

THE SORTING INDEX

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ABSTRACT. We consider a bivariate polynomial that generalizes both the length and reflection length generating functions in a finite Coxeter group. In seeking a combinatorial description of the coefficients, we are led to the study of a new Mahonian statistic, which we call the sorting index. The sorting index of a permutation and its type B and type D analogues have natural combinatorial descriptions which we describe in detail.

1. INTRODUCTION

The following is a well-known and elegant formula for the distribution of inversions over permutations of n :

$$(1) \quad \sum_{w \in S_n} q^{\text{inv}(w)} = (1+q)(1+q+q^2) \cdots (1+q+\cdots+q^{n-1}).$$

(Knuth [11, 5.1.1] attributes the formula to Rodrigues.) MacMahon [12] showed that another statistic, now called the *major index*, has the same distribution as inversion number. In his honor, any permutation statistic with this distribution is referred to as a *Mahonian statistic*. Similarly, there is a simple formula for the distribution of the number of cycles in a permutation (see, e.g., [14, Prop. 1.3.4]):

$$(2) \quad \sum_{w \in S_n} t^{\text{cyc}(w)} = t(t+1)(t+2) \cdots (t+n-1).$$

The coefficients of this polynomial are well-known as the (unsigned) *Stirling numbers of the first kind*.

Both Equations (1) and (2) can be given simple combinatorial proofs, and it is interesting that they both factor with the same number of terms. This suggests a natural bivariate polynomial:

$$S_n(q, t) := t(t+q)(t+q+q^2) \cdots (t+q+\cdots+q^{n-1}),$$

which clearly generalizes both polynomials above. This polynomial does not give the joint distribution of inversion number and cycle number. However, we find statistics $\text{sor}(w)$ and $\text{rl-min}(w)$ such that $S_n(q, t)$ is the generating function for both the pair of statistics $(\text{inv}, \text{rl-min})$ and the pair (sor, cyc) . In particular, $\text{sor}(w)$ is Mahonian, and $\text{rl-min}(w)$ has a ‘‘Stirling’’ distribution. While a variation of $\text{rl-min}(w)$ appears early in [14], $\text{sor}(w)$ seems not to have appeared in the literature. Given the variety of distinct Mahonian statistics (see [1], [3], [4], [7], [8], [13, A008302] and references therein) we find the novelty of $\text{sor}(w)$ noteworthy. We call $\text{sor}(w)$ the *sorting index* of w .

In Section 2 we make precise the story told above, and we follow that discussion with very similar results for signed permutations in Section 3. In the context of reflection groups, $\text{inv}(w)$

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corresponds to *length* and $n - \text{cyc}(w)$ corresponds to *reflection length*. Sections 4 and 5 discuss how the ideas of Sections 2 and 3 might be generalized to other Coxeter groups. Indeed, the discovery of the sorting index followed from the investigation of a common refinement of the length and reflection length generating functions in an arbitrary finite reflection group. The polynomial $S_n(q, t)$ given above is the type A version. See Section 5. Another avenue for generalization could be from permutations to *words* (with repeated letters) as in [12, Section III], though we do not pursue this line of inquiry here. In Section 6 we remark upon recent work of others related to the type A version of the sorting index.

Throughout the paper, we will let $[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}$. We refer the reader to [2, Sections 8.1 and 8.2] for facts about the combinatorics of Coxeter groups (in particular combinatorial descriptions for inversions of types B_n and D_n) that we use here without proof.

2. PERMUTATIONS (TYPE A)

One generating set for S_n is the set of transpositions:

$$T = \{(ij) : 1 \leq i < j \leq n\};$$

another is the set of adjacent transpositions:

$$S = \{(ii+1) : 1 \leq i \leq n-1\} \subset T.$$

We will usually write elements $w \in S_n$ as words on the set $\{1, 2, \dots, n\}$ or as products of transpositions in S or T . We will abbreviate transpositions by $t_{ij} = (ij)$ and adjacent transpositions by $s_i = (ii+1)$.

In what follows, we will give two different factorizations of the *diagonal sum*, i.e., $\sum_{w \in S_n} w$, in the group algebra $\mathbb{Z}[S_n]$. We then examine how two pairs of statistics, $(\text{inv}, \text{rl-min})$ and (sor, cyc) , behave under the factorizations. Because both factorizations lead to the expression

$$S_n(q, t) := \prod_{i=1}^n (t + [i]_q - 1),$$

we will conclude that the joint distribution of the pairs of the statistics are the same. We refer inv and sor as *Mahonian statistics*, rl-min and cyc as *Stirling statistics*.

2.1. Mahonian statistics. Given a permutation $w \in S_n$, the minimal number of adjacent transpositions needed to express w is called its *length*, $\ell(w)$. An *inversion* of w is a pair $i < j$ such that $w_i > w_j$. Let $\text{inv}(w)$ denote the number of inversions of w . It is well-known that the number of inversions in a permutation is equal to its length: $\text{inv}(w) = \ell(w)$. For example, $\ell(312) = 2$ since $312 = s_2s_1$, but also 312 has two inversions since $3 > 1$ and $3 > 2$. Any statistic with the same distribution as inversion number is called *Mahonian*.

