

EQUIVALENCE CLASSES OF PERMUTATIONS AVOIDING A PATTERN

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ABSTRACT. Given a permutation pattern \mathbf{p} and an equivalence relation on permutations, we study the corresponding equivalence classes all of whose members avoid \mathbf{p} . Four relations are studied: Conjugacy, order isomorphism, Knuth-equivalence and toric equivalence. Each of these produces a known class of permutations or a known counting sequence. For example, involutions correspond to conjugacy, and permutations whose insertion tableau is hook-shaped with 2 in the first row correspond to Knuth-equivalence. These permutations are equinumerous with certain congruence classes of graph endomorphisms. In the case of toric equivalence we find a class of permutations that are counted by the Euler totient function, with a subclass counted by the number-of-divisors function. We also provide a new symmetry for bivincular patterns that produces some new non-trivial Wilf equivalences

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1. INTRODUCTION

Let \mathfrak{S}_n be the permutation group on n letters. We will mostly use one-line notation for the elements of this group, i.e., the permutation in \mathfrak{S}_4 that sends $1 \mapsto 2, 2 \mapsto 4, 3 \mapsto 1, 4 \mapsto 3$ will be written 2413. This corresponds to the bottom line in the two-line notation

$$\begin{array}{cccc} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{array}$$

In usual pattern avoidance and matching one studies and counts the permutations in \mathfrak{S}_n that avoid or match a particular pattern. These permutations belong to two sets:

$$\begin{aligned} \mathfrak{A}_n(\mathbf{p}) &= \{\pi \in \mathfrak{S}_n \mid \pi \text{ avoids } \mathbf{p}\}, \\ \mathfrak{M}_n(\mathbf{p}) &= \{\pi \in \mathfrak{S}_n \mid \pi \text{ matches } \mathbf{p}\}. \end{aligned}$$

For example if the chosen pattern is $\mathbf{p} = 123$ (an increasing subsequence of three letters) we have

$$\begin{aligned}\mathfrak{A}_4(\mathbf{p}) &= \{1432, 2143, 2413, 2431, 3142, 3214, 3241, \\ &\quad 3412, 3421, 4132, 4213, 4231, 4312, 4321\}, \\ \mathfrak{M}_4(\mathbf{p}) &= \{\underline{1}234, \underline{1}243, \underline{1}324, \underline{1}342, \underline{1}423, \underline{2}134, \underline{2}341, \\ &\quad \underline{2}314, \underline{3}124, \underline{4}123\},\end{aligned}$$

where we have marked an occurrence of the pattern by underlining the letters. Note that we do not care if the pattern appears as a subsequence whose entries are adjacent or not in the permutation. In general the number of permutations in \mathfrak{S}_n that avoid 123 is the n -th Catalan number, see e.g., Simion and Schmidt (1985),

$$|\mathfrak{A}_n(\mathbf{p})| = C_n = \frac{1}{n+1} \binom{2n}{n}.$$

The permutations that match (or contain) the same pattern are the remaining permutations, giving $|\mathfrak{M}_n(\mathbf{p})| = n! - C_n$. This fact remains true for any pattern from \mathfrak{S}_3 .

In this paper we will explore how pattern avoidance interacts with equivalence relations. More precisely, given an equivalence relation and a pattern we will study and count the equivalence classes that do not contain any permutations that match the pattern. Similarly we will look at equivalence classes that only contain permutations that match the pattern. In short, we will be studying the two sets

$$\begin{aligned}\tilde{\mathfrak{A}}_n(\mathbf{p}) &= \{\pi \in \mathfrak{S}_n \mid \pi \text{ and every equivalent permutation avoids } \mathbf{p}\}, \\ \tilde{\mathfrak{M}}_n(\mathbf{p}) &= \{\pi \in \mathfrak{S}_n \mid \pi \text{ and every equivalent permutation matches } \mathbf{p}\}.\end{aligned}$$

If the relation can be extended to patterns we will also study the sets

$$\begin{aligned}\mathfrak{A}_n(\tilde{\mathbf{p}}) &= \{\pi \in \mathfrak{S}_n \mid \pi \text{ avoids } \mathbf{p} \text{ and every equivalent pattern}\}, \\ \mathfrak{M}_n(\tilde{\mathbf{p}}) &= \{\pi \in \mathfrak{S}_n \mid \pi \text{ matches } \mathbf{p} \text{ and every equivalent pattern}\}.\end{aligned}$$

The motivation for this work came about somewhat by accident when I was studying Knuth-equivalent permutations and occurrences of patterns in them. Two permutations are *Knuth-equivalent* if they have the same *insertion tableaux*, see e.g., Fulton (1997). I was trying to see how the occurrence of the pattern 231 could be seen on the tableau and because of a bug in the code I had written I generated permutations whose entire class avoids the pattern. So I had generated the set $\tilde{\mathfrak{A}}_n(231)$. It turns out that the permutations in this set correspond to hook-shaped tableaux, with $1, 2, \dots, k$ in the first line, such as

1	2	3	4
5			
6			

Now, the equivalence class of the pattern 231 is $\{231, 213\}$ and it is possible to show that a permutation avoids these two patterns if and only if it has an insertion tableau of the type described above. So we now have (see Proposition 5.9 and Corollary 5.11)

$$\tilde{\mathfrak{A}}_n(231) = \mathfrak{A}_n(231, 213) = \mathfrak{A}_n(\widetilde{231}).$$

A pattern with this property will be called *stable* below. Since the hook-length formula can be used to count the number of tableaux of a particular shape it can be used to count the sets above, giving

$$|\mathfrak{A}_n(231, 213)| = 2^{n-1},$$

which is well-known, and in fact (Reifeferste, 2004, Proposition 5.3) used a similar method to prove it.

Knuth-equivalence is one of four equivalence relations we will look at here, see the overview below for some examples of the results.

1.1. Overview and main results. The question of how patterns interact with equivalence relations seems to only become interesting when we consider so-called *bivincular patterns*, which generalize and subsume classical patterns. They are defined in section 2, but roughly speaking they are classical patterns with additional requirements on what positions and values are allowed in the occurrence of the pattern.

The next section deals with generalities that can be applied to any equivalence relation. The main result there is Proposition 2.5, which reduces the number of patterns one needs to look at. For example, let π^r be the permutation π read backwards. If for any permutations π and μ we have π is equivalent to μ if and only if π^r is equivalent to μ^r , then the proposition shows that $|\tilde{\mathfrak{A}}_n(\mathbf{p})| = |\tilde{\mathfrak{A}}_n(\mathbf{p}^r)|$, $|\tilde{\mathfrak{M}}_n(\mathbf{p})| = |\tilde{\mathfrak{M}}_n(\mathbf{p}^r)|$. This means that if we are interested in counting then there are half as many patterns to look at.

The next four sections deal with a particular equivalence relation each. The ordering of the sections is based on the difficulty of the proofs.

In section 3 we regard two permutations as equivalent if they have the same cycle type, or equivalently, $\pi \sim \mu$ if there exists a permutation σ such that $\pi = \sigma\mu\sigma^{-1}$. A particular result from that section is that permutations whose entire class avoids the bivincular pattern $\begin{smallmatrix} \overline{1} & \overline{2} & \overline{3} \\ \underline{2} & \underline{3} & \underline{1} \end{smallmatrix}$ are the involutions. (Recall that bivincular patterns are defined in section 2 below.)

In section 4 we regard two permutations as equivalent if they have the same order so the equivalence classes here will be unions of conjugacy classes. The results in this section will rely on results from the previous one. Here's an example: permutations whose entire class avoids $\begin{smallmatrix} \overline{1} & \overline{2} & \overline{3} & \dots & \overline{k} \\ \underline{2} & \underline{3} & \dots & \underline{k} & \underline{1} \end{smallmatrix}$ are permutations with order that can not be written $k \cdot m$ with $k \nmid m$.

In section 5 we regard two permutations as equivalent if they are Knuth-equivalent, meaning that they have the same insertion tableau. For another characterization of this equivalence relation, see Definition 5.1. The result mentioned in the introduction is from this section. Another is that permutations whose entire class avoids $\begin{smallmatrix} \overline{1} & \overline{2} & \overline{3} \\ \underline{2} & \underline{3} & \underline{1} \end{smallmatrix}$ are permutations whose insertion tableau is hook-shaped and has 2 in the first row. These permutations are counted with $1 + \frac{1}{2} \binom{2n-2}{n-1}$ and are equinumerous with certain congruence classes of graph endomorphisms, see Proposition 5.13 and Open Problem 4. Finally, in Corollary 5.11, we show that every classical pattern from \mathfrak{S}_3 is stable.

In section 6 we regard two permutations as equivalent if they are in the same *toric class*, which is defined rigorously below, but very roughly speaking this means that two permutations are equivalent if their permutation matrices become the same when they are wrapped around a torus. There are three main results in this section. The first is Proposition 6.8, which provides a new symmetry relation for certain bivincular patterns, which produces some new non-trivial Wilf-equivalences. Then there are two related results, the first showing that the permutations whose entire class avoids $\begin{smallmatrix} \overline{1} & \overline{2} & \overline{3} \\ \underline{2} & \underline{1} & \underline{3} \end{smallmatrix}$ have a rich structure and are enumerated by the Euler totient function, $\phi(n+1)$, see Theorem 6.28; and that the subset of permutations whose entire class avoids $\begin{smallmatrix} \overline{1} & \overline{2} & \overline{3} \\ \underline{2} & \underline{1} & \underline{3} \end{smallmatrix}$ are equinumerous with the divisors of n , see Theorem 6.31. These two results allow us to state a conjecture that is equivalent to the Riemann Hypothesis, see Conjecture 6.33.

In the last section, 7, we mention some other equivalence relations we have considered.

2. GENERALITIES

Below we will recall the definition of bivincular patterns, but first recall that given a pattern \mathbf{p} we say that it *occurs* in a permutation π , or that π *matches* \mathbf{p} , if π contains a subsequence whose letters are in the same order as the letters in the pattern. If a permutation does not match a pattern we say that it *avoids* the pattern.

Example 2.1. *The permutation $\pi = 241635$ has four occurrences of the pattern 123, given by the subsequences 246, 245, 135 and 235. It has two occurrences of the pattern 231, given by the subsequences 241 and 463. It avoids the pattern 321.*

A *vincular* pattern is like a classical pattern but it can have requirements on which letters must be adjacent in the occurrence. More precisely if two adjacent letters in the pattern are underlined then the corresponding letters in the occurrence must be adjacent. These were introduced in full generality by Babson and Steingrímsson (2000), although special cases had been considered earlier.

Example 2.2. *The permutation $\pi = 241635$ has two occurrences of the pattern $1\underline{2}\underline{3}$, given by the subsequences 135 and 235. These are also occurrences of the pattern $1\underline{2}\underline{3}$ since they appear at the end of the permutation. It avoids $\underline{1}\underline{2}\underline{3}$.*

A *bivincular* pattern is a further generalization where requirements on the values that must be in the occurrence are allowed. Here we use two-line notation and bars on adjacent letters in the top line mean that the corresponding letters in the occurrence must have adjacent values. These were introduced by Bousquet-Mélou et al. (2008).

