POINT COUNTS OF D_k AND SOME A_k AND E_k INTEGER LATTICES INSIDE HYPERCUBES

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ABSTRACT. Regular integer lattices are characterized by k unit vectors that build up their generator matrices. These have rank k for D-lattices, and are rank-deficient for A-lattices, for E_6 and E_7. We count lattice points inside hypercubes centered at the origin for all three types, as if classified by maximum infinity norm in the host lattice. The results assume polynomial format as a function of the hypercube edge length.

1. SCOPE

We consider infinite translationally invariant point lattices set up by generator matrices G

$$p_i = \sum_{j=1}^{k} G_{ij}\alpha_j \tag{1}$$

which select point coordinates p given a vector of integers α. In a purely geometric-enumerative manner we count all points that reside inside a hypercube defined by $|p_i| \le n$, $\forall i$. These numbers shall be called $A_k^b(n)$, $D_k^b(n)$ and $E_k^b(n)$ for the three lattice types dealt with. In the incremental version of boxing the hypercubes, the points that are on the surface of the hypercube are given the upper index s,

$$A_k^s(n) = A_k^b(n) - A_k^b(n-1),\ D_k^s(n) = D_k^b(n) - D_k^b(n-1), \text{ and } E_k^s(n) = E_k^b(n) - E_k^b(n-1), \tag{2}$$

the first differences of the "bulk" numbers with respect to the edge size n.

There is vague resemblance to volume computation of the polytope defined in α-space by other straight cuts in p-space [11, 10].

In all cases discussed, the generating functions $D_k^b(x)$, $A_k^b(x)$ or $E_k^b(x)$ are rational functions with a factor $(1-x)^k$ in the denominator. They count sequences starting with a value of 1 at $n = 0$. The generating functions of the first differences, $D_k^s(x)$ etc., are therefore obtained by decrementing the exponent of $1-x$ in these denominators by one [14, 19], and have not been written down individually for that reason.

The manuscript considers first the D-lattices D_6–D_4 in tutorial detail in sections 2–4, then the case of general k in Section 5. The points in A_2–A_4 are counted in sections 6–8 by examining sums over the α-coefficients, and the general value of k is addressed by summation over p-coordinates in Section 9. The cases E_6–E_8 are reduced to the earlier lattice counts in sections 10–12.

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2. Lattice D_2

In the D_2 lattice, the expansion coefficients α_i and Cartesian coordinates p_i are connected by

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}. \tag{3}$$

If we read the two lines of this system of equations separately, points inside the square $|p_i| \le n$ $(i = 1, 2)$ are constrained to α-coordinates inside a tilted square, as shown in Figure 1.

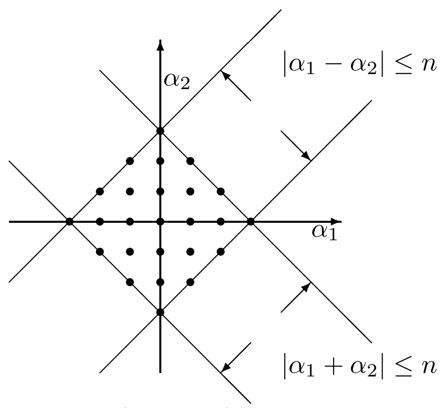

Figure 1. The conditions $|\alpha_1 \pm \alpha_2| \le n$ select two orthogonal diagonal stripes in the (α_1, α_2)-plane. Their intersection is a tilted square centered at the origin.

The point count inside the square is

$$D_2^b = \sum_{|\alpha_1 - \alpha_2| \le n} \sum_{|\alpha_1 + \alpha_2| \le n} 1. \tag{4}$$

Resummation considering the two non-overlapping triangles below and above the horizontal axis yields

$$\begin{aligned} D_2^b &= \sum_{\alpha_2=-n}^{0} \sum_{\alpha_1=-n-\alpha_2}^{\alpha_2+n} 1 + \sum_{\alpha_2=1}^{n} \sum_{\alpha_1=\alpha_2-n}^{n-\alpha_2} 1 \\ &= \sum_{\alpha_2=-n}^{0} (2\alpha_2 + 2n + 1) + \sum_{\alpha_2=1}^{n} (2n - 2\alpha_2 + 1). \end{aligned} \tag{5}$$

We will frequently sum over low order multinomials of this type with a basic formula in terms of Bernoulli Polynomials B, [9, (0.121)][20, (1.2.11)][7]

$$\sum_{m=1}^{j} m^k = \frac{B_{1+k}(j+1) - B_{1+k}(0)}{1+k}. \tag{6}$$

Application to (5) and its first differences yields essentially sequences A001844 and A008586 of the Online Encylopedia of Integer Sequences (OEIS) [17]:

Theorem 1. *(Lattice points in the bulk and on the surface of D_2)*

$$D_2^b(n) = 2n^2 + 2n + 1 = 1, 5, 13, 25, \ldots; \quad D_2^s(n) = \begin{cases} 1, & n = 0; \\ 4n, & n > 0. \end{cases} \tag{7}$$

3. Lattice D_3

The relation between expansion coefficients α_i and Cartesian coordinates p_i for the D_3 lattice is

$$\begin{pmatrix}1 & 1 & 0\\ 1 & -1 & 1\\ 0 & 0 & -1\end{pmatrix}\cdot\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}=\begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix}. \tag{8}$$

The determinant of the Generator Matrix is non-zero; by multiplication with the inverse matrix, a form more suitable to the counting problem results:

$$\begin{pmatrix}1/2 & 1/2 & 1/2\\ 1/2 & -1/2 & -1/2\\ 0 & 0 & -1\end{pmatrix}\cdot\begin{pmatrix}p_1\\ p_2\\ p_3\end{pmatrix}=\begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\end{pmatrix}. \tag{9}$$

