

HIGHER ORDER SPT-FUNCTIONS

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Dedicated to the memory of A.O.L. (Oliver) Atkin

ABSTRACT. Andrews' spt-function can be written as the difference between the second symmetrized crank and rank moment functions. Using the machinery of Bailey pairs a combinatorial interpretation is given for the difference between higher order symmetrized crank and rank moment functions. This implies an inequality between crank and rank moments that was only known previously for sufficiently large n and fixed order. This combinatorial interpretation is in terms of a weighted sum of partitions. A number of congruences for higher order spt-functions are derived.

1. INTRODUCTION

Andrews [3] defined the function $\text{spt}(n)$ as the number of smallest parts in the partitions of n . He related this function to the second rank moment. He also proved some surprising congruences mod 5, 7 and 13. Namely, he showed that

$$(1.1) \quad \text{spt}(n) = np(n) - \frac{1}{2}N_2(n),$$

where $N_2(n)$ is the second rank moment function and $p(n)$ is the number of partitions of n , and he proved that

$$\begin{aligned} \text{spt}(5n+4) &\equiv 0 \pmod{5}, \\ \text{spt}(7n+5) &\equiv 0 \pmod{7}, \\ \text{spt}(13n+6) &\equiv 0 \pmod{13}. \end{aligned}$$

As noted in [17], (1.1) can be rewritten as

$$\text{spt}(n) = \frac{1}{2}(M_2(n) - N_2(n)),$$

where $M_2(n)$ is the second crank moment function. Rank and crank moments were introduced by A. O. L. Atkin and the author [6]. Bringmann [12] studied analytic, asymptotic and congruence properties of the generating function for the second rank moment as a quasi-weak Maass form. Further congruence properties of Andrews' spt-function were found by the author [17], Folsom and Ono [14] and Ono [21]. In [17] it was conjectured that

$$(1.2) \quad M_{2k}(n) > N_{2k}(n),$$

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for all $k \geq 1$ and $n \geq 1$. Here $M_{2k}(n)$ and $N_{2k}(n)$ are the $2k$ -th crank and $2k$ -th rank moment functions. For each fixed k , the inequality was proved for sufficiently large n by Bringmann, Mahlburg and Rhoades [10], who determined the asymptotic behaviour for the difference $M_{2k}(n) - N_{2k}(n)$ (see Section 7). The first few cases of the conjecture were previously proved by Bringmann and Mahlburg [11]. In this paper we prove the inequality unconditionally for all n and k by finding a combinatorial interpretation for the difference between symmetrized crank and rank moments. Analytic and arithmetic properties of higher order rank moments were studied by Bringmann, Lovejoy and Osburn [9] and by Bringmann, the author and Mahlburg [8].

Andrews [2] defined the k -th symmetrized rank function by

$$\eta_k(n) = \sum_{m=-n}^n \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} N(m, n),$$

where $N(m, n)$ is the number of partitions of n with rank m . Andrews gave a new interpretation of the symmetrized rank function in terms of Durfee symbols. As a natural analog to the symmetrized rank function we define the k -th symmetrized crank function by

$$\mu_k(n) = \sum_{m=-n}^n \binom{m + \lfloor \frac{k-1}{2} \rfloor}{k} M(m, n),$$

where $M(m, n)$ is number of partitions of n with crank m , for $n \neq 1$. For $n = 1$ we define

$$M(-1, 1) = 1, M(0, 1) = -1, M(1, 1) = 1, \quad \text{and otherwise } M(m, 1) = 0.$$

One of our main results is the following identity

$$(1.3) \quad \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n = \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + n_2 + \dots + n_k}}{(1 - q^{n_k})^2 (1 - q^{n_{k-1}})^2 \dots (1 - q^{n_1})^2 (q^{n_1+1}; q)_{\infty}}.$$

When $k = 1$ this result reduces to (1.1). In equation (1.3) and throughout this paper we use standard q -notation [15]. We compare equation (1.3) with the identity

$$(1.4) \quad \sum_{n=1}^{\infty} \mu_{2k}(n) q^n = \frac{1}{(q)_{\infty}} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + n_2 + \dots + n_k}}{(1 - q^{n_k})^2 (1 - q^{n_{k-1}})^2 \dots (1 - q^{n_1})^2},$$

which is proved in Section 3. Some remarks about this identity are also given in Section 7.

In Section 2 we show that many of Andrews' results [2] for symmetrized rank moments can be extended to symmetrized crank moments. In Section 3 we prove a general result for Bailey pairs from which our main identity (1.3) follows. In Section 4, we use an analog of Stirling numbers of the second kind to show how ordinary moments can be expressed in terms of symmetrized moments and how our main identity implies the inequality (1.2). For each $k \geq 1$, we are able to define a higher-order spt function $\text{spt}_k(n)$ so that

$$\text{spt}_k(n) = \mu_{2k}(n) - \eta_{2k}(n),$$

for all $k \geq 1$ and $n \geq 1$. In Section 5 we give the combinatorial definition of $\text{spt}_k(n)$ in terms of a weighted sum over the partitions of n . We note that when $k = 1$, $\text{spt}_k(n)$ coincides with Andrews' spt-function.

In Section 6 we prove a number of congruences for the higher order spt-functions. In Section 7 we make some concluding remarks and close the paper with a table of $\text{spt}_k(n)$ for small n and k .

2. SYMMETRIZED CRANK MOMENTS

In this section we collect some results for symmetrized crank moments. Many of Andrews' results and proofs for symmetrized rank moments have analogs for symmetrized crank moments; thus we omit some details.

$$\begin{aligned}
C(z, q) &:= \sum_{n=1}^{\infty} \sum_{m=-n}^n M(m, n) z^m q^n \\
&= \frac{1}{(q)_{\infty}} \left(1 + \sum_{n=1}^{\infty} \frac{(1-z)(1-z^{-1})(-1)^n q^{n(n+1)/2} (1+q^n)}{(1-zq^n)(1-z^{-1}q^n)} \right) \\
&\hspace{15em} \text{(by [16, Eq.(7.15), p.70])} \\
&= \frac{1}{(q)_{\infty}} \left(1 + \sum_{n=1}^{\infty} (-1)^n q^{n(n+1)/2} \left(\frac{1-z}{1-zq^n} + \frac{1-z^{-1}}{1-z^{-1}q^n} \right) \right) \\
&= \frac{(1-z)}{(q)_{\infty}} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1-zq^n},
\end{aligned}$$

and

$$C^{(j)}(z, q) = \frac{-j!}{(q)_{\infty}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n q^{n(n-1)/2 + jn} (1-q^n)}{(1-zq^n)^{j+1}},$$

for $j \geq 0$.

