Dickman polylogarithms and their constants

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Abstract

The Dickman function $F(\alpha)$ gives the asymptotic probability that a large integer N has no prime divisor exceeding N^α. It is given by a finite sum of generalized polylogarithms defined by the exquisite recursion $L_k(\alpha) = -\int_\alpha^{1/k} \mathrm{d}x L_{k-1}(x/(1-x))/x$ with $L_0(\alpha) = 1$. The behaviour of these Dickman polylogarithms as $\alpha \to 0$ defines an intriguing series of constants, C_k. I conjecture that $\exp(\gamma z)/\Gamma(1-z)$ is the generating function for $\sum_{k\geq 0} C_k z^k$. I obtain high-precision evaluations of $F(1/k)$, for integers $k < 11$, and compare the Dickman problem with problems in condensed matter physics and quantum field theory.

1 Introduction

The fraction of positive integers less than N with no prime divisor greater than N^α tends to a finite limit $F(\alpha)$ as $N \to \infty$. Clearly $F(\alpha) = 1$ for $\alpha \geq 1$. For $0 < \alpha < 1$, the Dickman [21] function $F(\alpha)$ satisfies the remarkable differential equation [18, 35, 16, 26]

$$F'(\alpha) = \frac{1}{\alpha} F\left(\frac{\alpha}{1-\alpha}\right) \tag{1}$$

and may be computed as a finite sum of terms

$$F(\alpha) = \sum_{k=0}^{K(\alpha)} L_k(\alpha) \tag{2}$$

where $L_0(\alpha) = 1$, $K(\alpha)$ is the largest integer such that $\alpha K(\alpha) < 1$, and the recursion

$$L_k(t) = -\int_t^{\frac{1}{k}} L_{k-1}\left(\frac{x}{1-x}\right)\frac{\mathrm{d}x}{x} \tag{3}$$

defines the Dickman polylogarithm of weight $k > 0$ as an iterated integral. I shall prove two theorems that allow me to obtain at least 100 digits of $L_k(t)$ for all weights $k < 10$ and hence shall conjecture the generating function for the Dickman constants

$$C_k = \lim_{t\to 0}\left(L_k(t) - \sum_{j=1}^{k} \frac{C_{k-j}\log^j(t)}{j!}\right) \tag{4}$$

where $C_0 = 1$ and C_k is determined by the boundary condition $L_k(\frac{1}{k}) = 0$ for $k > 0$.

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2 Dickman trilogs from standard trilogs

Since $L_1(t) = \log(t)$, we have $C_1 = 0$ and $F(1/2) = 1 - \log(2)$. Moreover, we easily obtain [17]

$$L_2(t) = -\int_t^{\frac{1}{2}} \log\left(\frac{x}{1-x}\right)\frac{\mathrm{d}x}{x} = \mathrm{Li}_2(t) + \frac{1}{2}\log^2(t) - \frac{\pi^2}{12} \tag{5}$$

where $\mathrm{Li}_k(t) = \sum_{n>0} t^n/n^k$ is the standard polylogarithm of weight k and $C_2 = -\frac{\pi^2}{12}$ ensures that $L_2(\frac{1}{2}) = 0$. Thus

$$F(1/3) = 1 - \log(3) + \sum_{n=1}^{\infty}\frac{1}{n^2 3^n} + \frac{1}{2}\log^2(3) - \frac{\pi^2}{12} \tag{6}$$

$$\approx 0.048608388291131566907183039343407421354329580478141 \tag{7}$$

may be computed with great facility.

After considerable effort, I shall prove in Section 5 that

$$\begin{aligned} L_3(t) &= \mathrm{Li}_3(2t-1) - \mathrm{Li}_3(1-2t) - \mathrm{Li}_3(t) - \mathrm{Li}_3(2-1/t) \\ &\quad + \log\left(\frac{t}{1-2t}\right)L_2(t) + \frac{\pi^2}{6}\log(t) - \frac{1}{6}\log^3(t) + \frac{17}{12}\zeta(3) \end{aligned} \tag{8}$$

and hence obtain the Dickman constant

$$C_3 = \lim_{t\to 0}\left(L_3(t) + \frac{\pi^2}{12}\log(t) - \frac{1}{6}\log^3(t)\right) = -\frac{1}{3}\zeta(3). \tag{9}$$

