiv 1\pmod 3\ \&\ 4p=x^2+27y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3;\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{216^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{48^{2k}}\quad\left(\bmod\ p^{(5+(\frac{-6}{p}))/2}\right)$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24}\ \&\ p=x^2+6y^2\ (x,y\in\mathbb{Z}),\\ 8x^2-2p\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24}\ \&\ p=2x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1,\ \textit{i.e.},\ p\equiv 13,17,19,23\pmod{24};\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{28^{4k}}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=1\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }p\neq 7\text{ and }p\equiv 5,7\pmod 8;\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{12}3)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }12\mid p-1,\ p=x^2+y^2,\ 3\nmid x\text{ and }3\mid y,\\-(\frac{xy}{3})4xy\pmod{p^2}&\text{if }12\mid p-5\text{ and }p=x^2+y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }p\equiv 3\pmod 4.\end{cases}$$

Also, for any $a\in\mathbb{Z}^+$ we have

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{15k+4}{(-27)^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 4\left(\frac{p^a}{3}\right)+\left(\frac{p^{a-1}}{3}\right)\frac{4}{3}p^2B_{p-2}\left(\frac13\right)\pmod{p^3},$$
$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{5k+1}{(-192)^k}\binom{2k}{k}^2\binom{3k}{k}\equiv\left(\frac{p^a}{3}\right)+\left(\frac{p^{a-1}}{3}\right)\frac{5}{18}p^2B_{p-2}\left(\frac13\right)\pmod{p^3},$$
$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{6k+1}{216^k}\binom{2k}{k}^2\binom{3k}{k}\equiv\left(\frac{p^a}{3}\right)-\left(\frac{p^{a-1}}{3}\right)\frac{5}{12}p^2B_{p-2}\left(\frac13\right)\pmod{p^3},$$
$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{8k+1}{48^{2k}}\binom{2k}{k}^2\binom{4k}{2k}\equiv\left(\frac{p^a}{3}\right)-\left(\frac{p^{a-1}}{3}\right)\frac{5}{24}p^2B_{p-2}\left(\frac13\right)\pmod{p^3},$$
$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{40k+3}{28^{4k}}\binom{2k}{k}^2\binom{4k}{2k}\equiv 3\left(\frac{p^a}{3}\right)-\left(\frac{p^{a-1}}{3}\right)\frac{5p^2}{392}B_{p-2}\left(\frac13\right)\pmod{p^3}\ \text{if }p\neq 7,$$
$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 3\left(\frac{p^a}{3}\right)+\left(\frac{p^{a-1}}{3}\right)\frac{5}{24}p^2B_{p-2}\left(\frac13\right)\pmod{p^3}.$$

and

$$\frac{1}{p^a}\sum_{p^a/2<k<p^a}\frac{8k+1}{48^{2k}}\binom{2k}{k}^2\binom{4k}{2k}\equiv 0\pmod{p^2}.$$

Remark 5.2. (i) In view of Conjecture 5.6, we also have conjectures such as

$$\frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(15k+4)\binom{2k}{k}^2\binom{3k}{k}(-27)^{n-1-k}\in 3^{-\delta(2n+1)}\mathbb{Z}$$

and

$$\frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(40k+3)\binom{2k}{k}^2\binom{4k}{2k}28^{4(n-1-k)}\in\mathbb{Z},$$

where n is any integer greater than one. In addition, we guess that for any prime $p>3$ and $a\in\mathbb{Z}^+$ we have

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{5k+1}{(-144)^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv\left(\frac{p^a}{3}\right)+\left(\frac{p^{a-1}}{3}\right)\frac{5}{12}p^2B_{p-2}\left(\frac{1}{3}\right)\pmod{p^3}.$$

(ii) The following Ramanujan-type series are closely related to some congruences in Conj. 5.6.