By [2, Theorem 1] we know that $\eta_k(n) = 0$ if k is odd. In a similar fashion we find that $\mu_k(n) = 0$ if k is odd.

We will need

Theorem 2.1 (Andrews [2]).

$$(2.1) \quad \sum_{n=1}^{\infty} \eta_{2k}(n) q^n = \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2 + kn} \frac{(1+q^n)}{(1-q^n)^{2k}} = \frac{1}{(q)_{\infty}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(3n+1)/2 + kn}}{(1-q^n)^{2k}}.$$

This theorem has a crank analog.

Theorem 2.2.

$$(2.2) \quad \sum_{n=1}^{\infty} \mu_{2k}(n) q^n = \frac{1}{(q)_{\infty}} \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2 + kn} \frac{(1+q^n)}{(1-q^n)^{2k}} = \frac{1}{(q)_{\infty}} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2 + kn}}{(1-q^n)^{2k}}.$$

Proof. As in the proof of [2, Theorem 2] we have

$$\begin{aligned}
\sum_{n=1}^{\infty} \mu_{2k}(n) q^n &= \frac{1}{(2k)!} \left(\left(\frac{\partial}{\partial z} \right)^{2k} z^{k-1} C(z, q) \right) \Big|_{z=1} \\
&= \frac{1}{(2k)!} \sum_{j=0}^{k-1} \binom{2k}{j} (v-1) \cdots (v-j) C^{(2k-j)}(1, q)
\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{(q)_\infty} \sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n-1)/2 + (2k-j)n} (1-q^n)}{(1-q^n)^{2k-j+1}}, \\
&= \frac{1}{(q)_\infty} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n-1)/2 + 2kn}}{(1-q^n)^{2k}} \left(1 + \frac{q^{-n}}{(1-q^n)^{-1}}\right)^{k-1} \\
&= \frac{1}{(q)_\infty} \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2 + kn}}{(1-q^n)^{2k}}.
\end{aligned}$$

□

3. RANK MOMENTS, CRANK MOMENTS AND BAILEY CHAINS

In [22], Alexander Patkowski used a limiting form of Bailey's Lemma to obtain a partition identity analogous to (1.1), which relates an spt-like function to the second rank moment. We consider a similar limiting form that iterates Bailey's Lemma and obtain a general theorem for Bailey pairs (see Theorem 3.3 below). Then we show how our main identity (1.3) for the difference between symmetrized crank and rank moments follows from using well-known Bailey pairs. In this section we use the standard notation found in [15].

Definition 3.1. A pair of sequences $(\alpha_n(a, q), \beta_n(a, q))$ is called a Bailey pair with parameters (a, q) if

$$\beta_n(a, q) = \sum_{r=0}^n \frac{\alpha_r(a, q)}{(q; q)_{n-r} (aq; q)_{n+r}}$$

for all $n \geq 0$.

Theorem 3.2 (Bailey's Lemma). *Suppose $(\alpha_n(a, q), \beta_n(a, q))$ is a Bailey pair with parameters (a, q) . Then $(\alpha'_n(a, q), \beta'_n(a, q))$ is another Bailey pair with parameters (a, q) , where*

$$\alpha'_n(a, q) = \frac{(\rho_1, \rho_2; q)_n}{(aq/\rho_1, aq/\rho_2; q)_n} \left(\frac{aq}{\rho_1 \rho_2}\right)^n \alpha_n(a, q)$$

and

$$\beta'_n(a, q) = \sum_{k=0}^n \frac{(\rho_1, \rho_2; q)_k (aq/\rho_1 \rho_2; q)_{n-k}}{(aq/\rho_1, aq/\rho_2; q)_n (q; q)_{n-k}} \left(\frac{aq}{\rho_1 \rho_2}\right)^k \beta_k(a, q).$$

For more information on Bailey's Lemma and its applications see [1, Ch.3]. We will need the following limit which is an easy exercise.

$$(3.1) \quad \lim_{\rho_2 \rightarrow 1} \lim_{\rho_1 \rightarrow 1} \frac{1}{(1-\rho_1)(1-\rho_2)} \left(1 - \frac{(q)_k (q/\rho_1 \rho_2)_k}{(q/\rho_1)_k (q/\rho_2)_k}\right) = \sum_{j=1}^k \frac{q^j}{(1-q^j)^2}.$$

Theorem 3.3. *Suppose $(\alpha_n, \beta_n) = (\alpha_n(1, q), \beta_n(1, q))$ is a Bailey pair with $a = 1$, and $\alpha_0 = \beta_0 = 1$. Then*

$$\begin{aligned}
&\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q)_{n_1}^2 q^{n_1 + n_2 + \dots + n_k} \beta_{n_1}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \\
&= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1 + n_2 + \dots + n_k}}{(1-q^{n_k})^2 (1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} + \sum_{r=1}^{\infty} \frac{q^{kr} \alpha_r}{(1-q^r)^{2k}}.
\end{aligned}$$

Proof. From Bailey's Lemma we have

$$\begin{aligned} & \sum_{j=1}^n \frac{(\rho_1)_j (\rho_2)_j (q/\rho_1 \rho_2)_{n-j} (q/\rho_1 \rho_2)^j \beta_j}{(q)_{n-j}} \\ &= \frac{(q/\rho_1)_n (q/\rho_2)_n}{(q)_n^2} \left(1 - \frac{(q)_n (q/\rho_1 \rho_2)_n}{(q/\rho_1)_n (q/\rho_2)_n} \right) \\ & \quad + (q/\rho_1)_n (q/\rho_2)_n \sum_{r=1}^n \frac{(\rho_1)_r (\rho_2)_r}{(q)_{n-r} (q)_{n+r} (q/\rho_1)_r (q/\rho_2)_r} \left(\frac{q}{\rho_1 \rho_2} \right)^r \alpha_r. \end{aligned}$$

We divide both sides by $(1 - \rho_1)(1 - \rho_2)$, let $\rho_1 \rightarrow 1$, $\rho_2 \rightarrow 1$, and use (3.1) to obtain

$$\sum_{j=1}^n (q)_{j-1}^2 q^j \beta_j = \sum_{j=1}^n \frac{q^j}{(1 - q^j)^2} + (q)_n^2 \sum_{r=1}^n \frac{q^r \alpha_r}{(q)_{n-r} (q)_{n+r} (1 - q^r)^2}.$$

Letting $n \rightarrow \infty$ we have

$$\sum_{j=1}^{\infty} (q)_{j-1}^2 q^j \beta_j = \sum_{j=1}^{\infty} \frac{q^j}{(1 - q^j)^2} + \sum_{r=1}^{\infty} \frac{q^r \alpha_r}{(1 - q^r)^2},$$

which is the case $k = 1$ of the theorem.