Example 2.3. *The permutation $\pi = 241635$ has one occurrence of the pattern $\frac{1\underline{2}\underline{3}}{1\underline{2}\underline{3}}$, given by the subsequence 235. This is also an occurrence of $\frac{1\underline{2}\underline{3}}{1\underline{2}\underline{3}}$, with the hook in the top line signifying that the largest letter in the permutation is in the occurrence. It has one occurrence of $\frac{1\underline{2}\underline{3}}{1\underline{2}\underline{3}}$ given by the subsequence 245, which is also an occurrence of $\frac{1\underline{2}\underline{3}}{1\underline{2}\underline{3}}$. It avoids $\frac{1\underline{2}\underline{3}}{1\underline{2}\underline{3}}$.*

We will also use the notation of Bousquet-Mélou et al. (2008) to write bivincular patterns: A bivincular pattern consists of a triple (p, X, Y) where p is a permutation in \mathfrak{S}_k and X, Y are subsets of $\llbracket 0, k \rrbracket$. An occurrence of this bivincular pattern in a permutation $\pi = \pi_1 \cdots \pi_n$ in \mathfrak{S}_n is a subsequence $\pi_{i_1} \cdots \pi_{i_k}$ such that the letters in the subsequence are in the same relative order as the letters of p and

- for all x in X , $i_{x+1} = i_x + 1$; and
- for all y in Y , $j_{y+1} = j_y + 1$, where $\{\pi_{i_1}, \dots, \pi_{i_k}\} = \{j_1, \dots, j_k\}$ and $j_1 < j_2 < \cdots < j_k$.

By convention we put $i_0 = 0 = j_0$ and $i_{k+1} = n + 1 = j_{k+1}$.

The bivincular patterns behave well with respect to the operations reverse, complement and inverse: Given a bivincular pattern (p, X, Y) we define

$$\begin{aligned} (p, X, Y)^r &= (p^r, k - X, Y), \\ (p, X, Y)^c &= (p^c, X, k - Y), \\ (p, X, Y)^i &= (p^i, Y, X), \end{aligned}$$

where p^r is the usual reverse of the permutation of p , p^c is the usual complement of the permutation of p , and p^i is the usual inverse of the permutation of p . Here $k - M = \{k - m \mid m \in M\}$.

We get a very simple but useful Lemma:

Lemma 2.4. *Let a denote one of the operations above (or their compositions). Then a permutation π avoids the bivincular pattern \mathbf{p} if and only if the permutation π^a avoids the bivincular pattern \mathbf{p}^a . \square*

The following proposition will allow us to reduce the number of patterns that we need to look at.

Proposition 2.5. *Below a is one of r, c, i , or a composition of them.*

(1) *Assume that for any permutations π, μ ,*

$$\pi^a \sim \mu^a \text{ implies } \pi \sim \mu.$$

Then the maps $a: \tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow \tilde{\mathfrak{A}}_n(\mathbf{p}^a)$, $a: \tilde{\mathfrak{M}}_n(\mathbf{p}) \rightarrow \tilde{\mathfrak{M}}_n(\mathbf{p}^a)$ are injections.

(2) *Assume that for any permutations π, μ ,*

$$\pi \sim \mu \text{ implies } \pi^a \sim \mu^a.$$

Then the maps $a^{-1}: \tilde{\mathfrak{A}}_n(\mathbf{p}^a) \rightarrow \tilde{\mathfrak{A}}_n(\mathbf{p})$, $a^{-1}: \tilde{\mathfrak{M}}_n(\mathbf{p}^a) \rightarrow \tilde{\mathfrak{M}}_n(\mathbf{p})$ are injections.

(3) *Assume that for any permutations π, μ ,*

$$\pi \sim \mu \text{ if and only if } \pi^a \sim \mu^a.$$

Then the maps $a: \tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow \tilde{\mathfrak{A}}_n(\mathbf{p}^a)$, $a: \tilde{\mathfrak{M}}_n(\mathbf{p}) \rightarrow \tilde{\mathfrak{M}}_n(\mathbf{p}^a)$ are bijections.

Proof. It suffices to prove (1) for the avoiding classes, since the other cases are similar. We first prove that the image of the map a is actually in $\tilde{\mathfrak{A}}_n(\mathbf{p}^a)$. Take π in $\tilde{\mathfrak{A}}_n(\mathbf{p})$, so π and all equivalent permutations avoid the pattern \mathbf{p} . We must show that π^a and all equivalent permutations avoid \mathbf{p}^a . But if ρ is equivalent to π^a and contains \mathbf{p}^a then $\rho^{a^{-1}}$ is equivalent to $(\pi^a)^{a^{-1}} = \pi$ and contains $(\mathbf{p}^a)^{a^{-1}} = \mathbf{p}$, which is a contradiction. Since the map a is a bijection (on \mathfrak{S}_n) it follows that its restriction to $\tilde{\mathfrak{A}}_n(\mathbf{p})$ is an injection. \square

Often part (3) will be implied by using:

Lemma 2.6. *Assume that for any permutation π we have $\pi \sim \pi^a$. Then $\pi \sim \mu$ if and only if $\pi^a \sim \mu^a$ and the maps $a: \tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow \tilde{\mathfrak{A}}_n(\mathbf{p}^a)$, $a: \tilde{\mathfrak{M}}_n(\mathbf{p}) \rightarrow \tilde{\mathfrak{M}}_n(\mathbf{p}^a)$ are bijections.*

The proposition above is mostly useful when one is looking for patterns \mathbf{p} that give interesting sets $\tilde{\mathfrak{A}}_n(\mathbf{p})$. If for example we were interested in toric equivalence and decide to look at all bivincular patterns of length 3, without referring the proposition, we would have to go through the entire list of 1536 bivincular patterns. By Theorem 6.6 we can reduce by all the basic symmetries, which leaves us with only 212 patterns to look at, making the problem much more tractable.

For the rest of the section we assume that the equivalence relation has been extended to patterns.

Lemma 2.7. (1) *Let \mathbf{p} be a pattern and assume that for any $\mathbf{q} \sim \mathbf{p}$ we have*

$$\tilde{\mathfrak{A}}_n(\mathbf{p}) \subseteq \tilde{\mathfrak{A}}_n(\mathbf{q}).$$

Then $\tilde{\mathfrak{A}}_n(\mathbf{p}) \subseteq \tilde{\mathfrak{A}}_n(\tilde{\mathbf{p}})$.

(2) *Let \mathbf{p} be a pattern and assume that for any $\mathbf{q} \sim \mathbf{p}$ we have*

$$\tilde{\mathfrak{M}}_n(\mathbf{p}) \subseteq \tilde{\mathfrak{M}}_n(\mathbf{q}).$$

Then $\tilde{\mathfrak{M}}_n(\mathbf{p}) \subseteq \tilde{\mathfrak{M}}_n(\tilde{\mathbf{p}})$.

Proof. This is obvious. \square

Proposition 2.8. *Assume that for any permutations π, μ ,*

$$\pi \sim \mu \text{ if and only } \pi^a \sim \mu^a,$$

and that the same is still true if π, μ are replaced with patterns. Then the pattern \mathbf{p} is stable if and only if \mathbf{p}^a is stable.

Proof. We assume \mathbf{p} is stable. The statement follows from following the composition below from left to right.

$$\tilde{\mathfrak{A}}_n(\mathbf{p}^a) \rightarrow \tilde{\mathfrak{A}}_n(\mathbf{p}) = \mathfrak{A}_n(\tilde{\mathbf{p}}) \rightarrow \tilde{\mathfrak{A}}_n(\tilde{\mathbf{p}}^a) = \mathfrak{A}_n(\tilde{\mathbf{p}}^a). \quad \square$$

Whenever we reference integer sequences of the form Axxxxxx we are always referring to the Online Encyclopedia of Integer Sequences, Sloane (2010).

3. CONJUGACY

Here we will regard two permutations as equivalent if they have the same cycle type, or equivalently, $\pi \sim \mu$ if there exists a permutation σ such that $\pi = \sigma\mu\sigma^{-1}$. We start by examining which symmetries behave nicely with respect to this equivalence relation.

3.1. Symmetry. Since π and π^{-1} have the same cycle type (write π as a product of disjoint cycles, then obtain π^{-1} by reversing each cycle), we see that conjugacy is compatible with inverses.

It is also easy to see that π and π^{rc} have the same cycle type. This can be seen by representing permutations as matrices.

Example 3.1. *Let $\pi = 948167523$. Then the permutation matrix of π is*

$$M_\pi = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

To write π as a product of disjoint cycles we start in the first row and use the position of the 1 in that row to tell us which row to visit next. Stop when we loop around. This gives us

$$\pi = (193824)(567).$$

When we apply reverse and then complement it is like reading the matrix from the bottom up, from right to left. We see that we will get the same cycle structure, with i replaced by $n + 1 - i$.

By Proposition 2.5 we get the following theorem.

Theorem 3.2. *Let a be a composition of elements from the set $\{i, \text{rc}\}$. Then the map $a : \tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow \tilde{\mathfrak{A}}_n(\mathbf{p}^a)$ is a bijection for all n .*

3.2. Number of classes. The number of equivalence classes in \mathfrak{S}_n is

$$1, 2, 3, 5, 7, 11, 15, 22, 30 \dots \quad n = 1, 2, 3 \dots$$

This is just the number of conjugacy classes in \mathfrak{S}_n or equivalently the number of partitions of n , A000041.

3.3. Pattern avoidance. We start with a very easy result.

Proposition 3.3. (1) *The pattern $\mathbf{p} = 12$ gives*

$$\tilde{\mathfrak{A}}_n(\mathbf{p}) = \emptyset \quad (n \geq 3).$$

(2) *The pattern $\mathbf{p} = \overline{12}$ gives*

$$\tilde{\mathfrak{A}}_n(\mathbf{p}) = \begin{cases} \text{id}^r & \text{if } n = 2, \\ \text{id} & \text{otherwise.} \end{cases}$$

Proof.

- (1) Take any permutation that avoids this pattern. It must be $n(n-1) \cdots 321$. This pattern has cycle type $(2, 2, \dots, 2, 1)$ if n is odd and $(2, 2, \dots, 2, 2)$ if n is even. The permutations $2143 \cdots (n-1)(n-2)n$ (if n is odd) or $2143 \cdots n(n-1)$ (if n is even) have the same cycle type, but contain the pattern. Therefore the count is always zero. This argument works for $n \geq 3$ but not in \mathfrak{S}_1 and \mathfrak{S}_2 .
- (2) I claim that $\tilde{\mathfrak{A}}_n(\mathbf{p})$ consists of the identity permutation, except when $n = 2$ then it is replaced by 21. It is clear that these permutations are in the set, we just need to show that there are no others. To do this we must show that we can for any cycle type different from $(1, 1, \dots, 1)$ (for the identity) we can create a permutation $\pi = (n-1) \cdots n \cdots$ with that cycle type. So take a permutation π' that contains a cycle of length ≥ 2 . Let $(ab \cdots)$ be the start of that cycle. Now just conjugate with the $(1a)(b(n-1))$ and we have preserved the cycle type and put $n-1$ and the beginning in one-line notation. \square

The next result is a special case of Proposition 3.5).