$$\sum_{k=0}^{\infty}\frac{5k+1}{(-192)^k}\binom{2k}{k}^2\binom{3k}{k}=\frac{4\sqrt{3}}{\pi},\ \sum_{k=0}^{\infty}\frac{6k+1}{216^k}\binom{2k}{k}^2\binom{3k}{k}=\frac{3\sqrt{3}}{\pi},$$
$$\sum_{k=0}^{\infty}\frac{8k+1}{48^{2k}}\binom{2k}{k}^2\binom{4k}{2k}=\frac{2\sqrt{3}}{\pi},\quad\sum_{k=0}^{\infty}\frac{40k+3}{28^{4k}}\binom{2k}{k}^2\binom{4k}{2k}=\frac{49}{3\sqrt{3}\,\pi},$$

and

$$\sum_{k=0}^{\infty}\frac{28k+3}{(-2^{12}3)^k}\binom{2k}{k}^2\binom{4k}{2k}=\frac{16}{\sqrt{3}\,\pi}.$$

For the sake of brevity, below we will omit remarks like Remarks 5.1 and 5.2.

For $n\in\mathbb{N}$ the Euler polynomial of degree n is given by

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}.$$

Conjecture 5.7. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{10})^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,9\ (\mathrm{mod}\ 20)\ \&\ p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 2p-2x^2\pmod{p^2} & \text{if } p\equiv 3,7\ (\mathrm{mod}\ 20)\ \&\ 2p=x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-5}{p})=-1,\ i.e.,\ p\equiv 11,13,17,19\ (\mathrm{mod}\ 20).\end{cases}$$

and

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{20k+3}{(-2^{10})^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 3\left(\frac{-1}{p^a}\right)+3\left(\frac{-1}{p^{a-1}}\right)p^2E_{p-3}\pmod{p^3}\ \text{ for all } a\in\mathbb{Z}^+.$$

Provided $p>3$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{12^{4k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv 1,9,11,19\pmod{40}\ \&\ p=x^2+10y^2\ (x,y\in\mathbb{Z}),\\ 2p-8x^2\pmod{p^2}&\text{if } p\equiv 7,13,23,37\pmod{40}\ \&\ p=2x^2+5y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } (\frac{-10}{p})=-1,\ i.e.,\ p\equiv 3,17,21,27,29,31,33,39\pmod{40},\end{cases}$$

and

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{10k+1}{12^{4k}}\binom{2k}{k}^2\binom{4k}{2k}\equiv\left(\frac{-2}{p^a}\right)-\left(\frac{-2}{p^{a-1}}\right)\frac{p^2}{48}E_{p-3}\left(\frac14\right)\pmod{p^3}$$

for all $a\in\mathbb{Z}^+$. When $p>5$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-2^{14}3^45)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv 1,9\pmod{20},\ p=x^2+y^2,\ 5\nmid x \text{ and } 5\mid y,\\ 4xy\pmod{p^2}&\text{if } p\equiv 13,17\pmod{20},\ p=x^2+y^2 \text{ and } 5\mid x+y,\\ 0\pmod{p^2}&\text{if } p\equiv 3\pmod 4.\end{cases}$$

Conjecture 5.8. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}}{(-2^{15})^k}$$
$$\equiv\begin{cases}(\frac{-2}{p})(x^2-2p)\pmod{p^2}&\text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } (\frac{p}{11})=-1,\ i.e.,\ p\equiv 2,6,7,8,10\pmod{11}.\end{cases}$$

Also, for any $a\in\mathbb{Z}^+$ we have

$$\frac{1}{p^a}\sum_{k=0}^{p^a-1}\frac{154k+15}{(-2^{15})^k}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}$$
$$\equiv 15\left(\frac{-2}{p^a}\right)+\left(\frac{-2}{p^{a-1}}\right)\frac{15}{16}p^2E_{p-3}\left(\frac14\right)\pmod{p^3}.$$

Moreover, for each $n=2,3\ldots$ we have

$$\frac{1}{2n(2n+1)\binom{2n}{n}}\sum_{k=0}^{n-1}(154k+15)\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}(-2^{15})^{n-1-k}\in\mathbb{Z}.$$