We now define the *sorting index* of a permutation. Given a permutation $w \in S_n$, there is a unique expression

$$w = t_{i_1j_1} \cdots t_{i_kj_k}$$

as a product of transpositions with $j_1 < \cdots < j_k$. The sorting index, $\text{sor}(w)$, is given by

$$\text{sor}(w) = \sum_{s=1}^k j_s - i_s.$$

The transpositions in this factorization are precisely the transpositions used in the “straight selection sort” algorithm, which we now describe. Using a transposition we first move the

	$w = 6372451$	$w = 3715246$
$\text{inv}(w) = \ell(w)$	14	9
$\text{sor}(w)$	18	14
$\text{cyc}(w) = n - \ell'(w)$	1	2
$\text{rl-min}(w)$	1	4

TABLE 1. Examples of inversion number versus sorting index, and $\text{cyc}(w)$ versus $\text{rl-min}(w)$.

largest number that is out of order to the end, then we move the next largest number to its proper place, and so on. (See Knuth [11, 5.2.3].) For example, if $w = 2431756$, we have

$$2431756 \xrightarrow{(57)} 2431657 \xrightarrow{(56)} 2431567 \xrightarrow{(24)} \mathbf{2134567} \xrightarrow{(12)} 1234567,$$

and so

$$w = t_{12}t_{24}t_{56}t_{57}, \quad \text{sor}(w) = (2 - 1) + (4 - 2) + (6 - 5) + (7 - 5) = 6.$$

The sorting index, then, is the sum of the distances that all the letters had to move during the sorting. We will show that this statistic has the same distribution as inversion number.

2.2. Stirling statistics. For $w \in S_n$, the *reflection length* of w , $\ell'(w)$, is the minimal number of reflections (in T) needed to express w . Let $\text{cyc}(w)$ denote the number of cycles of w . Reflection length and cycle number are related by $\text{cyc}(w) = n - \ell'(w)$.

For $w \in S_n$ define the following statistic:

$$\text{rl-min}(w) = n - |\{i : w_i > w_j \text{ for some } j > i\}| = |\{i : w_i < w_j \text{ for all } i < j\}|,$$

that is, $\text{rl-min}(w)$ is the number of letters of w that are right-to-left minima. For example, $w = 2413756$ has $\text{rl-min}(w) = 4$. We will see that rl-min has the same distribution as cyc .

2.3. Factorizations of the diagonal sum. One of the simplest ways to generate all permutations of n is defined recursively. For each permutation of $n - 1$, simply insert the letter n in all possible positions. We now describe a way to encode this idea algebraically. Define elements Ψ_i of the group algebra of S_n as: $\Psi_1 = 1 + s_1$, $\Psi_2 = 1 + s_2 + s_2s_1 = 1 + s_2 \cdot \Psi_1$, and in general for $i \geq 1$, $\Psi_i = 1 + s_i \cdot \Psi_{i-1}$. It should be clear that each term in Ψ_i is a reduced expression in S_{i+1} . The following proposition shows how the product of these terms, $1 \leq i < n$, produces reduced expressions for all elements of S_n .

Proposition 1. *For $n \geq 2$ we have*

$$\Psi_1 \Psi_2 \cdots \Psi_{n-1} = \sum_{w \in S_n} w.$$

Proof. Clearly $\Psi_1 = \sum_{w \in S_2} w$. For induction, suppose $\Psi_1 \cdots \Psi_{n-2} = \sum_{w \in S_{n-1}} w$. We can identify the elements of S_{n-1} with the set

$$\{w \in S_n : w(n) = n\}.$$

(In fact, written as products of the simple transpositions above, they are identical.) Given such an element $w = w_1 \cdots w_{n-1}n$, the terms in Ψ_{n-1} dictate the location of n :

$$\begin{aligned} w \cdot \Psi_{n-1} &= w + ws_{n-1} + ws_{n-1}s_{n-2} + \cdots + ws_{n-1}s_{n-2} \cdots s_1 \\ &= w_1 \cdots w_{n-1}n + w_1 \cdots w_{n-2}nw_{n-1} + \cdots + nw_1 \cdots w_{n-1}. \end{aligned}$$

It is clear moreover that $w \cdot \Psi_{n-1} = v \cdot \Psi_{n-1}$ if and only if $w = v$. We have

$$\Psi_1 \cdots \Psi_{n-2} \Psi_{n-1} = \sum_{w \in S_n, w(n)=n} w \cdot \Psi_{n-1} = \sum_{u \in S_n} u,$$

as desired. \square

We can now state the following corollary to Proposition 1. For $t = 1$, this is Equation (1).

Corollary 1. *For $n \geq 1$ we have*

$$S_n(q, t) = \prod_{i=1}^n (t + [i]_q - 1) = \sum_{w \in S_n} q^{\text{inv}(w)} t^{\text{rl-min}(w)}.$$

This expression for the generating function of (inv, rl-min) can be seen as a special case of Theorem 1.2 of [5] (see Remark 2, page 17).

Proof. We find it convenient to prove an equivalent factorization, namely,

$$t^n S_n(q, 1/t) = \prod_{i=1}^n (1 + t[i]_q - t) = \sum_{w \in S_n} q^{\text{inv}(w)} t^{n-\text{rl-min}(w)}.$$

Define the linear map $\psi : \mathbb{Z}[S_n] \rightarrow \mathbb{Z}[q, t]$ given by $\psi(w) = q^{\text{inv}(w)} t^{n-\text{rl-min}(w)}$. By construction, $\psi(\Psi_i) = (1 + qt + q^2t + \cdots + q^i t) = (1 + t[i+1]_q - t)$. Suppose for induction that

$$\psi(\Psi_1 \cdots \Psi_{n-2}) = \psi(\Psi_1) \cdots \psi(\Psi_{n-2}) = \sum_{w \in S_{n-1}} q^{\text{inv}(w)} t^{\text{rl-min}(w)}.$$

It suffices to show that the following holds:

$$\psi(\Psi_1 \cdots \Psi_{n-2}) \psi(\Psi_{n-1}) = \psi(\Psi_1 \cdots \Psi_{n-2} \Psi_{n-1}).$$