Now we suppose that the theorem is true for $k = K - 1$, so that

$$\begin{aligned} & \sum_{n_K \geq n_{K-1} \geq \dots \geq n_2 \geq 1} \frac{(q)_{n_2}^2 q^{n_2 + \dots + n_K} \beta_{n_2}}{(1 - q^{n_K})^2 (1 - q^{n_{K-1}})^2 \dots (1 - q^{n_2})^2} \\ &= \sum_{n_K \geq n_{K-1} \geq \dots \geq n_2 \geq 1} \frac{q^{n_2 + \dots + n_K}}{(1 - q^{n_K})^2 (1 - q^{n_{K-1}})^2 \dots (1 - q^{n_2})^2} + \sum_{r=1}^{\infty} \frac{q^{(K-1)r} \alpha_r}{(1 - q^r)^{2K-2}}. \end{aligned}$$

We now replace (α_n, β_n) by the Bailey pair (α'_n, β'_n) in Bailey's Lemma to obtain

$$\begin{aligned} & \sum_{\substack{n_K \geq n_{K-1} \geq \dots \geq n_2 \geq 1 \\ n_2 \geq n_1 \geq 0}} \frac{(q)_{n_2}^2 q^{n_2 + \dots + n_K} (\rho_1)_{n_1} (\rho_2)_{n_1} (q/\rho_1 \rho_2)_{n_2 - n_1} (q/\rho_1 \rho_2)^{n_1} \beta_{n_1}}{(1 - q^{n_K})^2 (1 - q^{n_{K-1}})^2 \dots (1 - q^{n_2})^2 (q)_{n_2 - n_1} (q/\rho_1)_{n_2} (q/\rho_2)_{n_2}} \\ &= \sum_{n_K \geq n_{K-1} \geq \dots \geq n_2 \geq 1} \frac{q^{n_2 + \dots + n_K}}{(1 - q^{n_K})^2 (1 - q^{n_{K-1}})^2 \dots (1 - q^{n_2})^2} + \sum_{r=1}^{\infty} \frac{q^{(K-1)r} (\rho_1)_r (\rho_2)_r (q/\rho_1 \rho_2)^r \alpha_r}{(1 - q^r)^{2K-2} (q/\rho_1)_r (q/\rho_2)_r}, \end{aligned}$$

and

$$\begin{aligned} & \sum_{n_K \geq n_{K-1} \geq \dots \geq n_2 \geq n_1 \geq 1} \frac{(q)_{n_2}^2 q^{n_2 + \dots + n_K} (\rho_1)_{n_1} (\rho_2)_{n_1} (q/\rho_1 \rho_2)_{n_2 - n_1} (q/\rho_1 \rho_2)^{n_1} \beta_{n_1}}{(1 - q^{n_K})^2 (1 - q^{n_{K-1}})^2 \dots (1 - q^{n_2})^2 (q)_{n_2 - n_1} (q/\rho_1)_{n_2} (q/\rho_2)_{n_2}} \\ &= \sum_{n_K \geq n_{K-1} \geq \dots \geq n_2 \geq 1} \frac{q^{n_2 + \dots + n_K}}{(1 - q^{n_K})^2 (1 - q^{n_{K-1}})^2 \dots (1 - q^{n_2})^2} \left(1 - \frac{(q)_{n_2} (q/\rho_1 \rho_2)_{n_2}}{(q/\rho_1)_{n_2} (q/\rho_2)_{n_2}} \right) \\ & \quad + \sum_{r=1}^{\infty} \frac{q^{(K-1)r} (\rho_1)_r (\rho_2)_r (q/\rho_1 \rho_2)^r \alpha_r}{(1 - q^r)^{2K-2} (q/\rho_1)_r (q/\rho_2)_r}. \end{aligned}$$

We divide both sides by $(1 - \rho_1)(1 - \rho_2)$, let $\rho_1 \rightarrow 1$, $\rho_2 \rightarrow 1$, and use (3.1) to obtain

$$\sum_{n_K \geq n_{K-1} \geq \dots \geq n_1 \geq 1} \frac{(q)_{n_1}^2 q^{n_1 + n_2 + \dots + n_K} \beta_{n_1}}{(1 - q^{n_K})^2 (1 - q^{n_{K-1}})^2 \dots (1 - q^{n_1})^2}$$

$$= \sum_{n_K \geq n_{K-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_K}}{(1-q^{n_K})^2(1-q^{n_{K-1}})^2 \dots (1-q^{n_1})^2} + \sum_{r=1}^{\infty} \frac{q^{Kr} \alpha_r}{(1-q^r)^{2K}},$$

which is the result for $k = K$. The general result follows by induction. \square

Corollary 3.4.

(3.2)

$$\sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} = \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2+kn} \frac{(1+q^n)}{(1-q^n)^{2k}}.$$

Proof. The result follows from Theorem 3.3 using the well-known Bailey pair [1, pp.27-28]

$$\alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^n q^{n(n-1)/2} (1+q^n) & n \geq 1, \end{cases} \quad \beta_n = \begin{cases} 1 & n = 0, \\ 0 & n \geq 1. \end{cases}$$

\square

We note that we can rewrite (3.2) as

$$(3.3) \quad \sum_{n=1}^{\infty} \mu_{2k}(n) q^n = \frac{1}{(q)_{\infty}} \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2},$$

after using (2.2).

Corollary 3.5.

(3.4)

$$\begin{aligned} & \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{(q)_{n_1} q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} \\ &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2} + \sum_{n=1}^{\infty} (-1)^n q^{n(3n-1)/2+kn} \frac{(1+q^n)}{(1-q^n)^{2k}}. \end{aligned}$$

Proof. The result follows from Theorem 3.3 using the well-known Bailey pair [1, p.28]

$$\alpha_n = \begin{cases} 1 & n = 0, \\ (-1)^n q^{n(3n-1)/2} (1+q^n) & n \geq 1, \end{cases} \quad \beta_n = \frac{1}{(q)_n}.$$

\square

Corollary 3.6.

