

TWO NEW INTERPRETATIONS OF THE FISHBURN NUMBERS AND THEIR REFINED GENERATING FUNCTIONS

PAUL LEVANDE

ABSTRACT. We show that two classes of combinatorial objects—inversion tables with no subsequence of decreasing consecutive numbers and matchings with no 2-nestings—are enumerated by the Fishburn numbers. In particular, we give a simple bijection between matchings with no 2-nestings and inversion tables with no subsequence of decreasing consecutive numbers. We then prove using the involution principle that inversion tables with no subsequence of decreasing consecutive numbers have the same generating function as the Fishburn numbers. The Fishburn numbers have previously been shown by Bousquet-Mélou, Claesson, Dukes and Kitaev to enumerate **(2+2)**-avoiding posets, matchings with no left- or right-nestings, permutations avoiding a particular pattern, and so-called ascent sequences, and by Dukes and Parviainen to enumerate upper triangular matrices with non-negative entries and no empty rows or columns. Claesson and Linusson conjectured they also enumerated matchings with no 2-nestings. Using these new interpretations of the Fishburn numbers and another version of the involution, we prove the conjectured equality (also proven using matrices by Jelínek and by Yan) of two refinements by Remmel and Kitaev of the Fishburn generating function. In an appendix, we state and prove another conjecture of Claesson and Linusson giving the distribution of left-nestings over the set of all matchings.

1. INTRODUCTION

It is well-known that the famous Catalan numbers C_n enumerate, among many other classes of combinatorial objects, non-crossing and non-nesting matchings on $[2n]$, where a *matching* is an involution with no fixed points, or a partition of $[2n]$ into disjoint pairs, and a *nesting* in a matching X is a pair of pairs $(a, b), (c, d) \in X$ such that $a < c < d < b$. A superset of the set of non-nesting matchings on $[2n]$ is the set of non-neighbor-nesting matchings on $[2n]$, where a *neighbor nesting* is a nesting $(a, b), (c, d) \in X$ such that either $c = a + 1$ or $d = b - 1$. Zagier [11], following Stoimenow [9], showed:

$$(1) \quad \sum_{n=0}^{\infty} f_n t^n = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1-t)^i)$$

where f_n is the number of non-neighbor-nesting matchings on $[2n]$. Recently, Bousquet-Mélou et al [1]. showed that f_n also enumerates other seemingly-disparate sets, each of which can be seen to be a superset of a set enumerated by the Catalan numbers:

- **(2+2)**-avoiding posets with n elements, a superset of the set of **(2+2)**- and **(3+1)**-avoiding posets with n elements enumerated by the n -th Catalan numbers C_n ,
- Permutations π of $[n]$ such that, if $\pi = \pi_1 \pi_2 \dots \pi_n$, there are no $i < j$ such that $\pi_j = \pi_i - 1$ and $\pi_{i+1} > \pi_i$, a superset of the set of 231-avoiding permutations of $[n]$ enumerated by the n -th Catalan number C_n .

- ascent sequences of length n , where an ascent sequence is a sequence $x_1x_2\dots x_n$ such that $x_1 = 0$ and $0 \leq x_i \leq \text{asc}(x_1x_2\dots x_{i-1}) + 1$, where $\text{asc}(x_1x_2\dots x_{i-1})$ is the number of ascents of the sequence $x_1x_2\dots x_{i-1}$, a superset of the set of sequences $x_1\dots x_n$ such that $x_1 = 0$ and $x_{i+1} \leq x_i + 1$ enumerated by the n -th Catalan number C_n .

Claesson and Linusson [2] recently conjectured another interpretation of the sequence f_n , which they refer to as the *Fishburn numbers*: Matchings on $[2n]$ with no 2-nestings, where a k -nesting is a nesting $(a, b), (c, d)$ such that $a < c < d < b$ and $c - a \leq k$.

Remmel and Kitaev [6] recently gave the following refinement of (1):

$$(2) \quad \sum_{n=0}^{\infty} \sum_{d=1}^n f_{n,d} t^n z^d = 1 + \sum_{m=1}^{\infty} \frac{zt}{(1-zt)^m} \prod_{i=1}^{m-1} (1 - (1-t)^i),$$

where $f_{n,d}$ is (among other interpretations) the number of ascent sequences of length n with d zeroes. Remmel and Kitaev also conjectured that the bivariate generating function $\sum_{n=0}^{\infty} \sum_{d=1}^n f_{n,d} t^n z^d$ has the following simpler form:

$$(3) \quad \sum_{n=0}^{\infty} \sum_{d=1}^n f_{n,d} t^n z^d = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1-t)^{i-1} (1-zt)).$$

In the following paper, we solve both conjectures and show how they are related: First, we give a simple bijection between matchings on $[2n]$ with no 2-nestings and inversion tables of length n with no decreasing subsequence of consecutive numbers, i.e., sequences $a_1a_2\dots a_n$ such that $0 \leq a_i \leq i - 1$ for all i and such that there exist no $p < q$ with $a_p = j + 1$, $a_q = j$. We then prove using the involution principle that the generating function of inversion tables of length n with no decreasing subsequence of consecutive numbers is (1). We then show using the same proof that, if $f_{n,d}$ is now taken to be the number of such inversion tables $a_1a_2\dots a_n$ where $a_i = i - 1$ for precisely d distinct values of i , the bivariate generating function of $f_{n,d}$ is given by (3). Finally, we show using a variation of this proof that the the bivariate generating function of $f_{n,d}$, under this interpretation, is also given by (2). In an appendix, we give a brief solution to a related conjecture of Claesson and Linusson, that the distribution of *left-nestings*, or nestings $(a, b), (c, d)$ with $c = a + 1$, over all matchings is given by the second-order Eulerian triangle.

Note: Distinct proofs of the equality of (2) and (3), both using a matrix-based interpretation of the Fishburn numbers due to Dukes and Parviainen [3], were recently given independently by Jelínek [5] and by Yan [10]. This work was done independently from the work presented here.

2. FROM FACTORIAL MATCHINGS TO INVERSION TABLES

First, a note on visual notation: A matching X on $[2n]$ can be illustrated by a diagram of n semicircular arcs, where an arc connects a and b if and only if $(a, b) \in X$. For example, the matching $(1, 4)(2, 9)(3, 6)(5, 10)(7, 8)$ on $[10]$ is illustrated by the diagram of 5 semicircular arcs in Figure 1. Note that an arc diagram is equivalent to its matching, i.e., the arc diagram does not have to be labelled to specify a unique matching. Claesson and Linusson [2] proved the following claim about *left-nestings*, where a left-nesting is a nesting $(a, b), (c, d)$ with $c = a + 1$. (For example, $(2, 9)(3, 6)$ in Figure 1 is a left-nesting.)

