

Continued fractions constructed from prime numbers

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Abstract

We give 50 digits values of the simple continued fractions whose denominators are formed from a) prime numbers, b) twin primes, c) generalized d -twins, d) primes of the form $m^2 + n^4$, e) primes of the form $m^2 + 1$, f) Mersenne primes and g) primorial primes. All these continued fractions belong to the set of measure zero of exceptions to the theorems of Khinchin and Levy. We claim that all these continued fractions are transcendental numbers. Next we propose the conjecture which indicates the way to deduce the transcendence of some continued fractions from transcendence of another ones.

1 Introduction

Let a_0 be an integer and let a_k , $k = 1, 2, \dots, n$ are positive integers (in general a_k can be arbitrary complex numbers, see e.g. [30]). Then

$$r = [a_0; a_1, a_2, a_3, \dots, a_n] \equiv a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \dots + \frac{1}{a_n}}}} \quad (1)$$

is the simple (i.e. with all nominators equal to 1) finite continued fraction. The numbers a_k , $k = 1, 2, \dots, n$ are called partial quotients and

$$\frac{P_k}{Q_k} = [a_0; a_1, a_2, a_3, \dots, a_k], \quad k = 1, 2, \dots, n \quad (2)$$

is called the k -th convergent of r . If for the infinite continued fraction

$$[a_0; a_1, a_2, a_3, \dots] \quad (3)$$

the sequence of convergents P_n/Q_n converges to some limit r when $n \rightarrow \infty$ then we say that the infinite continued fraction $[a_0; a_1, a_2, a_3, \dots]$ is equal to r . The convergence of the continued fraction (3) is linked to the behavior of the sum of partial quotients a_n :

$$\text{sequence } \frac{P_n}{Q_n} \text{ is convergent to } r \Leftrightarrow \sum_{n=1}^{\infty} a_n \text{ is divergent} \quad (4)$$

see e.g. [29, Theorem 10, p.10]. It means that for convergence of the continued fraction it is necessary that both $P_n, Q_n \rightarrow \infty$ in such a way, that the ratio P_n/Q_n has a definite limit for $n \rightarrow \infty$. If the infinite continued fraction is convergent then the values of the convergents $P_k(r)/Q_k(r)$ approximate the value of r with accuracy at least $1/Q_k Q_{k+1}$ [29, Theorem 9, p.9]:

$$\left| r - \frac{P_k}{Q_k} \right| < \frac{1}{Q_k Q_{k+1}} < \frac{1}{Q_k^2 a_{k+1}} < \frac{1}{Q_k^2}. \quad (5)$$

Rational numbers have finite continued fractions, quadratic irrationals have periodic infinite continued fractions and vice versa: eventually periodic continued fractions represent quadratic surds. All remaining irrational numbers have non-periodic continued fractions.

Khinchin has proved that [29, p.93]

$$\lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n)^{\frac{1}{n}} = \prod_{m=1}^{\infty} \left\{ 1 + \frac{1}{m(m+2)} \right\}^{\log_2 m} \equiv K_0 \approx 2.685452001 \dots \quad (6)$$

is a constant for almost all real r , see also [39], [23, §1.8]. The exceptions are rational numbers, quadratic irrationals and some irrational numbers too, like for example the

Euler constant $e = 2.7182818285\dots$, but this set of exceptions is of the Lebesgue measure zero. The constant K_0 is called the Khinchin constant.

In 1935 Khinchin [28] has proved that for almost all real r the denominators $Q_n(r)$ of the convergents of the continued fraction expansions for r satisfy $\lim_{n \rightarrow \infty} \sqrt[n]{Q_n(r)} = L_0$ and in 1936 Paul Levy [32] found an explicit expression for this constant L_0 :

$$\lim_{n \rightarrow \infty} \sqrt[n]{Q_n(r)} = e^{\pi^2/12 \log(2)} \equiv L_0 = 3.27582291872\dots \quad (7)$$

All presented below continued fractions belong to this exceptional set of irrationals for which the geometric means of the denominators $(a_1 a_2 \dots a_n)^{1/n}$ and the n -th radical roots of the denominator $Q_n^{1/n}$ tend to infinity.

In this paper we will consider continued fractions with partial quotients given by an infinity of all primes as well as primes of special form belonging to families containing conjectured infinity of members. All these continued fractions are non-periodic, and thus are irrational, but we claim that all of them are also *transcendental*. In Sect. 3 we review some facts and theorems concerning the transcendentality of continued fractions. In Sect. 4 some experimental results regarding transcendentality of numbers constructed from primes are presented.

2 Seven examples

In consecutive sections we will discuss the following cases: the set of all primes $2, 3, 5, 7, \dots$, twin primes, generalized d -twins, i.e. pairs of adjacent primes separated by d , primes of the form $m^2 + n^4$, primes given by the quadratic form $m^2 + 1$. Next are considered sparse Mersenne primes and at the end even sparser primorials primes.

It is possible to consider other families of primes, like Sophie Germain primes (it is conjectured that there are infinitely many of them), irregular primes (Jensen in 1915 proved that there are infinitely many of them), regular primes of which it was

conjectured that $e^{-1/2} \approx 61\%$ of all prime numbers are regular, the Cullen numbers $n2^n + 1$ when they are primes etc. but we leave it for further studies.

Except Sections 2.1 and 2.4, where we will treat the families of primes containing rigorously proved infinity of members, all remaining consideration are performed under the assumption there is infinity of primes in each class of primes, although proofs of infinitude of all these sets of primes seems to be very far away. Thus many of our reasonings are heuristical.

The examples are in order of sparseness of each family of primes.

2.1 The set of all primes

Let us put $a_n = p_n$ where p_n denotes the n -th primes: $[0; 2, 3, 5, 7, 11, 13, \dots]$. As there is an infinity of primes the condition (4) is fulfilled and let us denote the limit of the continued fraction by

$$u = [0; 2, 3, 5, 7, 11, 13, \dots] = \frac{1}{2 + \frac{1}{3 + \frac{1}{5 + \frac{1}{7 + \frac{1}{11 + \ddots}}}}} \quad (8)$$

Using PARI system [46] and all 1229 primes up to 10000 it is possible to obtain over 8000 digits of the above continued fraction in just a few seconds because

$$[0; 2, 3, 5, 7, 11, 13, \dots, 9973] = \frac{3.38592889 \dots \times 10^{4297}}{7.83177791 \dots \times 10^{4297}} \quad (9)$$

and the product of $Q_k Q_{k+1}$ on the rhs of (5) is larger than 10^{8500} . The first 50 digits of u reads:

$$u = 0.43233208718590286890925379324199996370511089688 \dots \quad (10)$$

This number is not recognized at the Symbolic Inverse Calculator (<http://pi.lacim.uqam.ca/eng/>) maintained by Simone Plouffe. Accidentaly, it is very close to the one of Renyi's

parking constants $m_{\mathbb{R}} = (1 - e^{-2})/2 = 0.43233235838\dots$, see [23, pp. 278–283]:
 $m_{\mathbb{R}} - u = 2.712\dots \times 10^{-7}$.

It is possible to obtain analytically the geometrical means of the partial quotients in (8). It is well known (see e.g. [21, Chap.4]), that the Chebyshev function $\theta(x)$ behaves like:

$$\theta(x) \equiv \sum_{p \leq x} \log(p) = x + \mathcal{O}(\sqrt{x}). \quad (11)$$

Thus skipping the error term we have

$$\prod_{k=1}^n p_k = e^{p_n}. \quad (12)$$

It is well known that [37, Sect. 2.II.A] that

$$p_n = n \log(n) + n(\log \log(n) - 1) + o\left(\frac{n \log \log(n)}{n}\right). \quad (13)$$

For our purposes it suffices to know that

$$p_n > n \log(n) \quad \text{for } n > 1 \quad (14)$$

see e.g. [38]. Hence we can write for the geometrical means of the partial quotients the estimation:

$$(a_1 a_2 \dots a_n)^{\frac{1}{n}} = \left(\prod_{k=1}^n p_k \right)^{\frac{1}{n}} = (e^{p_n})^{\frac{1}{n}} > n \rightarrow \infty \quad (15)$$

thus the continued fraction u belongs to the set of measure zero of exceptions to the Khinchin Theorem (6). It is also an exception to the Levy Theorem, because from the general properties of continued fractions:

$$Q_{n+1} = a_n Q_n + Q_{n-1} \quad (16)$$

we have $Q_n > \prod_{k=1}^n p_k > n^n$ and thus $Q_n^{1/n} \rightarrow \infty$ in contrast to (7). It is an explicit example of the continued fraction with unbounded $(Q_n)^{1/n}$.