$D_3^b(n)$ is the number of integer solutions restricted to the cube $-n \le p_i \le n$. This is the full triple sum $(2n+1)^3$—where $2n+1$ sizes the edge length of the cube—minus the number of solutions of (9) that result in non-integer α_i. The structure of the three equations in (9) suggests to separate the cases according to the parities of p_3 and p_1+p_2:

$$D_3^b(n)=\sum_{\substack{|p_1|\le n,|p_2|\le n,|p_3|\le n\\ p_1+p_2+p_3\text{ even}}}1=\sum_{\substack{|p_1|\le n,|p_2|\le n\\ p_1+p_2\text{ even}}}\sum_{\substack{|p_3|\le n\\ p_3\text{ even}}}1+\sum_{\substack{|p_1|\le n,|p_2|\le n\\ p_1+p_2\text{ odd}}}\sum_{\substack{|p_3|\le n\\ p_3\text{ odd}}}1. \tag{10}$$

The auxiliary sums are examined separately for even and odd n [17, A109613,A052928]:

$$\sum_{\substack{|p_3|\le n\\ p_3\text{ even}}}1 = n+\frac{1+(-1)^n}{2}=1,1,3,3,5,5,7,7,9,9,\ldots; \tag{11}$$

$$\sum_{\substack{|p_3|\le n\\ p_3\text{ odd}}}1 = n+\frac{1-(-1)^n}{2}=0,2,2,4,4,6,6,8,8\ldots. \tag{12}$$

The parity-filtered double sum of (10) over the square in (p_1,p_2)-space selects points on lines parallel to the diagonal.

Definition 1. *(Order of even (g) and odd (u) point sets in k-dimensional hypercube planes)*

$$V_k^g(n)\equiv\sum_{\substack{|p_i|\le n\\ p_1+p_2+\cdots p_k\text{ even}}}1;\quad V_k^u(n)\equiv\sum_{\substack{|p_i|\le n\\ p_1+p_2+\cdots p_k\text{ odd}}}1. \tag{13}$$

This decomposition applies to higher dimensions recursively:

$$V_k^g(n) = V_{k-1}^u(n)V_1^u(n)+V_{k-1}^g(n)V_1^g(n); \tag{14}$$

$$V_k^u(n) = V_{k-1}^u(n)V_1^g(n)+V_{k-1}^g(n)V_1^u(n). \tag{15}$$

Starting from $V_1^g(n)$ and $V_1^u(n)$ given in (11)–(12), the recurrences provide Table 1. The two disjoint sets of lattice points complement the hypercube:

$$V_k^g(n)+V_k^u(n)=(2n+1)^k. \tag{16}$$

Table 1. Low-dimensional examples of the lattice sums (13).

index	value
$V_1^g(n)$	$n+\frac{1+(-)^n}{2}$
$V_1^u(n)$	$n+\frac{1-(-)^n}{2}$
$V_2^g(n)$	$2n^2+2n+1$
$V_2^u(n)$	$2n(n+1)$
$V_3^g(n)$	$4n^3+6n^2+3n+\frac{1+(-)^n}{2}$
$V_3^u(n)$	$4n^3+6n^2+3n+\frac{1-(-)^n}{2}$
$V_4^g(n)$	$8n^4+16n^3+12n^2+4n+1$
$V_4^u(n)$	$4n(n+1)(2n^2+2n+1)$
$V_5^g(n)$	$16n^5+40n^4+40n^3+20n^2+5n+\frac{1+(-)^n}{2}$
$V_5^u(n)$	$16n^5+40n^4+40n^3+20n^2+5n+\frac{1-(-)^n}{2}$
$V_6^g(n)$	$(2n^2+2n+1)(16n^4+32n^3+20n^2+4n+1)$
$V_6^u(n)$	$2n(n+1)(4n^2+2n+1)(4n^2+6n+3)$
$V_7^g(n)$	$64n^7+224n^6+336n^5+280n^4+140n^3+42n^2+7n+\frac{1+(-)^n}{2}$
$V_7^u(n)$	$64n^7+224n^6+336n^5+280n^4+140n^3+42n^2+7n+\frac{1-(-)^n}{2}$
$V_8^g(n)$	$128n^8+512n^7+896n^6+896n^5+560n^4+224n^3+56n^2+8n+1$
$V_8^u(n)$	$8n(n+1)(2n^2+2n+1)(8n^4+16n^3+12n^2+4n+1)$

Theorem 2. *(fcc lattice counts for edge measure $2n+1$)*

$$V_k^g(n) = \begin{cases} \dfrac{(2n+1)^k}{2}+\dfrac{1}{2}, & k \text{ even}; \\ \dfrac{(2n+1)^k}{2}+\dfrac{(-)^n}{2}, & k \text{ odd}. \end{cases} \tag{17}$$

Proof. The proof is simple by induction with the aid of (14) and (16), using $V_1^g(n)$ of (11) and $V_1^u(n)$ of (12). □

$D_3^b(n)$ in (10) equals $V_3^g(n)$ by definition. D_3^s and D_3^b are sequences A110907 and A175109 in the OEIS [17].

Theorem 3. *(Lattice points in the bulk and on the surface of D_3)*

$$D_3^b(n) = 4n^3+6n^2+3n+\frac{1+(-)^n}{2} = 1, 13, 63, 171, 365, 665\ldots; \tag{18}$$

$$D_3^s(n) = \begin{cases} 1, & n=0; \\ 12n^2+1+(-1)^n & n>0. \end{cases} = 1, 12, 50, 108, 194, 300, 434, \ldots \tag{19}$$

The corresponding recurrences and generating function are

$$D_3^b(n) = 3D_3^b(n-1) - 2D_3^b(n-2) - 2D_3^b(n-3) + 3D_3^b(n-4) - D_3^b(n-5); \tag{20}$$

$$D_3^b(x) = \frac{(1+6x+x^2)(1+4x+x^2)}{(1+x)(1-x)^4}; \tag{21}$$

$$D_3^s(n) = 2D_3^s(n-1) - 2D_3^s(n-3) + D_3^s(n-4); \quad (n>3). \tag{22}$$

4. Lattice D_4

The transformation between expansion coefficients and Cartesian coordinates in the D_4 case reads

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}. \tag{23}$$

The technique of counting points inside cubes is the same as in the previous section. Inversion of the 4×4 matrix yields

$$\begin{pmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 & -1/2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix}. \tag{24}$$

We wish to count all lattice points subject to the constraint $|p_i| \le n$ $(i = 1, \ldots 4)$, and the first two lines of the previous equation require in addition that the sum over all four p_i is even to keep all four α_i in the integer domain:

$$D_4^b(n) = \sum_{\substack{|p_1|\le n, |p_2|\le n, |p_3|\le n, |p_4|\le n \\ p_1+p_2+p_3+p_4 \text{ even}}} 1. \tag{25}$$

This expression is $V_4^g(n)$ already computed above. $D_4^s(n)$ is OEIS sequence A117216; $D_4^b(n)$ is A175110 [17].