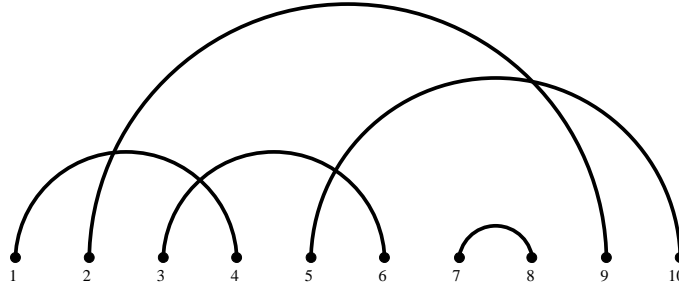


FIGURE 1. A matching on $[10]$.

Claim 2.1. *The number of matchings on $[2n]$ with no left-nestings is $n!$.*

Proof. This is perhaps easiest to see visually and inductively: Let X be a matching on $[2n]$ with no left-nestings. A matching X' on $[2n + 2]$ can be formed from X by adding an arc whose right endpoint is the rightmost part of the diagram, i.e., by adding $(a, 2n + 2)$ to X , and re-labeling (moving all endpoints greater than a over by one). If the left endpoint is placed immediately before the left endpoint of another arc, i.e., if $(a, b) \in X$ and $a < b$, this will form a left-nesting. Therefore, X' will no left-nestings if and only if $(a, b) \in X$ and $a > b$, i.e., if and only if the left endpoint of the newest arc is placed immediately before the right endpoint of an arc (including its own). Since there will be $n + 1$ right endpoints, there are $n + 1$ possible matchings on $[2n + 2]$ with no left-nestings formed by adding an arc to X .

Conversely, removing the pair $(a, 2n + 2)$ from some matching Z' on $[2n + 2]$ with no left-nestings and re-labeling will produce a matching Z with no left-nestings: The only left-nesting this could create in Z is $(a - 1, b), (a, c)$ with $a < c < b$, but then $(a + 1, c + 1), (a, 2n + 2) \in Z'$ would be a left-nesting. Therefore, every matching on $[2n + 2]$ with no left-nestings is formed from adding an arc in $n + 1$ possible ways to a matching on $[2n]$, and if there are $n!$ matchings on $[2n]$ with no left-nestings, there will be $(n + 1)!$ matchings on $[2n + 2]$ with no left-nestings. \square

We illustrate the inductive argument in Figure 2.

Note that we can (following Claesson and Linusson) refine the above proof: Let X be a matching with no left-nestings, and let $\phi(X) = a_1 a_2 \dots a_n$, where a_i is the number of right endpoints to the left of the left endpoint of the i -th arc, where the arcs are numbered from left to right by right endpoint. For example, if X is the matching on $[8]$ in Figure 2, $\phi(X) = 0021$.

Claim 2.2. *ϕ gives a bijection between matchings with no left-nestings and inversion tables, i.e., sequences $a_1 a_2 \dots a_n$ such that $0 \leq a_i \leq i - 1$ for all $1 \leq i \leq n$.*

Proof. This should be clear from the above: A new arc added to a matching on $[2n]$ corresponding to a pair $(a, 2n + 2)$ can have its left endpoint be placed only immediately before the first right endpoint, the second right endpoint, and so on, i.e., to the right of $0, 1, \dots, n$ right endpoints. \square

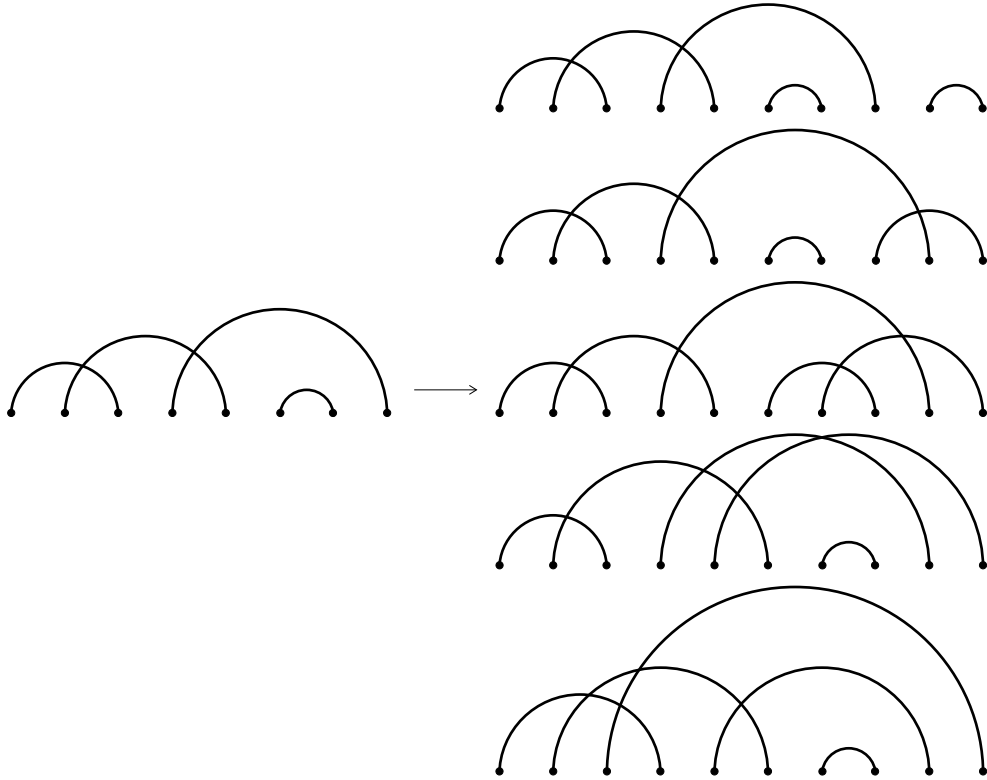


FIGURE 2. Left-nesting-avoiding matchings on $[8]$ and $[10]$

For example, when applied to the the matchings on $[10]$ in Figure 2, ϕ gives, from top to bottom, 00214, 00213, 00212, 00211, and 00210.