2.2 Twin primes

The twin prime conjecture states that there are infinitely many pairs of primes (t_n, t_{n+1}) differing by two: $t_{n+1} - t_n = 2$. Let $\pi_2(x)$ denote the number of pairs of twin primes (t_n, t_{n+1}) smaller than x . Then the conjecture B of Hardy and Littlewood [26] on the number of prime pairs $p, p+d$ applied to the case $d = 2$ gives, that

$$\pi_2(x) \sim C_2 \int_2^x \frac{u}{\log^2(u)} du = C_2 \frac{x}{\log^2(x)} + \dots, \quad (17)$$

where C_2 is called “twin constant” and is defined by the following infinite product:

$$C_2 \equiv 2 \prod_{p>2} \left(1 - \frac{1}{(p-1)^2} \right) = 1.32032363169\dots \quad (18)$$

If there is indeed (as everybody believes, see e.g. [31]) an infinity of twins, then the continued fraction

$$u_2 = [0; 3, 5, 5, 7, 11, 13, 17, 19, \dots] \quad (19)$$

should be infinite, non-periodic and convergent. We count here 5 two times as it is a customary way of defining the Brun’s constant [43] and it an only case of double appearance of a prime in the set of twins as for adjacent twin pairs $(p-2, p)$ and $(p, p+2)$ one of numbers $(p-2, p, p+2)$ always is divisible by 3. Again performing calculations in PARI and using primes < 10000 we found here 205 twin pairs (but only 409 different primes) and first 50 digits of the continued fraction (19) are

$$u_2 = 0.31323308098694591263078648647217280043925117451\dots \quad (20)$$

There is much less terms in u_2 up to 10000 than primes < 10000 in u , hence the value of u_2 was obtained with accuracy about 2900 digits. We have checked using Plouffe’s Symbolic Inverse Calculator (<http://pi.lacim.uqam.ca/eng/>), that this constant is not recognized as a combination of other mathematical quantities.

Because twin primes are sparser than all primes we have $t_n > p_n$ thus in view of (15) the geometrical means $(3 \cdot 5 \dots t_n)^{1/n}$ will diverge even faster, hence the continued fraction u_2 belongs to the set of exceptions to the Khinchin Theorem. It

is also a counterexample to the Levy Theorem, because denominator $Q_n(u_2)$ of the n -th convergent of u_2 is larger than the denominator $Q_n(u)$ of the n -th convergent of u .

2.3 Generalized d -twins

It is natural to consider the whole family of continued fractions u_d , $d = 2, 4, 6, 8, \dots$ formed from the *consecutive* primes separated by d : $p_{n+1} - p_n = d$. We put this example here after twins, although for sufficiently large d the primes $p_{n+1} - p_n = d$ will be even sparser than say Mersenne primes and from the other side $d = 2$ are less frequent than $d = 6$, see [35]. The consecutive primes separated by $d = 4$ are sometimes called Cousins, [50]. For example, in the case of $d = 6$ we have :

$$u_6 = [0; 23, 29, 31, 37, 47, 53, 53, 59, 61, 67, 73, 79, 83, 89, 131, 137, 151, 157, 157, 163, \dots]$$

and some primes p_n when $p_n - p_{n-1} = p_{n+1} - p_n$ do appear twice (in the case of u_2 only 5 appears two times). As in the case of twins it is conjectured that for each d there is an infinity of prime pairs (p_{n+1}, p_n) with $p_{n+1} - p_n = d$, see e.g. [10], [35]. From this conjecture it follows that the numbers u_d are irrational. Using PARI/GP we have calculated the values of u_d up to $d = 570$, what took four days of CPU time on the 64 bits AMD Opteron 2700 MHz processor. We have searched for primes up to $2^{44} = 1.759 \dots \times 10^{13}$ and the largest encountered gap between consecutive primes was $d = 706$ which appeared only once. We have calculated u_d if there was a number of gaps of given d sufficient to determine u_d with at least a few hundreds digits (for example, there were 17 pairs of 570-twins up to 2^{44}). The Table I gives a sample of obtained values with 50 digits accuracy. The whole file with 275 values of u_d given with more than 110 digits is available from the author webpage http://www.ift.uni.wroc.pl/~mwolf/u_d.dat.

For large d the value of u_d is practically determined by the first occurrence $p_f(d)$ of that gap — pairs of consecutive primes with gap $d \gg 2$ are separated by very large intervals, for example first $d = 540$ appears between $(738832927927, 738832928467)$

and next gap $d = 540$ is between $(3674657545087, 3674657545627)$. It was conjectured by Shanks in 1964 [42] that the gap d appears for the first time at the prime $p_f(d) \sim e^{\sqrt{d}}$. We have given heuristic arguments [49] that

$$p_f(d) \sim \sqrt{d} \exp\left(\frac{1}{2}\sqrt{\ln^2(d) + 4d}\right) \quad (21)$$

and for $d \gg 1$ simply $p_f(d) \sim \sqrt{de^{\sqrt{d}}}$. Thus we claim that for large d there should be the approximate formula:

$$u_d \approx [0; \sqrt{de^{\sqrt{d}}}, \sqrt{de^{\sqrt{d}}} + d] \approx \frac{1}{\sqrt{de^{\sqrt{d}}}}. \quad (22)$$

The plot of u_d and comparison with the Shanks and our conjecture is given in the Fig.1.

Again like u and u_2 continued fractions u_d belongs to the set of exceptions to the Khinchin Theorem and Levy Theorem.

2.4 Primes of the form $m^2 + n^4$

In the seminal paper [24] John Friedlander and Henryk Iwaniec have proved that there exists infinity of primes of the form $m^2 + n^4$. More precisely, if $\pi_{\text{FI}}(x)$ denotes the number of primes of the form $m^2 + n^4 < x$ then approximately

$$\pi_{\text{FI}}(x) \sim \frac{C_{\text{FI}}x^{3/4}}{\log(x)} \quad (23)$$

where the constant $C_{\text{FI}} = \sqrt{2}\Gamma(\frac{1}{4})^2/3\pi^{3/2} = 1.112835788898764 \dots$ and here Γ is the Euler Gamma function. Thus taking as partial quotients of the continued fraction primes of the form $m^2 + n^4$ for sure we will obtain an irrational number which we will denote u_{FI} :

$$u_{\text{FI}} = [0; 2, 5, 17, 17, 37, 41, 97, 97, \dots] \quad (24)$$

Like in previous examples some primes appear twice: $17 = 4^2 + 1^4 = 1^2 + 2^4$, $97 = 9^2 + 2^4 = 4^2 + 3^4$ etc. Looking for all primes of this form with $1 \leq m \leq 100$ and

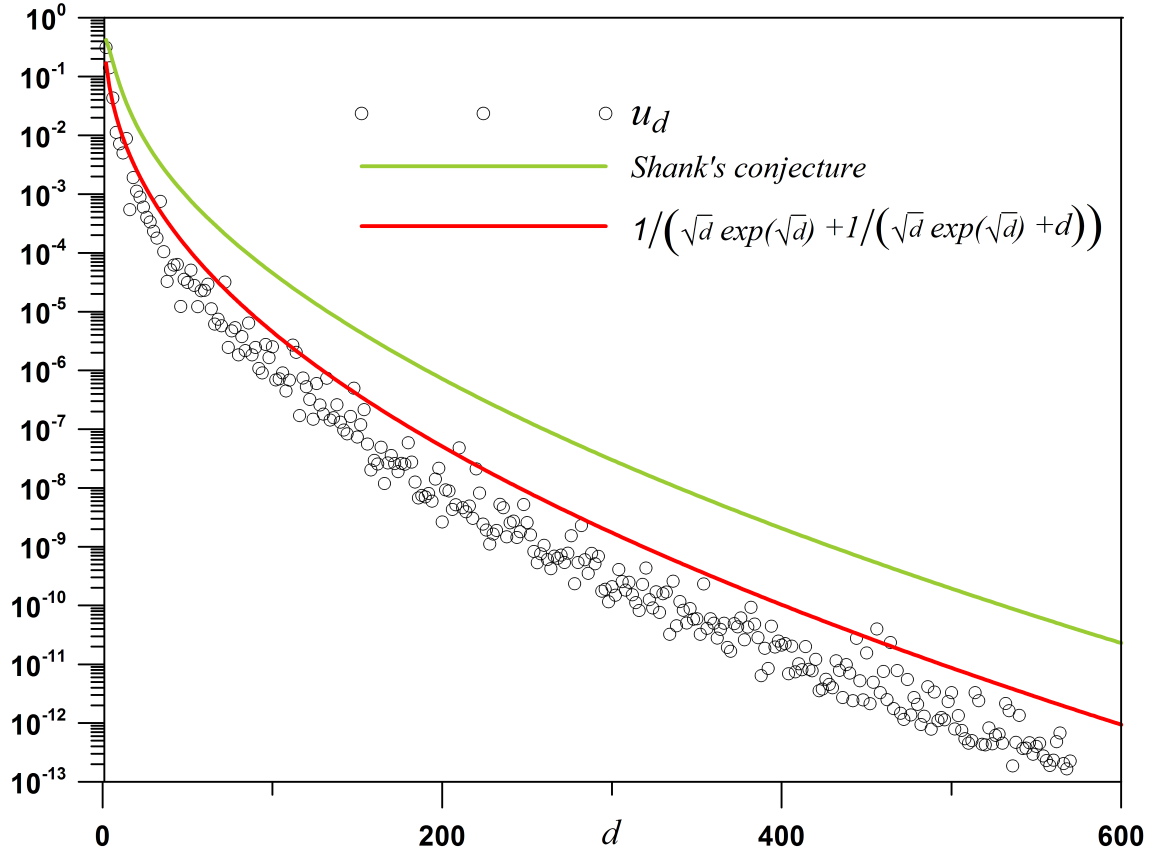


Fig.1 The plot of u_d and two approximations: in green the Shank's conjecture

$$\frac{1}{e^{\sqrt{d}} + \frac{1}{e^{\sqrt{d}} + d}} \quad \text{and in red our conjecture} \quad \frac{1}{\sqrt{d}e^{\sqrt{d}} + \frac{1}{\sqrt{d}e^{\sqrt{d}} + d}}$$