It follows from (2,5,8) that $F(\alpha)$ reduces to standard polylogs for $\alpha \geq \frac{1}{4}$. In particular, the asymptotic probability that an integer N has no prime divisor greater than $N^{\frac{1}{4}}$ is

$$\begin{aligned} F(1/4) &= 1 - 2\log(2) + \mathrm{Li}_2\left(\frac{1}{4}\right) + 2\log^2(2) - \frac{\pi^2}{12} \\ &\quad - \mathrm{Li}_3\left(\frac{1}{4}\right) - \mathrm{Li}_2\left(\frac{1}{4}\right)\log(2) - \frac{2}{3}\log^3(2) + \frac{13}{24}\zeta(3) \end{aligned} \tag{10}$$

$$\approx 0.0049109256477608323527391509236151860324842974176929 \tag{11}$$

with 800 good digits obtainable in less than 10 milliseconds.

3 Conjecture for the Dickman constants

I shall show, in later sections, how to evaluate $F(\alpha)$, for $\frac{1}{4} > \alpha \geq \frac{1}{10}$, and the intriguing constants C_k, for $3 < k < 10$, by one-dimensional numerical quadrature, at precisions well in excess of 100 decimal digits. This is amply sufficient to obtain very reliable conjectures for the analytic form of C_k. The complexity of these calculations is in marked contrast to the final simplicity of the conjectured results. For example,

$$\begin{aligned} C_4 &= \int_0^{\frac{1}{2}}\left(\log\left(\frac{x}{1+2x}\right)\mathrm{Li}_2(x) + \frac{1}{2}\log^2(x)\,\mathrm{Li}_1(-2x)\right)\frac{\mathrm{d}x}{x(1+x)} + 3\,\mathrm{Li}_4\left(\frac{1}{2}\right) \\ &\quad - \frac{3}{8}\mathrm{Li}_4\left(\frac{1}{4}\right) - \frac{3\log(2)}{4}\mathrm{Li}_3\left(\frac{1}{4}\right) + \frac{\pi^2 - 9\log^2(2)}{12}\mathrm{Li}_2\left(\frac{1}{4}\right) + \frac{21\log(2)\zeta(3)}{8} \\ &\quad + \frac{\pi^2\log^2(2)}{24} - \frac{\pi^2\log(2)\log(3)}{6} + \frac{\log^3(2)\log(3)}{2} - \frac{5\log^4(2)}{8} \end{aligned} \tag{12}$$

has a very simple conjectural evaluation as $\frac{1}{10}C_2^2$, which holds at 800-digit precision. The complexity of evaluation (12) results from delicate integration by parts, so as to remove divergent terms at the lower limit of integration as $t \to 0$ in (3) and leave a finite integral whose integrand contains no trilogs.

The constant C_8 is of particular interest, since that is the first case where a multiple zeta value (MZV) [40, 9, 10, 11, 6] might have occurred. However, after far more intricate numerical quadratures, the LLL [29] algorithm implemented by the Pari-GP [19] procedure `lindep` gave the conjectured evaluation

$$C_8 \stackrel{?}{=} \frac{\zeta(5)\zeta(3)}{15} - \frac{[\zeta(3)\pi]^2}{216} - \frac{67\pi^8}{29030400} \tag{13}$$

which is free of the irreducible MZV of lowest weight, namely $\zeta(5,3) = \sum_{m>n>0} 1/(m^5 n^3)$. After noticing that -67 is the first non-unit integer in the sequence $1, -1, 1, -1, -67, -1$, given by Neil Sloane [37] in OEIS entry A008991, for numerators of coefficients in the expansion of

$$\sqrt{\frac{\sin(x)}{x}} = 1 - \frac{x^2}{12} + \frac{x^4}{1440} - \frac{x^6}{24192} - \frac{67x^8}{29030400} - \frac{x^{10}}{5677056} + \mathrm{O}\left(x^{12}\right), \tag{14}$$

I was led to conjecture the wonderfully compact generating function

$$G(z) \equiv \frac{\exp(\gamma z)}{\Gamma(1-z)} \stackrel{?}{=} \sum_{k=0}^{\infty} C_k z^k \tag{15}$$

where γ is Euler's constant. The equivalent, yet more illuminating, formula

$$G(z) = \sqrt{\frac{\sin(\pi z)}{\pi z}} \exp\left(-\sum_{n>0} \frac{\zeta(2n+1)}{2n+1} z^{2n+1}\right) \tag{16}$$

neatly accounts for Sloane's tell-tale integer 67 in (13) and leads to the conjecture

$$C_9 \stackrel{?}{=} -\frac{\zeta(9)}{9} + \frac{\pi^2\zeta(7)}{84} - \frac{\pi^4\zeta(5)}{7200} + \frac{\pi^6\zeta(3)}{72576} - \frac{[\zeta(3)]^3}{162} \tag{17}$$

which was tested by numerical quadrature of a product of Dickman tetralogarithms, thanks to the following theorem.