In order to prove that the number of matchings on $[2n]$ with no 2-nestings is the n -th Fishburn number f_n , we restrict ϕ to matchings with no 2-nestings:

Claim 2.3. *ϕ gives a bijection between matchings with no 2-nestings and inversion tables $a_1 a_2 \dots a_n$ with no decreasing subsequences of consecutive integers, i.e., with no $p < q$ such that $a_p = j + 1$ and $a_q = j$ for some j .*

Proof. Recall that a 2-nesting is a nesting $(a, b), (c, d)$ with $c - a \leq 2$. In particular, if X is a matching with no left-nestings and $\phi(X) = a_1 a_2 \dots a_n$, then X will have a 2-nesting if and only if, for some a , $(a, b), (a + 2, d) \in X$, $d < b$, and, if $(a + 1, g) \in X$, $g < a$: if $a + 2 < g < b$, then (a, b) and $(a + 1, g)$ will be a left-nesting, and if $b < g$, $(a + 1, g), (a + 2, d)$ will be a left-nesting. Therefore X will have a 2-nesting if and only if its arc diagram gives precisely one right endpoint between the left endpoints of the arc corresponding to (a, b) and the arc corresponding to $(a + 2, d)$, or if and only if $a_p = j + 1$ and $a_q = j$, where d is the p -th right endpoint, b is the q -th right endpoint, and j is the number of right endpoints to the left of a .

Similarly, if, for some $p < q$, $a_p = j + 1$ and $a_q = j$, then the arc diagram of X will have precisely one right endpoint in between the left endpoint of the q -th arc and the left endpoint of the p -th arc. Let the q -th arc correspond to the pair $(e, f) \in X$ and the p -th arc correspond

to the pair $(g, h) \in X$, with the single arc with right endpoint in between correspond to the pair $(r, s) \in X$, so $e < s < g < h < f$. $e + 1, e + 2, \dots, s - 1, s + 2, s + 3, \dots, g - 1$ must all be left endpoints of arcs, since there is only one right endpoint in between e and g . In general, let $X(a)$ be the element paired with a in X . To avoid left-nestings, $X(e + 1) > X(e) = f$, $X(e + 2) > X(e + 1)$, and so on, so $X(s - 1) > X(s - 2) > \dots > X(e) = f$. Similarly, to avoid left-nestings, $X(g - 1) < X(g) = h$, $X(g - 2) < X(g - 1)$, and so on, so $X(s + 1) < X(s + 2) < \dots < X(g) = h$. Therefore $(s - 1, X(s - 1)), (s + 1, X(s + 1)) \in X$ and $X(s + 1) < h < f < X(s - 1)$, and so $(s - 1, X(s - 1)), (s + 1, X(s + 1)) \in X$ is a 2-nesting, and X has at least one 2-nesting if and only if $\phi(X)$ has at least one decreasing subsequence of consecutive integers. \square

Let T_n be the set of inversion tables of length n with no decreasing subsequences of consecutive integers.

3. THE MAIN PROOF

We will now prove that $|T_n| = f_n$, i.e., that the set of inversion tables of length n with no decreasing subsequences of consecutive integers is enumerated by the n -th Fishburn number. In particular, we will prove that:

$$(4) \quad \sum_{n=0}^{\infty} |T_n| t^n = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1 - t)^i).$$

First, we will define a class of diagrams to give a visual interpretation to the right-hand-side of (4). Given a staircase partition diagram $(1, 2, \dots, k)$, a *filling* of the diagram is a placement of dots into the squares of the diagram, with at most one dot per square. We say that a diagram with filling is a *Fishburn diagram* if every column has at least one dot. Let Y be the set of Fishburn diagrams of any size, with Y_n the set of Fishburn diagrams with n dots. Equivalently, let Y be the set of sequences of sets $A_1 A_2 \dots A_k$, where k can vary, such that $A_i \subset \{0, 1, \dots, i - 1\}$ and $A_i \neq \emptyset$, with Y_n the set of such sequences such that $\sum_{i=1}^k |A_i| = n$. We will refer to Fishburn diagrams and sequences of sets interchangeably; for the correspondence, given a sequence of sets $A_1 A_2 \dots A_k$ with the above condition, the corresponding diagram is a filling of $(1, 2, \dots, k)$ with a dot in the j -th row from the bottom in the i -th column from the left if and only if $j - 1 \in A_i$.

Claim 3.1. *The following identity of generating functions holds:*

$$(5) \quad \sum_{n=0}^{\infty} \sum_{A_1 A_2 \dots A_k \in Y_n} t^n (-1)^{n-k} = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1 - t)^i).$$

Proof. Let us first consider the weighted sum of Fishburn diagrams of length m , i.e., sequences $A_1 A_2 \dots A_m \in Y$, beginning by giving each column a weight of -1 . For a specific column A_i with $1 \leq i \leq m$, we can either place a dot, or not, each square. Let the weight of a dot be $-t$ and the weight of a square with no dot be 1. Then for each square, our weighted choice is $1 - t$, and there are i squares in this column. Since we must pick at least one dot

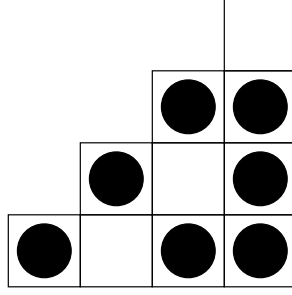


FIGURE 3. A Fishburn diagram in Y_7 with signed weight $(-1)^3 t^7$.

for the column, the weighted sum over all possible choices is $1 - (1 - t)^i$. Therefore,

$$\sum_{A_1 A_2 \dots A_m \in Y} (-t)^{|A_1|+|A_2|+\dots+|A_m|} (-1)^m = \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1 - t)^i),$$

and

$$(6) \quad \sum_{m=0}^{\infty} \sum_{A_1 A_2 \dots A_m \in Y} (-t)^{|A_1|+|A_2|+\dots+|A_m|} (-1)^m = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1 - t)^i).$$

It should be clear that (6) is equivalent to (5). \square

We can think of the left-hand-side of (5) as the signed weighted sum over all Fishburn diagrams of any size, where the weight is given by the number of dots and the sign is given by the number of “extra” dots, i.e., dots other than the minimal one per column. For example, Figure 3 shows the Fishburn diagram corresponding to $A_1 A_2 A_3 A_4 \in Y_7$, where $A_1 = \{0\}$, $A_2 = \{1\}$, $A_3 = \{0, 2\}$ and $A_4 = \{0, 1, 2\}$, which will have a signed weight of $(-1)^3 t^7$.

In order to prove (4), we will define an involution $\psi_n : Y_n \rightarrow Y_n$ such that:

- $A_1 A_2 \dots A_k \in \text{Fix}(\psi_n)$ if and only if, for all $1 \leq i \leq k$, $|A_i| = 1$ (so $k = n$) and there are no p, q, j with $p < q$, $A_p = \{j + 1\}$, $A_q = \{j\}$,
- If $A_1 A_2 \dots A_k \notin \text{Fix}(\psi_n)$ and $\psi_n(A_1 A_2 \dots A_k) = B_1 B_2 \dots B_r$, $r = k \pm 1$,

where $\text{Fix}(\psi_n) = \{A_1 A_2 \dots A_k \in Y_n : \psi_n(A_1 A_2 \dots A_k) = A_1 A_2 \dots A_k\}$. If we can define such an involution ψ_n , then we will have

$$\sum_{n=0}^{\infty} |\text{Fix}(\psi_n)| t^n = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1 - t)^i).$$

This will prove (4), since T_n will be in trivial bijection with $\text{Fix}(\psi_n)$.