$1 \leq n \leq 10$ (the largest prime was $19801 = 99^2 + 10^4$) we get the value of u_{FI} with over 1100 digits accuracy; the first 50 digits of it are:

$$u_{\text{FI}} = 0.455024816490170022369052808279744824105755548905 \dots \quad (25)$$

Let us notice that

$$\frac{1}{(2 + 1/(5 + 1/(17 + 1/(17 + 1/(37 + 1/(41 + 1/(97 + 1/98)))))))} = \frac{20993638525}{46137348479} = 0.455024816490170022369048157801049432084768331968 \dots$$

and the difference between this value and u_{FI} is less than 10^{-23} !

Table I

d	u_d
4	$1.4103814184127409729946079947661391024642878552250 \times 10^{-1}$
6	$4.3413245800886640441937906138426444157119875018764 \times 10^{-2}$
8	$1.1234653732060451418609230935360294984983811524705 \times 10^{-2}$
10	$7.1938972705064358418419102215951120335820544247877 \times 10^{-3}$
12	$5.0250059564863844924667112008186998625931272954692 \times 10^{-3}$
14	$8.8489409307271044901495673780577102976304420791245 \times 10^{-3}$
16	$5.4614948350881467294308534284337241698766002935218 \times 10^{-4}$
18	$1.9120391314299159400657740968697274305281924125799 \times 10^{-3}$
20	$1.1273943145526585257207207582991176443515379616999 \times 10^{-3}$
22	$8.8573891094929851372874303656530678911673854053699 \times 10^{-4}$
24	$5.9916096230989554005997263265407846890656053212565 \times 10^{-4}$
26	$4.0371410525148524468010569219212401713453876041188 \times 10^{-4}$
28	$3.3658696996531260967017397551173798914121748535404 \times 10^{-4}$
30	$2.3272049015980164345521674554989676374011829679698 \times 10^{-4}$
32	$1.7885887465418665415382499015390795012182483537844 \times 10^{-4}$
34	$7.5357908538425634007656299916322144807040843935028 \times 10^{-4}$
36	$1.0470107727765143055064789951193220804598138780293 \times 10^{-4}$
38	$3.2687215994929278130910770751451289367042590019431 \times 10^{-5}$
40	$5.1725029603788623137563671868924142637218125718293 \times 10^{-5}$
42	$6.1954029872477528100134249220879079074519481392595 \times 10^{-5}$
44	$6.3763310332564890009355447509046689625278954819441 \times 10^{-5}$
46	$1.2275511580096446939755547564625149207372813752259 \times 10^{-5}$
48	$3.5424563347877377649245656903453981296411399487963 \times 10^{-5}$
50	$3.1341084997626641267187094247975857118584579732840 \times 10^{-5}$
\vdots	\vdots
566	$2.0417988154535953561248601983565125430801657124094 \times 10^{-13}$
568	$1.6638019955234637865242752590874891539355008513604 \times 10^{-13}$
570	$2.2511824714719308536000694530283450847909292429681 \times 10^{-13}$

2.5 Primes of the form $m^2 + 1$

Now let us consider the set of prime numbers

$$\mathcal{Q} = \{2, 5, 17, 37, 101, 197, 257, 401, 577, 677, 1297, 1601, \dots\} \quad (26)$$

given by the quadratic polynomial $m^2 + 1$ and let q_n denote the n -th prime of this form. By the conjecture E of Hardy and Littlewood [26] the number $\pi_q(x)$ of primes $q_n < x$ of the form $q_n = m^2 + 1$ is given by

$$\pi_q(x) \sim C_q \frac{\sqrt{x}}{\log(x)}, \quad (27)$$

where

$$C_q = \prod_{p \geq 3} \left(1 - \frac{(-1)^{(p-1)/2}}{p-1} \right) = 1.372813462818246009112192696727 \dots \quad (28)$$

Comparing it with (23) we see that indeed primes $m^2 + 1$ are sparser than primes $m^2 + n^4$. For example up to 10^8 there are 65162 primes of the form $m^2 + n^4$ and only 841 primes of the form $m^2 + 1$. Although the conjecture (27) remains unproved there is no doubt in its validity. Thus let us create the presumedly infinite continued fraction by identifying $a_n = q_n, n \geq 1$:

$$u_q = [0; 2, 5, 17, 37, 101, 197, 257, 401, 577, 677, 1297, 1601, \dots]. \quad (29)$$

Using 841 primes of the form $m^2 + 1$ smaller than 10^8 and performing the calculations in PARI with precision set to 20000 digits we get over 11000 digits of u_q as the ratio on the rhs of (5) was $< 10^{-11700}$. First 50 digits of u_q reads:

$$u_q = 0.45502569980199468718020210263808421898137687948 \dots \quad (30)$$

Let us remark that $u_{\text{FI}} - u_q = 8.833 \dots \times 10^{-7}$.

There is no known formula analogous to (11) for primes of the form $m^2 + 1$, but because $q_n \geq p_n$ the geometrical means of $2 \cdot 5 \cdot 17 \dots q_n$ will diverge faster than (15).

It is possible to obtain very rough speed of divergence of $(2 \cdot 5 \cdot 17 \dots q_n)^{1/n}$. Namely, making use of (27) and inverting $\pi_q(q_n) = n$ we get:

$$q_n \sim \left(\frac{2n \log(n/C_q)}{C_q} \right)^2 + 2 \log \left(\frac{n}{C_q} \right) \log \log \left(\frac{n}{C_q} \right) \quad (31)$$

Because $2 > C_q$ it follows that $2 \cdot 5 \cdot 17 \dots q_n$ grows faster than $2^{2n}(n!)^2/C_q^{2n} > (n!)^2$ and the Stirling formula for $n!$ gives that $(2 \cdot 5 \cdot 17 \dots q_n)^{1/n}$ grows faster than n^2 and again u_q is the exception to the Khinchin Theorem as well as to the Levy Theorem.

2.6 Mersenne primes

The Mersenne primes \mathcal{M}_n are the primes of the form $2^p - 1$ where p must be a prime, see e.g. [37, Sect. 2.VII]. Only 47 primes of this form are currently known, see Great Internet Mersenne Prime Search (GIMPS) at www.mersenne.org. For many years the largest known primes are the Mersenne primes, as the Lucas–Lehmer primality test (applicable only to $\mathcal{M}_n = 2^p - 1$) needs just a multiple of p steps, thus the complexity of checking primality of \mathcal{M}_n is $\mathcal{O}(\log(\mathcal{M}_n))$. Let us remark that algorithm of Agrawal, Kayal and Saxena (AKS) for general prime p works in about $\mathcal{O}(\log^{7.5}(p))$ steps and modification by Lenstra and Pomerance in about $\mathcal{O}(\log^6(p))$ steps.

Again there is no proof of the infinitude of \mathcal{M}_n but a common belief is that as there are presumedly infinitely many even perfect numbers thus there is also an infinity of Mersenne primes.