Proposition 3.4. *The pattern $\mathbf{p} = \overline{1} = (1, \{0\}, \{0\})$ gives*

$$\tilde{\mathfrak{A}}_n(\mathbf{p}) = \text{derangements in } \mathfrak{S}_n$$

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, \dots, \quad n = 1, 2, 3, \dots$$

which is sequence A000166. This sequence has the generating function $\frac{e^{-x}}{1-x}$.

Proof. A permutation has no fixed points if and only its cycle type contains no 1. Call the set of derangements in \mathfrak{S}_n , D_n .

($D_n \subseteq \mathfrak{S}_n$): Take π not in D_n , then some permutation with the same cycle type as π contains the pattern. This implies that π has a 1 in its cycle type so π has a fixed point. Therefore π is not in D_n .

($\mathfrak{S}_n \subseteq D_n$): Take π not in \mathfrak{S}_n , so π has a fixed point and therefore its cycle type contains a 1. Let a be the fixed point. Swap 1 and a in the one-line notation for π and we turn 1 into a fixed point, while preserving the cycle type. \square

By (Stanley, 1986, Exercise 7, Chapter 2) we have that the number of permutations of $\llbracket n \rrbracket$ that have no k -cycle is given by the generating function

$$\frac{e^{-x^k/k}}{1-x}.$$

The proposition above is the special case $k = 1$ which means the permutations have no 1-cycles, i.e., fixed points. It turns out that the pattern that appears in the proposition is part of a family of patterns:

Proposition 3.5. *Let $k \geq 1$ The pattern*

$$\mathbf{p} = \overline{\begin{array}{cccc} 1 & 2 & 3 & \cdots & k \\ 2 & 3 & \cdots & k & 1 \end{array}}$$

gives

$\tilde{\mathfrak{A}}_n(\mathbf{p}) =$ *permutations in of $\llbracket n \rrbracket$ that do not contain a k -cycle,*

the count has generating function $\frac{e^{-x^k/k}}{1-x}$. For $k = 1, \dots, 7$ we get sequences A000166, A000266, A000090, A000138, A060725, A060726, A060727.

Proof. This is a slight generalization of the proof above and omitted. \square

The next two propositions have similar proofs to Proposition 3.4, so we omit the proofs.

Proposition 3.6. *The pattern $\mathbf{p} = \overline{\begin{array}{ccc} 1 & 2 & 3 \\ 2 & 3 & 1 \end{array}}$ gives*

$\tilde{\mathfrak{A}}_n(\mathbf{p}) =$ *involutions in \mathfrak{S}_n*

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = 1, 2, 4, 10, 26, 76, 232, 764, 2620, \dots, \quad n = 1, 2, 3, \dots$$

which is A000085. This sequence has the generating function $\exp(x + x^2/2)$.

It turns out that this pattern is also part of a family of patterns:

Proposition 3.7. *Let $k \geq 1$ The pattern*

$$\mathbf{p} = \overline{\begin{array}{cccc} 1 & 2 & 3 & \cdots & k \\ 2 & 3 & \cdots & k & 1 \end{array}}$$

gives

$\tilde{\mathfrak{A}}_n(\mathbf{p}) =$ *permutations in of \mathfrak{S}_n only containing cycles of length $< k$,*

the count has the generating function

$$\exp\left(x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{x^{k-1}}{k-1}\right).$$

For $k = 1, \dots, 7$ we get sequences A000004 (The zero sequence), A000012 (The all 1's sequence), A000085, A057693, A070945, A070946, A070947.

Proposition 3.8. *The pattern $\mathbf{p} = \overline{\begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array}}$ gives*

$\tilde{\mathfrak{A}}_n(\mathbf{p}) =$ *id and transpositions in \mathfrak{S}_n*

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = 1, 1, 4, 7, 11, 16, 22, 29, 37, \dots, \quad n = 1, 2, 3, \dots$$

If we count from $n = 3$ we get $T(n+1) + 1$ where $T(n) = \frac{n(n+1)}{2}$ is A000217, the triangular numbers.

Proof. I claim that $\tilde{\mathfrak{A}}_n(\mathbf{p})$ consists of permutations with cycle type $(1, 1, \dots, 1)$ or $(2, 1, \dots, 1)$. Call the set with those kinds of cycle type Y_n .

$(Y_n \subseteq \tilde{\mathfrak{A}}_n(\mathbf{p}))$: $\tilde{\mathfrak{A}}_n(\mathbf{p})$ clearly contains the identity permutation with the former cycle type. So consider any permutation π with the latter cycle type. This permutation will consist of a transposition and then everything else is fixed. A permutation of this type clearly avoids \mathbf{p} .

$(\tilde{\mathfrak{A}}_n(q) \subseteq Y_n)$: Take π that is not in Y_n . Then we must consider two cases

- (1) The cycle type of π has a cycle of length ≥ 3 : Let this cycle start with $(abc \cdots)$. Then just conjugate π with $(c1)(bn)((n-1)a)$, which preserves the cycle type but introduces an occurrence of the pattern.

- (2) The cycle type of π has two cycles, both of length ≥ 2 . We can assume these cycles have length exactly 2, since if they were larger we can just apply the argument above to either one of them. So let these cycles be (ab) and (cd) . Then just conjugate π with $(a1)(b(n-1))(c(n-2))(dn)$. \square

Proposition 3.9. *The pattern $\mathbf{p} = \frac{1}{2} \frac{2}{3} \frac{3}{1}$ gives*

$$\begin{aligned} \tilde{\mathfrak{A}}_n(\mathbf{p}) &= \text{permutations moving at most two letters} \\ |\tilde{\mathfrak{A}}_n(\mathbf{p})| &= 1, 2, 4, 7, 11, 16, 22, 29, 37, \dots, \quad n = 1, 2, 3, \dots \end{aligned}$$

This is A000124: Central polygonal numbers, $n(n-1)/2 + 1$.

Proof. Let X_n be the set on the right. It is clear that $X_n \subseteq \tilde{\mathfrak{A}}_n(\mathbf{p})$. The other implication is also simple: given a cycle of length 3 or more we can conjugate to produce a permutation in the same class that looks like $2n \cdots 1 \cdots$. Finally, given two 2-cycles we can conjugate to produce $3n1 \cdots 2$. \square

Open Problem 1. *The permutations above are equinumerous (but not equal to) permutations avoiding 132- and 321-avoiding permutations. It would be interesting to produce a bijection.*

The next three propositions have proofs that are similar to the proof of Proposition 3.9, so we omit their proofs.

Proposition 3.10. *The pattern $\mathbf{p} = \frac{1}{1} \frac{2}{3} \frac{3}{2}$ gives*

$$\begin{aligned} \tilde{\mathfrak{A}}_n(\mathbf{p}) &= \text{fixed point free involutions} \\ |\tilde{\mathfrak{A}}_n(\mathbf{p})| &= 1, 2, 3, 4, 1, 16, 1, 106, 1, \dots, \quad n = 1, 2, 3, \dots \\ &= \begin{cases} 1 & \text{if } n \text{ is odd,} \\ (n-1)!! + 1 & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

The even subsequence is A001147: Double factorial numbers.

Proposition 3.11. *The pattern $\mathbf{p} = \frac{1}{1} \frac{2}{3} \frac{3}{2}$ gives*

$$\begin{aligned} \tilde{\mathfrak{A}}_n(\mathbf{p}) &= \text{the identity and 3-cycles} \\ |\tilde{\mathfrak{A}}_n(\mathbf{p})| &= 1, 2, 3, 9, 21, 41, 71, 113, 169, \dots, \quad n = 1, 2, 3, \dots \\ &= 1 + 2 \binom{n}{3}. \end{aligned}$$

From $n = 3$ this is A007290 plus 1.

Proposition 3.12. *The pattern $\mathbf{p} = \frac{1}{2} \frac{2}{3} \frac{3}{1}$ gives*

$$\begin{aligned} \tilde{\mathfrak{A}}_n(\mathbf{p}) &= \text{the identity, 2-cycles and 3-cycles} \\ |\tilde{\mathfrak{A}}_n(\mathbf{p})| &= 1, 2, 4, 15, 31, 56, 92, 141, 205, \dots, \quad n = 1, 2, 3, \dots \end{aligned}$$

From $n = 4$ this is A000330 plus 1, where A000330 are the square pyramidal numbers, $\frac{n(n+1)(2n+1)}{6}$.

Open Problem 2. *The number of permutations avoiding $\underline{132}$ that contain the pattern $\underline{321}$ exactly once is counted by the square pyramidal numbers. Find a bijection to the permutations above (leaving out the identity).*

4. ORDER

Here we will regard two permutations as equivalent if they have the same order. Clearly the equivalence classes here will be unions of conjugacy classes and therefore the results in this section will rely on results from the previous one.

4.1. **Symmetry.** The following is a direct consequence of Theorem 3.2:

Theorem 4.1. *Let a be a composition of elements from the set $\{i, rc\}$. Then $a : \tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow \tilde{\mathfrak{A}}_n(\mathbf{p}^a)$ is a bijection for all n .*

4.2. **Number of classes.** The number of equivalence classes in \mathfrak{S}_n is

$$1, 2, 3, 4, 6, 6, 9, 11, 14 \quad n = 1, \dots, 9.$$

Of course this is just A009490: Number of distinct orders of permutations of n objects.

4.3. **Pattern avoidance.**

Proposition 4.2. *The pattern $\mathbf{p} = \overline{\overline{1}}_1$ gives*

$\tilde{\mathfrak{A}}_n(\mathbf{p}) =$ *permutations with order that can not be obtained from permutations with a fixed point (1-cycle).*

= permutations with order that can not be written $\prod a_i$

with $1 + \sum a_i = n$.

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = 0, 1, 2, 6, 44, 0, 1644, 7728, 84384, \dots, \quad n = 1, 2, 3, \dots$$

Proof. This follows directly from Proposition 3.4. □

This result can also be generalized in the same way as Proposition 3.4 above:

Proposition 4.3. *Let $k \geq 1$. The pattern $\mathbf{p} = \overline{\overline{1 \ 2 \ 3 \ \dots \ k}}_{2 \ 3 \ \dots \ k \ 1}$ gives*

$\tilde{\mathfrak{A}}_n(\mathbf{p}) =$ *permutations with order that can not be obtained from permutations with a k -cycle,*

= permutations with order that can not be written $k \cdot m$ with $k \nmid m$.