Given $A_1 A_2 \dots A_k \in Y_n$, let j be the smallest integer such that either:

- For some i , $j \in A_i$ and $|A_i| > 1$, or
- For some $p < q$, $j + 1 \in A_p$, $j \in A_q$.

If there is no such j , define $\psi_n(A_1 A_2 \dots A_k) = A_1 A_2 \dots A_k$. Otherwise, we divide into cases:

Case 1: If, for some i , $j \in A_i$ and $|A_i| > 1$, let I be the minimal such i . By the minimality of j , j is the smallest member of A_I . Let $j + R$ be the second-smallest member of A_I . Define sets B_L as follows for $1 \leq L \leq k + 1$.

$$B_L = \begin{cases} A_L & 1 \leq L \leq I - R \\ \{s - R + 1 : s \in A_I, s \neq j\} & L = I - R + 1 \\ \{s : s \in A_{L-1}, s < j + 1\} \cup \{s + 1 : s \in A_{L-1}, s \geq j + 1\} & I - R + 2 \leq L \leq I \\ \{j\} & L = I + 1 \\ \{s : s \in A_{L-1}, s < j\} \cup \{s + 1 : s \in A_{L-1}, s \geq j\} & I + 2 \leq L \leq k + 1 \end{cases}$$

It should be clear that $(B_1, B_2, \dots, B_{k+1}) \in Y_n$. Define $\psi_n(A_1, A_2, \dots, A_k) = (B_1, B_2, \dots, B_{k+1})$.

Note that, by construction, $B_{I+1} = \{j\}$ is the last set of B_1, B_2, \dots, B_{k+1} to contain j , and B_{I-R+1} is the last set of B_1, B_2, \dots, B_{I+1} to contain $j + 1$, which must be the smallest member of B_{I-R+1} . Note also that, since A_I was the first set of A_1, A_2, \dots, A_k to include j and have multiple members, no member of B_1, B_2, \dots, B_{k+1} will contain j and have multiple members.

Case 2: If there is no i such that $j \in A_i$ and $|A_i| > 1$, then, for some $p < q$, $j + 1 \in A_p$ and $j \in A_q$. Let Q be maximal such that $j \in A_Q$. By definition, $A_Q = \{j\}$. Let P be maximal such that $j + 1 \in A_P$ and $P < Q$. Therefore $j + 1 \notin A_L$ for $P + 1 \leq L \leq Q - 1$ and $j \notin A_L$ for $Q + 1 \leq L \leq k$. By the minimality of j , $j + 1$ must be the smallest member of A_P . Define sets B_L as follows for $1 \leq L \leq k - 1$:

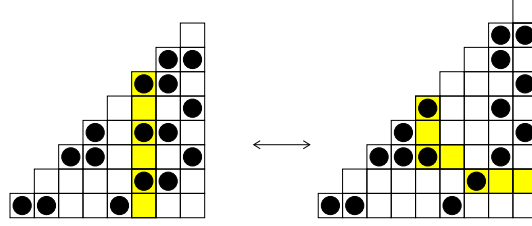
$$B_L = \begin{cases} A_L & 1 \leq L \leq P - 1 \\ \{s : s \in A_{L+1}, s < j + 1\} \cup \{s - 1 : s \in A_{L+1}, s > j + 1\} & P \leq L \leq Q - 2 \\ \{s + M - N - 1 : s \in A_P\} \cup \{j\} & L = Q - 1 \\ \{s : s \in A_{L+1}, s < j\} \cup \{s - 1 : s \in A_{L+1}, s > j\} & Q \leq L \leq k - 1 \end{cases}$$

It should be clear that $(B_1, B_2, \dots, B_{k-1}) \in Y_n$. Define $\psi_n(A_1, A_2, \dots, A_k) = (B_1, B_2, \dots, B_{k-1})$.

Note that, by construction, B_{Q-1} will be the first set of B_1, B_2, \dots, B_{k-1} to contain j and have multiple members, with $j + Q - P$ the second-smallest member.

Note as well that, in either case, ψ_n preserves the minimality of j , and that therefore ψ_n is an involution. This proves (4).

We illustrate the involution in Figure 3. The Fishburn diagram on the left corresponds to $A_1 A_2 A_3 A_4 A_5 A_6 A_7 A_8 \in Y_{16}$, where $A_1 = A_2 = A_5 = \{0\}$, $A_3 = \{2\}$, $A_4 = \{2, 3\}$, $A_6 = \{1, 3, 5\}$, $A_7 = \{1, 3, 5, 6\}$, and $A_8 = \{2, 4, 6\}$. It will have a signed weight of $(-1)^8 t^{16}$. The column corresponding to $\{1, 3, 5\}$ has been highlighted. The Fishburn diagram on the right corresponds to $B_1 B_2 B_3 B_4 B_5 B_6 B_7 B_8 B_9 \in Y_{16}$, where $B_1 = B_2 = B_6 = \{0\}$, $B_3 = \{2\}$, $B_4 = \{2, 3\}$, $B_5 = \{2, 4\}$, $B_7 = \{1\}$, $B_8 = \{2, 4, 6, 7\}$, and $B_9 = \{3, 5, 7\}$. It will have a signed weight of $(-1)^7 t^{16}$. Squares have been highlighted to indicate how $2, 4 \in B_5$, $2 \notin B_6$, $1 \in B_7$ and $1 \notin B_8, B_9$. Informally, the dots corresponding to 3 and 5 in A_6 have been “moved” into dots corresponding to 2 and 4 in what is now the column B_5 , leaving the dot corresponding to the 1 in A_6 now in a column B_7 one square taller, with squares added in the third row from the bottom in B_6 and the second row from the bottom in B_8 and B_9 .

FIGURE 4. $\psi_{11} : Y_{11} \rightarrow Y_{11}$

4. REFINEMENTS OF THE GENERATING FUNCTION OF THE FISHBURN NUMBERS

Let $T_{n,d}$ be the subset of T_n consisting of inversion tables $a_1 a_2 \dots a_n \in T_n$ such that $a_i = i - 1$ for exactly d distinct values of i . We will prove the following two generating function identities,

$$(7) \quad \sum_{n=0}^{\infty} \sum_{d=1}^n |T_{n,d}| t^n z^d = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1-t)^{i-1} (1-zt)),$$

and

$$(8) \quad \sum_{n=0}^{\infty} \sum_{d=1}^n |T_{n,d}| t^n z^d = 1 + \sum_{m=1}^{\infty} \frac{zt}{(1-zt)^m} \prod_{i=1}^{m-1} (1 - (1-t)^i),$$

using variations on the proof of (4). This will prove the equality of (2) and (3).