Theorem 4. *(Lattice points in the bulk and on the surface of D_4)*

$$\begin{aligned} D_4^b(n) &= 1 + 4n + 12n^2 + 16n^3 + 8n^4 \qquad (26)\\ &= 1, 41, 313, 1201, 3281, 7321, 14281, 25313, 41761, 65161, 97241 \ldots; \\ D_4^s(n) &= \begin{cases} 1, & n = 0; \\ 8n(1+4n^2) & n > 0; \end{cases} \qquad (27)\\ &= 1, 40, 272, 888, 2080, 4040, 6960, 11032, 16448, 23400, 32080 \ldots \end{aligned}$$

The associated generating function and recurrences are

$$D_4^b(x) = \frac{1 + 36x + 118x^2 + 36x^3 + x^4}{(1-x)^5}; \tag{28}$$

(29)
$$D_4^b(n) = 5D_4^b(n-1) - 10D_4^b(n-2) + 10D_4^b(n-3) - 5D_4^b(n-4) + D_4^b(n-5);$$

$$D_4^s(n) = 4D_4^s(n-1) - 6D_4^s(n-2) + 4D_4^s(n-3) - D_4^s(n-4); \quad (n > 4). \tag{30}$$

5. Lattices D_k, general k

No new aspect arises in comparison to the previous two sections [16]. The $D_k^b(n)$ equal the $V_k^g(n)$ and their first differences constitute the $D_k^s(n)$:

$$D_5^b(n) = 16n^5 + 40n^4 + 40n^3 + 20n^2 + 5n + \frac{1+(-)^n}{2}; \tag{31}$$

$$D_5^s(n) = \begin{cases} 1, & n = 0; \\ 1 + 40n^2 + 80n^4 + (-)^n, & n > 0; \end{cases} \tag{32}$$

$$D_6^b(n) = 32n^6 + 96n^5 + 120n^4 + 80n^3 + 30n^2 + 6n + 1; \tag{33}$$

$$D_6^s(n) = \begin{cases} 1, & n = 0; \\ 4n(1+12n^2)(3+4n^2), & n > 0; \end{cases}; \tag{34}$$

$$D_7^b(n) = 64n^7 + 225n^6 + 336n^5 + 280n^4 + 130n^3 + 43n^2 + 7n + \frac{1+(-)^n}{2}; \tag{35}$$

$$D_7^s(n) = \begin{cases} 1, & n = 0; \\ 1 + 84n^2 + 560n^4 + 448n^6 + (-)^n, & n > 0. \end{cases} \tag{36}$$

D_5 and D_6 are materialized as sequences A175111 to A175114 [17]. All cases are summarized in a Corollary to Theorem 2:

Corollary 1. *(D_k Lattice points inside the hypercube)*

$$D_k^b(n) = \begin{cases} \frac{(2n+1)^k}{2} + \frac{1}{2}, & k \text{ even}; \\ \frac{(2n+1)^k}{2} + \frac{(-)^n}{2}, & k \text{ odd}. \end{cases} \tag{37}$$

$$D_k^s(n) = \begin{cases} \frac{(2n+1)^k}{2} - \frac{(2n-1)^k}{2}, & k \text{ even},\ n > 0; \\ \frac{(2n+1)^k}{2} - \frac{(2n-1)^k}{2} + (-)^n, & k \text{ odd},\ n > 0. \end{cases} \tag{38}$$

The generating functions are

$$D_k^b(x) = \begin{cases} \dfrac{\sum_{i=0}^k \beta_i^g x^i}{(1-x)^{k+1}}, & k \text{ even}; \\ \dfrac{1 + \sum_{i=1}^k \beta_i^u x^i}{(1+x)(1-x)^{k+1}}, & k \text{ odd}; \end{cases} \tag{39}$$

where

$$2\beta_i^g \equiv \sum_{t=0}^{i} [(2i-2t+1)^k + 1] \binom{k+1}{t} (-)^t, \tag{40}$$

(41)

$$2\beta_i^u \equiv \sum_{t=0}^{i} [(2i-2t+1)^k + (-)^{i-t}] \binom{k+1}{t} (-)^t + \sum_{t=0}^{i-1} [(2i-2t-1)^k - (-)^{i-t}] \binom{k+1}{t} (-)^t.$$

Remark 1. *The D_k^* lattices are characterized by*

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1/2 \\ 0 & 1 & 0 & \cdots & 0 & 1/2 \\ 0 & 0 & 1 & \ddots & 0 & 1/2 \\ \vdots & \vdots & 0 & 1 & \ddots & 1/2 \\ \vdots & \vdots & \vdots & 0 & 1 & 1/2 \\ 0 & 0 & 0 & \cdots & 0 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \end{pmatrix}. \tag{42}$$

Matrix inversion gives

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & 0 & \vdots & -1 \\ \vdots & 0 & 1 & \vdots & -1 \\ 0 & \vdots & \ddots & 1 & -1 \\ 0 & 0 & \cdots & 0 & 2 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \vdots \end{pmatrix}. \tag{43}$$

which shows that there is no constraint on generating any p_i inside the regions $|p_i| \le n$: The number of lattice points up to infinity norm n is simply $D_k^{*b}(n) = (2n+1)^k$.