Following the proof of Proposition 1, we see that Ψ_{n-1} has the effect of positioning n in a permutation. If n is placed in the rightmost position it participates in no inversions and is a right-to-left minimum. If n is inserted so that there are $i > 0$ letters to its right, then because it is the greatest letter it can never be a right-to-left minimum, and it participates in i inversions. Thus it contributes weight $q^i t$. For a permutation $w = w_1 \cdots w_{n-1}n$, we have:

$$\begin{aligned} \psi(w \cdot \Psi_{n-1}) &= \psi(w_1 \cdots w_{n-1}n) + \psi(w_1 \cdots w_{n-2}nw_{n-1}) + \cdots + \psi(nw_1 \cdots w_{n-1}) \\ &= \psi(w) + qt\psi(w) + \cdots + q^{n-1}t\psi(w) \\ &= \psi(w)(1 + t[n]_q - t) = \psi(w)\psi(\Psi_{n-1}). \end{aligned}$$

Thus,

$$\begin{aligned}
\psi(\Psi_1 \cdots \Psi_{n-2} \Psi_{n-1}) &= \psi \left(\sum_{w \in S_n, w(n)=n} w \cdot \Psi_{n-1} \right) \\
&= \sum_{w \in S_n, w(n)=n} \psi(w \cdot \Psi_{n-1}) \\
&= \psi(\Psi_{n-1}) \sum_{w \in S_n, w(n)=n} \psi(w) \\
&= \psi(\Psi_{n-1}) \psi(\Psi_1 \cdots \Psi_{n-2}),
\end{aligned}$$

as desired. \square

We now present a different factorization of the diagonal sum. Intuitively, the procedure it describes is: given a permutation of $n - 1$, put letter n in any position $i = 1, \dots, n - 1$, and move the letter that was in position i to the end. Let $\Phi_1 = 1 + t_{12}$, $\Phi_2 = 1 + t_{23} + t_{13}$, and for $j \geq 2$ let $\Phi_j = 1 + \sum_{i \leq j} t_{i, j+1}$. The following proposition shows that the product of these terms produces a minimal length product of reflections for each element of S_n .

Proposition 2. *For $n \geq 2$, we have*

$$\Phi_1 \Phi_2 \cdots \Phi_{n-1} = \sum_{w \in S_n} w.$$

Proof. Obviously the formula holds for $n = 2$. For induction, suppose $\Phi_1 \cdots \Phi_{n-2} = \sum_{w \in S_{n-1}} w$. We observe that Φ_{n-1} is 1 plus the sum of all reflections involving the letter n . Thus, for $w = w_1 \cdots w_{n-1} n$ we have

$$\begin{aligned}
w \cdot \Phi_{n-1} &= w + wt_{n-1n} + wt_{n-2n} + \cdots + wt_{1n} \\
&= w_1 \cdots w_{n-1} n + w_1 \cdots n w_{n-1} + w_1 \cdots n w_{n-1} w_{n-2} + \cdots + n w_2 \cdots w_{n-1} w_1.
\end{aligned}$$

From these descriptions it is clear that $w \cdot \Phi_{n-1} = v \cdot \Phi_{n-1}$ if and only if $w = v$, and so we have the desired result:

$$\Phi_1 \cdots \Phi_{n-2} \Phi_{n-1} = \sum_{w \in S_n, w(n)=n} w \cdot \Phi_{n-1} = \sum_{w \in S_n} w.$$

\square

By looking at the combinatorics of the factorization in Proposition 2, we have the following.

Corollary 2. *For $n \geq 1$, we have*

$$S_n(q, t) = \prod_{i=1}^n (t + [i]_q - 1) = \sum_{w \in S_n} q^{\text{sor}(w)} t^{\text{cyc}(w)}.$$

Proof. We find it convenient to prove an equivalent factorization, namely,

$$t^n S_n(q, 1/t) = \prod_{i=1}^n (1 + t[i]_q - t) = \sum_{w \in S_n} q^{\text{sor}(w)} t^{n - \text{cyc}(w)} = \sum_{w \in S_n} q^{\text{sor}(w)} t^{\ell'(w)}.$$

Define the linear map $\phi : \mathbb{Z}[S_n] \rightarrow \mathbb{Z}[q, t]$ given by $\phi(w) = q^{\text{sor}(w)}t^{\ell(w)}$. It is clear that $\phi(\Phi_i) = (1 + qt + q^2t + \cdots + q^it) = (1 + t[i + 1]_q - t)$. It suffices to show

$$\phi(\Phi_1 \cdots \Phi_{n-2})\phi(\Phi_{n-1}) = \phi(\Phi_1 \cdots \Phi_{n-2}\Phi_{n-1}).$$

Following the proof of Proposition 2 we let $w = w_1 \cdots w_{n-1}n$. Notice that since $w_n = n$, then for $1 \leq i < n$, $\ell'(wt_{in}) = \ell'(w) + 1$. Hence we have:

$$\begin{aligned} \phi(w \cdot \Phi_{n-1}) &= \phi(w) + \phi(wt_{n-1n}) + \cdots + \phi(wt_{1n}) \\ &= \phi(w) + qt\phi(w) + \cdots + q^{n-1}t\phi(w) \\ &= \phi(w)(1 + t[n]_q - t) = \phi(w)\phi(\Phi_{n-1}). \end{aligned}$$

Thus,

$$\begin{aligned} \phi(\Phi_1 \cdots \Phi_{n-2}\Phi_{n-1}) &= \phi\left(\sum_{w \in S_n, w(n)=n} w \cdot \Phi_{n-1}\right) \\ &= \sum_{w \in S_n, w(n)=n} \phi(w \cdot \Phi_{n-1}) \\ &= \phi(\Phi_{n-1}) \sum_{w \in S_n, w(n)=n} \phi(w) \\ &= \phi(\Phi_{n-1})\phi(\Phi_1 \cdots \Phi_{n-2}), \end{aligned}$$

as desired. \square

Taken together, these corollaries give our first main result.