4.1. The simpler refinement. We begin with a proof of (7). Let $Y_{n,d}$ be the subset of Y_n consisting of Fishburn diagrams $A_1 A_2 \dots A_k \in Y_n$ such that $i - 1 \in A_i$ for exactly d distinct values of i .

Claim 4.1.

$$(9) \quad \sum_{n=0}^{\infty} \sum_{d=1}^n \sum_{A_1 A_2 \dots A_k \in Y_{n,d}} z^d t^n (-1)^{n-k} = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1-t)^{i-1} (1-zt)).$$

Proof. We follow the proof of (5) and first consider the weighted sum of Fishburn diagrams of length m , i.e., sequences $A_1 A_2 \dots A_m \in Y$, beginning by giving each column a weight of -1 . For a specific column A_i with $1 \leq i \leq m$, we can either place a dot, or not, each square. Let the weight of a dot below the i -th row from the bottom be $-t$, the weight of a dot in the i -th row from the bottom be $-zt$, and the weight of a square with no dot be 1. Then for each square, our weighted choice is $1 - t$, except for the top square in the column, where the weighted choice is $1 - zt$, and there are $i - 1$ other squares in this column. Since we must pick at least one dot for the column, the weighted sum over all possible choices is $1 - (1-t)^{i-1} (1-zt)$, and it should be clear this proves (9). \square

Claim 4.2. ψ_n restricts to an involution from $Y_{n,d} \rightarrow Y_{n,d}$, i.e., if $A_1 A_2 \dots A_k \in Y_{n,d}$, then $\psi_n(A_1 A_2 \dots A_k) \in Y_{n,d}$.

Proof. If $A_1 A_2 \dots A_k \in \text{Fix}(\psi_n)$, then this is clearly true. Otherwise, let j be minimal such that either some $j \in A_i$ and $|A_i| > 1$ or $j + 1 \in A_p, j \in A_q$ for $p < q$. Assume the former

holds, i.e., $j \in A_i$ and $|A_i| > 1$. Let I be the minimal such i , $j + R \in A_I$ be the second-smallest element of A_I , and let $\psi(A_1 A_2 \dots A_k) = B_1 B_2 \dots B_{k+1}$. From the definition of ψ_n , we see that $L - 1 \in B_L$ if and only if

$$\begin{aligned} L - 1 \in A_L & & 1 \leq L \leq I - R \\ I - 1 \in A_I & & L = I - R + 1 \\ L - 2 \in A_{L-1} & & I - R + 2 \leq L \leq k + 1, L \neq I + 1 \end{aligned}$$

Therefore $B_1 B_2 \dots B_{k+1} \in Y_{n,d}$. Since ψ_n is an involution, this suffices. \square

For example, both of the Fishburn diagrams in Figure 3 are in $Y_{11,5}$. Let $\psi_n|_{Y_{n,d}} = \psi_{n,d}$. Then $\psi_{n,d} : Y_{n,d} \rightarrow Y_{n,d}$ is an involution, and therefore

$$(10) \quad \sum_{n=0}^{\infty} \sum_{d=1}^n |\text{Fix}(\psi_{n,d})| z^d t^n = 1 + \sum_{m=1}^{\infty} \prod_{i=1}^m (1 - (1-t)^{i-1} (1-zt)).$$

It should be clear that (10) is equivalent to (7), since $\text{Fix}(\psi_{n,d}) = \text{Fix}(\psi_n) \cap Y_{n,d}$ is in trivial bijection with $T_{n,d}$.

4.2. The more complicated refinement. We will now prove (8). First, we will define a set $\widetilde{Y}_{n,d}$ which has the right-hand-side of (8) as a natural generating function. Second, we will show that the right-hand-side of (8) is equal to the bivariate generating function of the fixed point set of an involution $\widetilde{\psi}_{n,d} : \widetilde{Y}_{n,d} \rightarrow \widetilde{Y}_{n,d}$. Finally, we will define a natural bijection from $\text{Fix}(\widetilde{\psi}_{n,d})$ to $\text{Fix}(\psi_{n,d})$.

Let $\widetilde{Y}_{n,d}$ be the set of ordered pairs $(\lambda, A_1 A_2 \dots A_k)$ where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{k+1})$ is a composition of d into $k + 1$ non-negative parts, with $\lambda_1 > 0$, and $A_1 A_2 \dots A_k \in Y_{n-d}$.

Claim 4.3.

$$(11) \quad \sum_{n=0}^{\infty} \sum_{d=1}^{\infty} \sum_{(\lambda, A_1 A_2 \dots A_k) \in \widetilde{Y}_{n,d}} z^d t^n (-1)^{n-k} = 1 + \sum_{m=1}^{\infty} \frac{zt}{(1-zt)^m} \prod_{i=1}^{m-1} (1 - (1-t)^i).$$

Proof. Let us consider all $(\lambda, A_1 A_2 \dots A_{m-1})$, where m is fixed. We weight $(\lambda, A_1 A_2 \dots A_{m-1})$ by the product of the weight of $A_1 A_2 \dots A_{m-1}$, as in the proof of (5) (but not the weight from the proof of (9)) and the weight of λ , which we take to be $(zt)^{|\lambda|}$, where $|\lambda| = \lambda_1 + \lambda_2 + \dots + \lambda_m$. The generating function of compositions with m non-negative parts, where the first part must be positive, is $\frac{zt}{(1-zt)^m}$. Then the weighted sum over all $(\lambda, A_1 A_2 \dots A_{m-1})$ is $\frac{zt}{(1-zt)^m} \prod_{i=1}^{m-1} (1 - (1-t)^i)$, and (11) follows. \square

We will now define an involution $\widetilde{\psi}_{n,d} : \widetilde{Y}_{n,d} \rightarrow \widetilde{Y}_{n,d}$ such that

- $(\lambda, A_1 A_2 \dots A_k) \in \text{Fix}(\psi_n)$ if and only if, for all $1 \leq i \leq k$, $|A_i| = 1$ (so $k = n - d$) and if for some j , $A_p = \{j + 1\}$, $A_q = \{j\}$, and $p < q$, then $\lambda_{j+2} \neq 0$,
- If $(\lambda, A_1 A_2 \dots A_k) \notin \text{Fix}(\widetilde{\psi}_{n,d})$ and $\widetilde{\psi}_{n,d}(\lambda, A_1 A_2 \dots A_k) = (\mu, B_1 B_2 \dots B_r)$, $r = k \pm 1$.