6. Lattice A_2

$A_2^b(n)$ is the number of integer solutions to

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \tag{44}$$

in the range $|p_i| \le n$. The three requirements from the three lines of this equation become

$$A_2^b = \sum_{|\alpha_1| \le n} \sum_{\substack{|-\alpha_1+\alpha_2| \le n \\ |-\alpha_2| \le n}} 1. \tag{45}$$

As outlined in Figure 2, decomposition of the conditions allows resummation over the quadrangles above and below the α_1 axis:

$$A_2^b(n) = \sum_{\alpha_2=-n}^{0} \sum_{\alpha_1=-n}^{n+\alpha_2} 1 + \sum_{\alpha_2=1}^{n} \sum_{\alpha_1=\alpha_2-n}^{n} 1 = \sum_{\alpha_2=-n}^{0} (2n+1+\alpha_2) + \sum_{\alpha_2=1}^{n} (2n+1-\alpha_2), \tag{46}$$

further evaluated with (6).

Theorem 5. *(Lattice points in the bulk and on the surface of A_2,* [17, A003215]*)*

$$A_2^b(n) = 1+3n(n+1) = 1, 7, 19, 37, 61, 91, 127, 169, 217, 271, 331, 397, 469, \ldots \tag{47}$$

The first differences are [17, A008458]

$$A_2^s(n) = \begin{cases} 1, & n = 0; \\ 6n, & n > 0. \end{cases} \tag{48}$$

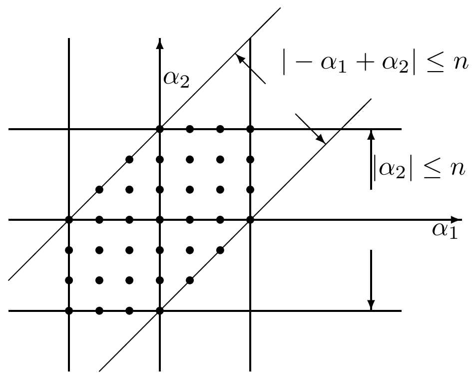

FIGURE 2. The conditions $|\alpha_1| \le n$ and $|\alpha_2| \le n$ select a square in the (α_1, α_2)-plane. The requirement $|-\alpha_1+\alpha_2| \le n$ admits only values inside a diagonal stripe. The intersection is the dotted hexagon.

7. Lattice A_3

The generator matrix sets

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{pmatrix}. \tag{49}$$

This translates the four bindings $|p_i| \le n$ into four constraints on the three α:

$$A_3^b(n) = \sum_{|\alpha_1|\le n} \sum_{\substack{|-\alpha_1+\alpha_2|\le n \\ |-\alpha_2+\alpha_3|\le n}} \sum_{|-\alpha_3|\le n} 1. \tag{50}$$

Figure 3 illustrates resummation of the format

$$\sum_{|\alpha_1|\le n} \sum_{|-\alpha_1+\alpha_2|\le n} 1 = \sum_{\alpha_2=-2n}^{0} \sum_{\alpha_1=-n}^{\alpha_2+n} 1 + \sum_{\alpha_2=1}^{2n} \sum_{\alpha_1=\alpha_2-n}^{n} 1. \tag{51}$$

This is applied twice (note this factorization generates quad-sums which are a

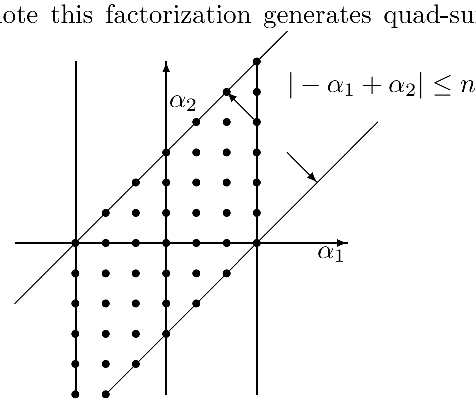

FIGURE 3. The conditions $|\alpha_1| \le n$ and $|-\alpha_1+\alpha_2| \le n$ select points in the dotted parallelogram.

convenient notation to keep track of the limits. The sums actually remain triple sums):

(52)
$$
\begin{aligned}
A_3^b(n) = (\sum_{\alpha_2=-2n}^{0} \sum_{\alpha_1=-n}^{\alpha_2+n} 1 + \sum_{\alpha_2=1}^{2n} \sum_{\alpha_1=\alpha_2-n}^{n} 1)(\sum_{\alpha_2=-2n}^{0} \sum_{\alpha_3=-n}^{\alpha_2+n} 1 + \sum_{\alpha_2=1}^{2n} \sum_{\alpha_3=\alpha_2-n}^{n} 1) \\
= \sum_{\alpha_2=-2n}^{0} \sum_{\alpha_1=-n}^{\alpha_2+n} \sum_{\alpha_3=-n}^{\alpha_2+n} 1 + \sum_{\alpha_2=1}^{2n} \sum_{\alpha_1=\alpha_2-n}^{n} \sum_{\alpha_3=\alpha_2-n}^{n} 1 \\
= \sum_{\alpha_2=-2n}^{0} (2n+1+\alpha_2)^2 + \sum_{\alpha_2=1}^{2n} (2n+1-\alpha_2)^2.
\end{aligned}
$$

After binomial expansion, both remaining sums are reduced with (6):

Theorem 6. *(Lattice points in the bulk and on the surface of A_3)*

(53) $$A_3^b(n) = 1 + \frac{2}{3}n(7+12n+8n^2) = 1, 19, 85, 231, 489, 891, 1469, 2255, 3281, \ldots$$

(54) $$A_3^s(n) = \begin{cases} 1, & n = 0 \\ 2+16n^2, & n > 0 \end{cases} = 1, 18, 66, 146, 258, 402, 578, \ldots$$

These are sequences A063496 and A010006 in the OEIS [17].