(3.5)

$$\begin{aligned} & \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n \\ &= \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2 (q^{n_1+1}; q)_{\infty}}. \end{aligned}$$

Proof. After dividing both sides of (3.4) by $(q)_{\infty}$ and using (3.2) we have

$$\begin{aligned} & \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2 (q^{n_1+1}; q)_{\infty}} \\ &= \frac{1}{(q)_{\infty}} \left(\sum_{n=1}^{\infty} (-1)^{n-1} q^{n(n-1)/2+kn} \frac{(1+q^n)}{(1-q^n)^{2k}} - \sum_{n=1}^{\infty} (-1)^{n-1} q^{n(3n-1)/2+kn} \frac{(1+q^n)}{(1-q^n)^{2k}} \right) \end{aligned}$$

$$= \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n,$$

by (2.2) and (2.1). □

4. RANK AND CRANK MOMENT INEQUALITIES

In this section we prove the conjectured inequality (1.2) for rank and crank moments. We need to relate ordinary and symmetrized moments. This is achieved by defining an analog of Stirling numbers of the second kind. This approach was suggested by Mike Hirschhorn.

We define a sequence of polynomials

$$g_k(x) = \prod_{j=0}^{k-1} (x^2 - j^2),$$

for $k \geq 1$. We want a sequence of numbers $S^*(n, k)$ such that

$$x^{2n} = \sum_{k=1}^n S^*(n, k) g_k(x),$$

for $n \geq 1$.

Definition 4.1. We define the sequence $S^*(n, k)$ ($1 \leq k \leq n$) recursively by

- (1) $S^*(1, 1) = 1$,
- (2) $S^*(n, k) = 0$ if $k \leq 0$ or $k > n$, and
- (3) $S^*(n + 1, k) = S^*(n, k - 1) + k^2 S^*(n, k)$, for $1 \leq k \leq n + 1$.

Below is a table of $S^*(n, k)$ for small n :

				1					
				1		1			
			1	5		1			
		1	21	14		1			
	1	85	147	30		1			
1	341	1408	649	55		1			

We note that if we replace k^2 by k in the recurrence we obtain the Stirling numbers of the second kind. The numbers $S^*(n, k)$ first occur in a paper of MacMahon [20, p.106]. Miklós Bóna reminded me that Neil Sloane’s Online Encyclopedia of Integer Sequences [23] can also handle 2-dimensional sequences. One just needs to input the first few terms of

$$(4.1) \quad \{ \{ S^*(n, k) \}_{k=1}^n \}_{n=1}^{\infty} = 1, 1, 1, 1, 5, 1, 1, 21, 14, 1, 1, 85, 147, 30, 1, \dots,$$

to find the sequence labelled **A036969** [24], where more references can be found.

We have

Lemma 4.2. For $n \geq 1$,

$$x^{2n} = \sum_{k=1}^n S^*(n, k) g_k(x).$$

Proof. We proceed by induction on n . The result is true for $n = 1$ since $S^*(1, 1) = 1$ and $g_1(x) = x^2$. We now suppose the result is true for $n = m$, so that

$$x^{2m} = \sum_{k=1}^m S^*(m, k) g_k(x).$$

We have $g_{k+1}(x) = (x^2 - k^2)g_k(x)$ and

$$x^2 g_k(x) = g_{k+1}(x) + k^2 g_k(x),$$

for $k \geq 1$. Thus

$$\begin{aligned} x^{2m+2} &= \sum_{k=1}^m S^*(m, k) x^2 g_k(x) \\ &= \sum_{k=1}^m S^*(m, k) (g_{k+1}(x) + k^2 g_k(x)) \\ &= \sum_{k=1}^{m+1} (S^*(m, k-1) + k^2 S^*(m, k)) g_k(x) \\ &= \sum_{k=1}^m S^*(m+1, k) g_k(x), \end{aligned}$$

and the result is true for $n = m + 1$ and true for all n by induction. \square

We can now express ordinary moments in terms of symmetrized moments.

Theorem 4.3. For $k \geq 1$

$$(4.2) \quad \mu_{2k}(n) = \frac{1}{(2k)!} \sum_{m=-n}^n g_k(m) M(m, n),$$

$$(4.3) \quad \eta_{2k}(n) = \frac{1}{(2k)!} \sum_{m=-n}^n g_k(m) N(m, n),$$

$$(4.4) \quad M_{2k}(n) = \sum_{j=1}^k (2j)! S^*(k, j) \mu_{2j}(n),$$

$$(4.5) \quad N_{2k}(n) = \sum_{j=1}^k (2j)! S^*(k, j) \eta_{2j}(n).$$

Proof. Suppose $k \geq 1$. Then

$$\begin{aligned} \mu_{2k}(n) &= \sum_{m=-n}^n \binom{m+k-1}{2k} M(m, n) \\ &= \frac{1}{(2k)!} \sum_{m=-n}^n (m+k-1)(m+k-2) \cdots (m-k) M(m, n) \\ &= \frac{1}{(2k)!} \sum_{m=-n}^n (m^2 - (k-1)^2)(m^2 - (k-2)^2) \cdots (m^2 - 1) m(m-k) M(m, n) \\ &= \frac{1}{(2k)!} \sum_{m=-n}^n g_k(m) M(m, n), \end{aligned}$$

since $M(-m, n) = M(m, n)$ for all m . This gives (4.2) and similarly (4.3). Using Lemma 4.2 and (4.2) we see that

$$M_{2k} = \sum_{m=-n}^n m^{2k} M(m, n)$$

$$\begin{aligned}
&= \sum_{m=-n}^n \left(\sum_{j=1}^k S^*(k, j) g_j(m) \right) M(m, n) \\
&= \sum_{j=1}^k (2j)! S^*(k, j) \mu_{2j}(n),
\end{aligned}$$

which is (4.4). Equation (4.5) follows similarly. \square

We can now deduce our crank-rank moment inequality.