Let us define the supposedly infinite and convergent continued fraction $u_{\mathcal{M}}$ by taking $a_n = \mathcal{M}_n$:

$$u_{\mathcal{M}} = [0; 3, 7, 31, 127, 8191, 131071, 524287, 2147483647, \dots] \quad (32)$$

Using all 47 Mersenne primes $3, 7, 31, \dots, 2^{43112609} - 1$ in a couple of minutes we have calculated $u_{\mathcal{M}}$ with the precision better than $10^{-121949117}$; first 50 digits of $u_{\mathcal{M}}$ are:

$$u_{\mathcal{M}} = 0.31824815840584486942596202748140694243806236564 \dots \quad (33)$$

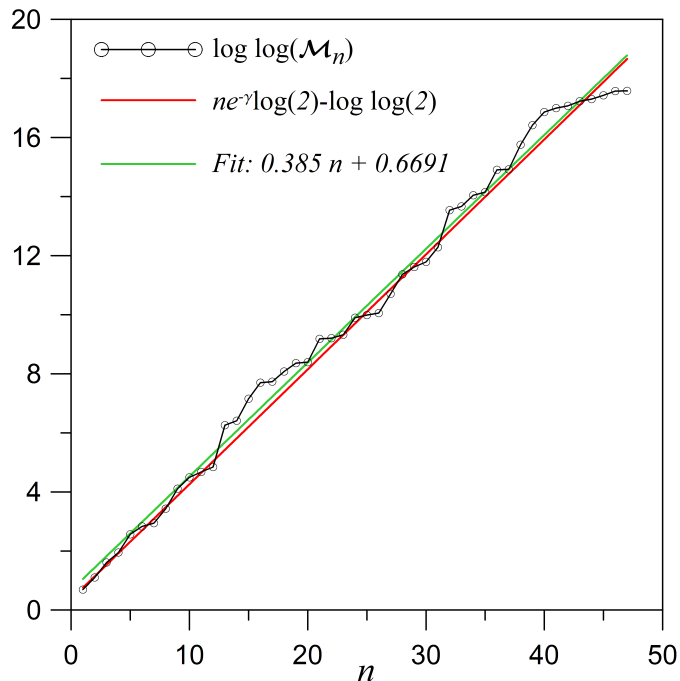


Fig.2 The plot of $\log \log(\mathcal{M}_n)$ and the Wagstaff conjecture (34). The fit was made to all known \mathcal{M}_n and it is $0.3854n + 0.6691$, while $ne^{-\gamma} \log(2)n - \log \log(2) \approx 0.3892n + 0.3665$. The rather good coincidence of $\log \log(\mathcal{M}_n)$ and (34) is seeming, as to get original \mathcal{M}_n 's the errors are amplified to huge values by double exponentiation.

Of course $u_{\mathcal{M}}$ is also the exception to the Khinchin and Levy Theorems in view of the very fast growth of $u_{\mathcal{M}}$ — Wagstaff conjectured [48], that \mathcal{M}_n grow doubly exponentially:

$$\log_2 \log_2 \mathcal{M}_n \sim ne^{-\gamma}. \tag{34}$$

where $\gamma = 0.57721566\dots$ is the Euler–Mascheroni constant. In the Fig. 2 we compare the Wagstaff conjecture with all 47 presently known Mersenne primes.

2.7 Primorial primes

If p_n is the n -th prime number then numbers of the form $2 \times 3 \times 5 \cdots \times p_n \equiv p_n\sharp$ are called primorials and \sharp stands here by analogy of exclamation mark in the factorial. The primorials are expressed directly by the Chebyshev function $\theta(x)$:

$$p_n\sharp = e^{\theta(p_n)} = e^{(1+o(1))p_n}. \tag{35}$$

For some primes r_n the numbers $r_n\# \pm 1$ are primes. They are called primorial primes and are even sparser than Mersenne primes as we will see below. Despite this rareness of primorial primes it was conjectured that there is infinity of them [13] and thus the continued fractions

$$u_{r_+} = [0; 3, 7, 31, 211, \dots] \quad (36)$$

obtained from primorial primes of the form $r_n\# + 1$ will be at least irrational number, as well as the continued fraction build from primorial primes of the form $r_n\# - 1$:

$$u_{r_-} = [0; 5, 29, 2309, \dots]. \quad (37)$$

From the known presently only 22 (see sequence A005234 in OEIS) primorial primes $r_n\# + 1$ we get the continued fraction

$$u_{r_+} = [0; 3, 7, 31, 211, \dots, 42209\# + 1, 145823\# + 1, 366439\# + 1, 392113\# + 1] = \quad (38)$$

$$0.318248165083690124777685589996787844788657122331533049467 \dots$$

with the error less than $10^{-914474}$. Let us remark that $u_{r_+} - u_{\mathcal{M}} = 6.678 \dots \times 10^{-9}$, although only three first primes (3, 7, 37) are the same.

From all 18 presently known (see sequence A006794 in OEIS) primes of the form $r_n\# - 1$ we get

$$u_{r_-} = [0; 5, 29, 2309, \dots, 15877\# - 1] = \quad (39)$$

$$0.198630157303503810875201233614346862875870630898479777625647 \dots$$

with the error less than 10^{-48415} . Chris K. Caldwell and Yves Gallot gave heuristic arguments [13] that there is infinity of primorial primes of both kinds. More precisely, they claim that the expected numbers of primorial primes of each of the forms $r\# \pm 1$ with $r < x$ are both approximately $e^\gamma \log(x)$. From $n = e^\gamma \log(r_n)$ we get that

$$r_n \sim e^{n/e^\gamma}, \quad (40)$$

where r_n stands for n -th prime giving the primorial prime of the form $r_n\# \pm 1$. Then the n -th primorial prime, and hence a_n of $u_{r\pm}$, will be

$$a_n = r_n\# = \exp\left(e^{-\gamma} \sum_{p \leq r_n} p\right). \quad (41)$$

From the formula

$$\sum_{p \leq n} f(p) = \int_2^n \frac{f(x)}{\log(x)} dx + f(2)li(2) + f(n)(\pi(n) - li(n)) - \int_2^n \{\pi(x) - li(x)\} dx \quad (42)$$

where $\pi(x)$ is the number of primes $< x$ and $li(x) = \int_2^x dx/\log(x)$ is the logarithmic integral, see [38, eq.(2.26)] we get

$$\sum_{p \leq n} p = li(n^2) + error = \frac{n^2}{2 \log(n)} + error'. \quad (43)$$

Here $error'$ besides expressions on rhs in (42) contains also higher terms coming from the asymptotic expansion of $li(x^2)$. Finally we obtain

$$r_n\# \approx \exp\left(\frac{1}{2n} \exp(2e^{-\gamma}n)\right). \quad (44)$$

From this it follows that the n -th primorial prime is much larger than the n -th Mersenne prime $\mathcal{M}_n \sim 2^{2^{n/e^\gamma}}$. Indeed, the ratio:

$$\frac{\log(r_n\#)}{\log(\mathcal{M}_n)} = \frac{1}{2 \log(2)n} \left(\frac{e^2}{2}\right)^{ne^{-\gamma}} \quad (45)$$

grows with n .

3 Continued fractions and transcendence

There is a vast literature concerning the transcendentality of continued fractions. The Theorem of H. Davenport and K.F. Roth [16] asserts, that if the denominators Q_n of convergents of the continued fraction $r = [a_0; a_1, a_2, \dots]$ fulfill

$$\limsup_n \frac{\sqrt{\log(n)} \log(\log(Q_n(r)))}{n} = \infty \quad (46)$$

then r is transcendental. This theorem requires for the transcendence of r very fast increase of denominators of the convergents: at least doubly exponential growth is required for 46. The set of continued fractions which can satisfy the Theorem of H. Davenport and K.F. Roth is of measure zero, as it follows from the Theorem 31 from the Khinchin's book [29], which asserts there exists an absolute constant B such that for *almost all* real numbers r and sufficiently large n the denominators of its continued fractions satisfy:

$$Q_n(r) < e^{Bn}. \quad (47)$$

The paper of A. Baker [6] from 1962 contains a few theorems on the transcendentality of Maillet type continued fractions [34], i. e. continued fractions with bounded partial quotients which have transcendental values. In the paper [3] B. Adamczewski and Y. Bugeaud, among others, have improved (46) to the form:

$$\text{If } \limsup_n \frac{\log(\log(Q_n(r)))}{n^{2/3}(\log(Q_n(r)))^{2/3} \log(\log(Q_n(r)))} = \infty \quad (48)$$

then r is transcendental.

Besides Maillet continued fractions there are some specific families of other continued fractions of which it is known that they are transcendental. In the papers [36], [4] it was proved that the Thue–Morse continued fractions with bounded partial quotients are transcendental. Quite recently there appeared the preprint [12] where the transcendence of the Rosen continued fractions was established. For more examples see [5].