4 Integration by parts

I define an auxiliary family of functions by the recursion

$$f_{n+1}(t) = -\int_t^{\frac{1}{n+1}} \frac{f_n(x)\,\mathrm{d}x}{x(1-nx)} \tag{18}$$

with $f_0(t) = 1$. Then $f_1(t) = \log(t) = L_1(t)$ and the dilogarithm

$$f_2(t) = -\int_t^{\frac{1}{2}} \frac{\log(x)\,\mathrm{d}x}{x(1-x)} = \log\left(\frac{t}{1-t}\right)\log(t) - L_2(t) \tag{19}$$

is easily related to $L_2(t)$, using integration by parts.

Theorem 1: Let

$$M_{k,n}(t) \equiv -\int_t^{\frac{1}{k}} L_{k-n-1}\left(\frac{x}{1-(n+1)x}\right)\frac{f_n(x)\,\mathrm{d}x}{x(1-nx)} \tag{20}$$

for any pair of integers $k > n \geq 0$. Then this integral evaluates to

$$M_{k,n}(t) = \sum_{m=0}^{n} (-1)^{n-m} L_{k-m}\left(\frac{t}{1-mt}\right) f_m(t). \tag{21}$$

Proof: At $n = 0$, definition (20) gives $M_{k,0}(t) = L_k(t)$, by virtue of recursion (3) for $L_k(t)$. Hence (21) holds at $n = 0$. For $k > n+1 > 0$, recursion (18) for $f_{n+1}(t)$ allows an integration by parts in definition (20), to obtain

$$M_{k,n}(t) = L_{k-n-1}\left(\frac{t}{1-(n+1)t}\right) f_{n+1}(t) - M_{k,n+1}(t) \tag{22}$$

with a vanishing constant term, since the L_{k-n-1} term vanishes at $t = 1/k$, where its argument $t/(1-(n+1)t)$ evaluates to $1/(k-n-1)$. Hence I prove (21) by induction, for all $k > n \geq 0$.

Comment: This is a very powerful result, peculiar to the Dickman problem. For example, it allows us to compute the Dickman heptalogarithm $L_7(t)$ very accurately, as a single integral of a product of trilogarithms, instead of having to evaluate a four-fold iterated integral of trilogs.

The key to the method is to observe that $f_{n+1}(t) = M_{n+1,n}(t)$. To prove this, I set $k = n+1$ in the definition (20) of $M_{k,n}(t)$ and then use recursion (18) for $f_{n+1}(t)$. I thus prove the claimed result (19) for $f_2(t)$ by setting $k = 2$ and $n = 1$ in the evaluation (21) of Theorem 1. Similarly, yet much more importantly, I obtain

$$f_3(t) = \log\left(\frac{t}{1-2t}\right) f_2(t) - L_2\left(\frac{t}{1-t}\right)\log(t) + L_3(t) \tag{23}$$

by setting $k = 3$ and $n = 2$, thereby avoiding duplication of the considerable effort expended in obtaining the trilogarthmic result (8) for $L_3(t)$. Then, by setting $k = 7$ and $n = 3$ in the theorem, I obtain the Dickman heptalogarithm

$$\begin{aligned} L_7(t) \quad &= \quad L_6\left(\frac{t}{1-t}\right)\log(t) - L_5\left(\frac{t}{1-2t}\right) f_2(t) + L_4\left(\frac{t}{1-3t}\right) f_3(t) \\ &\quad + \int_t^{\frac{1}{7}} L_3\left(\frac{x}{1-4x}\right)\frac{f_3(x)\,\mathrm{d}x}{x(1-3x)} \end{aligned} \tag{24}$$

with a one-dimensional quadrature of products of known trilogs, in the final term. Moreover, there are two methods of evaluating $L_6(t)$ as an integral of products of known dilogs and trilogs, using the pair (L_2, f_3) or the pair (L_3, f_2) in the final integrand. For $L_5(t)$ there are three methods, using (L_3, f_1), (L_2, f_2) or (L_1, f_3). Of these, the (L_2, f_2) pair is the most efficient. For $L_4(t)$ there are four methods, using (L_3, f_0), (L_2, f_1), (L_1, f_2) or (L_0, f_3), where $L_0(t) = f_0(t) = 1$. Efficiency dictates that one should use either the second or third pair; caution suggests that one should use all four methods to check the accuracy of numerical quadrature.