Corollary 4.4. For $1 \leq k \leq n$

$$M_{2k}(n) > N_{2k}(n).$$

Proof. Suppose $k \geq 1$. Then from (3.5) we have

$$\sum_{n=1}^{\infty} (\mu_{2j}(n) - \eta_{2j}(n)) q^n = \frac{q^j}{(1-q)^{2j} (q^2; q)_{\infty}} + \cdots,$$

and we see that

$$\mu_{2j}(n) > \eta_{2j}(n),$$

for all $n \geq j \geq 1$. Now using (4.4), (4.5) and the fact that the coefficients $S^*(k, j)$ are positive integers we have

$$\begin{aligned}
M_{2k}(n) - N_{2k}(n) &= \sum_{j=1}^k (2j)! S^*(k, j) (\mu_{2j}(n) - \eta_{2j}(n)) \\
&\geq 2(\mu_2(n) - \eta_2(n)) > 0,
\end{aligned}$$

for all $n \geq 1$. \square

5. HIGHER ORDER SPT-FUNCTIONS

In this section we define a higher-order spt function $\text{spt}_k(n)$ so that

$$\text{spt}_k(n) = \mu_{2k}(n) - \eta_{2k}(n),$$

for all $k \geq 1$ and $n \geq 1$. The idea is to interpret the right side of (3.5) in terms of partitions.

Definition 5.1. For a partition π with m different parts

$$n_1 < n_2 < \cdots < n_m,$$

we define $f_j = f_j(\pi)$ to be the frequency of part n_j for $1 \leq j \leq m$.

We note that $f_1 = f_1(\pi)$ is the number of smallest parts in the partition π and Andrews' function

$$\text{spt}(n) = \sum_{\pi \vdash n} f_1(\pi).$$

Definition 5.2. Let $k \geq 1$. For a partition π we define a weight

$$\omega_k(\pi) = \sum_{\substack{m_1 + m_2 + \cdots + m_r = k \\ 1 \leq r \leq k}} \binom{f_1 + m_1 - 1}{2m_1 - 1} \sum_{2 \leq j_2 < j_3 < \cdots < j_r} \binom{f_{j_2} + m_2}{2m_2} \binom{f_{j_3} + m_3}{2m_3} \cdots \binom{f_{j_r} + m_r}{2m_r},$$

and

$$\text{spt}_k(n) = \sum_{\pi \vdash n} \omega_k(\pi).$$

We note that the outer sum above is over all compositions $m_1 + m_2 + \cdots + m_r$ of k .

Example 5.3 ($k = 1$). There is only one composition of 1, $\omega_1(\pi) = f_1(\pi)$ and

$$\text{spt}_1(n) = \text{spt}(n).$$

Example 5.4 ($k = 2$). There are two compositions of 2, namely 2 and 1 + 1,

$$\omega_2(\pi) = \binom{f_1 + 1}{3} + f_1 \sum_{2 \leq j} \binom{f_j + 1}{2},$$

and

$$\text{spt}_2(n) = \sum_{\pi \vdash n} \omega_2(\pi).$$

We calculate $\text{spt}_2(4)$. There are five partitions of 4:

4	$f_1 = 1$	$\omega_2 = 0$
3 + 1	$f_1 = f_2 = 1$	$\omega_2 = 1$
2 + 2	$f_1 = 2$	$\omega_2 = 1$
2 + 1 + 1	$f_1 = 2, f_2 = 1$	$\omega_2 = 1 + 2 = 3$
1 + 1 + 1 + 1	$f_1 = 4$	$\omega_2 = 10$

Hence $\text{spt}_2(4) = 0 + 1 + 1 + 3 + 10 = 15$.

Example 5.5 ($k = 3$). There are four compositions of 3, namely 3, 2 + 1, 1 + 2 and 1 + 1 + 1 + 1. Hence the definition of $\omega_3(\pi)$ has four terms:

$$\omega_3(\pi) = \binom{f_1 + 2}{5} + \binom{f_1 + 1}{3} \sum_{2 \leq j} \binom{f_j + 1}{2} + f_1 \sum_{2 \leq j} \binom{f_j + 2}{4} + f_1 \sum_{2 \leq j < k} \binom{f_j + 1}{2} \binom{f_k + 1}{2},$$

and

$$\text{spt}_3(n) = \sum_{\pi \vdash n} \omega_3(\pi).$$

To illustrate, we calculate $\text{spt}_3(5)$. There are seven partitions of 5:

5	$f_1 = 1$	$\omega_3 = 0$
4 + 1	$f_1 = f_2 = 1$	$\omega_3 = 0$
3 + 2	$f_1 = f_2 = 1$	$\omega_3 = 0$
3 + 1 + 1	$f_1 = 2, f_2 = 1$	$\omega_3 = 1$
2 + 2 + 1	$f_1 = 1, f_2 = 2$	$\omega_3 = 1$
2 + 1 + 1 + 1	$f_1 = 3, f_2 = 1$	$\omega_3 = 1 + 4 = 5$
1 + 1 + 1 + 1 + 1	$f_1 = 5$	$\omega_3 = 21$

Hence $\text{spt}_3(5) = 0 + 0 + 0 + 1 + 1 + 5 + 21 = 28$.

Our goal in this section is to prove

Theorem 5.6. For $1 \leq k \leq n$

$$\text{spt}_k(n) = \mu_k(n) - \eta_k(n).$$

Proof. First we need the elementary identities

$$\sum_{n=j}^{\infty} \binom{n+j-1}{2j-1} x^n = \frac{x^j}{(1-x)^{2j}} \quad \text{and} \quad \sum_{n=j}^{\infty} \binom{n+j}{2j} x^n = \frac{x^j}{(1-x)^{2j+1}}.$$

To give the idea of the proof we first consider the case $k = 4$. From (3.5) we have