Conjecture 5.9. *Let p be an odd prime and let $a\in\mathbb{Z}^+$. If $p\equiv 1\pmod 3$, then*
$$\sum_{k=0}^{p^a-1}\frac{9k+2}{108^k}\binom{2k}{k}^2\binom{3k}{k}\equiv 0\pmod{p^{2a}}.$$
If $p\equiv 1,3\pmod 8$, then
$$\sum_{k=0}^{p^a-1}\frac{16k+3}{256^k}\binom{2k}{k}^2\binom{4k}{2k}\equiv 0\ (\mathrm{mod}\ p^{2a+\delta_{p,3}}).$$
If $p\equiv 1\pmod 4$, then
$$\sum_{k=0}^{p^a-1}\frac{4k+1}{64^k}\binom{2k}{k}^3\equiv 0\ (\mathrm{mod}\ p^{2a})$$
and
$$\sum_{k=0}^{p^a-1}\frac{36k+5}{12^{3k}}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}\equiv 0\ (\mathrm{mod}\ p^{2a+\delta_{p,5}}).$$

Remark 5.3. The reader may consult conjectures of Rodriguez-Villegas [RV] (see also [M05]) on $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/108^k$, $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}/256^k$ and $\sum_{k=0}^{p-1}\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k}/12^{3k}$ modulo p^2, where $p>3$ is a prime.

We will give more conjectures similar to Conjectures 5.1-5.9 in the forthcoming survey [S11b]. Now we turn to sums of products of two binomial coefficients.

Conjecture 5.10. (i) *For any prime $p\equiv 1\pmod 3$ and positive integer a, we have*
$$\sum_{k=0}^{p^a-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{54^k}\equiv 0\pmod{p^{a+1}}.$$

(ii) *Let $p\equiv 1,3\pmod 8$ be a prime and write $p=x^2+2y^2$ with $x\equiv 1\pmod 4$. Then*
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{128^k}\equiv(-1)^{\lfloor(p+5)/8\rfloor}\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$
Also, for any $a\in\mathbb{Z}^+$ we have
$$\sum_{k=0}^{p^a-1}\frac{k\binom{2k}{k}\binom{4k}{2k}}{128^k}\equiv 0\pmod{p^{a+1+\delta_{p,3}}}.$$

(iii) *Let* $p\equiv 1\pmod 4$ *be a prime and write* $p=x^2+y^2$ *with* $x\equiv 1\pmod 4$ *and* $y\equiv 0\pmod 2$. *Then*

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}\equiv\begin{cases}(-1)^{\lfloor x/6\rfloor}(2x-p/(2x))\pmod{p^2}&\text{if } p\equiv 1\pmod{12},\\ (\frac{xy}{3})(2y-p/(2y))\pmod{p^2}&\text{if } p\equiv 5\pmod{12}.\end{cases}$$

Also, for any $a\in\mathbb{Z}^+$ *we have*

$$\sum_{k=0}^{p^a-1}\frac{k\binom{6k}{3k}\binom{3k}{k}}{864^k}\equiv 0\pmod{p^{a+1}}\quad\text{and}\quad\frac{1}{5^{a+2}}\sum_{k=0}^{5^a-1}\frac{k\binom{6k}{3k}\binom{3k}{k}}{864^k}\equiv 3\pmod 5.$$

Conjecture 5.11. *Let* $p>3$ *be a prime.*

(i) *If* $p\equiv 7\pmod{12}$ *and* $p=x^2+3y^2$ *with* $y\equiv 1\pmod 4$, *then*

$$\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-3)/4}\left(4y-\frac{p}{3y}\right)\pmod{p^2}$$

and we can determine y *mod* p^2 *via the congruence*

$$\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)\frac{k\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p+1)/4}y\pmod{p^2}.$$

(ii) *If* $p\equiv 1\pmod{12}$, *then*

$$\sum_{k=0}^{p-1}\binom{p-1}{k}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^2}{16^k}\equiv 0\pmod{p^2}.$$

If $p\equiv 11\pmod{12}$, *then*

$$\sum_{k=0}^{p-1}\left(\frac{k}{3}\right)\frac{\binom{2k}{k}^2}{(-16)^k}\equiv 0\pmod p.$$

Remark 5.4. The author could prove that $\sum_{k=0}^{p-1}(\frac{k}{3})\binom{2k}{k}^2/(-16)^k\equiv 0\pmod{p^2}$ for any prime $p\equiv 1\pmod 4$.