8. Lattice A_4

A_4 is characterized by a quad-sum over α_i with five constraints on the p_i set up by the hypercube:

(55) $$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \end{pmatrix}.$$

(56) $$A_4^b(n) = \sum_{|\alpha_1|\le n} \sum_{|-\alpha_1+\alpha_2|\le n} \sum_{\substack{|-\alpha_2+\alpha_3|\le n \\ |-\alpha_3+\alpha_4|\le n}} \sum_{|\alpha_4|\le n} 1.$$

The resummation (51) is separately applied to (α_1, α_2) and (α_3, α_4); the entanglement between α_2 and α_3 is noted in the second factor:

$$A_4^b(n) = \left(\sum_{\alpha_2=-2n}^{0} \sum_{\alpha_1=-n}^{\alpha_2+n} 1 + \sum_{\alpha_2=1}^{2n} \sum_{\alpha_1=\alpha_2-n}^{n} 1 \right) \times \left(\sum_{\substack{\alpha_3=-2n \\ |-\alpha_2+\alpha_3|\le n}}^{0} \sum_{\alpha_4=-n}^{\alpha_3+n} 1 + \sum_{\substack{\alpha_3=1 \\ |-\alpha_2+\alpha_3|\le n}}^{2n} \sum_{\alpha_4=\alpha_3-n}^{n} 1 \right)$$
$$= \left(\sum_{\alpha_2=-2n}^{0} (2n+1+\alpha_2) + \sum_{\alpha_2=1}^{2n} (2n+1-\alpha_2) \right) \times \left(\sum_{\substack{\alpha_3=-2n \\ |-\alpha_2+\alpha_3|\le n}}^{0} (2n+1+\alpha_3) + \sum_{\substack{\alpha_3=1 \\ |-\alpha_2+\alpha_3|\le n}}^{2n} (2n+1-\alpha_3) \right). \quad (57)$$

Product expansion generates 4 terms. The coupling between α_2 and α_3 is rewritten individually in their 4 different quadrants facilitated with Figure 4.

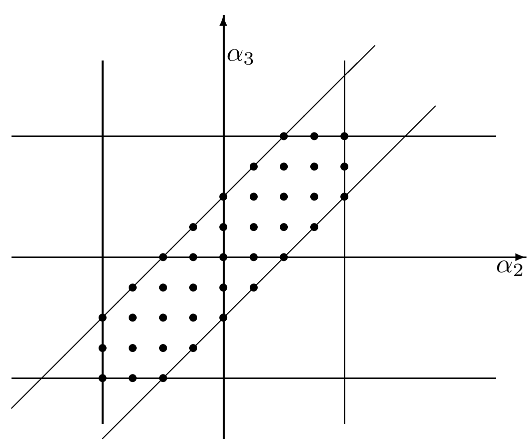

FIGURE 4. The conditions $|\alpha_2| \le 2n$ and $|\alpha_3| \le 2n$ define the large square, and $|-\alpha_2+\alpha_3| \le n$ narrows the region down to the dotted hexagon.

$$\sum_{\alpha_2=-2n}^{0} \sum_{\substack{\alpha_3=-2n \\ |-\alpha_2+\alpha_3|\le n}}^{0} = \sum_{\alpha_3=-2n}^{-n} \sum_{\alpha_2=-n}^{\alpha_3+n} + \sum_{\alpha_3=-n+1}^{0} \sum_{\alpha_2=\alpha_3-n}^{0}; \tag{58}$$

$$\sum_{\alpha_2=-2n}^{0} \sum_{\substack{\alpha_3=1 \\ |-\alpha_2+\alpha_3|\le n}}^{2n} = \sum_{\alpha_3=1}^{n} \sum_{\alpha_2=\alpha_3-n}^{0}; \tag{59}$$

$$\sum_{\alpha_2=1}^{2n} \sum_{\substack{\alpha_3=-2n \\ |-\alpha_2+\alpha_3|\le n}}^{0} = \sum_{\alpha_3=-n+1}^{0} \sum_{\alpha_2=1}^{\alpha_3+n}; \tag{60}$$

$$\sum_{\alpha_2=1}^{2n} \sum_{\substack{\alpha_3=1 \\ |-\alpha_2+\alpha_3|\le n}}^{2n} = \sum_{\alpha_3=1}^{n} \sum_{\alpha_2=1}^{\alpha_3+n} + \sum_{\alpha_3=n+1}^{2n} \sum_{\alpha_2=\alpha_3-n}^{2n}. \tag{61}$$

So A_4^b in (57) translates into six elementary double sums over products of the form $(2n+1\pm\alpha_2)(2n+1\pm\alpha_3)$, eventually handled with (6).

Theorem 7. *(Lattice points in the bulk and on the surface of* A_4 [17, A083669]*)*

$$A_4^b(n) = 1 + \frac{5}{12}n(n+1)(14+23n+23n^2) = 1, 51, 381, 1451, 3951, 8801, \ldots \tag{62}$$

$$A_4^s(n) = \frac{5}{3}(7+23n^2) = 1, 50, 330, 1070, 2500, 4850, 8350 \ldots \tag{63}$$

9. Lattices A_k, $k > 4$

Direct summation over the polytopes in α_i-space becomes increasingly laborious in higher dimensions; we switch to summation in p_i-space based on the alternative

$$A_k^b(n) = \sum_{\substack{|p_i|\le n,\, i=1,\ldots,k+1 \\ \sum_{i=1}^{k+1} p_i = 0}} 1. \tag{64}$$

This is derived by adding a $k+1$-st unit vector with all components equal to zero—with the exception of the last component–as a final column to the generator matrix. (This simple format suffices; laminations are not involved.) An associated coefficient α_{k+1} embeds the lattice into full space, while the condition $\alpha_{k+1} = 0$ is maintained for the counting process. Inversion of the matrix generator equation demonstrates that this zero-condition translates into the requirement on the sum over the p_i shown above. This point of view is occasionally used to *define* the A-lattices.

Counting the points subjected to some fixed $\sum_i p_i = m$ is equivalent to computation of the multinomial coefficient

$$[x^m](1 + x + x^{-1} + x^2 + x^{-2} + \cdots + x^n + x^{-n})^k. \tag{65}$$

Balancing the accumulated powers as required for A_k^b necessarily ties them to the central multinomial numbers [3, 4]:

$$A_{k-1}^b(n) = \binom{k}{nk}_{2n} = \sum_{j=0}^{\lfloor nk/(2n+1)\rfloor} (-)^j \binom{k}{j}\binom{nk - j(2n+1) + k - 1}{k-1}. \tag{66}$$

Selecting values for the p_i is equivalent to a Motzkin-path, picking one term of each of the k instances of the $1 + x + x^{-1}$ of the trinomial, for example [5]. First, the formula is a route to quick numerical evaluation (Table 2). Second, it proves that $A_k^b(n)$ is a polynomial of order $\le k$ in n, because each of the binomial factors in the j-sum is a polynomial of order $k-1$. This is easily made more explicit by invocation of the Stirling numbers of the first kind [13][1, (24.1.3)].