Proof. This follows directly from Proposition 3.5. □

Proposition 4.4. *The pattern $\mathbf{p} = \overline{\overline{1 \ 2 \ 3}}_{2 \ 3 \ 1}$ gives*

$\tilde{\mathfrak{A}}_n(\mathbf{p}) =$ *involutions in $\llbracket n \rrbracket$*

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = 1, 2, 4, 10, 26, 76, 232, 764, 2620, \dots, \quad n = 1, 2, 3, \dots$$

which is A000085 in OEIS. This sequence has the generating function $\exp(x+x^2/2)$.

Proof. This follows directly from Proposition 3.6. □

This generalizes in a similar way we saw with Proposition 4.2 above, but we omit it.

5. KNUTH-EQUIVALENCE

Definition 5.1. Let A be an alphabet with an ordering.

- (1) An *elementary Knuth-transformation* on a word with letters from A applies one of the transformations below, or their inverses, to three consecutive letters in the word.

K1 $yxz \mapsto yxz$ if $x < y \leq z$,

K2 $xzy \mapsto zxy$ if $x \leq y < z$.

- (2) Two words w and w' are said to be *Knuth-equivalent* if they can be changed into each other by a sequence of elementary Knuth-transformations. We write $w \equiv w'$ if this is the case.

In this paper we will only consider the alphabet $A = \mathbb{N} = \{1, 2, 3, \dots\}$ so K1 means we can interchange zx if the next letter to the left fits between them; and K2 means that we can interchange xz if the next letter to the right fits between them. E.g., $24135 \equiv 21435 \equiv 21453$.

As is shown in Fulton (1997), two permutations π and μ are Knuth-equivalent if and only if they have the same insertion tableau. Below we will denote the insertion tableau of π with $P(\pi)$ and the recording tableau with $Q(\pi)$.

5.1. Symmetry. It is easy to see from Definition 5.1 that if $a = r$ or $a = c$ then $\pi \sim \mu$ if and only if $\pi^a \sim \mu^a$. Then Proposition 2.5 implies the following theorem.

Theorem 5.2. *Let a be a composition of elements from the set $\{r, c\}$. Then a $:\tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow \mathfrak{A}_n(\mathbf{p}^a)$ is a bijection for all n .*

5.2. Number of classes. The number of equivalence classes in \mathfrak{S}_n is

$$1, 2, 4, 10, 26, 76, 232, 764, 2620 \dots \quad n = 1, \dots$$

This is A000085: number of Young tableaux with n cells.

5.3. Pattern avoidance. Here we extend the relation to patterns by saying that two patterns (p, X, Y) , (p', X', Y') are equivalent if and only if p and p' are equivalent, and $X = X'$, $Y = Y'$.

Proposition 5.3. *For $k \geq 1$ we have that the pattern $\mathbf{p} = 12 \dots k$ satisfies*

$$\tilde{\mathfrak{A}}_n(\mathbf{p}) = \mathfrak{A}_n(\mathbf{p}).$$

The members of these sets are permutations whose longest increasing subsequence is of length less than k . For $n = 1, \dots, 7$ we get the sequences A000004 (zero sequence), A000012 (the 1's sequence), A000108 (the Catalan numbers), A005802 (number of vexillary permutations), A047889, A047890, A052399. These sequences are studied in Bergeron and Gascon (2000).

Proof. We always have $\tilde{\mathfrak{A}}_n(\mathbf{p}) \subseteq \mathfrak{A}_n(\mathbf{p})$ so take a permutation $\pi \in \mathfrak{A}_n(\mathbf{p})$. This implies that the first row of the tableau $P(\pi)$ has length $\leq k$. The same is then true for every equivalent permutation which implies that every equivalent permutation also avoids the pattern. \square

Corollary 5.4. *The pattern $\mathbf{p} = 12 \dots k$ is stable for $k \geq 1$.*

Proof. Since $\tilde{\mathbf{p}} = \{\mathbf{p}\}$ for these patterns the result follows from Proposition 5.3. \square

Proposition 5.5. *For $k \geq 1$ we have that*

$$\tilde{\mathfrak{A}}_n(\underline{12 \dots k}) = \tilde{\mathfrak{A}}_n(12 \dots k).$$

This implies that the patterns $\underline{12 \dots k}$ are not stable, unless $k = 1, 2$.

Proof. We obviously have $\tilde{\mathfrak{A}}_n(\underline{12 \dots k}) \supseteq \tilde{\mathfrak{A}}_n(12 \dots k)$, so assume π is a permutation that is not in the set on the right. This means that the first row of $P(\pi)$ is of length $\geq k$. To construct an equivalent permutation that contains the consecutive pattern, apply the inverse RSK-correspondance to the tableaux-pair $(P(\pi), Q)$ where Q is filled in trivially (by reading from left to right, the top row first). \square

Proposition 5.6. *For $k \geq 1$ we have that the pattern $\mathbf{p} = \overline{1 \ 2 \dots k}$ gives*

$$\tilde{\mathfrak{A}}_n(\mathbf{p}) = \text{permutations whose insertion tableau does not start with } \mathbf{p}$$

these are permutations whose longest increasing subsequence is of length less than k .

Proof. Similar to the proof above. \square

Open Problem 3. *The count for the permutations above seems to be given by $n! - \frac{n!}{k!}$ for $k \geq n$. Prove this.*

Proposition 5.7. *For $k \geq 1$ we have that the pattern $\mathbf{p} = \overline{12 \dots k}$ satisfies*

$$\tilde{\mathfrak{A}}_n(\mathbf{p}) = \mathfrak{A}_n(\mathbf{p}).$$

For $n = 1, \dots, 7$ we get the sequences A000004 (zero sequence), A000012 (the 1's sequence), A049774, A117158, A177523, A177533, A177553. These sequences have been studied in by Hardin (2010)

Proof. We obviously have $\tilde{\mathfrak{A}}_n(\mathbf{p}) \subseteq \mathfrak{A}_n(\mathbf{p})$, so assume π is a permutation that is not in the set on the left. Let π' be an equivalent permutation that contains the pattern. This means that there is a row of $P(\pi')$ that contains the sequence $\ell, \ell + 1, \dots, \ell + k - 1$ in adjacent boxes. Since $P(\pi) = P(\pi')$ we see that π must also contain the pattern. \square

Corollary 5.8. *The pattern $\mathbf{p} = \overline{12 \dots k}$ is stable for $k \geq 1$.*

Proof. Since $\tilde{\mathbf{p}} = \{\mathbf{p}\}$ for these patterns the result follows from Proposition 5.7. \square

Proposition 5.9. *The pattern $\mathbf{p} = 231$ gives*

$\tilde{\mathfrak{A}}_n(\mathbf{p}) = \text{permutations with hook-shaped insertion tableaux, filled in trivially}$

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = 1, 2, 4, 8, 16, 32, 64, 128, 256 \dots, \quad n = 1, 2, 3, \dots$$

which is A000079: 2^{n-1} .

Before we start proving this we make a technical definition for convenience:

Definition 5.10. Let \mathbf{p} be a pattern. Given an occurrence of \mathbf{p} in a permutation π we define the *area* of the occurrence as the distance from the first letter in the occurrence to the last. A *minimal occurrence* of a pattern \mathbf{p} in a permutation π is an occurrence of \mathbf{p} with the least possible area.

Proof. We begin by showing that the permutations in the set on the left are exactly the permutations that avoid 231 and 213. Then (Reifegerste, 2004, Proposition 5.3) implies the result.

Now, let π be a permutation belonging to the set on the left. We need to show that π avoids the pattern 213. First note that π must avoid the pattern 213 since an occurrence of 213 can be changed into an occurrence of 231 by an elementary transformation. It therefore suffices to show that an occurrence of 213 can be changed into an occurrence of 213 by elementary transformations. Let

$$\pi = \dots \check{x} \dots \check{y} \dots \check{z} \dots$$

be a minimal occurrence of 213; so $y < x < z$, every letter w between x and y satisfies $w > z$ and every letter w between y and z satisfies $y < w < x$. Now any w between x and y with $w > z$ would give an occurrence of 231 so x and y are actually adjacent. If y and z are adjacent we are done, otherwise let w be the first letter to the left of y .

$$\pi = \dots \check{x} \check{y} w \dots \check{z} \dots$$

We can swap x and y by an elementary transformation and we still have an occurrence of 213

$$\pi \equiv \dots y \check{x} \check{w} \dots \check{z} \dots,$$

and the number of letters between w and z is now one less than the number of letters that were between y and z in π . We can now perform elementary transformations until we have gotten rid of all the letters between x and z , but one, and this will be an occurrence of 213.

Now let π be a permutation belonging to the union on the right. We need to show that any permutation Knuth-equivalent to π also avoids 231. It suffices to consider a permutation ρ that differs from π by one elementary transformation. We must look at two cases:

(1)

$$\rho = \cdots xyz \cdots \equiv \cdots yxz \cdots = \pi,$$

with $x < z < y$. If ρ has an occurrence of 231 then it would either have to be

$$\rho = \cdots \tilde{x}y\tilde{z} \cdots \tilde{w} \cdots,$$

or

$$\rho = \cdots \tilde{x}\tilde{y}z \cdots \tilde{w} \cdots.$$

The first scenario would immediately give an occurrence of 231 in π , which is a contradiction, and the second scenario we would have also get an occurrence of 231:

$$\pi = \cdots y\tilde{x}\tilde{z} \cdots \tilde{w} \cdots.$$

(2)

$$\rho = \cdots zxy \cdots \equiv \cdots zyx \cdots = \pi,$$

This case is similar to the previous one and left to the reader. \square

Corollary 5.11. *Every classical pattern in \mathfrak{S}_3 is stable.*

Proof. The case $p = 123$ follows from Proposition 5.4 and the proof of the Proposition above implies the case $p = 231$. By taking reverse and complement and using Proposition 2.8 we get the rest of \mathfrak{S}_3 . \square

We will need an obvious lemma regarding hook-shaped tableaux in the proof of the next proposition:

Lemma 5.12. *If the tableau $P(\pi)$ is hook-shaped then in each step of its construction when an element is bumped from row i , it is smaller than the current element in row $i + 1$.*

Proposition 5.13. *The pattern $\mathbf{p} = \overline{\mathbb{1}}_{2\ 3\ 1}^{\mathbb{1}}^{\mathbb{2}}$ gives*

$\tilde{\mathfrak{A}}_n(\mathbf{p}) =$ *permutations with hook-shaped insertion tableaux*
with 2 in the first row, as well as $\text{id}^{\mathbb{1}}$,

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = 1, 2, 4, 11, 36, 127, 463, 1717, 6436 \dots, \quad n = 1, 2, 3, \dots, 9$$

which is (from $n = 3$) A112849: Number of congruence classes (epimorphisms/vertex partitionings induced by graph endomorphisms) of undirected cycles of even length: $|C(C_{2n})|$. See Michels and Knauer (2009).