Given $(\lambda, A_1 A_2 \dots A_k) \in \widetilde{Y}_{n,d}$, let j be the smallest integer such that either, for some i , $j \in A_i$ and $|A_i| > 1$ or, for some p, q , $j+1 \in A_p$, $j \in A_q$, and $\lambda_{j+2} = 0$. If there is no such j , define $\widetilde{\psi}_{n,d}(\lambda, A_1 A_2 \dots A_k) = (\lambda, A_1 A_2 \dots A_k)$. Otherwise,

Case 1: If, for some i , $j \in A_i$ and $|A_i| > 1$, define $\widetilde{\psi}_{n,d}(\lambda, A_1 A_2 \dots A_k) = (\mu, \psi_{n-d}(A_1 A_2 \dots A_k))$, where $\mu = (\lambda_1, \lambda_2, \dots, \lambda_{j+1}, 0, \lambda_{j+2}, \lambda_{j+3}, \dots, \lambda_{k+1})$. Note that $\psi_{n-d}(A_1 A_2 \dots A_k) = B_1 B_2 \dots B_{k+1} \in Y_{n-d}$ by the definition of ψ_{n-d} , so $(\mu, \psi_{n-d}(A_1 A_2 \dots A_k)) \in \widetilde{Y}_{n,d}$.

Case 2: If there is no i such that $j \in A_i$ and $|A_i| > 1$, then, for some $p < q$, $j+1 \in A_p$, $j \in A_q$, and $\lambda_{j+2} = 0$. Define $\widetilde{\psi}_{n,d}(\lambda, A_1 A_2 \dots A_k) = (\mu, \psi_{n-d}(A_1 A_2 \dots A_k))$, where $\mu = (\lambda_1, \lambda_2, \dots, \lambda_{j+1}, \lambda_{j+3}, \dots, \lambda_{k+1})$. Note that $\psi_{n-d}(A_1 A_2 \dots A_k) = B_1 B_2 \dots B_{k-1} \in Y_{n-d}$ by the definition of ψ_{n-d} , so $(\mu, \psi_{n-d}(A_1 A_2 \dots A_k)) \in \widetilde{Y}_{n,d}$.

Therefore,

$$(12) \quad \sum_{n=0}^{\infty} \sum_{d=1}^n |\text{Fix}(\widetilde{\psi}_{n,d})| t^n z^d = 1 + \sum_{m=1}^{\infty} \frac{zt}{(1-zt)^m} \prod_{i=1}^{m-1} (1 - (1-t)^i).$$

To finish the proof of (8), we need only define a bijection from $\text{Fix}(\widetilde{\psi}_{n,d})$ to $\text{Fix}(\psi_{n,d})$.

In fact, we claim that $\widetilde{Y}_{n,d}$ is in natural fixed-point-preserving bijection with a subset of $Y_{n,d}$. Define an injection $f : \widetilde{Y}_{n,d} \rightarrow Y_{n,d}$ as follows: Given $(\lambda, A_1 A_2 \dots A_k) \in \widetilde{Y}_{n,d}$, where $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_{k+1})$, begin with a staircase diagram $(1, 2, \dots, d+k)$. Place an X in the top of the column L if $L = \lambda_1 + 1, \lambda_1 + \lambda_2 + 2, \lambda_1 + \lambda_2 + \lambda_3 + 3$, and so on up to $\lambda_1 + \lambda_2 + \dots + \lambda_k + k$. Place a dot in the top of all other columns; note that you have placed d dots at the top of their columns, with k other squares dividing the top diagonal into the composition $(\lambda_1, \lambda_2, \dots, \lambda_{k+1})$. Place an X in each square in the same column as a dot at the top of its column, and in each square to the right of, and in the row immediately below, a dot at the top of its column. The squares that now have neither a dot nor an X will form a staircase sub-diagram with shape $(1, 2, \dots, k)$. Place the Fishburn diagram $A_1 A_2 \dots A_k$, which has $n-d$ dots, into this staircase sub-diagram. Let $D_1 D_2 \dots D_{d+k}$ be the resulting Fishburn diagram, and define $f(\lambda, A_1 A_2 \dots A_k) = D_1 D_2 \dots D_{d+k}$.

By construction, $D_1 D_2 \dots D_{d+k}$ is a Fishburn diagram with $n-d+d = n$ dots, d of which are at the tops of their columns, so $D_1 D_2 \dots D_{d+k} \in Y_{n,d}$. For example, if $\lambda = (1, 2, 0, 3, 0)$, $A_1 = \{0\}$, $A_2 = \{1\}$, $A_3 = \{1, 2\}$, and $A_4 = \{0, 1, 3\}$, then λ is a composition of 6 into 5 parts and $A_1 A_2 A_3 A_4 \in Y_7$. Therefore $(\lambda, A_1 A_2 A_3 A_4) \in \widetilde{Y}_{13,6}$, and $f(\lambda, A_1 A_2 A_3 A_4) \in Y_{13,6}$ is the Fishburn diagram in Figure 5.

(In Figure 5 we used white for the seven dots from $A_1 A_2 A_3 A_4$ and black for the six dots from λ . Note how the 10 squares with neither an X nor a black dot form a subdiagram with staircase shape $(1, 2, 3, 4)$.)

Claim 4.4. *f is a bijection between $\widetilde{Y}_{n,d}$ and the subset of $Y_{n,d}$ consisting of Fishburn diagrams $D_1 D_2 \dots D_k$ such that, if $i-1 \in D_i$, $|D_i| = 1$, and $i-2 \notin D_{i+1}, D_{i+2}, \dots, D_k$. In particular, $(\lambda, A_1 A_2 \dots A_k) \in \text{Fix}(\widetilde{\psi}_{n,d})$ if and only if $f(\lambda, A_1 A_2 \dots A_k) \in \text{Fix}(\psi_{n,d})$.*

Proof. The first part of the claim should be clear from the construction of f . For the second part of the claim, consider the operation of f on a particular $(\lambda, A_1 A_2 \dots A_k) \in \widetilde{Y}_{n,d}$: d

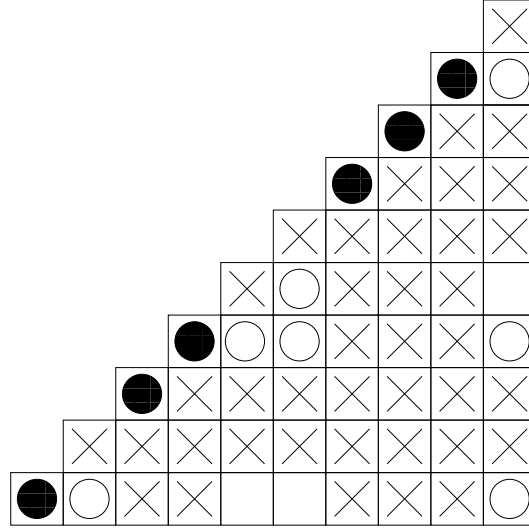


FIGURE 5. A Fishburn diagram in $Y_{13,6}$ corresponding to an element of $\widetilde{Y}_{13,6}$.