Corollary 3. *The pairs of statistics (inv, rl-min) and (sor, cyc) are equidistributed over S_n :*

$$\sum_{w \in S_n} q^{\text{inv}(w)}t^{\text{rl-min}(w)} = \sum_{w \in S_n} q^{\text{sor}(w)}t^{\text{cyc}(w)}.$$

In particular, sor is a Mahonian statistic, and rl-min is a Stirling statistic.

We remark that there is a bijection that maps a permutation w with k cycles to a permutation \hat{w} with k right-to-left minima, and thus $\text{cyc}(w) = \text{rl-min}(\hat{w})$. (This is a variation of an idea given by Stanley [14, Proposition 1.3.1].) We will illustrate the bijection with an example. First, we write w in cycle notation, so that each cycle has its least element written last, and the cycles are in increasing order by least element. With $w = 685942317$, we write it as $w = (6281)(54973)$. We simply remove parentheses to achieve the one-line notation for \hat{w} :

$$\hat{w} = 628154973.$$

To reverse the process, simply insert “)” to the right of each right-to-left minimum, and put “(” at the far left and following any internal right parentheses. One can check with this example that $\text{cyc}(w) = 2 = \text{rl-min}(\hat{w})$.

While the map $w \mapsto \hat{w}$ shows that cyc and rl-min are equidistributed, it does not carry $\text{sor}(w)$ to $\text{inv}(w)$ (in the example above $\text{sor}(w) = 21$ and $\text{inv}(\hat{w}) = 17$), and so it does not give a bijective proof of Corollary 3. It would be interesting to find a bijective proof of Corollary 3, or to find a simple bijection that carries inv to sor. While one can define such a map inductively via our factorizations, a more “natural” description would be better.

3. TYPE B

The hyperoctahedral group B_n is the set of permutations of $\{1, \dots, n, \bar{1}, \dots, \bar{n}\}$ (where $\bar{i} = -i$) that are centrally symmetric, i.e., elements

$$w = w_{\bar{n}} \cdots w_{\bar{1}} w_1 \cdots w_n$$

for which $w(-i) = -w(i)$. Thus w is determined by the word $w = w_1 \cdots w_n$. That is, elements of B_n are *signed permutations*. These may be generated by the following transpositions:

$$T^B = \{(ij) : 1 \leq i < j \leq n\} \cup \{(\bar{i}\bar{j}) : 1 \leq i \leq j \leq n\},$$

where, in order to maintain symmetry, the transposition (ij) means to swap both i with j and \bar{i} with \bar{j} (provided $i \neq \bar{j}$). The simple transpositions we denote

$$S^B = \{(\bar{1}1)\} \cup \{(i\ i+1) : 1 \leq i \leq n-1\},$$

and these form a minimal generating set for B_n . As before, we abbreviate the transpositions by $t_{ij} = (ij)$ and $s_i = (i\ i+1)$, with $s_0 = (\bar{1}1)$.

3.1. Mahonian statistics. For a signed permutation $w \in B_n$, the minimal number of terms in S^B needed to express w is called its *length*, $\ell_B(w)$. As with permutations, the length of an element in B_n (as a reflection group) is equal to its inversion number: $\ell_B(w) = \text{inv}_B(w)$, which we now define. Let $N(w)$ denote the number of bars in $w_1 \cdots w_n$, and define the type B_n inversion number as follows:

$$\text{inv}_B(w) = |\{1 \leq i < j \leq n : w(i) > w(j)\}| + |\{1 \leq i < j \leq n : -w(i) > w(j)\}| + N(w).$$

For example, $\text{inv}_B(2\bar{4}5\bar{1}\bar{3}) = (3 + 2 + 1) + (2 + 2 + 1) + 3 = 14$. By analogy with the type A case, we call any statistic with the same distribution as length a Mahonian statistic.

We will define a “type B” straight selection sort algorithm below. The algorithm achieves a unique factorization as a product of signed transpositions:

$$w = t_{i_1 j_1} \cdots t_{i_k j_k},$$

with $0 < j_1 < \cdots < j_k$. The *type B_n sorting index*, $\text{sor}_B(w)$, is defined to be

$$\text{sor}_B(w) = \sum_{s=1}^k j_s - i_s - \chi(i_s < 0).$$

For example, if $w = 2\bar{4}5\bar{1}\bar{3}$, we have

$$31\bar{5}4\bar{2}2\bar{4}5\bar{1}\bar{3} \xrightarrow{(35)} \bar{5}134\bar{2}2\bar{4}\bar{3}\bar{1}5 \xrightarrow{(\bar{2}4)} \bar{5}\bar{4}\bar{3}\bar{1}\bar{2}21\bar{3}45 \xrightarrow{(\bar{3}3)} \bar{5}\bar{4}\bar{3}\bar{1}\bar{2}2\mathbf{1}345 \xrightarrow{(12)} \bar{5}\bar{4}\bar{3}\bar{2}\bar{1}12345$$

and so:

$$w = t_{12}t_{33}t_{24}t_{35} \quad \text{sor}_B(w) = (2 - 1) + (3 - (-3) - 1) + (4 - (-2) - 1) + (5 - 3) = 13.$$

As before, the sorting index in some sense keeps track of the distances traveled by the elements being sorted. We will show that the type B sorting index is Mahonian over B_n .