Proof. Assume π is not in the set on the left, and let π' be an equivalent permutation that matches the pattern. We can write

$$\pi' = \cdots j \cdots k \cdots 1 \cdots, \quad j \leq k.$$

We can assume that j and k are adjacent. If 2 appears before 1 in π' then it will be bumped from the first row (by 1), and then π' and π would not be in the set on the right. So assume that 2 appears after 1. Then the tableaux $P(\pi') = P(\pi)$ will not be hook-shaped by Lemma 5.12.

Now assume π is not in the set on the right. We need to look at two cases:

- (1) 2 is not in the first row of $P(\pi)$. This means that 1 appears after 2 in π . If 1 and 2 are not adjacent in π then we get an occurrence of the pattern so we can assume that they are adjacent. If the block 21 is not at the end of the permutation then an elementary Knuth-swap can be made to produce the pattern, so we can assume that $\pi = \cdots 21$. It is now clear that π avoids the pattern if and only if $\pi = \text{id}^r$.
- (2) Now assume that that $P(\pi)$ is not hook-shaped. The permutation we get from

$$P(\pi), \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdot & \cdot & \cdot & \cdot \\ \hline 3 & 4 & \cdot & \cdot & & \\ \hline \cdot & \cdot & & & & \\ \hline \cdot & & & & & \\ \hline \cdot & & & & & \\ \hline \end{array}$$

by the RSK-correspondence will be equivalent to π . It is easy to see that it will contain the pattern. \square

Michels and Knauer (2009) give a formula for their sequence:

$$1 + \frac{1}{2} \binom{2n-3}{n-2} + \frac{1}{2} \binom{2n-3}{n-1}.$$

It is more natural to count the tableaux that appear with

$$1 + \sum_{i=2}^n \binom{n-2}{i-2} \cdot \binom{n-1}{i-1} = 1 + \frac{1}{n-1} \sum_{i=2}^n (i-1) \binom{n-1}{i-1}^2 = 1 + \frac{1}{2} \binom{2n-2}{n-1},$$

which gives the same numbers as their formula.

Open Problem 4. Find a bijection between the permutations above and the class enumerated in Michels and Knauer (2009).

5.4. Pattern matching. Before moving to the next equivalence we state and prove a result on equivalence classes that contain only permutations matching a certain pattern.

Proposition 5.14. The pattern $\mathbf{p} = \overline{12} \overline{2}$ gives

$$\begin{aligned} \widetilde{\mathfrak{M}}_n(\mathbf{p}) &= \text{permutations of the form } (n-1)\rho \text{ where } \rho \text{ avoids } 123 \\ |\widetilde{\mathfrak{M}}_n(\mathbf{p})| &= 1, 2, 5, 14, 42, 132, 429, 1430, \dots, \quad n = 2, \dots \end{aligned}$$

which is A000108 in OEIS, the Catalan numbers, except this is shifted, so we get 2 permutations in \mathfrak{S}_3 , for example.

Proof. The enumeration is obvious if we can prove the equality of the two sets. Assume that π is not in the set on the left, and that π does start with $n-1$. We can find a π' , equivalent to π , that does not match the pattern, and we can assume that π' is just one elementary swap from π . We have

$$\begin{aligned} \pi &= (n-1)k\ell \cdots n \cdots \\ \pi' &= k(n-1)\ell \cdots n \cdots, \end{aligned}$$

where $k < \ell < n-1$ (we can not have $k, \ell = n$). Then π contains 123.

Now assume that π is not in the set on the right and that π starts with $n-1$. Since π contains the pattern 123 we see that the first row of $P(\pi)$ has at least three boxes. The equivalent permutation we get from RSK of

$$P(\pi), \begin{array}{|c|c|c|c|c|} \hline 1 & 2 & \cdot & \cdot & \cdot & k \\ \hline \cdot & \cdot & & & & \\ \hline \cdot & \cdot & & & & \\ \hline \cdot & & & & & \\ \hline \end{array}$$

does not start with $n - 1$, so π is not in the set on the left. \square

It is well-known that the Catalan numbers enumerate tableaux of shape $(2, n)$ so that explains what tableaux appear above.

6. TORIC EQUIVALENCE

I learned about this equivalence relation from Anthony Labarre, who responded to my question on MO (2010).

If λ is a circular permutation of $\llbracket 0, n \rrbracket$ then λ_\circ denotes the permutation in \mathfrak{S}_n we get by reading λ from 0. E.g., if $\lambda = 130254$ then $\lambda_\circ = 25413$.

Our definition of toric equivalence follows Eriksson et al. (2001), but as noted there, an equivalent class of objects was studied by Steggall (1907). Here the relation can be roughly viewed as declaring two permutations to be equivalent if their permutation matrices become equal when they are wrapped around a torus. More precisely, given a permutation π of \mathfrak{S}_n we define π° as the circular permutation 0π of $\llbracket 0, n \rrbracket$. Then for any $m = 0, 1, \dots, n$ we define a new circular permutation

$$\pi^\circ \oplus m = (0 + m)(\pi_1 + m)(\pi_2 + m) \cdots (\pi_n + m) \bmod (n + 1).$$

(Here every letter is reduced modulo $n + 1$). It is also convenient to define

$$\pi \oplus m = (\pi^\circ \oplus m)_\circ.$$

Then the *toric class* of the original permutation π is defined as the set

$$\pi^\circ_\circ = \{\pi \oplus m \mid m = 0, 1, \dots, n\}.$$

Example 6.1. Let $\pi = 1243$. Then $\pi^\circ = 01243$ and

$$\begin{aligned} \pi^\circ \oplus 0 &= 01243, & \pi^\circ \oplus 1 &= 12304, & \pi^\circ \oplus 2 &= 23410, \\ \pi^\circ \oplus 3 &= 34021, & \pi^\circ \oplus 4 &= 40132. \end{aligned}$$

The toric class of π is $\pi^\circ_\circ = \{1243, 4123, 2341, 2143, 1324\}$.

6.1. Symmetry.

Lemma 6.2. If λ is a circular permutation of $\llbracket 0, n \rrbracket$ and π, ρ are two representations then $\pi^i \equiv \rho^i$.

Note that when we take the complement of a permutation of $\llbracket 0, n \rrbracket$ (modulo $n + 1$) then 0 and stays fixed, and it is the only letter that does so if n is even; if n is odd then $(n + 1)/2$ also stays fixed.

Lemma 6.3. For any permutation π of $\llbracket 1, n \rrbracket$ we have:

(1) Taking reverse commutes with the \circ -operator:

$$(\pi^r)^\circ \equiv (\pi^\circ)^r, \text{ or equivalently } ((\pi^\circ)^r)_\circ = \pi^r.$$

Also, for any circular permutation λ of $\llbracket 0, n \rrbracket$ we have

$$\lambda^r \oplus m \equiv (\lambda \oplus m)^r.$$

(2) Taking complement commutes with the \circ -operator:

$$(\pi^c)^\circ \equiv (\pi^\circ)^c, \text{ or equivalently } ((\pi^\circ)^c)_\circ = \pi^c.$$

Also, for any circular permutation λ of $\llbracket 0, n \rrbracket$ we have

$$\lambda^c \oplus m \equiv (\lambda \ominus m)^c.$$

(3) Taking inverse commutes with the \circ -operator:

$$(\pi^i)^\circ \equiv (\pi^\circ)^i, \text{ or equivalently } ((\pi^\circ)^i)_\circ = \pi^i.$$

Also, for any circular permutation λ of $\llbracket 0, n \rrbracket$ we have

$$\lambda^i \oplus x \equiv (\lambda \oplus 1)^i,$$

where x is the distance (counter-clock-wise) from n to 0 in λ . More generally, we have

$$\lambda^i \oplus mx \equiv (\lambda \oplus m)^i,$$

where x has the same value as above.

Proof. Parts (1) and (2) are left to the reader, and we only prove part (3). It suffices to prove that part in the case $m = 1$ as the other cases can be deduced by induction. The first element after 0 on the right-hand side is the location of 1 (from 0) in $\lambda \oplus 1$. This equals the location of 0 (from n) in λ .

The first element after 0 on the left-hand side equals (the first element after $n + 1 - x$ in λ^i) $+x$. Now the first element after $n + 1 - x$ in λ^i records the location of the element in λ that is one larger than the element at location $n + 1 - x$. But the element at location $n + 1 - x$ is n , so the first element after $n + 1 - x$ records the location of 0, which is at location 0. When we add x to this we get x back. This argument generalizes to every location from 0. \square

Example 6.4. Let $\lambda = 04372156$. Here the distance from n to 0 is $x = 5$. Then $\lambda \oplus 1 = 15403267$ and $(\lambda \oplus 1)^i = 63405217$. This is the right-hand side of part (3) above. To calculate the left-hand side we first have $\lambda^i = 05421673$ and then $\lambda^i \oplus 5 = 52176340$.

Some of the equations in Lemma 6.3 are also true when regarding the permutations as usual permutations of $\llbracket 0, n \rrbracket$, and not just as circular.

Proposition 6.5. Let a be any composition of the operations r, c, i . The permutations π and μ in \mathfrak{S}_n are torically equivalent if and only if π^a and μ^a are torically equivalent.

Proof. It clearly suffices to prove this for $a \in \{r, c, i\}$, and since any such a is its own inverse, it suffices to prove just one direction.

We start with $a = r$. Since π and μ are torically equivalent there exists $m \in \llbracket 0, n \rrbracket$ such that $\pi^\circ \oplus m = \mu^\circ$. But by Lemma 6.3, part (1), we have

$$(\pi^r)^\circ \oplus m \equiv (\pi^\circ)^r \oplus m \equiv (\pi^\circ \oplus m)^r \equiv (\mu^\circ)^r \equiv (\mu^r)^\circ,$$

so π^r and μ^r are torically equivalent.

Now consider $a = c$. Since π and μ are torically equivalent there exists $m \in \llbracket 0, n \rrbracket$ such that $\pi^\circ \oplus m = \mu^\circ$. But by Lemma 6.3, part(2), we have

$$(\pi^c)^\circ \ominus m \equiv (\pi^\circ)^c \ominus m \equiv (\pi^\circ \oplus m)^c \equiv (\mu^\circ)^c \equiv (\mu^c)^\circ,$$

so π^c and μ^c are torically equivalent.

Now consider $a = i$. Since π and μ are torically equivalent there exists $m \in \llbracket 0, n \rrbracket$ such that $\pi^\circ \oplus m = \mu^\circ$. Let x be the distance (counter-clock-wise) from n to 0 in π° . Then by Lemma 6.3, part(3), we have

$$(\pi^i)^\circ \oplus mx \equiv (\pi^\circ)^i \oplus mx \equiv (\pi^\circ \oplus m)^i \equiv (\mu^\circ)^i \equiv (\mu^i)^\circ,$$

so π^i and μ^i are torically equivalent. \square

Then Proposition 2.5 implies the following theorem.