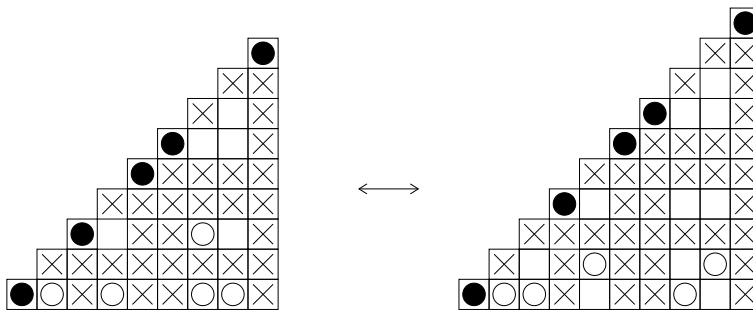
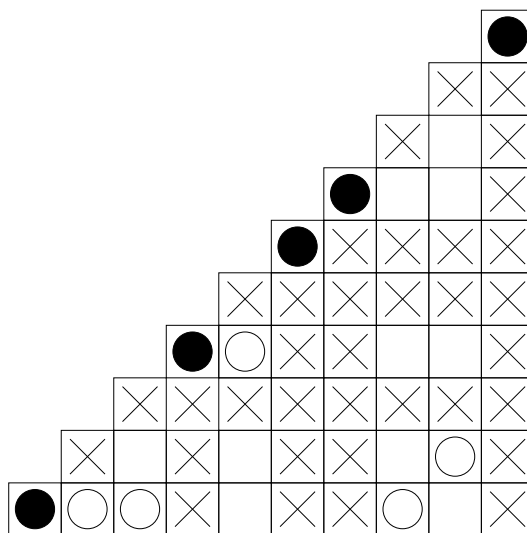
dots are placed at the tops of their columns, and X 's are placed in the columns directed below them and in the rows immediately below them and to their right, with a staircase subdiagram of side length k formed by the remaining free squares. There will be λ_{j+2} rows of X 's in between the $j + 1$ -st and $j + 2$ -nd rows of this subdiagram, since there will be one row of X 's for each dot at the top of its column, and the dots at the top of their columns are placed in blocks of lengths λ_1 (below the first row of the subdiagram), λ_2 (between the first and second rows of the subdiagram) and so on.

Therefore $j' + 1$ precedes j' in $D_1 D_2 \dots D_{k+d} = f(\lambda, A_1 A_2 \dots A_k)$ for some j' if and only if, for some j , $j + 1$ precedes j in $A_1 A_2 \dots A_k$ and $\lambda_{j+2} = 0$. Also, $|D_i| = 1$ for all $1 \leq i \leq k + d$ if and only if $|A_i| = 1$ for all $1 \leq i \leq k$. Therefore, $D_1 D_2 \dots D_{k+d} \in \text{Fix}(\psi_{n,d})$ if and only if $|A_i| = 1$ for all $1 \leq i \leq k$ and, for all $p < q$ with $A_p = \{j + 1\}$, $A_q = \{j\}$, $\lambda_{j+2} \neq 0$, or if and only if $f(\lambda, A_1 A_2 \dots A_k) \in \text{Fix}(\widetilde{\psi}_{n,d})$. \square

We illustrate this in Figure 6 and Figure 7. The Fishburn diagram on the left-hand-side of Figure 6 corresponds to $f(\lambda, A_1 A_2 A_3 A_4)$, where $\lambda = (1, 1, 2, 0, 1)$, $A_1 = A_2 = A_4 = \{0\}$, and $A_3 = \{0, 1\}$. Therefore $(\lambda, A_1 A_2 A_3 A_4) \in \widetilde{Y}_{10,5}$. However, $(\lambda, A_1 A_2 A_3 A_4) \notin \text{Fix}(\widetilde{\psi}_{10,5})$, since $|A_3| > 1$, and $\widetilde{\psi}_{10,5}(\lambda, A_1 A_2 A_3 A_4) = (\mu, B_1 B_2 B_3 B_4 B_5)$, with $\mu = (1, 0, 1, 2, 0, 1)$, $B_1 = B_2 = B_4 = \{0\}$, $B_3 = B_5 = \{1\}$. Note that $1 \in B_3$, $0 \in B_4$, and 0 is the second element of μ .

By contrast, the Fishburn diagram in Figure 7 corresponds to $f(\mu, E_1 E_2 E_3 E_4 E_5)$, where μ is again equal to $(1, 0, 1, 2, 0, 1)$, $E_1 = E_2 = E_4 = \{0\}$, $E_3 = \{2\}$, $E_5 = \{1\}$. Although $2 \in E_3$ and $1 \in E_5$, the third element of μ is not equal to zero, and so $(\mu, E_1 E_2 E_3 E_4 E_5) \in \text{Fix}(\widetilde{\psi}_{10,5})$. As we can see, $f(\mu, E_1 E_2 E_3 E_4 E_5) \in \text{Fix}(\widetilde{\psi}_{10,5})$.

Note that $\widetilde{\psi}_{n,d} \neq f^{-1} \circ \psi_{n,d} \circ f$, i.e., $\widetilde{\psi}_{n,d}$ is not simply an alternative description of $\psi_{n,d}$. In fact, $f^{-1} \circ \psi_{n,d} \circ f$ is not even well-defined, since applying $\psi_{10,5}$ to the Fishburn diagram on the left-hand-side of Figure 6 will give a Fishburn diagram outside of the image of f .