Taking as the partial quotients a_n different sequences of numbers leads to real numbers which very often turn out be transcendental. For example, the continued fraction s for which $a_n = n$:

$$s = [0; 1, 2, 3, 4, \dots] = 1/(1 + 1/(2 + 1/(3 + 1/(4 + \dots)))) = 0.697774657964 \dots \quad (49)$$

is transcendental. Let us mention that the continued fraction s' with all partial quotients equal to consecutive odd numbers:

$$s' = [0; 1, 3, 5, 7, 9, \dots] = -i \tan(i) = \frac{e - e^{-1}}{e + e^{-1}} = 0.761594155955764888119 \dots \quad (50)$$

is also transcendental. All these facts are special cases of the results obtained by Carl Ludwig Siegel in 1929 in a long paper [44]. In particular, the continued fractions $s_D = [0; a_1, a_2, a_3, \dots]$ are transcendental when the a_n 's are rational and form an arithmetical sequence of the difference D and first element A : $a_k = A + kD$, $k = 1, 2, 3, \dots$. Siegel mentioned explicitly the continued fraction (49), see [44, or p.231 in *Gesammelte Abhandlungen* vol. I]. He obtained these results as corollaries from the continued fraction expansion of the ratio of Bessel's functions $J_\lambda(x)$ (see also [1, formula 9.1.73]):

$$i \frac{J_{\lambda-1}(2ix)}{J_\lambda(2ix)} = \frac{\lambda}{x} + \frac{1}{\frac{\lambda+1}{x} + \frac{1}{\frac{\lambda+2}{x} + \frac{1}{\frac{\lambda+3}{x} + \dots}}} \quad (51)$$

which Siegel has shown to be transcendental for rational λ and algebraic $x \neq 0$ and where the Bessel function of first order is given by

$$J_\lambda(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m! \Gamma(m + \lambda + 1)} \left(\frac{x}{2}\right)^{2m+\lambda}. \quad (52)$$

where $\Gamma(x)$ is the Gamma function, see [44, first formula on p.231 in *Gesammelte Abhandlungen* vol. I] or [1, formula 9.1.10]. For $\lambda = 0$ and $x = 1$ and taking into account the relation $J_{-n}(x) = (-1)^n J_n(x)$, see e.g. [1, formula 9.1.5], we get the value of the continued fraction (49):

$$s = [0; 1, 2, 3, 4, \dots] = -i \frac{J_1(2i)}{J_0(2i)}. \quad (53)$$

The awkward form (51) can be written in more pleasant form in terms of modified Bessel functions of the first kind $I_\nu(x)$ defined by the series:

$$I_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{k=0}^{\infty} \frac{(x/2)^{2k}}{k! \Gamma(\nu + k + 1)}. \quad (54)$$

There is a following relation between $I_\nu(x)$ and $J_\nu(ix)$:

$$I_\nu(x) = (-i)^\nu J_\nu(ix), \quad (55)$$

see [1, formula 9.6.3]. Writing $A = \lambda/x$, $D = 1/x$, i.e. $\lambda = A/D$, $x = 1/D$ we turn (51) to the more elegant form

$$[A; A + D, A + 2D, \dots, A + nD, \dots] = \frac{I_{A/D-1}\left(\frac{2}{D}\right)}{I_{A/D}\left(\frac{2}{D}\right)} \quad (56)$$

For $A = -1$, $D = 2$ (or for $\lambda = -1/2$ and $x = 1/2$ in (51)) we obtain the value of the continued fraction s' defined by the formula (50):

$$s' = [0; 1, 3, 5, 7, 9, \dots] = 1 + \frac{I_{-3/2}(1)}{I_{-1/2}(1)} = 1 + i \frac{J_{-3/2}(i)}{J_{-1/2}(i)}. \quad (57)$$

The transcendence of (50) follows for $x = i$, ($i = \sqrt{-1}$) from the formula known already to Lambert and Euler [22]:

$$\tan(x) = \frac{x}{1 - \frac{x^2}{3 - \frac{x^2}{5 - \ddots}}}. \quad (58)$$

and the fact that $\tan(x)$ takes transcendental values at algebraic arguments. From this and from (57) as a byproduct we have the identity:

$$\tan(i) = i - \frac{J_{-3/2}(i)}{J_{-1/2}(i)}. \quad (59)$$

Another possibility for partial quotients is the geometrical series : $a_n = q^n$ and we believe that the corresponding continued fractions:

$$G_q = [0; q, q^2, q^3, \dots] \quad (60)$$

are transcendental for all natural $q \geq 2$. This continued fraction is linked to the famous Rogers–Ramanujan continued fraction defined by

$$RR(q) = \frac{q^{1/5}}{1 + \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \ddots}}}}. \quad (61)$$

From the general transformations of continued fractions rules see [30, p.9] we have the relation:

$$q^{4/5}RR(q) = \frac{q}{1 + \frac{q^2}{1 + \frac{q^3}{1 + \frac{q^4}{1 + \frac{q^5}{1 + \frac{q^6}{\ddots}}}}}} = \frac{1}{q^{-1} + \frac{1}{q^{-1} + \frac{1}{q^{-2} + \frac{1}{q^{-2} + \frac{1}{q^{-3} + \frac{1}{q^{-3} + \ddots}}}}}} \quad (62)$$

In [19] [20] it was proved that $RR(q)$ is transcendental for all algebraic $|q| < 1$, but it needs some further work to infer from this the transcendence of G_q . Let us mention, that quite recently K. Dilcher and K. B. Stolarsky [18] have proved that the continued fractions

$$g_q = [q; q^2, q^4, q^8, \dots, q^{2^n}, \dots] \quad (63)$$

are transcendental for all integer $q \geq 2$ — it follows immediately from (46) and the double exponential growth: $Q_n(g_q) > \prod_{k=1}^n q^{2^k}$. Adamczewski [2] extended this to all complex $|q| > 1$ which are algebraic numbers. Another (family) class of transcendental continued fractions can be found in [17].

Next we can construct a number f where partial quotients are factorials $a_n = n!$:

$$f = [0; 1, 2, 6, 24, 120, \dots, n!, \dots] = 0.6840959001066225003396337\dots \quad (64)$$

Even these partial quotients increase too slowly to apply the Theorem of Adamczewski and Bugeaud (48). For large n we have approximately $Q_n(f) \sim \prod_{k=1}^n k!$. This last product is called superfactorial and denoted by $n\$,$ see also [25, exercise 4.55]. We prefer the notation $n! = \prod_{k=1}^n k!$. Superfactorial can be expressed by the Barnes G -function for complex z defined by

$$G(z+1) = (2\pi)^{z/2} e^{-(z(z+1)+\gamma z^2)/2} \prod_{n=1}^{\infty} \left[\left(1 + \frac{z}{n}\right)^n e^{-z+z^2/2n} \right]. \quad (65)$$

It satisfies the functional equation

$$G(z + 1) = \Gamma(z)G(z) \tag{66}$$

and from this we have that

$$n! = G(n - 2). \tag{67}$$

The analog of the Stirling formula for $G(z)$ gives [47]:

$$\log G(z + 1) = z^2 \left(\frac{\log(z)}{2} - \frac{3}{4} \right) + \frac{z}{2} \log(2\pi) - \dots \tag{68}$$

From this we obtain

$$n! \sim e^{n^2(\log(n)/2 - 3/4)} \tag{69}$$

and unfortunately

$$\frac{\sqrt{\log(Q_n(f))} \log \log(Q_n(f))}{n} \rightarrow 0 \tag{70}$$

hence we do not get transcendentality of f via the Theorem of Adamczewski and Bugeaud.

The continued fraction build from Fibonacci numbers $a_n = F_n$

$$F = [0; 1, 1, 2, 3, 5, 8, \dots, F_n, \dots] = 0.588873952548933507671231121246787384 \dots \tag{71}$$

appears at the Sloane The On-Line Encyclopedia of Integer Sequences as the entry A073822.

Apparently both f and F also should be transcendental, but we are not aware of the proof of this fact. The factorial over Fibonacci numbers behaves as

$$\prod_{k=1}^n F_k = \phi^{n(n+1)/2} 5^{-n/2} C + \mathcal{O}(\phi^{n(n-3)/2} 5^{-n/2}) \tag{72}$$

where $\phi = (1 + \sqrt{5})/2$ and $C \approx 1.226742$, see [25, Exercise 9.41] and it is too slow to use the Davenport – Roth Theorem.

Let us quote at the end of this Section the following remarks from the [7, p. 104]: “And the latter recalls to mind another outstanding question in Diophantine

approximation, namely whether every continued fraction with unbounded partial quotients is necessarily transcendental; this too seems very difficult". Now there is a common believe that also algebraic numbers of degree ≥ 3 have unbounded partial quotients, see e.g. [41], [3].

4 Transcendence of $u, u_2, u_d, u_{\text{FI}}, u_q, u_M, u_{r\pm}$

Because all considered above continued fractions are non-periodic (if there exist really infinity of twins, Mersenne primes etc) they can not be solutions of a polynomial equations with rational coefficients of the degree 2, but we believe this statement remains true for rational polynomials of all degrees. Namely we are convinced that all considered above continued fractions $u, u_2, u_d, u_{\text{FI}}, u_q, u_M, u_{r\pm}$ are transcendental, however we were not able to prove it and this problem seems to be extremely difficult. But if say u or u_2 is not transcendental what the particular polynomial equation with very special (mysterious) integer coefficients should it satisfy?