Theorem 6.6. Let a be a composition of elements from the set $\{r, c, i\}$. Then $a : \tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow \tilde{\mathfrak{A}}_n(\mathbf{p}^a)$ is a bijection for all n .

We further define, for a pattern $\mathbf{p} = (p, X, Y)$, of rank r , such that the letter r is at position ℓ in p (so ℓ equals the last letter of p^i)

$$\mathbf{p} \oplus 1 = (p \oplus 1, X \oplus (r + 1 - \ell), Y \oplus 1) = (p \oplus 1, X \ominus \ell, Y \oplus 1),$$

where the addition on the sets is done modulo $r + 1$ as usual. Here is the reason for making this definition:

Example 6.7. Let $\pi = 76128543$ and

$$\mathbf{p} = (3421, \{2, 3\}, \{1, 2, 4\}) = \overline{\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 3 & 4 & 2 & 1 \\ \hline \end{array}}.$$

Then π has an occurrence of \mathbf{p} shown here $\underline{76128543}$. Now $\pi \oplus 1 = 65418723$ has an occurrence of

$$\mathbf{p} \oplus 1 = (3214, \{0, 1\}, \{0, 2, 3\}) = \overline{\begin{array}{|c|c|c|c|} \hline 1 & 2 & 3 & 4 \\ \hline 3 & 2 & 1 & 4 \\ \hline \end{array}}$$

shown here $\underline{65418723}$.

This example is a special case of the following:

Proposition 6.8. Let $\mathbf{p} = (p, X, Y)$ be a pattern of rank r such that $r \in Y$. The permutation π avoids the pattern \mathbf{p} if and only if the permutation $\pi \oplus 1$ avoids the pattern $\mathbf{p} \oplus 1$.

Proof. Assume the permutation π contains the pattern \mathbf{p} . Then when we add 1 modulo $n + 1$ the permutation and the pattern are split up and reassembled at the same spot, so the permutation $\pi \oplus 1$ will contain $\mathbf{p} \oplus 1$. Reverse this argument to show the other implication. \square

The last Proposition gives a new way to prove Wilf-equivalences for bivincular patterns:

Corollary 6.9. Let $\mathbf{p} = (p, X, Y)$ be a pattern of rank r such that $r \in Y$. The map

$$\oplus 1: \mathfrak{A}_n(\mathbf{p}) \rightarrow \mathfrak{A}_n(\mathbf{p} \oplus 1)$$

is a bijection, and $\tilde{\mathfrak{A}}_n(\mathbf{p}) = \tilde{\mathfrak{A}}_n(\mathbf{p} \oplus 1)$.

This can be iterated as long as we have $r \in Y$ and also combined with the basic symmetries as the following Example shows.

Example 6.10.

(1) Consider the two patterns

$$\mathbf{p} = (12, \emptyset, \{0, 2\}) = \overline{\begin{array}{|c|} \hline 1 & 2 \\ \hline \end{array}}, \quad \mathbf{q} = (12, \emptyset, \{0, 1\}) = \overline{\begin{array}{|c|} \hline 1 & 2 \\ \hline \end{array}}$$

that were shown to be Wilf-equivalent in (Parviainen, 2009, subsection 3.4). This can not be done only in terms of the basic symmetries, but follows from $\mathbf{p} \oplus 1 = \mathbf{q}$.

(2) Consider the two patterns

$$\mathbf{p} = (132, \emptyset, \{0, 1, 2\}) = \overline{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & 2 \\ \hline \end{array}}, \quad \mathbf{q} = (132, \emptyset, \{0, 2, 3\}) = \overline{\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 1 & 3 & 2 \\ \hline \end{array}}$$

that were shown to be Wilf-equivalent in (Parviainen, 2009, subsection 5.17). This can not be done only in terms of the basic symmetries, but using the $\oplus 1$ map we get

$$\begin{aligned} (\mathbf{p}^c \oplus 1)^r &= ((312, \emptyset, \{1, 2, 3\}) \oplus 1)^r = (231, \emptyset, \{0, 2, 3\})^r \\ &= (132, \emptyset, \{0, 2, 3\}) = \mathbf{q}. \end{aligned}$$

(3) Consider the two patterns

$$\mathbf{p} = (123, \{0\}, \{0, 3\}) = \overline{\overline{\overline{1}23}}, \quad \mathbf{q} = (123, \{1\}, \{0, 1\}) = \overline{\overline{\overline{1}23}}$$

that were shown to be Wilf-equivalent in (Parviainen, 2009, subsection 5.18). This can not be done only in terms of the basic symmetries, but follows from $\mathbf{p} \oplus 1 = \mathbf{q}$.

(4) Here is an example showing that we really need to have r in Y . Let $\mathbf{p} = 1\underline{3}24 = (1324, \{2\}, \emptyset)$. Then $\mathbf{p} \oplus 1 = (1243, \{3\}, \emptyset) = 12\underline{4}3$ and

$$|\mathfrak{A}_6(\mathbf{p})| = 549, \quad |\mathfrak{A}_6(\mathbf{p} \oplus 1)| = 550.$$

(5) The patterns in Example 6.7 above are Wilf-equivalent by Corollary 6.9 and this is very likely a new result (since to my knowledge the question of Wilf-equivalence of bivincular patterns has only been considered up to length 3).

Although the $\oplus 1$ map isn't as useful when we don't have $\text{rank}(p)$ in the vertical set it still gives us:

Lemma 6.11. Fix an occurrence of a bivincular pattern \mathbf{p} in a permutation π . Assume \mathbf{p} has rank r and let k be the letter in the occurrence that corresponds to r . Then the permutation $\pi \ominus k$ contains $\mathbf{p} \oplus 1$.

Theorem 6.12. For any pattern \mathbf{p} we have $\tilde{\mathfrak{A}}_n(\mathbf{p}) \subseteq \mathfrak{A}_n(\tilde{\mathbf{p}})$

Proof. Assume π is not in the set on the right, so π contains a pattern \mathbf{p}' that is torically equivalent to \mathbf{p} . We can assume that $\mathbf{p} = \mathbf{p}' \oplus 1$. By Lemma 6.11 we see that a permutation π' that is torically equivalent to π contains \mathbf{p} . This shows that π is not in the set on the left. \square

This theorem will be crucial when we prove Theorems 6.27, 6.28, 6.31 below.

6.2. Number and size of classes. The number of equivalence classes in \mathfrak{S}_n are

$$1, 2, 3, 8, 24, 108, 640, 4492, \dots \quad n = 0, 1, 2, 3, \dots$$

This is A002619: Number of 2-colored patterns on an $(n+1) \times (n+1)$ board. Instead of proving that this is indeed true, let us prove a stronger result, that enumerates the number of classes of a particular size:

For each n we can write $n! = \sum (\text{number of classes of size } i) \cdot i$. For $n = 1, \dots, 8$ this gives us the following equations,

$$\begin{aligned} 1! &= 1 \cdot 1, \\ 2! &= 1 \cdot 2, \\ 3! &= 1 \cdot 2 + 4 \cdot 1, \\ 4! &= 1 \cdot 4 + 5 \cdot 4, \\ 5! &= 1 \cdot 2 + 2 \cdot 2 + 3 \cdot 2 + 6 \cdot 18, \\ 6! &= 1 \cdot 6 + 7 \cdot 102, \\ 7! &= 1 \cdot 4 + 2 \cdot 2 + 4 \cdot 10 + 8 \cdot 624, \\ 8! &= 1 \cdot 6 + 3 \cdot 10 + 9 \cdot 4476. \end{aligned}$$

The following theorem gives us a way to explain the number that appear above, i.e., how many classes of size i there are.

Theorem 6.13. The number of classes in \mathfrak{S}_n of size k is

$$\frac{1}{(n+1)k} \sum_{d|k} \mu(d) U(n+1, \frac{k}{d})$$

if $k|(n+1)$ but zero otherwise. Here $U(n+1, \ell) = \phi\left(\frac{n+1}{\ell}\right) \left(\frac{n+1}{\ell}\right)^\ell \ell!$

Before proving this, say that a permutation π in \mathfrak{S}_n is *cyclically invariant under $\oplus k$* if π and $\pi \oplus k$ are the same when considered as circular permutations.

Proof. Everything needed to prove this is in Steggall (1907), except the last step where we apply the Möbius inversion formula. We also adapt Steggall's arguments to our notation. Steggall showed that the number of permutations in \mathfrak{S}_n that are cyclically invariant under $\oplus k$ is

$$U(n, k) = \begin{cases} \phi\left(\frac{n}{k}\right) \left(\frac{n}{k}\right)^k k! & \text{if } k|n, \\ 0 & \text{otherwise.} \end{cases}$$

This number includes permutations that are cyclically invariant under $\oplus d$, where d is factor of k . Let $u(n, d)$ count the permutations that are cyclicly invariant under d , but no smaller factor. We therefore have

$$U(n, k) = \sum_{d|k} u(n, d).$$

Steggall inverts this formula, writing $u(n, d)$ in terms of $U(n, k)$ for some special cases. We can combine these by applying the Möbius inversion formula

$$u(n, k) = \sum_{d|k} \mu(d) U\left(n, \frac{k}{d}\right),$$

where μ is the Möbius function. We now have a formula that counts the number of permutations in \mathfrak{S}_n that are cyclicly invariant under $\oplus k$ and no smaller factor. Turning these into cyclic permutations their number is $u(n, k)/n$ and the action of $\oplus 1$ breaks this set into orbits of size k , each one corresponding to a toric class of size k in \mathfrak{S}_{n-1} . \square

Here is the array we get by calculating a few values of the number of classes in \mathfrak{S}_n , starting from $n = 0$:

1									
1	0								
2	0	0							
2	0	0	1						
4	0	0	0	4					
2	2	2	0	0	18				
6	0	0	0	0	0	102			
4	2	0	10	0	0	0	624		
6	0	10	0	0	0	0	0	4476	

It is easy to verify that for $k = 1$ we get $\phi(n+1)$ classes of size 1 from the formula in Theorem toricthm:fromSteggall. The permutations that lie in these one element classes will be important later; see Lemma 6.24 below, which gives another proof of the enumeration of these classes.

I should note that in Vella (2002/03) toric classes appear in a disguise as $(n+1)$ -orbits. The author shows that the total number of these orbits is

$$\frac{1}{(n+1)^2} \sum_{kp=n+1} \phi(p)^2 k! p^k,$$

by the orbit counting lemma. This gives A002619 in OEIS, and is equal to the row sums of the array above.

6.3. Pattern avoidance.

Patterns and modular sequences. A modular k sequence is sequence of letters $\ell_1, \ell_2, \dots, \ell_k$ such that $\ell_{i+1} = \ell_i + 1 \pmod{(n+1)}$.