Conjecture 5.12. *Let p be an odd prime and let $a \in \mathbb{Z}^+$.*

(i) *We have*

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} \equiv \left(\frac{-2}{p^a}\right) - \left(\frac{-2}{p^{a-1}}\right)\frac{3p^2}{16}E_{p-3}\left(\frac{1}{4}\right) \pmod{p^3}$$

and

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}C_{2k}}{64^k} \equiv \left(\frac{-1}{p^a}\right) - \left(\frac{-1}{p^{a-1}}\right)3p^2E_{p-3} \pmod{p^3},$$

where C_n stands for the Catalan number $\frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1}$.

(ii) *Suppose $p > 3$. Then*

$$\sum_{k=0}^{p^a-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{432^k} \equiv \left(\frac{-1}{p^a}\right) - \left(\frac{-1}{p^{a-1}}\right)\frac{25}{9}p^2E_{p-3} \pmod{p^3}$$

and

$$\sum_{k=0}^{p^a-1} \frac{\binom{6k}{3k}C_k^{(2)}}{432^k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^2},$$

where

$$C_k^{(2)} = \frac{\binom{3k}{k}}{2k+1} = \binom{3k}{k} - 2\binom{3k}{k-1}$$

is a second-order Catalan number.

(iii) *Assume $p > 3$. Then*

$$\sum_{k=1}^{p^a-1} \frac{\binom{2k}{k+1}\binom{3k}{k+1}}{27^k} \equiv 2\left(\frac{p^a}{3}\right) - 7 \pmod{p},$$

$$\sum_{k=1}^{p^a-1} \frac{\binom{2k}{k-1}\binom{3k}{k-1}}{27^k} \equiv \left(\frac{p^a}{3}\right) - p^a \pmod{p^2},$$

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} \equiv \left(\frac{p^a}{3}\right) - \left(\frac{p^{a-1}}{3}\right)\frac{p^2}{3}B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3},$$

$$\sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}C_k^{(2)}}{27^k} \equiv \left(\frac{p^a}{3}\right) - \left(\frac{p^{a-1}}{3}\right)\frac{2}{3}p^2B_{p-2}\left(\frac{1}{3}\right) \pmod{p^3}.$$

Furthermore,

$$\sum_{k=0}^{p^a-1} (4k+1)\frac{\binom{2k}{k}C_k^{(2)}}{27^k} \equiv \left(\frac{p^a}{3}\right) \pmod{p^4}.$$

Remark 5.5. For a prime $p>3$, Rodriguez-Villegas' conjecture (cf. [RV]) on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k},\ \sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{432^k}$$

modulo p^2 were proved by Mortenson [M03b]. By Gosper's algorithm (cf. [PWZ]) we find that

$$\sum_{k=0}^{n}\frac{9k+2}{27^k}\binom{2k}{k}\binom{3k}{k}=\frac{(3n+1)(3n+2)}{27^n}\binom{2n}{n}\binom{3n}{n}$$

and

$$\sum_{k=0}^{n}\frac{36k+5}{432^k}\binom{6k}{3k}\binom{3k}{k}=\frac{(6n+1)(6n+5)}{432^n}\binom{6n}{3n}\binom{3n}{n}.$$

Conjecture 5.13. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{24^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-216)^k}\equiv\begin{cases}\binom{2(p-1)/3}{(p-1)/3}\pmod{p^2} & \text{if } p\equiv1\pmod 3,\\ p/\binom{2(p+1)/3}{(p+1)/3}\pmod{p^2} & \text{if } p\equiv 2\pmod 3.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{3k}{k}C_k}{24^k}\equiv\frac19\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{3k}{k}C_k}{(-216)^k}\equiv\frac12\binom{2(p-(\frac{p}{3}))/3}{(p-(\frac{p}{3}))/3}\pmod p.$$

When $p\equiv1\pmod 3$ and $4p=x^2+27y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv2\pmod 3$, we have

$$x\equiv\sum_{k=0}^{p-1}\frac{k+2}{24^k}\binom{2k}{k}\binom{3k}{k}\equiv\sum_{k=0}^{p-1}\frac{9k+2}{(-216)^k}\binom{2k}{k}\binom{3k}{k}\pmod{p^2}.$$

Conjecture 5.14. *Let $p>3$ be a prime.*

(i) *We always have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k+1}}{48^k}\equiv0\pmod{p^2}.$$