It is well known that the Champernowne constant [14] built by concatenating consecutive numbers in the base b is transcendental:

$$C_b = (\gamma_1)_b(\gamma_2)_b(\gamma_3)_b(\gamma_4)_b \dots \quad (73)$$

where $(\gamma_k)_b$ denotes number k expressed in the base b (e.g. in the common in computer science notation the twelfth number in the hexadecimal base $b = 16$ is denoted C). In the human base $b = 10$ the C_{10} is given by:

$$C_{10} = 0.12345678910111213141516171819202122232425262728293031 \dots$$

Transcendentality of C_b is the corollary from the theorem proved by Kurt Mahler in paper [33] published in 1937. In fact in this paper [33] Mahler has proved more general result: the number σ obtained by concatenating the values of the positive, integer-valued increasing polynomial $f(k)$ in the base b :

$$\sigma = f(1)_b f(2)_b f(3)_b \dots$$

is transcendental, where $f(k)_b$ denotes the digits of the value of $f(k)$ in the base system b . The case of Champernowne constant is not mentioned in [33] explicitly but it follows for $f(k) = k$. Let us remark, that the continued fraction expansion of C_{10} behaves very erratically, with sporadic partial quotients of enormous size, for example the 19-th term is of the order 10^{169} , what is the typical behaviour for the Liouville numbers, i.e. such numbers that for each n there will be infinity of rationals A/B such that $|C_{10} - A/B| < 1/B^n$, see spikes in the Fig. 8.

The number C_{CE} obtained by concatenation of 0. with the base 10 representations of the prime numbers in order

$$C_{CE} = 0.235711131719232931374143475153 \dots$$

is known as Copeland–Erdős constant [15]. In this paper Arthur Herbert Copeland and Paul Erdős have shown that C_{CE} is normal, but apparently it is not proved that C_{CE} is transcendental.

We have mentioned in the Sect.3 that $s = [0; 1, 2, 3, 4, 5, 6, \dots]$ is transcendental. Thus we have the correspondence $s \leftrightarrow C_b$ and $u \leftrightarrow C_{CE}$, where both elements of the former pair are shown to be transcendental and both members of the latter pair are conjectured to be transcendental. Of course we have $p_n > n$, $t_n > n$, $q_n > n$ etc. but we do not know how the transcendence of u , u_2 , u_q follows from these inequalities.

One of the transcendence criterion is the Thue-Siegel-Roth Theorem, which we recall here in the following form:

Thue-Siegel-Roth Theorem: If there exist such $\epsilon > 0$ that for infinitely many fractions A_n/B_n the inequality

$$\left| r - \frac{A_n}{B_n} \right| < \frac{1}{B_n^{2+\epsilon}} \quad n = 1, 2, 3, \dots, \quad (74)$$

holds, then r is transcendental.

Let us stress, that ϵ here does not depend on n — it has to be the same for all fractions A_n/B_n . This theorem suggests the following definition of the measure of irrationality $\mu(r)$: For a given real number r let us consider the set Δ of all such

exponents δ that

$$0 < \left| r - \frac{P}{Q} \right| < \frac{1}{Q^\delta} \quad (75)$$

has at most finitely many solutions (P, Q) where P and $Q > 0$ are integers. Then $\mu(r) = \inf_{\delta \in \Delta} \delta$ is called the irrationality measure of r (sometimes any δ fulfilling (75) is called irrationality measure and then the smallest $\delta = \mu$ is called the irrationality exponent). If the set Δ is empty, then $\mu(r)$ is defined to be ∞ and r is called a Liouville number. If r is rational then $\mu(r) = 1$ and if r is algebraic of degree ≥ 2 then $\mu(r) = 2$ by the Thue-Siegel-Roth Theorem. There exist real numbers of arbitrary irrationality measure $2 \leq \mu < \infty$. Namely, the number

$$[a] + \frac{1}{[a^b] + \frac{1}{[a^{b^2}] + \frac{1}{[a^{b^3}] + \dots}}} \quad (76)$$

where $a > 1$, $b = \mu - 1$, has the irrationality measure μ , see [11]. For the constant $e = \lim_{n \rightarrow \infty} (1 + 1/n)^n$, which has the continued fraction of a regular form:

$$e = [2; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, 1, 1, 10, 1, 1, 12, 1, 1, \dots, 1, 1, 2n, \dots], \quad (77)$$

it is known that $\mu(e) = 2$, see [8, pp.362-365]. For π it is known that $2 \leq \mu(\pi) < 7.6304$, see [40], and it is conjectured [9, p.203] that $\mu(\pi) = 2$. There is a bound $\delta(n) > 2$ for infinitely many n following from the fact that of any two consecutive convergents to r at least one satisfies the inequality

$$\left| r - \frac{P_n}{Q_n} \right| < \frac{1}{2Q_n^2}, \quad (78)$$

see [29, Theorem 18] or [27, Theorem 183] and further: of any three consecutive convergents to r , one at least satisfies

$$\left| r - \frac{P_n}{Q_n} \right| < \frac{1}{\sqrt{5}Q_n^2}, \quad (79)$$

see [29, Theorem 20] or [27, Theorem 195]. Thus writing for convergents satisfying (78) or (79) appropriately $\epsilon(n) = \log(2)/\log Q_n$ and $\epsilon'(n) = \log(5)/2\log Q_n$ the inequality appearing in the Thue-Siegel-Roth Theorem will be satisfied for a given specific n . Of course fractions P_n/Q_n constructed in this way will have $\lim_{n \rightarrow \infty} \epsilon(n) = 0$, because Q_n increase monotonically and there will be no exponent of Q_n on the r.h.s. of (74) strictly larger than 2 and common for all n . In fact, Khinchin [29] has proved that almost all reals r have $\mu(r) = 2$.

The partial quotients of $u, u_d, u_q, u_{\text{FI}}$ grow too slow to use the Davenport—Roth Theorem, but if the behaviour of the Mersenne primes $\mathcal{M}_n \sim 2^{2^{ne^{-\gamma}}}$ mentioned at the end of Sect. 6 is valid, then we obtain for large n

$$Q_n > 2^{c2^{(n+1)e^{-\gamma}}}, \quad c = \frac{1}{2^{e^{-\gamma}} - 1} = 2.101893933\dots \quad (80)$$

and transcendence of $u_{\mathcal{M}}$ will follow from the Davenport—Roth Theorem (46). We illustrate the inequality (80) in the Figure 3 — the values of labels on the y -axis give an idea of the order of $Q_n(\mathcal{M}_n)$: the largest for $n = 47$ is of the order $Q_{47} = e^{1.9984\dots \times 10^8} = 2.32928\dots \times 10^{86789810!}$

Usually the number r in question (for example $e, \pi, \zeta(3)$, etc.) is given by some definition not involving continued fractions, but here we have expressions of $u, \dots, u_{\mathcal{M}}$ **only** by continued fractions and we can not calculate directly the differences $|r - P_n/Q_n|$, like it is possible for example for Liouville transcendental numbers or for e . For this last case, as mentioned earlier, the possibility of explicit calculation of the difference $|e - P_n/Q_n|$ gives that $\mu(e) = 2$ see [8, pp. 351–371]. We do not have any ideas now how to express $g_A, f, u, u_d, u_q, \dots$ independently by means of formulas not involving continued fractions. Nevertheless we have made the plot of the exponent $\delta(n)$ in the difference:

$$\left| U - \frac{P_n}{Q_n} \right| = \frac{1}{Q_n^{\delta(n)}} \quad n = 1, 2, 3, \dots, \quad (81)$$

where U stands for u, u_d, u_q, \dots and P_n/Q_n are convergents of continued fractions for U — it is a well known fact that convergents of continued fractions are the best rational approximations.

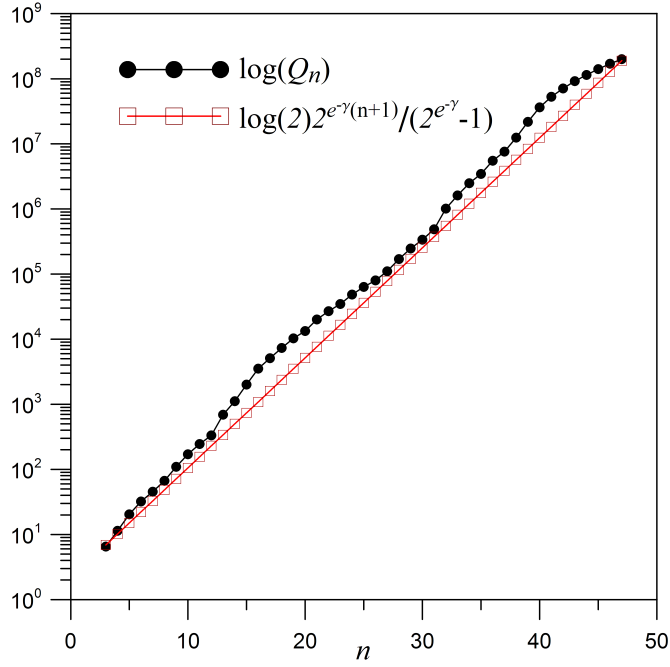


Fig.3 Illustration of the inequality (80) for $3 \leq n \leq 47$. Although the last points seem to coincide in fact $Q_{47} = 2.32928 \dots \times 10^{86789810}$, while $2^{e^{248e^{-\gamma}}} = 1.21513 \dots \times 10^{82034318}$ — hundreds thousands orders of difference!