Proposition 6.14. *If $\mathbf{p} = \overline{1}$ then the set $\tilde{\mathfrak{A}}_n(\mathbf{p})$ is equinumerous with the set $C'_{\llbracket 0, n \rrbracket}$ of circular permutations of $\llbracket 0, n \rrbracket$ that have no modular 2-sequences, i.e., i is never followed by $i + 1 \pmod{(n+1)}$.*

Modular 2-sequences are sometimes called *successor pairs*. The enumeration corresponds to sequence A000757:

$$0, 1, 1, 8, 36, 229, 1625, 13208, \dots, \quad n = 1, 2, 3, \dots$$

Proof. We will construct a bijection between $\tilde{\mathfrak{A}}_n(\mathbf{p})$ and the set of circular permutations of $\llbracket 0, n \rrbracket$ without successor pairs. We do this as follows:

$$f: \tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow C'_{\llbracket 0, n \rrbracket}$$

We let $f(\pi) = \pi^\circ = 0\pi$. We first need to show that the image of f is in the set on the right. So assume $f(\pi)$ has a successor pair, $i(i+1) \pmod{(n+1)}$. Then $(f(\pi) \oplus (n+1-i))_\circ$ is torically equivalent to π and starts with 1, but that is impossible, since all permutations torically equivalent to π avoid $\mathbf{p} = (1, \{0\}, \{0\})$.

Now there is a map in the other direction $g(\lambda) = \lambda_\circ$. A similar argument to the one above shows that the image of g is actually in the set on the left. The maps are obviously inverses of each other so we are done. \square

Proposition 6.15. *If $\mathbf{p} = \overline{12}$ then the set $\tilde{\mathfrak{A}}_n(\mathbf{p})$ is equinumerous with the set $C''_{\llbracket 0, n \rrbracket}$ of circular permutations of $\llbracket 0, n \rrbracket$ that have no modular 3-sequences, i.e., i is never followed by $i+1$ and $i+2 \pmod{(n+1)}$.*

The enumeration is

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = 1, 1, 5, 18, 95, 600, 4307, 35168, \quad (n = 1, 2, 3, 4, 5, 6, 7, 8).$$

This corresponds to sequence A165962.

Proof. We will construct a bijection between $\tilde{\mathfrak{A}}_n(\mathbf{p})$ and the set of circular permutations of $\llbracket 0, n \rrbracket$ without modular 3-sequences. We do this as follows:

$$f: \tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow C''_{\llbracket 0, n \rrbracket}$$

We let $f(\pi) = \pi^\circ = 0\pi$. We first need to show that the image of f is in the set on the right. So assume $f(\pi)$ has a modular 3-sequence, $i(i+1)(i+2) \pmod{(n+1)}$. Then $(f(\pi) \oplus (n+1-i))_\circ$ is torically equivalent to π and starts with 12, but that is impossible, since all permutations torically equivalent to π avoid \mathbf{p} .

Now there is a map in the other direction $g(\lambda) = \lambda_\circ$. A similar argument to the one above shows that the image of g is actually in the set on the left. The maps are obviously inverses of each other so we are done. \square

It then turns out that you can keep going (the proof is an easy generalization of the proofs above and we omit it):

Proposition 6.16. *For $k \geq 1$, let $\mathbf{p} = \overline{12 \dots k}$. The set $\tilde{\mathfrak{A}}_n(\mathbf{p})$ is equinumerous with the set of circular permutations of $\llbracket 0, n \rrbracket$ that have no modular $(n+1)$ -sequences.*

Patterns with connections to number theory. Before we look at the next pattern we have some definitions to make.

Definition 6.17. Let $n \geq 1$ be fixed.

- (1) For any k coprime to $n+1$ we define a permutation $\nu_{k,n}$ first by constructing a circular permutation $\lambda_{k,n}$ of $\llbracket 0, n \rrbracket$ as follows: Place 0 anywhere, then place 1 by moving k steps from 0 (so there are $k-1$ empty positions between 0 and 1), then place 2 by moving k steps from 1 and keep going until you place n . Then define $\nu_{k,n} = (\lambda_{k,n})_\circ$. We call the permutation constructed in this way the *natural permutation* (corresponding to k) in \mathfrak{S}_n .
- (2) If k is a divisor of n we write $\delta_{k|n} = \nu_{k,n}$ and call $\delta_{k|n}$ the *divisor permutation* (corresponding to $k|n$) in \mathfrak{S}_n .

The condition that k be coprime to $n+1$ is a necessary and sufficient condition for constructing $\nu_{k,n}$. The reason for calling these permutations *natural* is that the permutation $\nu_{k,n}$ behaves like the natural number k when multiplied with other natural permutations (see Proposition 6.25 below). The construction of the natural permutations is slightly reminiscent of the Josephus problem, see e.g., Shams-Baragh (2002).

Example 6.18. Let $n = 6$. The coprime integers to $n+1 = 7$ are 1, 2, 3, 4, 5 and 6. We construct $\lambda_{1,6}$ as follows:

$$0 _____ = 01 _____ = 012 _____ = \dots = 0123456,$$

so $\nu_{1,6} = 123456 = \delta_{1|6}$. Next we construct $\lambda_{2,6}$:

$$0 _____ = 0 _1 _____ = 0 _1 _2 _____ = 0 _1 _2 _3 _____ = 041 _2 _3 _____ = 04152 _3 _____ = 0415263,$$

so $\nu_{2,6} = 415263 = \delta_{2|6}$. Similarly we get

$$\nu_{3,6} = 531642 = \delta_{3|6},$$

$$\nu_{4,6} = 246135,$$

$$\nu_{5,6} = 362514,$$

$$\nu_{6,6} = 123456 = \delta_{6|6}.$$

Below we give list of all the natural permutations in \mathfrak{S}_1 to \mathfrak{S}_{10} . Note that by definition the location of 1 in $\nu_{k,n}$ and $\delta_{k|n}$ is k , or equivalently, the first letter of inverse of these permutations is k . They also have other good properties:

Lemma 6.19.

- (1) For a natural permutation $\nu = \nu_{k,n}$ the difference between subsequent letters,

$$j = \nu(\ell+1) - \nu(\ell)$$

is independent of ℓ and equals the first letter of ν .

- (2) If ν is a divisor permutation then $j = -n/k \pmod{n+1}$. In general this difference is the smallest positive integer j such that $kj = 1 \pmod{n+1}$.

Proof. (1) The letter that lands at ℓ equals

$$\frac{\ell + (n+1)s_\ell}{k}$$

where s_ℓ is chosen as the least integer to make this an integer. The letter that lands at $\ell+1$ equals

$$\frac{\ell + 1 + (n+1)s_{\ell+1}}{k}$$

where $s_{\ell+1}$ is chosen as the least integer to make this an integer. The difference is

$$\frac{1 - (n+1)(s_{\ell+1} - s_{\ell})}{k}$$

which is independent of ℓ . This difference must equal the first letter since if we add 0 at the front the difference is the same as the first element.

- (2) This can be done by examining the difference above: If $k|n$ then we choose $s_{\ell+1} = s_{\ell} - 1$ and the difference becomes n/k . The general case is similar. \square

In view of the last Lemma we will call the first letter of a natural permutation ν the *increment* of the permutation and denote it by j_{ν} or just j if there are no other natural permutations around.

Lemma 6.20.

- (1) $\nu_{k,n}^r = \nu_{n+1-k,n} = \nu_{k,n}^c$.
(2) $\nu_{k,n}^j = \nu_{j,n}$, where j is the increment of $\nu_{k,n}$.

Proof. This follows directly from Definition 6.17 and Lemma 6.19. Note that k is coprime to $n+1$ if and only if $n+1-k$ is coprime to $n+1$. \square

Lemma 6.21. *Let $\nu = \nu_{k,n}$ be a natural permutation.*

- (1) $\nu_1 + \nu_n = n+1$.
(2) If $k|n$ then $\nu_n = n/k$, and in general $\nu_n = -j \pmod{n+1}$, where j is the increment of ν .

Proof. Note that $\nu_n = (\nu^r)_1$ and the first letter of ν^r is the smallest positive solution of $(n+1-k)x \equiv 1 \pmod{n+1}$. Clearly this is $n+1-j$. \square

Lemma 6.22. *If $\delta_{k|n}$ is a divisor permutation then it consists of k increasing subsequences of length n/k . These sequences are*

$$\begin{aligned} &1, 2, \dots, n/k, \\ &n/k + 1, n/k + 2, \dots, 2n/k, \\ &\dots, \\ &n - n/k + 1, n - n/k + 2, \dots, n. \end{aligned}$$

They lie inside the permutation in such a way that the ℓ -th one starts at $k - \ell + 1$ and is placed at locations $k - \ell + 1 + dk$, where $d = 0, \dots, n/k - 1$.

Proof. This is obvious from the construction of the divisor permutations. \square

Example 6.23. *Consider the divisor permutation*

$$\delta_{3|12} = \underline{9}5\underline{1}(10)\underline{6}2(11)\underline{7}3(12)\underline{8}4,$$

it consists of 3 increasing subsequences of length $12/3 = 4$. We underline the second one.

Lemma 6.24. *The natural permutations are the permutations with a toric class of size 1.*

Proof. It is easy to see from the construction of the natural permutations that they have a toric class of size 1. So assume π is a permutation with toric class size 1, which is equivalent to $\pi \oplus 1 = \pi$, which is equivalent to $\pi^{\circ} \oplus 1 \equiv \pi^{\circ}$. Now let ℓ be the distance between 0 and 1 in π° . This becomes the distance between 1 and 2 in $\pi^{\circ} \oplus 1 = \pi^{\circ}$. By iterating this we have shown that the distance between k and $k+1 \pmod{n+1}$ is fixed. This implies that π is a natural permutation. \square

Proposition 6.25. *The natural permutations multiply like the natural numbers modulo $n + 1$, i.e., $\nu k, n \circ \nu_{k', n} = \nu_{kk', n}$, where kk' has been reduced modulo $n + 1$.*

Proof. This is easy to see using the increment. \square

Example 6.26. *For example*

$$\delta_{2|6} \circ \delta_{2|6} = \nu_{4,6} \quad \text{and} \quad \nu_{4,6} \circ \nu_{5,6} = \delta_{6|6}.$$

Proposition 6.27. *The set $\tilde{\mathfrak{A}}_n(213)$ consists of the identity and its reverse.*

Proof. It is clear that the identity and its reverse are in this set. By Theorem 6.12 a permutation in this set must avoid the classical patterns 2–1–3, 1–3–2, 3–1–2 and 2–3–1, so the only possibilities are the identity and its reverse. \square

Theorem 6.28. *Let $\mathbf{p} = \overline{123}$. The set $\tilde{\mathfrak{A}}_n(\mathbf{p})$ consists of the natural permutations in \mathfrak{S}_n . In particular*

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = \phi(n + 1),$$

where ϕ is Euler's totient function, and the map $\nu_{k,n} \mapsto k, \tilde{\mathfrak{A}}_n(\mathbf{p}) \rightarrow U_{n+1}$ is a homomorphism of groups, where U_{n+1} is the group of units in $\mathbb{Z}/(n + 1)\mathbb{Z}$.