$$\begin{aligned}
& \sum_{n=4}^{\infty} (\mu_8(n) - \eta_{8k}(n)) q^n \\
&= \sum_{1 \leq m \leq j \leq k \leq n} \frac{q^{m+j+k+n}}{(1-q^m)^2(1-q^j)^2(1-q^k)^2(1-q^n)^2(q^{m+1}; q)_{\infty}} \\
&= \sum_{1 \leq m=j=k=n} + \sum_{1 \leq m=j=k < n} + \sum_{1 \leq m=j < k=n} + \sum_{1 \leq m < j=k=n} + \\
&\quad \sum_{1 \leq m=j < k < n} + \sum_{1 \leq m < j=k < n} + \sum_{1 \leq m < j < k < n} + \sum_{1 \leq m < j < k < n} \\
&= \sum_{m=1}^{\infty} \frac{q^{4m}}{(1-q^m)^8} \prod_{i>m} \frac{1}{(1-q^i)} + \sum_{1 \leq m < n} \frac{q^{3m}}{(1-q^m)^6} \frac{q^n}{(1-q^n)^3} \prod_{\substack{i>m \\ i \neq n}} \frac{1}{(1-q^i)} \\
&+ \sum_{1 \leq m < n} \frac{q^{2m}}{(1-q^m)^4} \frac{q^{2n}}{(1-q^n)^5} \prod_{\substack{i>m \\ i \neq n}} \frac{1}{(1-q^i)} + \sum_{1 \leq m < n} \frac{q^m}{(1-q^m)^2} \frac{q^{3n}}{(1-q^n)^7} \prod_{\substack{i>m \\ i \neq n}} \frac{1}{(1-q^i)} \\
&+ \sum_{1 \leq m < k < n} \frac{q^{2m}}{(1-q^m)^4} \frac{q^k}{(1-q^k)^3} \frac{q^n}{(1-q^n)^3} \prod_{\substack{i>m \\ i \neq k, n}} \frac{1}{(1-q^i)} \\
&+ \sum_{1 \leq m < k < n} \frac{q^m}{(1-q^m)^2} \frac{q^{2k}}{(1-q^k)^5} \frac{q^n}{(1-q^n)^3} \prod_{\substack{i>m \\ i \neq k, n}} \frac{1}{(1-q^i)} \\
&+ \sum_{1 \leq m < k < n} \frac{q^m}{(1-q^m)^2} \frac{q^k}{(1-q^k)^3} \frac{q^{2n}}{(1-q^n)^5} \prod_{\substack{i>m \\ i \neq k, n}} \frac{1}{(1-q^i)} \\
&+ \sum_{1 \leq m < j < k < n} \frac{q^m}{(1-q^m)^2} \frac{q^j}{(1-q^j)^3} \frac{q^k}{(1-q^k)^3} \frac{q^n}{(1-q^n)^3} \prod_{\substack{i>m \\ i \neq j, k, n}} \frac{1}{(1-q^i)}.
\end{aligned}$$

There are eight compositions of 4: 4, 3+1, 2+2, 1+3, 2+1+1, 1+2+1, 1+1+2, and 1+1+1+1. Each of the eight sums above has the form

$$\sum_{1 \leq n_1 < n_{j_2} < \dots < n_{j_r}} \frac{q^{m_1 n_1}}{(1-q^{n_1})^{2m_1}} \frac{q^{m_2 n_{j_2}}}{(1-q^{n_{j_2}})^{2m_2+1}} \dots \frac{q^{m_r n_{j_r}}}{(1-q^{n_{j_r}})^{2m_r+1}} \prod_{i \notin \{n_{j_2}, \dots, n_{j_r}\}} \frac{1}{(1-q^i)},$$

where $m_1 + m_2 + \dots + m_r$ is a composition of $k = 4$. This sum can be written as

$$\begin{aligned}
& \sum_{\substack{1 \leq n_1 < n_{j_2} < \dots < n_{j_r} \\ f_1 \geq m_1, f_{j_2} \geq m_2, \dots, f_{j_r} \geq m_r}} \binom{f_1 + m_1 - 1}{2m_1 - 1} \binom{f_{j_2} + m_2 - 1}{2m_2} \dots \binom{f_{j_r} + m_r - 1}{2m_r} \\
& \cdot q^{f_1 n_1 + f_{j_2} n_{j_2} + \dots + f_{j_r} n_{j_r}} \prod_{\substack{i > n_1 \\ i \notin \{n_{j_2}, \dots, n_{j_r}\}}} \frac{1}{(1-q^i)}.
\end{aligned}$$

We see that this is the generating function for certain weighted partitions in which n_1 is the smallest part, $n_1 < n_{j_2} < \dots < n_{j_r}$ is an r -subset of the parts of the partition, and f_j is the frequency of

part n_j for each j . It follows that

$$\sum_{n=1}^{\infty} (\mu_{8k}(n) - \eta_{8k}(n)) q^n = \sum_{\pi \vdash n} \omega_4(\pi) = \sum_{n=4}^{\infty} \text{spt}_4(n) q^n.$$

The proof of the general case is completely analogous. Now suppose $k \geq 1$. From (3.5) we have

$$\begin{aligned} & \sum_{n=1}^{\infty} (\mu_{2k}(n) - \eta_{2k}(n)) q^n \\ &= \sum_{1 \leq n_1 \leq n_2 \leq \dots \leq n_k} \frac{q^{n_1 + n_2 + \dots + n_k}}{(1 - q^{n_1})^2 (1 - q^{n_2})^2 \dots (1 - q^{n_k})^2 (q^{n_1+1}; q)_{\infty}}. \end{aligned}$$

We partition this sum into 2^{k-1} subsums by changing each “ \leq ” in the general inequality $n_1 \leq n_2 \leq \dots \leq n_k$ to either “ $=$ ” or “ $<$ ”. In this way each subsum corresponds to a unique composition $m_1 + m_2 + \dots + m_r$ of k (where $1 \leq r \leq k$). We proceed just as in the case $k = 4$ and the general result follows. \square

6. CONGRUENCES FOR HIGHER ORDER SPT-FUNCTIONS

In [8] it was shown that given any prime $\ell > 3$ with k and j fixed there are infinitely many arithmetic progressions $An + B$ such that

$$\eta_{2k}(An + B) \equiv 0 \pmod{\ell^j}.$$

Using known results for crank moments [8, §7] and standard techniques [8], [12] we may deduce the analog of this result for higher order spt-functions. In this section we prove a number of nice explicit congruences for higher order spt-functions. Many of the congruences follow from known results for rank and crank moments [6].

Theorem 6.1.

$$(6.1) \quad \text{spt}_2(n) \equiv 0 \pmod{5}, \quad \text{if } n \equiv 0, 1, 4 \pmod{5}$$

$$(6.2) \quad \text{spt}_2(n) \equiv 0 \pmod{7}, \quad \text{if } n \equiv 0, 1, 5 \pmod{7}$$

$$(6.3) \quad \text{spt}_2(n) \equiv 0 \pmod{11}, \quad \text{if } n \equiv 0 \pmod{11}.$$

Proof. By definition,

$$\text{spt}_2(n) = \mu_4(n) - \eta_4(n) = \frac{1}{24}(M_4(n) - M_2(n) - N_4(n) + N_2(n)).$$

From [6, (5.6)] we have

$$N_4(n) = -\frac{2}{3}(3n+1)M_2(n) + \frac{8}{3}M_4(n) + (1-12n)N_2(n),$$

and

$$(6.4) \quad 24\text{spt}_2(n) = (2n - \frac{1}{3})M_2(n) - \frac{5}{3}M_4(n) + 12nN_2(n).$$