In the Figures 4–8 we present plots of $\delta(n)$ for u, u_q and for π, e as well as for C_{10} for comparison. First we have calculated u, u_q, \dots with 150000 digits accuracy from the generic definition by constructing the continued fractions with a many thousands partial denominators. Next we have calculated P_n and Q_n for n until the difference $|U - P_n/Q_n|$ was zero in prescribed accuracy. From the differences $|U - \frac{P_n}{Q_n}|$ we calculated the values of $\delta(n) = -\log |U - \frac{P_n}{Q_n}| / \log(Q_n)$ and the sample of results is plotted on Fig.4 and 5 for u and u_q . The bound following from (78) is fulfilled for all $n < 2100$. In the next Figures we present the plot of $\delta(n)$ for π (Fig.6), e (Fig.7) and C_{10} (Fig.8). For C_{10} we have plotted $\delta(n) - 2$ because values of this difference changes by many orders, in contrast to smooth behavior seen in the Figs. 4-7. The spikes seen in the Fig.8 are similar to the behavior of the Liouville transcendental numbers, but the last statement in [33] asserts that C_{10} is not the Liouville number.

In [45] J. Sondow has proved that:

$$\mu(r) = 1 + \limsup_{n \rightarrow \infty} \frac{\log Q_{n+1}}{\log Q_n} = 2 + \limsup_{n \rightarrow \infty} \frac{\log a_{n+1}}{\log Q_n} \quad (82)$$

From this we have for u as $a_n = p_n \sim n \log(n)$ and for large n $Q_n \sim n^n$ that

$\mu(u) = 2$ and the same for u_2, u_q , but for Mersenne primes we get from the Wagstaff conjecture:

$$\mu(u_{\mathcal{M}}) < 2 + 2^{e^{-\gamma}} - 1 = 2.47477\dots \quad (83)$$

But if there is only finite number of Mersenne primes (and hence finitely many even perfect numbers), then $\mu(u_{\mathcal{M}}) = 1$. In the Fig.9 we present the plot of $\delta(\mathcal{M}_n; n) = -\log|u_{\mathcal{M}} - P_n/Q_n|/\log(Q_n)$ and indeed the values oscillate around $1 + 2^{e^{-\gamma}} = 2.47477\dots$

We propose the conjecture which indicates the way to deduce the transcendence of some continued fractions from transcendence of another ones:

Conjecture (\star): Let $\alpha = [a_0; a_1, a_2, \dots]$, $\lim_{n \rightarrow \infty} a_n = \infty$, and $\beta = [b_0; b_1, b_2, \dots]$, where $a_n, b_n \in \mathbb{N}$. Suppose there exists such n_0 that for all $n > n_0$ the inequality $b_n > a_n$ holds. If α is transcendental then β is also transcendental.

The condition $\lim_{n \rightarrow \infty} a_n = \infty$ is necessary: if a_n is bounded, say $a_n < A$ for $\forall n$, $A \in \mathbb{N}$, then $\beta = [A; 2A, 2A, \dots] = \sqrt{1 + A^2}$. Also for transcendental b_n the above conjecture probably is not true. When the Conjecture (\star) will be proved it will suffice for our purposes to invoke the transcendence of the continued fraction $s = [0; 1, 2, 3, 4, \dots]$ (49), as for all examples from Sect.2 we have $a_n > n$ (then also some examples from Sect.3 will be transcendental, like f with $a_n = n!$ and F with $a_n = F_n$).

5 Final remarks

We have raised above some questions concerning the transcendence of continued fractions with partial quotients given by prime numbers of a few special forms. We hope that the experimental results reported above will stimulate further research in the field.

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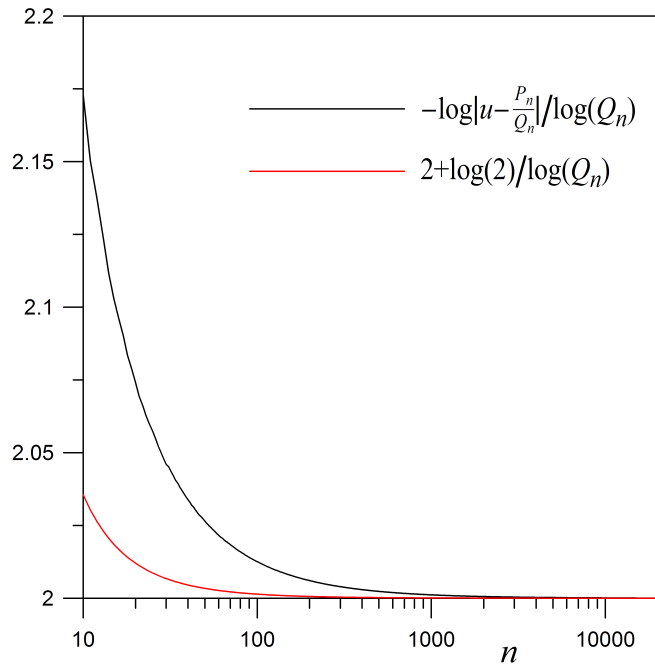


Fig.4 The plot of $\delta(n)$ (black) and the bound $2 + \log(2)/\log Q_n$ (red) for u following from the (78) up to $n = 22380$ (computations were done in precision 150000 digits and the value of $|u - P_{22380}/Q_{22380}|$ was zero with accuracy 150000 digits). For each n the bound (78) (as well as bound (79)) is fulfilled.

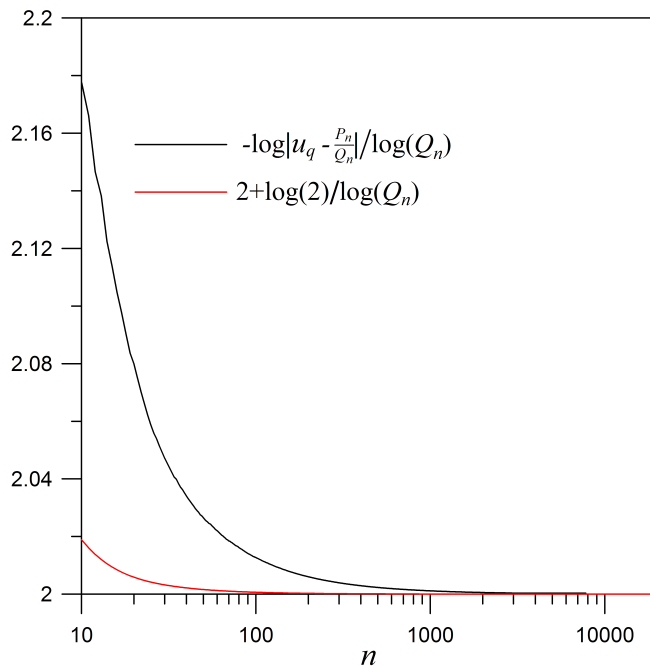


Fig.5 The plot of $\delta(n)$ (black) and the bound $2 + \log(2)/\log Q_n$ for u_q following from the (78). For each n the bound (78) is fulfilled.

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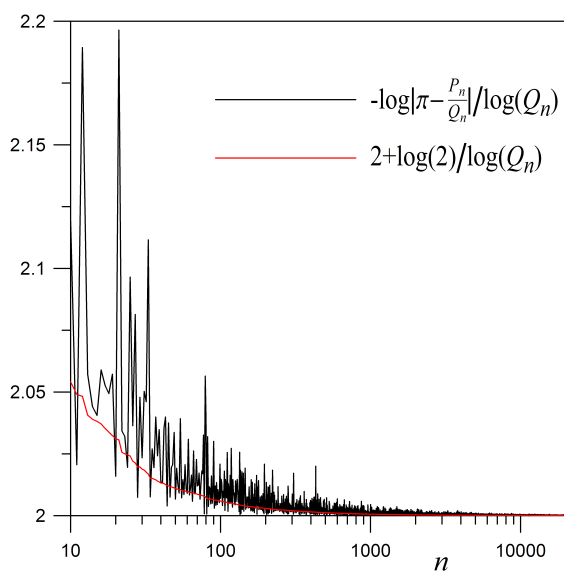


Fig.6 The plot of $\delta(n)$ (black) and the bound $2 + \log(2)/\log Q_n$ (red) for π following from the (78) up to $n = 20997$ (computatuions were done in precision 50000 digits and the value of $|\pi - P_{20997}/Q_{20997}|$ was zero with prescribed accuracy).