3.2. Stirling statistics. The minimal number of transpositions in T^B needed to express w is the *reflection length*, $\ell'_B(w)$. We know of no simple way to compute the reflection length of an element of B_n (apart from sorting). It would be nice to find a formula for reflection length involving, say, the cycle structure of $w \in B_n$ (perhaps viewed as an element of S_{2n}). Nevertheless, the coefficients of the polynomial

$$\sum_{w \in B_n} t^{\ell'_B(w)} = (1+t)(1+3t) \cdots (1+(2n-1)t) = \sum_{k=1}^n c(n,k)t^{n-k},$$

are known as the “type B” Stirling numbers of the first kind.

Define another statistic,

$$\text{nmin}(w) = |\{i : w_i > |w_j| \text{ for some } j > i\}| + N(w).$$

For example, if $w = 34\bar{1}87\bar{6}25$, then $\text{nmin}(w) = 4 + 2 = 6$, while one can check $\ell'_B(w) = 7$. We will show nmin is equidistributed with reflection length.

3.3. Factorizations of the diagonal sum. Just as we can generate all of S_n inductively by inserting the letter n into a permutation of $n-1$ in all possible positions, so we can generate B_n from B_{n-1} by inserting n or \bar{n} in all possible ways. Similarly to Section 2.3, define elements Ψ_i of the group algebra of B_n as follows: $\Psi_1 = 1 + s_0$, $\Psi_2 = 1 + s_1 + s_1s_0 + s_1s_0s_1 = 1 + s_1 \cdot \Psi_1 + s_1s_0s_1$, and in general for $i > 1$ define $\Psi_i = 1 + s_{i-1} \cdot \Psi_{i-1} + s_{i-1} \cdots s_1s_0s_1 \cdots s_{i-1}$.

Proposition 3. *We have*

$$\Psi_1 \Psi_2 \cdots \Psi_n = \sum_{w \in B_n} w.$$

Proof. Clearly $\Psi_1 = \sum_{w \in B_1} w$. For induction, suppose $\Psi_1 \cdots \Psi_{n-1} = \sum_{w \in B_{n-1}} w$. We can identify the elements of B_{n-1} with the set

$$\{w \in B_n : w(n) = n\}.$$

Given such an element $w = w_1 \cdots w_{n-1}n$, the terms in Ψ_n dictate the location and sign of n :

$$\begin{aligned} w \cdot \Psi_n &= w + ws_{n-1} + ws_{n-1}s_{n-2} + \cdots + w(s_{n-1} \cdots s_1) \\ &\quad + w(s_{n-1} \cdots s_1s_0) + w(s_{n-1} \cdots s_1s_0s_1) + \cdots + w(s_{n-1} \cdots s_1s_0s_1 \cdots s_{n-1}) \\ &= w_1 \cdots w_{n-1}n + w_1 \cdots w_{n-2}nw_{n-1} + w_1 \cdots nw_{n-2}w_{n-1} + \cdots + nw_1 \cdots w_{n-1} \\ &\quad + \bar{n}w_1 \cdots w_{n-1} + w_1\bar{n}w_2 \cdots w_{n-1} + \cdots + w_1 \cdots w_{n-1}\bar{n}. \end{aligned}$$

Moreover, from this description it is clear that $w \cdot \Psi_n = v \cdot \Psi_n$ if and only if $w = v$. We have

$$\Psi_1 \cdots \Psi_{n-1} \Psi_n = \sum_{w \in B_n, w(n)=n} w \cdot \Psi_n = \sum_{u \in B_n} u,$$

as desired. □

We will now describe how this product formula relates to signed permutation statistics.

Corollary 4. *For $n \geq 1$, we have*

$$B_n(q, t) := \prod_{i=1}^n (1 + t[2i]_q - t) = \sum_{w \in B_n} q^{\text{inv}_B(w)} t^{\text{nmin}(w)}.$$

As with Corollary 1, this generating function for $(\text{inv}_B, \text{nmin})$ is obtainable from [5, Theorem 1.2].

Proof. Define the linear map $\psi : \mathbb{Z}[B_n] \rightarrow \mathbb{Z}[q, t]$ given by $\psi(w) = q^{\text{inv}_B(w)} t^{\text{nmin}(w)}$. It is clear that $\psi(\Psi_i) = (1 + qt + q^2t + \cdots + q^{2i-1}t) = (1 + t[2i]_q - t)$. Assume for induction that

$$\psi(\Psi_1 \cdots \Psi_{n-1}) = \psi(\Psi_1) \cdots \psi(\Psi_{n-1}) = \sum_{w \in B_{n-1}} \psi(w).$$

It suffices to show that the following holds:

$$\psi(\Psi_1 \cdots \Psi_{n-1})\psi(\Psi_n) = \psi(\Psi_1 \cdots \Psi_{n-1}\Psi_n).$$

Following the proof of Proposition 3, we see that Ψ_n has the effect of positioning n in a signed permutation and determining its sign. Following the characterizations of nmin and inv_B above, we have, for a signed permutation $w = w_1 \cdots w_{n-1}n$:

$$\begin{aligned} \psi(w \cdot \Psi_n) &= \psi(w_1 \cdots w_{n-1}n) + \psi(w_1 \cdots w_{n-2}nw_{n-1}) + \cdots + \psi(nw_1 \cdots w_{n-1}) \\ &\quad + \psi(\bar{n}w_1 \cdots w_{n-1}) + \psi(w_1\bar{n} \cdots w_{n-1}) + \cdots + \psi(w_1 \cdots w_{n-1}\bar{n}), \\ &= \psi(w) + qt\psi(w) + \cdots + q^{n-1}t\psi(w) \\ &\quad + q^n t\psi(w) + q^{n+1}t\psi(w) + \cdots + q^{2n-1}t\psi(w), \\ &= (1 + t[2n]_q - t)\psi(w) = \psi(\Psi_n)\psi(w). \end{aligned}$$