If $p\equiv1\pmod 3$ and $p=x^2+3y^2$ with $x\equiv1\pmod 3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{48^k}\equiv2x-\frac{p}{2x}\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{k+1}{48^k}\binom{2k}{k}\binom{4k}{2k}\equiv x\pmod{p^2}.$$

If $p\equiv 2 \pmod 3$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{48^k}\equiv\frac{3p}{2\binom{(p+1)/2}{(p+1)/6}}\pmod{p^2}.$$

(ii) *If* $(\frac{p}{7})=1$ *and* $p=x^2+7y^2$ *with* $(\frac{x}{7})=1$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{63^k}\equiv\left(\frac{p}{3}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{k+8}{63^k}\binom{2k}{k}\binom{4k}{2k}\equiv 8\left(\frac{p}{3}\right)x\pmod{p^2}.$$

If $(\frac{p}{7})=-1$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{63^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}^2}{63^k}\equiv 0\pmod p.$$

(iii) *If* $p\equiv 1 \pmod 4$ *and* $p=x^2+y^2$ *with* $x\equiv 1 \pmod 4$ *and* $y\equiv 0 \pmod 2$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{72^k}\equiv\left(\frac{6}{p}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{1-k}{72^k}\binom{2k}{k}\binom{4k}{2k}\equiv\left(\frac{6}{p}\right)x\pmod{p^2}.$$

If $p\equiv 3 \pmod 4$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{72^k}\equiv\left(\frac{6}{p}\right)\frac{2p}{3\binom{(p+1)/2}{(p+1)/4}}\pmod{p^2}.$$

Remark 5.6. Let $p>3$ be a prime. Concerning part (i) the author could prove that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(-192)^k}\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{48^k}\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{4k}{2k}}{(-192)^k}\equiv\frac{1}{4}\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{4k}{2k}}{48^k}\pmod{p^2}.$$

We have similar things related to parts (ii) and (iii) of Conj. 5.14.

Finally, we mention that we also have some conjectural super congruences involving quadratic polynomials and sums of products of more than three binomial coefficients. For example, inspired by the identity

$$\sum_{k=1}^{\infty}\frac{(-1)^k(205k^2-160k+32)}{k^5\binom{2k}{k}^5}=-2\zeta(3)$$

due to T. Amdeberhan and D. Zeilberger [AZ], on April 4, 2010 the author (cf. [S10a]) conjectured that

$$\sum_{k=0}^{(p-1)/2}(205k^2+160k+32)(-1)^k\binom{2k}{k}^5\equiv 32p^2+\frac{896}{3}p^5B_{p-3}\pmod{p^6}$$

for any prime $p>3$. Also, (1.22) and (1.23) in Conjecture 1.4 were motivated by the first and the second congruences in our following conjecture.

Conjecture 5.15. (i) *For any odd prime p, we have*

$$\sum_{n=0}^{p-1}\frac{18n^2+7n+1}{(-128)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{-1/4}{k}^2\binom{-3/4}{n-k}^2\equiv p^2\left(\frac{2}{p}\right)\pmod{p^3},$$

$$\sum_{n=0}^{p-1}\frac{40n^2+26n+5}{(-256)^n}\binom{2n}{n}^2\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}\equiv 5p^2\pmod{p^3},$$

$$\sum_{n=0}^{p-1}\frac{12n^2+11n+3}{(-32)^n}\sum_{k=0}^{n}\binom{n}{k}^4\binom{2k}{k}\binom{2(n-k)}{n-k}\equiv 3p^2+\frac{7}{4}p^5B_{p-3}\pmod{p^6}.$$

(ii) *If $p>3$ is a prime, then*

$$\sum_{n=0}^{p-1}\frac{3n^2+n}{16^n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}\equiv -4p^4q_p(2)+6p^5q_p(2)^2\pmod{p^6}.$$

For any integer $m>1$, we have

$$a_m:=\frac{1}{2m^3(m-1)}\sum_{n=0}^{m-1}(3n^2+n)16^{m-1-n}\sum_{k=0}^{n}\binom{n}{k}^2\binom{2k}{k}\binom{2(n-k)}{n-k}\in\mathbb{Z};$$

moreover, a_m is odd if and only if m is a power of two.

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