Proof. We start by showing that the natural permutations avoid this pattern. Fix such a permutation $\nu = \nu_{k,n}$ with k coprime to $n + 1$. It suffices to show that given an integer $2 \leq \ell \leq n - 1$ such that $\ell - 1$ occurs after ℓ in the permutation, then n appears before $\ell - 1$. Let j be the increment of ν so we can write

$$\nu = j, 2j, 3j, \dots, mj, \dots, (m + h)j, \dots, nj \pmod{n + 1}$$

where $mj = \ell \pmod{n + 1}$ and $(m + h)j = \ell - 1 \pmod{n + 1}$. Then we must have $hj = -1 \pmod{n + 1}$ which implies $hj = n \pmod{n + 1}$ so n is at location h in ν and therefore appears before $\ell - 1$.

We must now show that any permutation that avoids this pattern must be a natural permutation. This relies on Lemmas 6.29 and 6.30. They show that $\pi = x(2x)(3x)\dots$ and is therefore natural. \square

Lemma 6.29. *If π is in the set $\tilde{\mathfrak{A}}_n(\overline{123})$ and $\pi = x\dots 1y\dots$ then $y = 1 + x \pmod{n + 1}$.*

Proof. Note that if $x = n$ then $\pi = \text{id}^\Gamma$, so we can assume that $x \neq n$. Assume $y \neq 1 + x \pmod{n + 1}$. Then $1 + x$ must appear after y in π , otherwise we have the pattern $\mathbf{p} \oplus 3$.

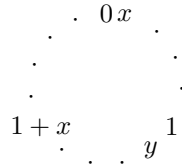


FIGURE 1. The circular permutation π° .

Consider what values y could take. It can not be 2 since that would produce the pattern in $\mathbf{p} \oplus 1$ in $\pi \ominus 1$. It can not be 3 since that would mean 2 was between x and 1 in π and $\pi \ominus 2$ would contain $\mathbf{p} \oplus 1$. Similarly for any $k < x$ we can show $y \neq k$. Therefore we must have $y > 1 + x$. If $y = 2 + x$ then $1, 2 + x, 1 + x$ in π give the pattern $\mathbf{p} \oplus 1$. The letter $2 + x$ must therefore be between $1 + x$ and 0 in π° above. We can now see that y can not be $3 + x$ either. By induction we can show

that y can not be any letter larger than $1 + x$. This gives us a contradiction so we must have $y = 1 + x$. \square

Lemma 6.30. *If π is in the set $\tilde{\mathfrak{A}}_n \left(\begin{smallmatrix} \overline{12} & 3 \\ 2 & 1 & 3 \end{smallmatrix} \right)$ and $\pi = x \cdots zy \cdots$ then $y = z + x \bmod (n + 1)$.*

Proof. We do this by induction on z . This is known for $z = 1$ from Lemma 6.29. Assume it is true for $z - 1$ and consider $\pi = x \cdots zy \cdots$. Then $\pi \ominus 1 = \cdots n(x - 1) \cdots (z - 1)(y - 1) \cdots$. This permutation must then start with $y - z$. So in π we have $\cdots 1(y - z + 1) \cdots$. But π starts with x so we get $y - z + 1 = 1 + x$ or $y = z + x$ as we wanted. \square

Theorem 6.31. *Let $\mathbf{p} = \begin{smallmatrix} \overline{12} & 3 \\ 2 & 1 & 3 \end{smallmatrix}$. The set $\tilde{\mathfrak{A}}_n(\mathbf{p})$ consists of the divisor permutations in \mathfrak{S}_n . In particular*

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = d(n),$$

where $d(n)$ counts the number of divisors in n .

Proof. If a divisor permutation contains the pattern $\begin{smallmatrix} \overline{12} \\ 2 & 1 \end{smallmatrix}$ then the letter corresponding to 1 must be at the end of a subsequence and the letter corresponding to 2 at the start of the next subsequence (see Lemma 6.22). But everything after the end of a subsequence is smaller than the elements in the subsequence so the pattern can not be completed to $\begin{smallmatrix} \overline{12} & 3 \\ 2 & 1 & 3 \end{smallmatrix}$. Since a divisor permutation is alone in its toric class we are done.

Now suppose we have a permutation that is in the set on the left. Then it must be a natural permutation, $\nu_{k,n}$, by Theorem 6.28. To finish the proof we need to show that $k|n$. Suppose not. Then the first subsequence in $\nu_{k,n}$ consisting of $1, 2, 3, \dots, \ell$ does not terminate at the last position of the permutation, meaning that something larger than ℓ appears there. This clearly gives us an occurrence of the pattern, which is a contradiction. \square

Recall that the location of 1 in $\nu_{k,n}$ and $\delta_{k|n}$ is k , or equivalently, the first letter of inverse of these permutations is k . This property combined with Theorem 6.31 gives us a way to write the *sum-of-divisors* function $\sigma(n)$ as

$$\sigma(n) = \sum_{\delta \in \tilde{\mathfrak{A}}_n \left(\begin{smallmatrix} \overline{12} & 3 \\ 2 & 1 & 3 \end{smallmatrix} \right)} (\text{location of 1 in } \delta).$$

Now consider the following theorem due to Robin (1984).

Theorem 6.32 (Robin's theorem). *Let $\sigma(n)$ denote the sum of the divisors of n . The Riemann Hypothesis is true if and only if*

$$\sigma(n) < e^\gamma \log \log n,$$

holds for all $n \geq 5041$. Here γ is the Euler-Mascheroni constant.

This allows us to state the Riemann Hypothesis in terms of pattern avoidance:

Conjecture 6.33 (Equivalent to RH). *The inequality*

$$\sum_{\delta \in \tilde{\mathfrak{A}}_n \left(\begin{smallmatrix} \overline{12} & 3 \\ 2 & 1 & 3 \end{smallmatrix} \right)} (\text{location of 1 in } \delta) < e^\gamma \log \log n,$$

holds for all $n \geq 5041$.

The largest known n for which the inequality in Robin's theorem is violated is 5040, so it *suffices* to start exploring in \mathfrak{S}_{5041} . I should mention that permutations have been shown before to have connections with the Riemann Hypothesis, for example using the Redheffer matrix in Wilf (2004/06), probabilistic methods in Aldous and Diaconis (1999) (see also Stopple), and group theory in Massias et al. (1988).

Natural permutations in \mathfrak{S}_1 to \mathfrak{S}_{10} . In \mathfrak{S}_1 we have

$$\nu_{1,1} = 1 = \delta_{1|1}$$

In \mathfrak{S}_2 we have

$$\nu_{1,2} = 12 = \delta_{1|2}$$

$$\nu_{2,2} = 21 = \delta_{2|2}$$

In \mathfrak{S}_3 we have

$$\nu_{1,3} = 123 = \delta_{1|3}$$

$$\nu_{3,3} = 321 = \delta_{3|3}$$

In \mathfrak{S}_4 we have

$$\nu_{1,4} = 1234 = \delta_{1|4}$$

$$\nu_{2,4} = 3142 = \delta_{2|4}$$

$$\nu_{3,4} = 2413$$

$$\nu_{4,4} = 4321 = \delta_{4|4}$$

In \mathfrak{S}_5 we have

$$\nu_{1,5} = 12345 = \delta_{1|5}$$

$$\nu_{5,5} = 54321 = \delta_{5|5}$$

In \mathfrak{S}_6 we have

$$\nu_{1,6} = 123456 = \delta_{1|6}$$

$$\nu_{2,6} = 415263 = \delta_{2|6}$$

$$\nu_{3,6} = 531642 = \delta_{3|6}$$

$$\nu_{4,6} = 246135$$

$$\nu_{5,6} = 362514$$

$$\nu_{6,6} = 654321 = \delta_{6|6}$$

In \mathfrak{S}_7 we have

$$\nu_{1,7} = 1234567 = \delta_{1|7}$$

$$\nu_{3,7} = 3614725$$

$$\nu_{5,7} = 5274163$$

$$\nu_{7,7} = 7654321 = \delta_{7|7}$$

In \mathfrak{S}_8 we have

$$\nu_{1,8} = 12345678 = \delta_{1|8}$$

$$\nu_{2,8} = 51627384 = \delta_{2|8}$$

$$\nu_{4,8} = 75318642 = \delta_{4|8}$$

$$\nu_{5,8} = 24681357$$

$$\nu_{7,8} = 48372615$$

$$\nu_{8,8} = 87654321 = \delta_{8|8}$$

In \mathfrak{S}_9 we have

$$\begin{aligned}\nu_{1,9} &= 123456789 = \delta_{1|9} \\ \nu_{3,9} &= 741852963 = \delta_{3|9} \\ \nu_{7,9} &= 369258147 \\ \nu_{9,9} &= 987654321 = \delta_{9|9}\end{aligned}$$

In \mathfrak{S}_{10} we have

$$\begin{aligned}\nu_{1,10} &= 123456789(10) = \delta_{1|10} \\ \nu_{2,10} &= 61728394(10)5 = \delta_{2|10} \\ \nu_{3,10} &= 4815926(10)37 \\ \nu_{4,10} &= 369147(10)258 \\ \nu_{5,10} &= 97531(10)8642 = \delta_{5|10} \\ \nu_{6,10} &= 2468(10)13579 \\ \nu_{7,10} &= 852(10)741963 \\ \nu_{8,10} &= 73(10)6295184 \\ \nu_{9,10} &= 5(10)49382716 \\ \nu_{10,10} &= (10)987654321 = \delta_{10|10}\end{aligned}$$

7. OTHER EQUIVALENCE RELATIONS

I have looked at many other equivalence relations besides the one we study above. Below are just two conjectures concerning two of them.

Conjecture 7.1. *Regard two permutations as equivalent if their descents appear at the same positions. The pattern $\mathbf{p} = 321$ gives*

$$\begin{aligned}\tilde{\mathfrak{A}}_n(\mathbf{p}) &= \text{Grassmannian permutations} \\ |\tilde{\mathfrak{A}}_n(\mathbf{p})| &= 1, 2, 5, 12, 27, 58, 121, 248, 503, \dots, \quad n = 1, 2, 3, \dots \\ &= 2^n - n.\end{aligned}$$

This is A000325.

Recall that a *Grassmannian permutation* has at most one descent

Conjecture 7.2. *Regard two permutations as equivalent if they give the same volume partition, see Young (2010). The pattern $\mathbf{p} = \begin{smallmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{smallmatrix}$ gives*

$$|\tilde{\mathfrak{A}}_n(\mathbf{p})| = 1, 2, 5, 18, 84, 480, 3240, 25200, \dots, \quad n = 1, 2, 3, \dots$$

which is (from $n = 2$) A038720: Next-to-last diagonal of A038719, which counts chains in a specific poset.

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