Thus,

$$\begin{aligned} \psi(\Psi_1 \cdots \Psi_{n-1}\Psi_n) &= \psi\left(\sum_{w \in B_n, w(n)=n} w \cdot \Psi_n\right) \\ &= \sum_{w \in B_n, w(n)=n} \psi(w \cdot \Psi_n) \\ &= \psi(\Psi_n) \sum_{w \in B_n, w(n)=n} \psi(w) \\ &= \psi(\Psi_n)\psi(\Psi_1 \cdots \Psi_{n-1}), \end{aligned}$$

as desired. \square

We now present a different factorization of the sum $\sum_{w \in B_n} w$. Let $\Phi_1 = 1 + t_{\bar{1}1}$, $\Phi_2 = 1 + t_{12} + t_{\bar{1}2} + t_{\bar{2}2}$, and for $j \geq 2$ let $\Phi_j = 1 + \sum_{j > i \geq \bar{j}} t_{ij}$.

Proposition 4. *For $n \geq 1$, we have*

$$\Phi_1 \Phi_2 \cdots \Phi_n = \sum_{w \in B_n} w.$$

Proof. Obviously the formula holds for $n = 1$. For induction, suppose $\Phi_1 \cdots \Phi_{n-1} = \sum_{w \in B_{n-1}} w$. We observe that Φ_n is 1 plus the sum of all transpositions involving the letter n . Thus, for

$w = w_1 \cdots w_{n-1} n$ we have

$$\begin{aligned} w \cdot \Phi_n &= w + wt_{n-1n} + wt_{n-2n} + \cdots + wt_{1n} + wt_{\bar{1}n} + wt_{\bar{2}n} + \cdots + wt_{\bar{n}n} \\ &= w_1 \cdots w_{n-1} n + w_1 \cdots n w_{n-1} + w_1 \cdots n w_{n-1} w_{n-2} + \cdots + n w_2 \cdots w_{n-1} w_1 \\ &\quad + \bar{n} w_2 \cdots w_{n-1} \bar{w}_1 + w_1 \bar{n} \cdots w_{n-1} \bar{w}_2 + \cdots + w_1 \cdots w_{n-1} \bar{n}. \end{aligned}$$

Clearly $w \cdot \Phi_n = v \cdot \Phi_n$ if and only if $w = v$, and so we have the desired result:

$$\Phi_1 \cdots \Phi_{n-1} \Phi_n = \sum_{w \in B_n, w(n)=n} w \cdot \Phi_n = \sum_{w \in B_n} w.$$

□

Next we use the factorization to obtain the statistical distribution.

Corollary 5. *We have*

$$B_n(q, t) = \sum_{w \in B_n} q^{\text{sor}_B(w)} t^{\ell'_B(w)}.$$

Proof. This proof is nearly identical to previous arguments. We show that the linear map $\phi : \mathbb{Z}[B_n] \rightarrow \mathbb{Z}[q, t]$ with $\phi(w) = q^{\text{sor}_B(w)} t^{\ell'_B(w)}$ obeys

$$\phi(\Phi_1 \cdots \Phi_{n-1}) \phi(\Phi_n) = \phi(\Phi_1 \cdots \Phi_{n-1} \Phi_n).$$

This follows by induction on n and the observation that Φ_n is all transpositions involving n and hence the products of transpositions generated are indeed reduced. □

We state our desired equidistribution result as follows.

Corollary 6. *The pairs of statistics $(\text{inv}_B, \text{nmin})$ and (sor_B, ℓ'_B) are equidistributed over B_n :*

$$\sum_{w \in B_n} q^{\text{inv}_B(w)} t^{\text{nmin}(w)} = \sum_{w \in B_n} q^{\text{sor}_B(w)} t^{\ell'_B(w)}.$$

In particular, sor_B is Mahonian and nmin is Stirling.

4. TYPE D

The type D_n Coxeter group can be defined as the subgroup of B_n ($n \geq 4$) given by elements with an even number of minus signs. The minimal generating set for D_n is

$$S^D := \{(\bar{1}2)\} \cup \{(i i + 1) : 1 \leq i \leq n - 1\}.$$

We denote these simple transpositions by $s_i = (i i + 1)$ and $s_{\bar{1}} = (\bar{1}2)$.

We will use the same notation for transpositions as in the type B_n case:

$$T^D = \{(ij) : 1 \leq i < j \leq n\} \cup \{(\bar{i}j) : 1 \leq i \leq j \leq n\},$$

with $t_{ij} = (ij)$. However, we caution that as opposed to the type B case, the transpositions $t_{\bar{i}i}$ are *not* reflections in D_n . (Recall that a reflection in a Coxeter system (W, S) is any conjugate of an element of S , i.e., t is a reflection if and only if $t = wsw^{-1}$ for some $s \in S$ and $w \in W$.) In particular, in D_n we have $t_{\bar{1}1} = 1$ and for $1 < i \leq n$ and $w = w_1 \cdots w_n$,

$$wt_{\bar{i}i} = \bar{w}_1 w_2 \cdots w_{i-1} \bar{w}_i w_{i+1} \cdots w_n.$$

The type D sorting index will be defined in terms of a certain factorization by transpositions in T^D , but, as not all elements of T^D are reflections, it does not share the same close

relationship with reflection length that was seen in types A and B. Thus we consider only the Mahonian statistics in this case.