Remark 2. *This scheme of polynomial extension has been used for coordination sequences before* [6]*, and is found in growth series as well* [2]*.*

TABLE 2. $A_k^b(n)$ displaying columns of central 3-nomial, 5-nomial, 7-nomial etc. numbers [17, A002426,A005191,A025012,A025014,A163269]

$k\backslash n$	0	1	2	3	4	5	6	7	8
1	1	3	5	7	9	11	13	15	17
2	1	7	19	37	61	91	127	169	217
3	1	19	85	231	489	891	1469	2255	3281
4	1	51	381	1451	3951	8801	17151	30381	50101
5	1	141	1751	9331	32661	88913	204763	418503	782153
6	1	393	8135	60691	273127	908755	2473325	5832765	12354469
7	1	1107	38165	398567	2306025	9377467	30162301	82073295	197018321
8	1	3139	180325	2636263	19610233	97464799	370487485	1163205475	3164588407

By computing the initial terms of any A_k numerically, the others follow by the recurrence obeyed by k-th order polynomials [8]:

$$A_k^b(n) = \sum_{j=1}^{k+1} \binom{k+1}{j} (-)^{j+1} A_k^b(n-j). \tag{67}$$

Theorem 8. *(Lattice points in the bulk and on the surface of A_5)*

$$A_5^b(n) = \frac{1}{5}(2n+1)(5+27n+71n^2+88n^3+44n^4); \tag{68}$$

$$A_5^s(n) = \begin{cases} 1, & n=0, \\ 2+50n^2+88n^4, & n>0, \end{cases} = 1, 140, 1610, 7580, 23330, \ldots \tag{69}$$

A_5^b is a bisection of sequence A071816 of the OEIS [17]. A_6^b is a bisection of sequence A133458 [17].

Theorem 9. *(A_6 and A_7 point counts)*

$$A_6^b(n) = 1 + \frac{7}{180} n(n+1)(222+727n+1568n^2+1682n^3+841n^4). \tag{70}$$

(71)

$$A_6^s(n) = \begin{cases} 1, & n=0, \\ \frac{7}{30} n(74+765n^2+841n^4), & n>0, \end{cases} = 1, 392, 7742, 52556, 212436 \ldots$$

(72)

$$A_7^b(n) = \frac{2n+1}{315}(315+2568n+10936n^2+26400n^3+37360n^4+28992n^5+9664n^6).$$

Remark 3. *The $A^b_k(n)$ can be phrased as k-th order polynomials of $L \equiv 2n+1$ with the same parity as k:*

$$A^b_1(L) = L; \tag{73}$$

$$A^b_2(L) = \frac{1}{4} + \frac{3}{4}L^2; \tag{74}$$

$$A^b_3(L) = \frac{1}{3}L + \frac{2}{3}L^3; \tag{75}$$

$$A^b_4(L) = \frac{9}{64} + \frac{25}{96}L^2 + \frac{115}{192}L^4; \tag{76}$$

$$A^b_5(L) = \frac{1}{5}L + \frac{1}{4}L^3 + \frac{11}{20}L^5; \tag{77}$$

$$A^b_6(L) = \frac{25}{256} + \frac{539}{2304}L^4 + \frac{5887}{11520}L^6; \tag{78}$$

$$A^b_7(L) = \frac{1}{7}L + \frac{7}{45}L^3 + \frac{2}{9}L^5 + \frac{151}{315}L^7; \tag{79}$$

$$A^b_8(L) = \frac{1225}{16384} + \frac{3229}{28672}L^2 + \frac{6063}{40960}L^4 + \frac{867}{4096}L^6 + \frac{259723}{573440}L^8. \tag{80}$$

If we rewrite (66) [15]

$$A^b_{k-1}(n) = \sum_{j=0}^{\lfloor k/(2+1/n)\rfloor} (-1)^j \frac{k}{j!} \frac{\Gamma[k(n+1)-j(2n+1)]}{\Gamma(k-j+1)\Gamma[kn-j(2n+1)+1]}, \tag{81}$$

the multiplication formula of the Γ-function converts this to terminating Saalschützian Hypergeometric Series:

$$A^b_{k-1}(1) = \frac{\Gamma(2k)}{\Gamma(k)\Gamma(k+1)}\, {}_4F_3\left(\begin{array}{c} -k, -\frac{k}{3}, -\frac{k-1}{3}, -\frac{k-2}{3} \\ -\frac{2k-1}{3}, -\frac{2k-2}{3}, -\frac{2k}{3}+1 \end{array} \mid 1\right), \tag{82}$$

(83)

$$A^b_{k-1}(n) = \frac{\Gamma[(n+1)k]}{\Gamma(k)\Gamma(nk+1)}\, {}_{2n+2}F_{2n+1}\left(\begin{array}{c} -k, -\frac{nk}{2n+1}, -\frac{nk-1}{2n+1}, -\frac{nk-2}{2n+1}, \cdots, -\frac{nk-2n}{2n+1} \\ -\frac{(n+1)k-1}{2n+1}, -\frac{(n+1)k-2}{2n+1}, \cdots, -\frac{(n+1)k-2n-1}{2n+1} \end{array} \mid 1\right).$$

The functional equation $\Gamma(m+1) = m\Gamma(m)$ presumably induces a non-linear recurrence along each column of Table 2, as shown by Sulanke for column $n=1$ [18]. Numerical experimentation rather than proofs [12] suggest:

Conjecture 1. *(Recurrences of centered 3-nomial, 5-nomial, 7-nomial coefficients)*

$$(k+1)A^b_k(1) - (2k+1)A^b_{k-1}(1) - 3kA^b_{k-2}(1) = 0; \tag{84}$$

(85)

$$2(k+1)(2k+1)A^b_k(2) + (k^2-49k-2)A^b_{k-1}(2) + 5(-21k^2+37k-18)A^b_{k-2}(2)$$
$$-25(k-1)(k-4)A^b_{k-3}(2) + 125(k-1)(k-2)A^b_{k-4}(2) = 0.$$

(86)

$$3(3k+2)(3k+1)(k+1)A^b_k(3) + (41k^3-600k^2-191k-6)A^b_{k-1}(3)$$
$$+7(-383k^3+1458k^2-1927k+840)A^b_{k-2}(3) + 49(-83k^3+1068k^2-4321k+5040)A^b_{k-3}(3)$$
$$+343(199k^3-1890k^2+6017k-6390)A^b_{k-4}(3) + 2401(k-3)(43k^2-351k+722)A^b_{k-5}(3)$$
$$-16807(k-3)(k-4)(5k-19)A^b_{k-6}(3) - 117649(k-5)(k-4)(k-3)A^b_{k-7}(3) = 0.$$

TABLE 3. Binomial coefficients $\eta_{k,j}$ of (88).

$k\backslash j$	1	2	3	4	5	6	7	8
1	1							
2	3	3						
3	9	24	16					
4	25	140	230	115				
5	70	735	2250	2640	1056			
6	196	3675	18732	38801	35322	11774		
7	553	17976	143696	468160	728448	541184	154624	
8	1569	87024	1052352	5067288	11994354	14906484	9350028	2337507

Remark 4. *Inverse binomial transformations of the $A_k^b(n)$ define coefficients $\eta_{k,j}$ via*

$$A_k^b(n) \equiv 1 + 2\sum_{j=1}^{n} \binom{n}{j} \eta_{k,j}, \tag{87}$$

$$\eta_{k,j} = \frac{1}{2}\sum_{l=0}^{j} (-)^{j+l} \binom{j}{l} \binom{k+1}{l(k+1)}_{2l}, \tag{88}$$

as demonstrated in Table 3. They are related to the partial fractions of the rational generating functions :

$$A_k^b(x) = \frac{1}{1-x} + 2\sum_{j=1}^{k} \eta_{k,j} \frac{x^j}{(1-x)^{j+1}} \equiv \frac{\sum_{l=0}^{k} \gamma_{k,l} x^l}{(1-x)^{k+1}}. \tag{89}$$

The first column and the diagonal of Table 3 appear to be sequences A097861 and A011818 of the OEIS, respectively [17].

Remark 5. *From* (66) *we deduce the numerator coefficients defined in* (89)*:*

$$\gamma_{k,l} = \sum_{n=0}^{l} \binom{k+1}{l-n} (-)^{l-n} \binom{k+1}{n(k+1)}_{2n}. \tag{90}$$

Some of these are shown in Table 4. Caused by the mirror symmetry of the coefficients, -1 is a root of the polynomial $\sum_l \gamma_{k,l} x^l$ if k is odd; a factor $1+x$ may then be split off.

Formula (2) converts Table 2 into Table 5. And similar to Conjecture 1 we formulate recurrences along columns of this derived table:

Conjecture 2. *(Recurrences of A_k^s)*

(91)

$$(k+1)(k-1)A_k^s(1) - (3k^2-k-1)A_{k-1}^s(1) - k(k-2)A_{k-2}^s(1) + 3k(k-1)A_{k-3}^s(1) = 0,$$

TABLE 4. Synopsis of the numerators $\gamma_{k,l}$ of the generating functions (89).

$k\backslash l$	0	1	2	3	4	5	6	7
1	1	1						
2	1	4	1					
3	1	15	15	1				
4	1	46	136	46	1			
5	1	135	920	920	135	1		
6	1	386	5405	11964	5405	386	1	
7	1	1099	29337	124187	124187	29337	1099	1
8	1	3130	152110	1126258	2112016	1126258	152110	3130
9	1	8943	767460	9371472	29836764	29836764	9371472	767460
10	1	25642	3809367	73628622	372715542	626734120	372715542	73628622

TABLE 5. $A^s_k(n)$ derived from Table 2, building differences between adjacent columns [17, A175197].

$k\backslash n$	0	1	2	3	4	5	6	7	8
1	1	2	2	2	2	2	2	2	2
2	1	6	12	18	24	30	36	42	48
3	1	18	66	146	258	402	578	786	1026
4	1	50	330	1070	2500	4850	8350	13230	19720
5	1	140	1610	7580	23330	56252	115850	213740	363650
6	1	392	7742	52556	212436	635628	1564570	3359440	6521704
7	1	1106	37058	360402	1907458	7071442	20784834	51910994	114945026
8	1	3138	177186	2455938	16973970	77854566	273022686	792717990	2001382932

$$
\begin{aligned}
(92)\quad & 2(k-1)(2k+1)(k+1)(65576k-74745)A^s_k(2) \\
&+ (262304k^4-10212201k^3+21353744k^2-8959001k-149490)A^s_{k-1}(2) \\
&+ 2(-6440305k^4+44418225k^3-87651471k^2+52631106k-4105233)A^s_{k-2}(2) \\
&+ 20(811225k^4-3988621k^3+5814523k^2+2441684k-8566578)A^s_{k-3}(2) \\
&+ 2(24847058k^4-190384802k^3+480247197k^2-462996527k+158679414)A^s_{k-4}(2) \\
&- (k-3)(20387704k^3-72824267k^2-29485137k+331041750)A^s_{k-5}(2) \\
&- 10(k-3)(k-4)(3707581k^2-5729012k+3352341)A^s_{k-6}(2) \\
&+ 150(k-3)(k-4)(k-5)(26006k+104375)A^s_{k-7}(2) = 0.
\end{aligned}
$$

10. Lattice E_6

The task is to sum over the 6-dimensional representation with limits set by the 8-dimensional cube:

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/2 \\ -1 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & -1/2 \\ 0 & 0 & 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & -1/2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{pmatrix}. \tag{93}$$