5 Dickman heptalogs from standard trilogs

To justify this methodology, I must first prove the claimed result (8), which reduces the Dickman trilog L_3 to standard trilogs and establishes my claim that $C_3 = -\frac{1}{3}\zeta(3)$. I begin by proving the rather simple identity

$$\begin{aligned} L_3(t) &= C_3 + C_2 \log(t) + \frac{1}{6}\log^3(t) - \mathrm{Li}_3(t) + \mathrm{Li}_2(t)\log(t) \\ &\quad + \int_0^t \left(\mathrm{Li}_2\left(\frac{x}{1-x}\right) + \frac{1}{2}\,\mathrm{Li}_1^2(x)\right)\frac{\mathrm{d}x}{x} \end{aligned} \tag{25}$$

where $C_2 = -\frac{\pi^2}{12}$ was determined by (5) and C_3 is an integration constant, to be determined later by the requirement that $L_3(\frac{1}{3}) = 0$. The proof of (25) is symptomatic: we simply differentiate with respect to t and check that $L_3'(t) = L_2(t/(1-t))/t$, as required by (3). Here, as ever, I use the relation $\mathrm{Li}_k'(t) = \mathrm{Li}_{k-1}(t)/t$, with $\mathrm{Li}_1(t) = -\log(1-t)$.

Next comes a harder part, namely to perform the integration in (25). Using integration by parts, I was able to reduce the problem to an instance of equation 8.111 in the fascinating and enormously informative book by Leonard Lewin [30], long since sadly out of print. Here, I am content to state and then to prove that

$$\begin{aligned} \frac{7}{4}\zeta(3) &= \int_0^t \left(\mathrm{Li}_2\left(\frac{x}{1-x}\right) + \frac{1}{2}\,\mathrm{Li}_1^2(x)\right)\frac{\mathrm{d}x}{x} \\ &\quad + \mathrm{Li}_3(1-2t) - \mathrm{Li}_3(2t-1) + \mathrm{Li}_3\left(\frac{-t}{1-2t}\right) \\ &\quad + \left(-\mathrm{Li}_2(t) + \frac{1}{6}\,\mathrm{Li}_1^2(2t) - \frac{1}{2}\,\mathrm{Li}_1(2t)\,\mathrm{Li}_1(1-t) + \frac{3}{2}\,\mathrm{Li}_2(1)\right)\mathrm{Li}_1(2t)\,. \end{aligned} \tag{26}$$

This is clearly true at $t = 0$, since $\mathrm{Li}_3(1) = \zeta(3)$ and $\mathrm{Li}_3(-1) = -\frac{3}{4}\zeta(3)$. Thus it is sufficient to show that the right hand side of (26) has a vanishing derivative. This derivative is of the form $D_1(t)/t + D_2(t)/(1-2t)$, where D_1 and D_2 are rather complicated combinations of dilogs and products of logs. However, it is easy to show that $D_1(0) = D_2(0) = 0$. Hence, to prove (26), it is sufficient to show that $D_1'(t) = D_2'(t) = 0$, which may be done by elementary manipulation of logs.

Inverting the argument of $\mathrm{Li}_3(-t/(1-2t))$, in (26), and imposing the boundary condition $L_3(\frac{1}{3}) = 0$, in (25), I arrive at the claimed result (8) for $L_3(t)$ and the claimed evaluation $3\,C_3 = -\zeta(3)$, provided that

$$3\left(2\,\mathrm{Li}_3\left(\frac{1}{3}\right) - \mathrm{Li}_3(-3)\right) - \log^3(3) = \frac{13}{2}\,\zeta(3). \tag{27}$$