4.1. Mahonian statistics. The length of an element $w \in D_n$, denoted $\ell_D(w)$, is the minimal number of adjacent transpositions from S^D needed to express w . The type D_n inversion number is defined as:

$$\text{inv}_D(w) = |\{1 \leq i < j \leq n : w(i) > w(j)\}| + |\{1 \leq i < j \leq n : -w(i) > w(j)\}|.$$

For example, $\text{inv}_D(\bar{3}24\bar{5}1) = (1 + 2 + 2) + (3 + 1 + 1 + 1) = 11$. We have $\ell_D(w) = \text{inv}_D(w)$ for $w \in D_n$.

Although it is not necessarily a product of reflections, there is nonetheless a unique factorization

$$w = t_{i_1 j_1} \cdots t_{i_k j_k},$$

with $1 < j_1 < \cdots < j_k$, and the *type D_n sorting index*, $\text{sor}_D(w)$, is defined to be

$$\text{sor}_D(w) = \sum_{s=1}^k j_s - i_s - 2 \cdot \chi(i_s < 0).$$

The sorting index of type D differs from the type B version in two key ways. First, we use only the transpositions (ij) with $j \geq 2$ (in particular $(\bar{1}1)$ is not used), and second, when i is negative the distance between i and j is $j - i - 2$. This is intuitively nice when we think of elements of D_n not as symmetric chains, but as symmetric posets with a “fork” in the middle:

$$w_{\bar{n}} \cdots w_{\bar{2}} \begin{matrix} w_1 \\ w_{\bar{1}} \end{matrix} w_2 \cdots w_n.$$

For example, consider the following sorting of the element $w = \bar{3}24\bar{5}1 \in D_5$. We have

$$\bar{1}5\bar{4}\bar{2} \begin{matrix} \bar{3} \\ 3 \end{matrix} 24\bar{5}1 \xrightarrow{(\bar{4}5)} \bar{5}1\bar{4}\bar{2} \begin{matrix} \bar{3} \\ 3 \end{matrix} 24\bar{1}5 \xrightarrow{(34)} \bar{5}\bar{4}1\bar{2} \begin{matrix} \bar{3} \\ 3 \end{matrix} 2\bar{1}45 \xrightarrow{(\bar{1}3)} \bar{5}\bar{4}\bar{3}\bar{2} \begin{matrix} 1 \\ \bar{1} \end{matrix} 2345$$

and so $\text{sor}_D(w) = (5 - (-4) - 2) + (4 - 3) + (3 - (-1) - 2) = 10$.

4.2. Factorizations of the diagonal sum. As with ordinary and signed permutations, there are at least two natural ways to generate elements of D_n given the elements of D_{n-1} . For the first of these, define $\Psi_1 = 1 + s_1 + s_{\bar{1}} + s_1 s_{\bar{1}}$, and for $i \geq 2$, define $\Psi_i = 1 + s_i \Psi_{i-1} + s_i \cdots s_2 s_1 s_{\bar{1}} s_2 \cdots s_i$. We have the following factorization of the diagonal sum.

Proposition 5. *For $n \geq 4$, we have*

$$(3) \quad \Psi_1 \Psi_2 \cdots \Psi_{n-1} = \sum_{w \in D_n} w.$$

Proof. One can verify (preferably with computer assistance) that $\Psi_1 \Psi_2 \Psi_3 = \sum_{w \in D_4} w$. For induction, we simply examine the effect that Ψ_{n-1} has on an element $w = w_1 \cdots w_{n-1} \in D_{n-1}$. We have:

$$\begin{aligned} w \cdot \Psi_{n-1} &= w + w s_{n-1} + \cdots + w s_{n-1} \cdots s_2 + w(s_{n-1} \cdots s_2 s_1) + w(s_{n-1} \cdots s_2 s_{\bar{1}})w \\ &\quad + w(s_{n-1} \cdots s_2 s_1 s_{\bar{1}}) + \cdots + w(s_{n-1} \cdots s_2 s_1 s_{\bar{1}} s_2 \cdots s_{n-1}) \\ &= w_1 \cdots w_{n-1} n + w_1 \cdots w_{n-2} n w_{n-1} + \cdots + w_1 n \cdots w_{n-1} + n w_1 \cdots w_{n-1} \\ &\quad + \bar{n} \bar{w}_1 \cdots w_{n-1} + \bar{w}_1 \bar{n} \cdots w_{n-1} + \cdots + \bar{w}_1 \cdots w_{n-1} \bar{n}. \end{aligned}$$

The result now follows as in the proof of Proposition 3. \square

Let $\psi : \mathbb{Z}[D_n] \rightarrow \mathbb{Z}[q]$ by $\psi(w) = q^{\text{inv}_D(w)}$, so that $\psi(\Psi_i) = (1 + q + \cdots + q^{i-1} + 2q^i + q^{i+1} + \cdots + q^{2i}) = (1 + q^i)[i + 1]_q$. By applying ψ to both sides of (3), we have the following. (The second equality follows from the fact that $[i]_q(1 + q^i) = [2i]_q$.)

Corollary 7. *We have*

$$D_n(q) := \sum_{w \in D_n} q^{\text{inv}_D(w)} = \prod_{i=1}^{n-1} (1 + q^i)[i + 1]_q = [n]_q \cdot \prod_{i=1}^{n-1} [2i]_q.$$

We omit the proof as it closely follows the related argument for type B.

We now give a factorization analogous to Propositions 2 and 4. Let $\Phi_j = 1 + \sum_{j > i \geq \bar{j}} t_{ij}$ be as defined in Proposition 4.