The congruence (6.1) now follows from

$$M_2(n) = 2np(n), \quad ([6, (1.27)])$$

$$N_2(n) \equiv (n+4)p(n), \quad \text{for } n \not\equiv 0, 3 \pmod{5} \quad ([17, \text{p.285}]),$$

$$p(5n+4) \equiv 0 \pmod{5}.$$

To begin the proof of (6.2) we use (6.4) to obtain

$$\text{spt}_2(n) \equiv M_4(n) + 3(n+1)M_2(n) + 4nN_2(n) \pmod{7}.$$

From [17, p.285]

$$(6.5) \quad N_2(n) \equiv (6n+1)p(n) \pmod{7}, \quad \text{for } n \not\equiv 0, 2, 6 \pmod{7}$$

so that

$$(6.6) \quad \text{spt}_2(n) \equiv M_4(n) + 3(n+1)M_2(n) \pmod{7}, \quad \text{for } n \equiv 0, 1, 5 \pmod{7}.$$

From [6, (1.21)] we have

$$M_4(7n+5) \equiv M_2(7n+5) \equiv 0 \pmod{7}, \quad \text{and} \quad \text{spt}_2(7n+5) \equiv 0 \pmod{7}.$$

From [6, (6.5)]

$$(6.7) \quad (n+2)M_4(n) \equiv -(6n^2+4n+1)M_2(n) \pmod{7},$$

so that

$$(6.8) \quad M_4(7n) \equiv 3M_2(7n) \equiv 0 \pmod{7}, \quad (\text{since } M_2(n) = 2np(n)),$$

$$(6.9) \quad M_4(7n+1) \equiv M_2(7n+1) \pmod{7},$$

and

$$\text{spt}_2(7n) \equiv \text{spt}_2(7n+1) \equiv 0 \pmod{7},$$

by (6.6).

The proof of (6.3) is similar to that of (6.1) and (6.2). From (6.4) we have

$$\text{spt}_2(n) \equiv M_4(n) + (n+9)M_2(n) + 6nN_2(n) \pmod{11}.$$

From [6, (6.6)]

$$(n+5)^3 M_4(n) \equiv (5n^4 + 10n^3 + 8n^2 + 8n + 9)M_2(n) \pmod{11},$$

so that

$$M_4(11n) \equiv M_2(11n) \equiv 0 \pmod{11},$$

and

$$\text{spt}_2(11n) \equiv 0 \pmod{11}.$$

□

Theorem 6.2.

$$(6.10) \quad \text{spt}_3(n) \equiv 0 \pmod{7}, \quad \text{if } n \not\equiv 3, 6 \pmod{7},$$

$$(6.11) \quad \text{spt}_3(n) \equiv 0 \pmod{2}, \quad \text{if } n \equiv 1 \pmod{4}.$$

Proof. From [6, (5.6)-(5.7)] and the definition of $\text{spt}_3(n)$ we have

$$(6.12) \quad \begin{aligned} \text{spt}_3(n) = & -\frac{7}{7920}M_6(n) + \frac{1}{1584}(60n+13)M_4(n) + \frac{1}{3960}(7-78n-108n^2)M_2(n) \\ & - \frac{1}{20}n(1+3n)N_2(n), \end{aligned}$$

and

$$(6.13) \quad \text{spt}_3(n) \equiv n(5n+4)M_2(n) + (3+2n)M_4(n) + n(3n+1)N_2(n) \pmod{7}.$$

This implies that

$$\text{spt}_3(7n+2) \equiv 0 \pmod{7}.$$

Known results for the rank and crank [6, (1.18),(1.21)] imply that

$$\text{spt}_3(7n+5) \equiv 0 \pmod{7}.$$

The congruences (6.5), (6.8), (6.9) and (6.13) imply that

$$\text{spt}_3(7n) \equiv \text{spt}_3(7n+1) \equiv 0 \pmod{7}.$$

The congruences (6.7) and (6.13) imply that

$$\text{spt}_3(7n+4) \equiv 2M_2(7n+4) + 3N_2(7n+4) \pmod{7}.$$

From (6.5) and the fact that $M_2(n) = 2np(n)$ we have

$$M_2(7n+4) \equiv p(7n+4), \quad N_2(7n+4) \equiv 4p(7n+4) \pmod{7}$$

and

$$\text{spt}_3(7n+4) \equiv 0 \pmod{7}.$$

We now turn to the congruence (6.11). First we note that the term

$$\frac{1}{20}n(1+3n)N_2(n) \equiv 0 \pmod{2},$$

when $n \equiv 1 \pmod{4}$ since $N_2(n) \equiv 0 \pmod{2}$. We define

$$s_3(n) = -\frac{7}{7920}M_6(n) + \frac{1}{1584}(60n+13)M_4(n) + \frac{1}{3960}(7-78n-108n^2)M_2(n)$$

so that

$$\text{spt}_3(4n+1) \equiv s_3(4n+1) \pmod{2}.$$

By [6, Theorem 4.2], the function

$$S_3(q) := \sum_{n=1}^{\infty} s_3(n)q^n \in PW_3,$$

where \mathcal{W}_n is a space of quasimodular forms of weight bounded by $2n$ defined in [6, (3.7)], and

$$(6.14) \quad P = P(q) = \frac{1}{(q)_{\infty}}.$$

We define the functions

$$P_3(q) = \sum_{n=1}^{\infty} p_3(n)q^n := P(q) \sum_{n=1}^{\infty} \sigma_3(n)q^n,$$

and

$$P_5(q) = \sum_{n=1}^{\infty} p_5(n)q^n := P(q) \sum_{n=1}^{\infty} \sigma_5(n)q^n.$$

Let $\delta_q = q \frac{d}{dq}$. By [6, (3.29) and Lemma 4.1] the functions $\delta_q(P)$, $\delta_q^2(P)$, $\delta_q^3(P)$, P_3 , $\delta_q(P_3)$, and $P_5 \in PW_3$. Since $\dim \mathcal{W}_3 = 6$ by [6, Cor.3.6], there is a linear relation between these functions and $S_3(q)$. A calculation gives that

$$s_3(n) = \frac{n}{270}(5-12n-147n^2)p(n) + \frac{1}{12}(6n+1)p_3(n) - \frac{7}{540}p_5(n)$$

and

$$4s_3(n) \equiv 6n(1+n^2)p(n) + (3+2n)p_3(n) + 7p_5(n) \pmod{8}.$$

Since $d^3 \equiv d^5 \pmod{8}$ it follows that

$$\sigma_3(n) \equiv \sigma_5(n) \pmod{8} \quad \text{and} \quad p_3(n) \equiv p_5(n) \pmod{8}.$$