The final hurdle of proving (27) was the most challenging. Spencer Bloch and Herbert Gangl told me that they expected the combination $2\,\mathrm{Li}_3(\frac{1}{3}) - \mathrm{Li}_3(-3)$ to evaluate to some rational multiple of $\zeta(3)$, "modulo logarithms". Yet it appears that such rational numbers are as distressingly hard to derive from first principles as they are disturbingly easy to guess from low precision numerical computation. My claim that $C_3 = -\frac{1}{3}\zeta(3)$ requires this rational multiple to be $\frac{13}{6}$, with a denominator divisible by 3. It was rather hard to see how a simple functional equation for trilogs of a single variable might produce such a denominator. Accordingly, I resorted to the Spence–Kummer functional relation for 9 trilogs of a pair of variables, given in equation 6.107 of Lewin's book [30]. Setting $x = -1$

and $y = \frac{1}{3}$ in that ornate identity and inverting the argument of $\mathrm{Li}_3(-\frac{1}{3})$, I proved (27), obtaining the combination $2\,\mathrm{Li}_3(1) - 6\,\mathrm{Li}_3(-1)$ on the right hand side. Thus the "morally rational" coefficient of $\zeta(3)$ for the combination $2\,\mathrm{Li}_3(\frac{1}{3}) - \mathrm{Li}_3(-3)$ is now proven to be $\frac{1}{3}(2-6(-\frac{3}{4})) = \frac{13}{6}$, thereby rescuing this particular problem from what Sasha Beilinson [4] memorably referred to as "the burdock thicket of generalities".

Having thus proven the trilogarthmic input for the method of Theorem 1, I am able to compute $F(\alpha)$ for $\alpha \geq \frac{1}{8}$, with great ease, and have provided 800 good digits for

$$F(1/5) \approx 3.5472470045603972983389451077062356095164361057262 \times 10^{-4} \quad (28)$$
$$F(1/6) \approx 1.9649696353955289651754986129204522894596719809623 \times 10^{-5} \quad (29)$$
$$F(1/7) \approx 8.7456699532939166955802835727699721733804719764580 \times 10^{-7} \quad (30)$$
$$F(1/8) \approx 3.2320693042261037725997853617282161576194751628024 \times 10^{-8} \quad (31)$$

at the web page `http://physics.open.ac.uk/~dbroadhu/cert/smoctic.txt` .

A significant amount of LLL analysis produced no simple integer relation of $F(1/5)$ to values of standards polylogs and their products. It might be interesting to investigate this issue more intensively, using David Bailey's parallelization [3] of Helaman Ferguson's PSLQ [22] algorithm, since more than 800 digits of $L_4(1/5)$ may now be computed from (3), using the explicit trilogarithms in (8). My own opinion, however, is that the Dickman probability $F(1/5)$ does not reduce to standard polylogs.

6 Dickman octalogs from Dickman tetralogs

It was rather frustrating that Theorem 1 did not provide a method for computing Dickman octalogs as single integrals of products of standard polylogs, since it was precisely at weight 8 that I wished to have high-quality numerical data with which to determine whether the first MZV, namely $\zeta(5,3)$, might show up in C_8. It was moral support from Mike Oakes [34] that determined me to push the investigation above weight 7.

The barrier that must now be surmounted stems from the fact that

$$M_4(y) \equiv \int_0^y \left(\log\left(\frac{x}{1+2x}\right)\mathrm{Li}_2(x) + \frac{1}{2}\log^2(x)\,\mathrm{Li}_1(-2x)\right)\frac{\mathrm{d}x}{x(1+x)} \quad (32)$$

appears in the Dickman tetralog, since $L_4(t) - C_4 + M_4(t/(1-2t))$ may be reduced to standard polylogs and their products. It is an easy matter to evaluate a single instance of (32) to high precision. For example, $M_4(\frac{1}{2})$ in (12) was computed to 800 digits, in order to determine the value of C_4 that results from the boundary condition $L_4(\frac{1}{4}) = 0$. However, it is quite another matter to obtain good results for C_8 and C_9, which are integrals with M_4 in their integrands. When t is significantly less than $\frac{1}{4}$, one may efficiently evaluate $M_4(t/(1-2t))$ by using Taylor series for $\log(1+2x)$ and $1/(1+x)$ under the integral sign in (32). Yet to compute C_8 and C_9 we need $L_4(t)$ for all arguments $0 < t < \frac{1}{4}$, inside the integrals for L_8 and L_9. So the remaining problem involves an investigation of the behaviour of $L_4(t)$ in the neighbourhood of $t = \frac{1}{4}$.

Theorem 2: Let $g_n(z) \equiv L_n(\frac{1}{n} - z)$ for $n > 0$. Then $g_n(z)$ has a Taylor series with rational coefficients, beginning with $g_n(z) = G_n z^n + O(z^{n+1})$, where $G_n = (-1)^n(n^n/n!)^2$.