This is extended to an 8-dimensional representation

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1/2 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{pmatrix}. \tag{94}$$

maintaining the count of E_6^b by adding the condition $\alpha_7 = \alpha_8 = 0$ to the lattice sum. Inversion of this matrix equation yields

$$\begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 3 & -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 2 & -1 & -1 & -1 & -1 & 0 & 0 & 0 \\ 1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{pmatrix}. \tag{95}$$

The first but last equation of this linear system argues that 6 components of p_i are confined to $\sum_{i=2,\ldots,6} p_i = 0$ while summing over $|p_i| \le n$ to ensure $\alpha_7 = 0$; the same sum regulated the 6-dimensional cube A_5^b. The last equation represents the confinement $p_1 + p_8 = 0$ to ensure $\alpha_8 = 0$. Since this is not entangled with the requirement on the other 6 components, the associated double sum emits a factor $2n + 1$. (Imagine counting points in a square of edge size $2n + 1$ along two coordinates p_1 and p_8, where $p_1 + p_8 = 0$ admits only points on the diagonal.)

Theorem 10. *(Point counts of E_6)*

$$E_6^b(n) = (2n+1)A_5^b(n) = \frac{1}{5}(1+2n)^2(5+27n+71n^2+88n^3+44n^4) \tag{96}$$
$$= 1, 423, 8755, 65317, 293949, 978043, 2661919, 6277545, 13296601, \ldots;$$

$$E_6^s(n) = \begin{cases} 1, & n = 0; \\ \frac{2}{5}n(47+480n^2+528n^4), & n > 0; \end{cases} \tag{97}$$
$$= 1, 422, 8332, 56562, 228632, 684094, 1683876, 3615626, 7019056, \ldots;$$

$$E_6^b(x) = \frac{1+416x+5815x^2+12880x^3+5815x^4+416x^5+x^6}{(1-x)^7}. \tag{98}$$

11. Lattice E_7

The E_7 lattice is spanned by

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1/2 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{pmatrix}. \tag{99}$$

Again we consider only the sublattice with even α_7, that is, integer p_i.

Theorem 11. *(Point counts of E_7)*

$$E_7^b(n) = A_7^b(n). \tag{100}$$

Proof. We reach out into a direction of the p_8 axis adding a unit vector with axis section α_8: $E_7^b(n)$ counts only points with $\alpha_8 = 0$.

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1/2 & 1 \end{pmatrix} \cdot \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{pmatrix} = \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{pmatrix}. \tag{101}$$

The inverse of this equation is

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 2 & 2 & 2 & 2 & 2 & 0 \\ 2 & 2 & 2 & 3 & 3 & 3 & 3 & 0 \\ 3 & 3 & 3 & 3 & 4 & 4 & 4 & 0 \\ 2 & 2 & 2 & 2 & 2 & 3 & 3 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 \\ 2 & 2 & 2 & 2 & 2 & 2 & 2 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ p_5 \\ p_6 \\ p_7 \\ p_8 \end{pmatrix} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \\ \alpha_5 \\ \alpha_6 \\ \alpha_7 \\ \alpha_8 \end{pmatrix}, \tag{102}$$

and—reading the last line—the restriction on the α_8 coordinate implied by the embedding translates into $\sum_i p_i = 0$. In comparison, we can also embed the A_7 lattice into its 8-dimensional host,

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{pmatrix} \cdot \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6\\ \alpha_7\\ \alpha_8\end{pmatrix} = \begin{pmatrix}p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\\ p_7\\ p_8\end{pmatrix}, \tag{103}$$

and invert this representation, too:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0\\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0\\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0\\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix}p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\\ p_7\\ p_8\end{pmatrix} = \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6\\ \alpha_7\\ \alpha_8\end{pmatrix}. \tag{104}$$

The implied slice $\alpha_8 = 0$ and the last line of this equation leads to the same condition $\sum_i p_i = 0$ as derived from (102). Since both cases select from the $(2n+1)^8$ points in the hypercube subject to the same condition, both counts are the same. □

12. Lattice E_8

The E_8 coordinates are mediated by

$$\begin{pmatrix} 2 & -1 & 0 & 0 & 0 & 0 & 0 & 1/2\\ 0 & 1 & -1 & 0 & 0 & 0 & 0 & 1/2\\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 1/2\\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & 1/2\\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & -1/2\\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & -1/2\\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1/2\\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1/2 \end{pmatrix} \cdot \begin{pmatrix}\alpha_1\\ \alpha_2\\ \alpha_3\\ \alpha_4\\ \alpha_5\\ \alpha_6\\ \alpha_7\\ \alpha_8\end{pmatrix} = \begin{pmatrix}p_1\\ p_2\\ p_3\\ p_4\\ p_5\\ p_6\\ p_7\\ p_8\end{pmatrix}. \tag{105}$$

Explicit numbers are found with the formula in Theorem 2:

Theorem 12. *(Lattice points in the bulk and on the surface of E_8)*

$$E_8^b(n) = V_8^g(n) = 1, 3281, 195313, 2882401, 21523361\ldots \tag{106}$$

$$E_8^s(n) = \begin{cases} 1, & n = 0;\\ 16n(4n^2+1)(16n^4+24n^2+1), & n > 0; \end{cases} \tag{107}$$
$$= 1, 3280, 192032, 2687088, 18640960, 85656080, \ldots$$

Proof. The inverse of the generator matrix in (105) has exactly one row filled with the value 1/2, all other entries are integer. As already argued for the D-lattices in sections 3–4, this leads to the constraint that the sum over the p_i must remain even, which matches Definition 1. □

13. Summary

For D_k lattices, the number of lattice points inside a hypercube is essentially a k-th order polynomial of the edge length, summarized in Eq. (37). For A_k lattices, explicit polynomials have been computed for $k \le 5$ in Eqs. (47), (53), (62) and (68). For higher dimensions, the numbers are centered multinomial coefficients (66) which can be quickly converted to k-th order polynomials in n. The counts for E_6, E_7 and E_8 are closely associated with the counts for A_5, A_7 and D_8, respectively.

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