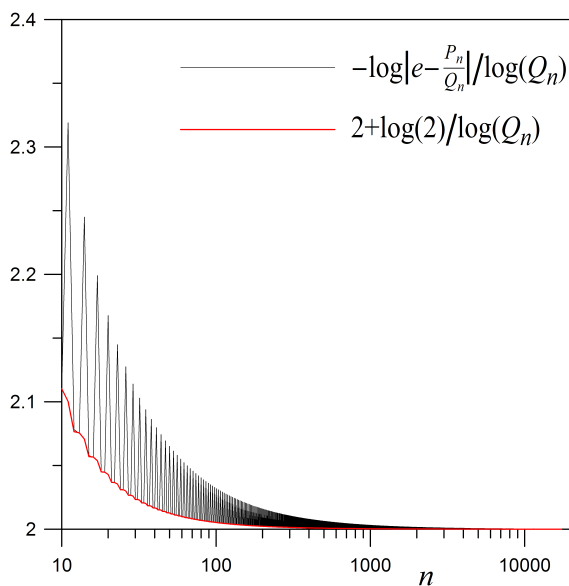


Fig.7 The plot of $\delta(n)$ (black) and the bound $2 + \log(2)/\log Q_n$ (red) for e following from the (78) up to $n = 17365$ (computatuions were done in precision 45500 digits and at this n the value of $|e - P_{17365}/Q_{17365}|$ was zero). The periodic structure of the continued fraction expansion for e is clearly seen.

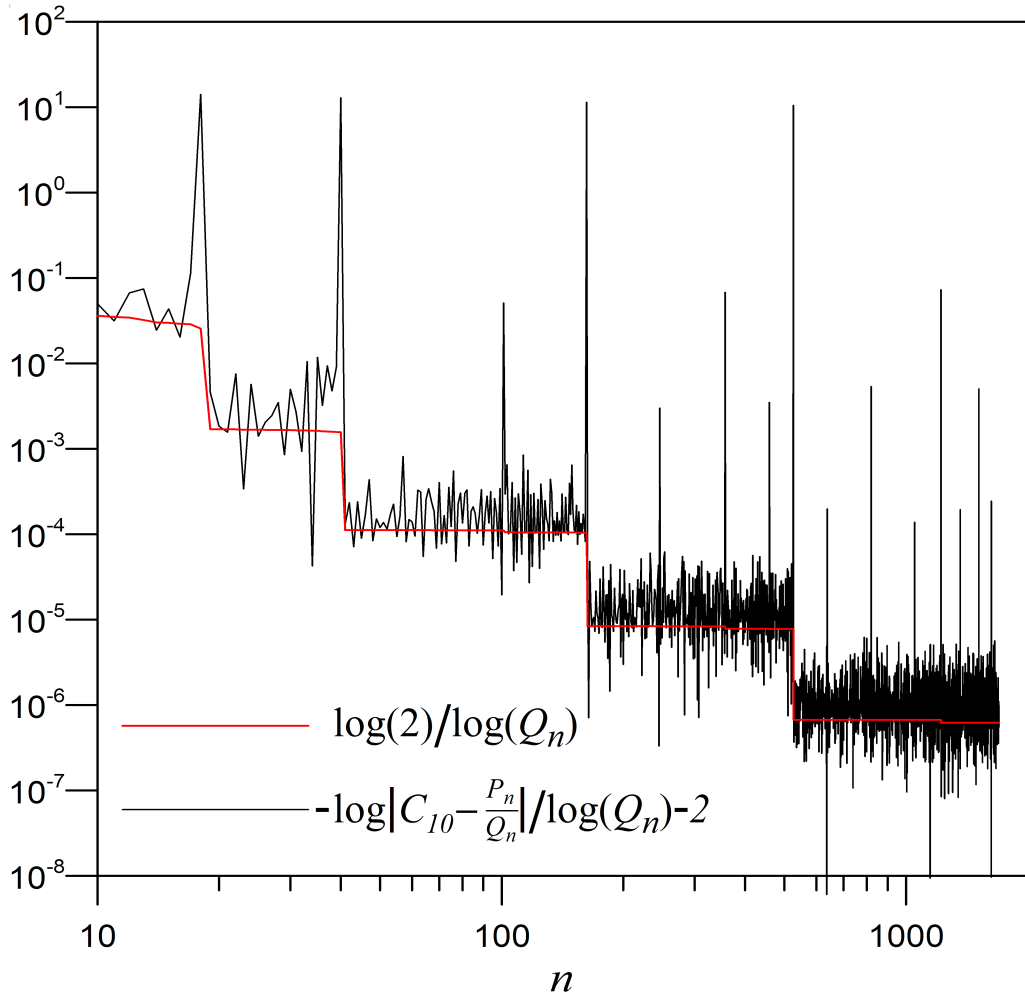


Fig.8 The plot of $\delta_{C_{10}}(n) - 2$ (black) and the bound $\log(2)/\log Q_n$ (red) for C_{10} following from the (78). Because of the weird behavior of the partial quotients of continued fraction expansion for C_{10} we have subtracted 2 from $\delta(n)$ and plotted the graph with the y axis in the logarithmic scale. After each extremely large partial quotient a_n there is an abrupt drop in the values of $\delta(n)$ and the bound $\log(2)/\log Q_n$ with accompanying spike for $n - 1$, see (5). It took almost 4 days CPU time to get data for this plot. Collecting data was done in a few separate runs with different precisions. Because the partial quotient $a_{526} > 10^{411100}$ and $Q_{527} > 10^{449994}$ the calculations for $526 \leq n < 1708$ was performed with 1,000,000 digits precision, see eq.(5). We stopped at $n = 1707$ because $a_{1708} > 10^{4911098}$. Spikes of $\delta_{C_{10}}(n)$ many orders higher than neighboring values suggest that C_{10} may be the transcendental number of Liouville type, but it is in contradiction with the last statement of the paper [33].

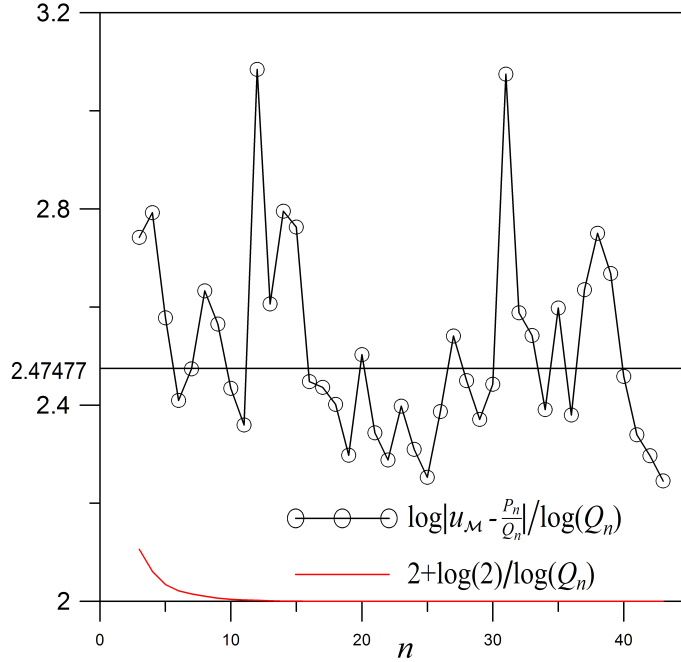


Fig.9 The plot of $-\log |u_{\mathcal{M}} - P_n/Q_n|/\log(Q_n)$ (black) and the bound $2 + \log(2)/\log Q_n$ (red) following from the (78) for $3 \leq n \leq 43$. Here the value of $u_{\mathcal{M}}$ was obtained from all 47 known Mersenne primes with more than 120 millions digits: the accuracy was better than $10^{-121949117}$. The denominators Q_n grow very fast and the bound $2 + \log(2)/\log(Q_n)$ tends quickly to 2. It took 12 days CPU time on the AMD Opteron 2700 MHz processor to collect data for $n \leq 40$: the point $n = 40$ needed precision of almost 40,000,000 digits, as $|u_{\mathcal{M}} - P_{40}/Q_{40}| = 1.5033 \times 10^{-38789567}$, while $1/Q_{40}^2 = 4.501 \dots \times 10^{-31553835}$. To calculate the difference $|u_{\mathcal{M}} - P_n/Q_n|$ for $n = 41, 42, 43$ the precision of 100000000 digits was needed and one point took 6 days on the same processor, as for example $|u_{\mathcal{M}} - P_{43}/Q_{43}| < 10^{-89770217}$.

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