Proof: We know that $g_1(z) = \log(1-z) = -z+O(z^2)$ has a Taylor series with rational coefficients. Setting $k = n+1$ in recursion (3) and transforming variables, I obtain

$$g_{n+1}(z) = -\int_0^z g_n\left(\frac{(n+1)^2 y}{n^2 + n(n+1)y}\right)\frac{(n+1)\mathrm{d}y}{1-(n+1)y}. \tag{33}$$

Now suppose that the claim is true for g_n. Then, by binomial expansion under the integral sign in (33), it is also true for g_{n+1}, since $G_{n+1}/G_n = -((n+1)^2/n^2)^n$. Hence, by induction, the claim is true for all $n > 0$.

Comment: Starting with $g_1(z) = \log(1-z)$, I used three iterations of recursion (33) to compute sufficient terms in the rational Taylor series for $g_4(z) = \frac{1024}{9}z^4 + O(z^5)$ to achieve at least 240 digits of precision for $L_4(t) = g_4(\frac{1}{4} - t)$ in the region $\frac{1}{4} > t > t_4 = 0.2358$. Then, for $t < t_4$, I proceeded as follows.

Let $a_n = (-b_n + (-1)^n c_n)/n$, where

$$b_n = -\frac{3}{n^3} + \sum_{k=1}^{n} \frac{2^k k + (-1)^{k-1} n}{k^2 n^2} \tag{34}$$

and c_n is the coefficient of z^n in the Taylor series for $\log(1+2z)\mathrm{Li}_2(z)/(1+z)$. Then, with $x \equiv t/(1-t)$ and $y \equiv t/(1-2t)$, the summation in

$$\begin{aligned} L_4(t) - C_4 \quad = \quad & \sum_{n=2}^{\infty}\{a_n + b_n \log(y)\}(-y)^n + L_3(x)\log(t) + \mathrm{Li}_2(-y)\log^2(y) \\ & + \left(\mathrm{Li}_2(-x) + \frac{1}{2}\log^2(x)\right)\left(\frac{\pi^2}{12} - \frac{1}{2}\log^2(y)\right) + \frac{1}{8}\log^4(y) \end{aligned} \tag{35}$$

provides a viable method for computing $L_4(t)$ for $t < t_4 = 0.2358$.

A sanity check was provided by comparing the two strategies at $t = t_4$ and verifying that 240 good digits of the conjectured value $\frac{1}{10}C_2^2$ were obtained for C_4. To achieve this, I expanded $g_4(z)$ up to $O(z^{300})$ and took 4800 terms in the summation over powers of $-y = -t/(1-2t)$ in (35). The relatively small number of terms used for g_4 resulted from the fact that it is rather laborious to perform three rational binomial iterations of recursion (33). By storing coefficients of the expansions, I am able to evaluate 240 good digits of the Dickman tetralog L_4 in less than 50 milliseconds and hence can efficiently compute the Dickman octalog, L_8, and the nonalog, L_9, as one-dimensional quadratures with integrands that call the procedure for L_4. By these means, I arrived at conjecture (13) for C_8 and then, thanks to OEIS sequence A008991, inferred the wonderfully compact conjectured generating function $\exp(\gamma z)/\Gamma(1-z)$ for $\sum_{k\geq 0} C_k z^k$, which then gave conjecture (17) for C_9, also now verified at high precision. Finally, I record 50 good digits of the Dickman probabilities

$$F(1/9) \approx 1.0162482827378365465348539356956957838244399586581 \times 10^{-9} \tag{36}$$

$$F(1/10) \approx 2.7701718377259589887581212006343423263430066501156 \times 10^{-11} \tag{37}$$

noting that 6 good digits were given in [32], which corrected serious errors in [5].

7 Context and conclusion

Thus far, I have spared the gentle reader explicit reference to quantum field theory (QFT) or to condensed matter physics. I trust that s/he will now permit me to reveal the physics context for investigating Dickman's mathematical [21, 18, 35, 16, 27, 26, 1, 12] problem.