Proposition 6. *For $n \geq 4$, we have*

$$(4) \quad \Phi_2 \Phi_3 \cdots \Phi_n = \sum_{w \in D_n} w.$$

Proof. One can verify the claim when $n = 4$. The general case follows by induction as with the earlier proofs. For consistency, we describe the effect of Φ_n on an element $w = w_1 \cdots w_{n-1}n$. We have:

$$\begin{aligned} w \cdot \Phi_n &= w + wt_{n-1}n + wt_{n-2}n + \cdots + wt_{1n} + wt_{\bar{1}n} + wt_{\bar{2}n} + \cdots + wt_{\bar{n}n} \\ &= w_1 \cdots w_{n-1}n + w_1 \cdots nw_{n-1} + w_1 \cdots nw_{n-1}w_{n-2} + \cdots + nw_2 \cdots w_{n-1}w_1 \\ &\quad + \bar{n}w_2 \cdots w_{n-1}\bar{w}_1 + w_1\bar{n} \cdots w_{n-1}\bar{w}_2 + \cdots + \bar{w}_1 \cdots w_{n-1}\bar{n}, \end{aligned}$$

from which the result follows. \square

Define $\phi : \mathbb{Z}[D_n] \rightarrow \mathbb{Z}[q]$ with $\phi(w) = q^{\text{sor}_D(w)}$. By construction, $\phi(\Phi_i) = (1 + q + \cdots + q^{i-2} + 2q^{i-1} + q^i + \cdots + q^{2i-2}) = (1 + q^{i-1})[i]_q$. Applying ϕ to (4), we have the following, for which we again omit the proof.

Corollary 8. *For $n \geq 4$, we have*

$$D_n(q) = \sum_{w \in D_n} q^{\text{sor}_D(w)} = \prod_{i=1}^{n-1} (1 + q^i)[i + 1]_q = [n]_q \cdot \prod_{i=1}^{n-1} [2i]_q.$$

From Corollaries 8 and 7 we have shown that sor_D is Mahonian.

Corollary 9. *The statistics inv_D and sor_D are equidistributed over D_n :*

$$\sum_{w \in D_n} q^{\text{inv}_D(w)} = \sum_{w \in D_n} q^{\text{sor}_D(w)}.$$

That is, sor_D is Mahonian.

W	e_1, \dots, e_n
A_n	$1, 2, 3 \dots, n$
B_n	$1, 3, 5 \dots, 2n - 3, 2n - 1$
D_n	$1, 3, 5 \dots, 2n - 3, n - 1$

TABLE 2. Exponents for the Coxeter groups of types A_n , B_n , and D_n .

5. FINITE COXETER GROUPS

Given a Coxeter system (W, S) and an element $w \in W$, we define the *length*, $\ell(w)$, to be the minimal number of terms needed to express w as a product of elements in the set S . The set of all reflections of (W, S) is the set $T := \{wsw^{-1} : w \in W, s \in S\}$. The *reflection length*, $\ell'(w)$, is the minimal number of terms needed to express w as a product of elements in the set T . We have the following classical results on the distribution of these statistics for finite Coxeter groups W . See [10, Sections 3.9 and 3.15].

Theorem 5.1. *For a finite Coxeter group W ,*

(a) *(Solomon)*

$$\sum_{w \in W} q^{\ell(w)} = \prod_{i=1}^n [e_i + 1]_q,$$

and

(b) *(Shepard-Todd)*

$$\sum_{w \in W} t^{\ell'(w)} = \prod_{i=1}^n (1 + e_i t),$$

where the e_i are the exponents of W .

Thus, both the length and reflection length generating functions have a common refinement, defined as follows:

$$W(q, t) := \prod_{i=1}^n (1 + t[e_i + 1]_q - t),$$

and implicitly there are statistics m and h such that:

$$\begin{aligned} W(q, t) &= \sum_{w \in W} q^{h(w)} t^{\ell'(w)} \\ &= \sum_{w \in W} q^{\ell(w)} t^{m(w)}. \end{aligned}$$

For $W = A_n$ and $W = B_n$ we have given explicit descriptions for both h^1 and m . It would be nice to have a type-independent description of these statistics. Notice that the i th factors Propositions 1, 2, 3, and 4 each have $e_i + 1$ terms. This is not the case for Propositions 5 and 6. In fact, one can show for D_4 that such a factorization (at least one that is well-behaved with respect to length or reflection length) does not exist.

¹We use h to suggest *height*, as used in the theory of root systems. Each reflection t (in a Weyl group) corresponds to a unique positive root β . As we have defined things, in both A_n and B_n , $h(\beta) = \text{sor}(t)$, and if $w = t_1 \cdots t_k$ is the factorization produced by sorting, then $h(\beta_1) + \cdots + h(\beta_k) = \text{sor}(w)$.

6. FINAL REMARKS

Upon completion of this work, the author was made aware of independent work of Wilson [15, Sec. 2.2], in which the type A version of the sorting index is described. Wilson has explored the sorting index further in a recent paper [16]. This paper establishes in particular that the sorting index is not trivially equivalent to many well-known Mahonian statistics. Nonetheless, work of Galovich and White [9] gives a recursive method for constructing a bijection $f : S_n \rightarrow S_n$ such that $\text{sor}(w)$ to $\text{inv}(f(w))$. This follows because the sorting index is what Galovich and White call a “splittable Mahonian” statistic.

Another connection to the type A sorting index appears in recent work of Foata and Han [6]. In particular, their “B-code” (b_1, \dots, b_n) of $w \in S_n$ is given by $b_{j_s} = j_s - i_s$ for each j_s that appears appears in the factorization $w = t_{i_1 j_1} \cdots t_{i_k j_k}$ of Section 2.1 ($b_j = 0$ otherwise). Moreover, the final paragraph of [6, Section 6] describes a related statistic “env” which gives rise to an equidistribution result similar to Corollary 3.

It is perhaps unsurprising that several researchers have independently discovered connections with the type A sorting index, as “straight selection sort” is such a natural operation.

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