FIGURE 6. $\widetilde{\psi}_{10,5} : \widetilde{Y}_{10,5} \rightarrow \widetilde{Y}_{10,5}$.FIGURE 7. An element of $Fix(\widetilde{\psi}_{10,5})$

5. FURTHER RESEARCH DIRECTIONS

Our initial approach to proving that matchings with no 2-nestings were enumerated by the Fishburn numbers was to try to find a bijection between such matchings and matchings with no left- or right-nestings. Our second approach was to find a bijection between matchings with no 2-nestings and ascent sequences—in particular, a bijection between matchings X on $[2n]$ on with no 2-nestings such that the inversion table $\phi(X)$ used k distinct integers and ascent sequences of length n with $k-1$ non-ascents. (Inductively, this would make sense: An inversion table of length n , with no decreasing subsequence of consecutive integers, that used k distinct integers could have $n+2-k$ integers added to the end to obtain an inversion table of length $n+1$ with no decreasing subsequence of consecutive integers. An ascent sequence of length n with $k-1$ non-ascents would have $n+1-k$ ascents, so it could have $n+2-k$ integers added to the end to obtain an ascent sequence). Presumably, bijections between the inversion tables discussed in this paper and other interpretations of the Fishburn numbers could be found by comparing the involutions used to prove the various interpretations have

the generating function (1) of the Fishburn numbers, as discussed in Stanley [8]. It would be interesting to see if any more direct or elegant bijections are possible.

A second interesting question is whether or not it is possible to define bounce, area, or dinv statistics on the various sets enumerated by the Fishburn numbers in such a way as to generalize the bounce, area, and dinv statistics defined on the Catalan sets (see Haglund [4]). In particular, it would be extremely interesting if any such statistics could be defined so bounce and area or area and dinv gave a symmetric bivariate generating function in general, as the combinatorics of the symmetry of the q, t -Catalan polynomial is famously poorly-understood. Inspired by our proof of the equality of (2) and (3), we were able to define statistics similar to area and dinv on the inversion tables discussed in this paper as well as slightly different statistics on ascent sequences. In both cases we obtained symmetric generating functions for all $n \leq 5$, but not for $n = 6$. (We thank Jason Bandlow for his assistance with checking the $n = 6$ cases by computer). The polynomial obtained from ascent sequences was closer to being symmetric than the polynomial obtained from inversion tables with no decreasing subsequence of consecutive integers; in both cases, $f(q, t) - f(t, q)$ was only a few terms long. It is possible that working with another interpretation of the Fishburn numbers would inspire statistics that were, in fact, symmetric.

6. APPENDIX

In this Appendix, we will prove the following conjecture of Claesson and Linusson's:

Claim 6.1. *The distribution of left-nestings over the set of all matchings on $[2n]$ is given by the "Second-order Eulerian triangle", entry A008517 in OEIS [7].*

Proof. According to the OEIS, the Second-order Eulerian triangle $T(n, k)$ is defined by the following recurrence relation: $T(n, k) = 0$ if $n < k$, $T(1, 1) = 1$, $T(n, -1) = 0$, $T(n, k) = kT(n-1, k) + (2n-k)T(n-1, k-1)$.

Assume X is a matching on $[2n-2]$ with precisely j left-nestings. If we add an arc corresponding to a pair $(a, 2n)$ to achieve a matching X' on $[2n]$, there are three possibilities:

- Case 1:** If the left-endpoint of this new arc is placed immediately before a right-endpoint of X , or immediately before its own right-endpoint, this will not change the number of left-nestings, so X' will have j left-nestings. There are n right-endpoints (including $2n$ itself), so there are n ways to do this.
- Case 2:** If the left-endpoint of this new arc is placed immediately before a left-endpoint of an arc that is *not* the inner arc of a left-nesting of X , this will contribute one new left-nesting to X' , without eliminating an existing left-nesting of X . Then X' will have $j+1$ left-nestings, and there are $n-1-j$ ways to do this.
- Case 3:** If the left-endpoint of this new arc is placed immediately before a left-endpoint of an arc that *is* the inner arc of a left-nesting of X , this will add one new left-nesting to X' , but also eliminate the original left-nesting from X . The net effect is therefore to contribute no new left-nestings to X' , which will therefore have j left-nestings. There are j ways to do this.

Therefore, of the $2n-1$ matchings on $[2n]$ formed by adding a rightmost arc to X , $n+j$ will have j left-nestings, and $n-1-j$ will have $j+1$ left-nestings. To put it another way,

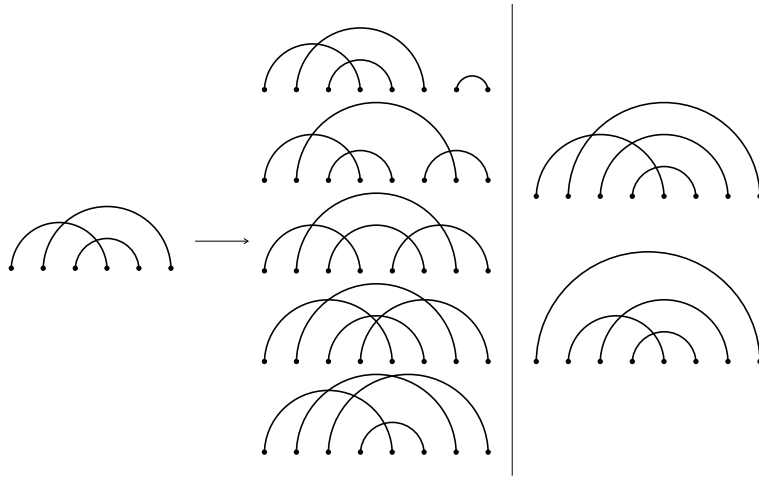


FIGURE 8. Matchings on [6] and [8]

if $L(n, j)$ is defined to be the number of matchings on $[2n]$ with j left-nestings, we now have that:

$$(13) \quad L(n, j) = (n + j)L(n - 1, j) + (n - j)L(n - 1, j - 1),$$

or, replacing j with $n - k$,

$$(14) \quad L(n, n - k) = (2n - k)L(n - 1, n - 1 - (k - 1)) + kL(n - 1, n - 1 - k).$$

Since $L(1, 1 - 1) = 1$, $L(n, n - (-1)) = 0$, and $L(n, n - k) = 0$ if $n < k$, we see inductively that $L(n, n - k) = T(n, k)$. \square

We illustrate the three cases in the above inductive proof in Figure 8, beginning with the matching $(1, 4)(2, 6)(3, 5)$ on [6], which has 1 left-nesting. Adding the arcs corresponding to the pairs $(7, 8)$, $(6, 8)$, $(5, 8)$, $(4, 8)$ and re-labeling falls into Case 1 and results in matchings on [8] with 1 left-nesting. Adding the arc corresponding to the pair $(3, 8)$ and re-labelling falls into case 3 and also results in a matching on [8] with 1 left-nesting. Adding the arcs corresponding to the pairs $(2, 8)$ and $(1, 8)$ and re-labelling falls into Case 3 and results in a matching on [8] with 2 left-nestings. Note that the above proof can be seen as a generalization of the inductive proof of Claim (2.1).

7. ACKNOWLEDGEMENTS

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