The collision of a pair of protons in the large hadron collider (LHC) [31], at energies never before achieved in a particle accelerator, is described by QFT in terms of the collision of quarks, or gluons, carrying a fraction x of the momentum of a proton. The key thing that we need to know, to make sense of the possible outcomes of these LHC collisions, is the probability distribution function (PDF) in x. This is not known a priori; rather it is inferred from electron-proton collisions at lower energies, where half of the problem, namely the electron, is already well understood. Yet the input PDF, from electron-proton collisions, is not immediately usable at the LHC. Rather, it must be "evolved", up to the higher LHC energy. The procedure for doing this is completely understood, in principle, yet involves formidable mathematical challenges, in practice [2, 39]. It is performed as a perturbation expansion in the coupling constant of the strong interaction between quarks and gluons. At successive orders in this compelling expansion, generalized polylogarithms [36, 24] in x make their appearance.

From the perspective of QFT, the Dickman problem might seem to be rather routine: as soon as the polylogarithmic structure of the recursion (3) for L_k is exposed, it is clear that the formidable technical machinery of QFT [36, 24, 25, 33, 6, 38, 20, 28] has much to offer to the computation of this problem in number theory. The fascinating circumstance is that each iteration (3) of the Dickman polylog L_k brings its own distinctive constant C_k in (4). This also mimics QFT, where limiting values of generalized polylogarithms are likewise zeta values, until one reaches weight 8, when a multiple zeta value appears [14, 13, 15, 6]. The Dickman problem would have shared even more features with QFT had C_8 contained this first irreducible MZV, namely $\zeta(5,3) = \sum_{m>n>0} 1/(m^5 n^3)$. In this paper I have shown, with overwhelming probability, that $\zeta(5,3)$ does not appear in C_8. Moreover, I have conjectured that C_k is, most wonderfully, the coefficient of z^k in the expansion of $\exp(\gamma z)/\Gamma(1-z)$ and hence zeta-valued for all k.

This does not, however, mean that the Dickman problem now lacks interest for physicists. On the contrary, it resembles fascinating problems in condensed matter physics. In the study of quantum spin chains [7], hugely demanding calculations gave zeta-valued results up to weight 7. When Dirk Kreimer and I asked Valdimir Korepin to climb the next cliff, up to weight 8, we genuinely did not know whether an MZV would appear. In fact it did not [8]. Thus the Dickman problem sits very well with the study of spin chains.

In conclusion, I suggest that the current investigation has added to our understanding of the Dickman function and has also confirmed how much more demanding is the polylogarithmic structure of QFT, which appears to surpass the Dickman function of number theory, and the spin chains of condensed matter physics, both in its grand challenge and, I believe, in its eventually to be comprehended great beauty [23].

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References

[1] E. Bach and R. Peralta, Asymptotic semismoothness probabilities, Math. Comp., 65 (1996), 1701–1715.

[2] S. D. Badger and E. W. N. Glover, Two-loop splitting functions in QCD, J. High Energy Phys., 7 (2004), 40.

[3] D. H. Bailey and D. J. Broadhurst, Parallel integer relation detection: techniques and applications, Math. Comp., 70 (2001), 1719–1736.

[4] A. Beilinson, `http://www.mathacademy.com/pr/quotes/index.asp?ACTION=AUT&VAL=beilinsonalexander` .

[5] R. Bellman and B. Kotkin, On the numerical solution of a differential-difference equation arising in analytic number theory, Math. Comp., 16 (1962), 473–475.

[6] J. Blümlein, D. J. Broadhurst and J. A. M. Vermaseren, The multiple zeta value data mine, Comput. Phys. Commun., 181 (2010), 582–625.

[7] H. E. Boos and V. E. Korepin, Quantum spin chains and Riemann zeta function with odd arguments, J. Phys., A34 (2001), 5311–5316.

[8] H. E. Boos, V. E. Korepin, Y. Nishiyama and M. Shiroishi, Quantum correlations and number theory, J. Phys., A35 (2002), 4443–4452.

[9] J. M. Borwein, D. M. Bradley and D. J. Broadhurst, Evaluation of k-fold Euler/Zagier sums: a compendium of results for arbitrary k, Elec. J. Combin., 4 (1997), R5.

[10] J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisonek, Combinatorial aspects of multiple zeta values. Elec. J. Combin., 5 (1998), R38.

[11] J. M. Borwein, D. M. Bradley, D. J. Broadhurst and P. Lisonek, Special values of multiple polylogarithms, Trans. Amer. Math. Soc., 353 (2001), 907–941.

[12] R. P. Brent, Factorization of the tenth Fermat number, Math. Comp., 68 (1999), 429–451.

[13] D. J. Broadhurst, J. A. Gracey and D. Kreimer, Beyond the triangle and uniqueness relations: non-zeta counterterms at large N from positive knots, Zeit. Phys., C75 (1997), 559–574.

[14] D. J. Broadhurst and D. Kreimer, Knots and numbers in ϕ^4 theory to 7 loops and beyond, Int. J. Mod. Phys., C6 (1995), 519–524.

[15] D. J. Broadhurst and D. Kreimer, Association of multiple zeta values with positive knots via Feynman diagrams up to 9 loops, Phys. Lett., B393 (1997), 403–412.

[16] N. G. de Bruijn, On the number of positive integers $\leq x$ and free of prime factors $> y$, Ned. Akad. Wetensch. Proc., A54 (1951), 50–60.

[17] J. Chamayou, A probabilistic approach to a differential-difference equation arising in analytic number theory, Math. Comp., 27 (1973), 197–203.

[18] S. D. Chowla and T. Vjayaraghvan, On the largest prime divisors of numbers, J. Indian Math. Soc. (N.S.), 11 (1947), 31–37.

[19] H. Cohen, `http://pari.math.u-bordeaux.fr/` .

[20] V. Del Duca, C. Duhr and V. A. Smirnov, The two-loop hexagon Wilson loop in $N = 4$ SYM, March 2010, arXiv:1003.1702 [hep-th].

[21] K. Dickman, On the frequency of numbers containing prime factors of a certain relative magnitude, Arkiv Mat., Astron. Fys., 22 (1930), 1–14.

[22] H. R. P. Ferguson, D. H. Bailey and S. Arno, Analysis of PSLQ, an integer relation finding algorithm, Math. Comp., 68 (1999), 351–369.

[23] R. P. Feynman, The character of physical law, (BBC, 1965), final sentence.

[24] T. Gehrmann and E. Remiddi, Numerical evaluation of harmonic polylogarithms, Comput. Phys. Commun., 141 (2001), 296–312.

[25] T. Gehrmann and E. Remiddi, Numerical evaluation of two-dimensional harmonic polylogarithms, Comput. Phys. Commun., 144 (2002), 200–223.

[26] A. Hildebrand and G. Tenenbaum, Integers without large prime factors, Journal de théorie des nombres de Bordeaux, 5 (1993), 411–484.

[27] D. E. Knuth, The art of computer programming, Vol. 2, Seminumerical algorithms, (Addison-Wesley, 1998), 382–384.

[28] R. N. Lee, A. V. Smirnov, V. A. Smirnov, Analytic results for massless three-loop form factors, arXiv:1001.2887 [hep-ph].

[29] A. K. Lenstra, H. W. Lenstra, and L. Lovasz, Factoring polynomials with rational coefficients, Math. Ann., 261 (1982), 515–534.

[30] L. Lewin, Polylogarithms and associated functions, (North Holland, 1981).

[31] LHC, `http://lhc.web.cern.ch/lhc/` .

[32] J. van de Lune and E. Wattel, On the numerical solution of a differential-difference equation arising in analytic number theory, Math. Comp., 23 (1969), 417–421.

[33] S. Moch, P. Uwer and S. Weinzierl, Nested sums, expansion of transcendental functions and multi-scale multi-loop integrals, J. Math. Phys., 43 (2002), 3363–3386.

[34] M. Oakes, February 2010, messages to PrimeForm list, `http://tech.groups.yahoo.com/group/primeform/msearch?query=octically` .

[35] V. Ramaswami, On the number of positive integers less than x and free of prime divisors greater than x^c, Bull. Amer. Math. Soc., 55 (1949), 1122–1127.

[36] E. Remiddi and J. A. M. Vermaseren, Harmonic polylogarithms, Int. J. Mod. Phys., A15 (2000), 725–754.

[37] N. J. A. Sloane, `http://www.research.att.com/~njas/sequences/A008991` .

[38] A. V. Smirnov, V. A. Smirnov and M. Steinhauser, Full result for the three-loop static quark potential, February 2010, arXiv:1001.2668 [hep-ph].

[39] J. A. M. Vermaseren, A. Vogt and S. Moch, The third-order QCD corrections to deep-inelastic scattering by photon exchange, Nucl. Phys., B724 (2005) 3–182.

[40] D. Zagier, Values of zeta functions and their applications, First European Congress of Mathematics, Vol. II, (Paris, 1992), Progr. Math., 120 (1994), 497–512.