Hence

$$4s_3(n) \equiv 6n(1+n^2)p(n) + (10+2n)p_3(n) \pmod{8},$$

and

$$s_3(4n+1) \equiv p(4n+1) + p_3(4n+1) \pmod{2}.$$

It is well known that

$$\delta_q(P) = \sum_{n=1}^{\infty} np(n)q^n = P(q) \sum_{n=1}^{\infty} \sigma(n)q^n.$$

Since $\sigma(n) \equiv \sigma_3(n) \pmod{2}$ it follows that

$$np(n) \equiv p_3(n) \pmod{2},$$

$$p(4n+1) \equiv p_3(4n+1) \pmod{2},$$

and

$$s_3(4n+1) \equiv 0 \pmod{2},$$

which completes the proof of (6.11). \square

Theorem 6.3.

$$(6.15) \quad \text{spt}_4(3n) \equiv 0 \pmod{3}.$$

Proof. From (2.1) and (2.2) we have

$$\begin{aligned} \sum_{n=1}^{\infty} \text{spt}_4(n)q^n &= \frac{1}{(q)_{\infty}} \left(\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2+4n}}{(1-q^n)^8} - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(3n+1)/2+4n}}{(1-q^n)^8} \right) \\ &\equiv \frac{1}{(q)_{\infty}} \left(\sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(n+1)/2+4n} (1-q^n)}{(1-q^{9n})} \right. \\ &\quad \left. - \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^{n-1} q^{n(3n+1)/2+4n} (1-q^n)}{(1-q^{9n})} \right) \pmod{3}. \end{aligned}$$

Before we can proceed we need some results for the rank and crank mod 9. We define

$$S_k(b) = S_k(b, t) := \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{(-1)^n q^{n(kn+1)/2+bn}}{(1-q^{tn})},$$

so that

$$\sum_{n=1}^{\infty} \text{spt}_4(n)q^n \equiv \frac{1}{(q)_{\infty}} (-S_1(4, 9) + S_1(5, 9) + S_3(4, 9) - S_3(5, 9)) \pmod{9}.$$

Now let $M(r, t, n)$ denote the number of partitions of n with crank congruent to $r \pmod{t}$ and let $N(r, t, n)$ denote the number of partitions of n with rank congruent to $r \pmod{t}$. Then by [7, (2.13)] and [13, (2.5)] we have

$$\sum_{n=0}^{\infty} N(r, t, n)q^n = \frac{1}{(q)_{\infty}} (S_3(r, t) + S_3(t-r, r))$$

and

$$\sum_{n=0}^{\infty} M(r, t, n)q^n = \frac{1}{(q)_{\infty}} (S_1(r, t) + S_1(t-r, r)).$$

From [13, (2.3)] and [7, (6.2)]

$$S_k(b, t) = -S_k(t-1-b),$$

for $k = 1, 3$. Hence

$$\sum_{n=0}^{\infty} M(4, 9, n)q^n = \frac{1}{(q)_{\infty}}(S_1(4, 9) + S_1(5, 9)) = \frac{1}{(q)_{\infty}}S_1(5, 9)$$

and

$$\sum_{n=0}^{\infty} N(4, 9, n)q^n = \frac{1}{(q)_{\infty}}(S_3(4, 9) + S_3(5, 9)) = \frac{1}{(q)_{\infty}}S_3(5, 9)$$

since

$$S_1(4, 9) = S_3(4, 9) = 0.$$

It follows that

$$\text{spt}_4(n) \equiv M(4, 9, n) - N(4, 9, n) \pmod{3}.$$

Lewis [19, (1a)] has shown that

$$M(4, 9, 3n) = N(4, 9, 3n)$$

and our congruence (6.15) follows. \square

If we try the approach of using quasimodular forms to prove the congruence (6.15) we are led to a congruence for the Ramanujan tau-function.

Corollary 6.4.

$$(6.16) \quad \begin{aligned} \tau(n) &\equiv (588 + 297n + 258n^2 + 9n^3 + 108n^4 + 486n^5) \sigma_1(n) \\ &+ (60 + 255n + 189n^2 + 612n^3 + 162n^4) \sigma_3(n) \\ &+ (306 + 297n + 540n^2 + 180n^3) \sigma_5(n) + (177 + 576n + 454n^2) \sigma_7(n) \\ &+ (201 + 690n) \sigma_9(n) + 117 \sigma_{11}(n) \pmod{3^6}. \end{aligned}$$

Proof. From [6, (5.6)-(5.8)] and the definition of $\text{spt}_3(n)$ we see that

$$(6.17) \quad \begin{aligned} \text{spt}_4(n) &= -\frac{67}{7362432}M_8(n) + \frac{1}{2629440}(491 + 1176n)M_6(n) \\ &- \frac{1}{1051776}(1309 + 8400n + 5856n^2)M_4(n) \\ &+ \frac{1}{3067680}(-851 + 10966n + 21204n^2 + 12162n^3)M_2(n) \\ &+ \frac{1}{140}(n + 4n^2 + 3n^3)N_2(n). \end{aligned}$$

We define

$$\begin{aligned} s_4(n) &= -\frac{67}{7362432}M_8(n) + \frac{1}{2629440}(491 + 1176n)M_6(n) \\ &- \frac{1}{1051776}(1309 + 8400n + 5856n^2)M_4(n) \\ &+ \frac{1}{3067680}(-851 + 10966n + 21204n^2 + 12162n^3)M_2(n) \end{aligned}$$

so that

$$\text{spt}_4(3n) \equiv s_4(3n) \pmod{3}.$$

By [6, Theorem 4.2], the function

$$S_4(q) := \sum_{n=1}^{\infty} s_4(n)q^n \in PW_4,$$

$$S_4^*(q) := (\delta_q^2 - 1) S_4(q) = \sum_{n=1}^{\infty} (n^2 - 1) s_4(n) q^n \in P\mathcal{W}_6,$$

and

$$(6.18) \quad S_4^*(q) \equiv 0 \pmod{3},$$

by Theorem 6.3. By [6, (3.29)] the functions $\delta_q^j(\Phi_{2k+1})$ ($0 \leq j \leq 5 - k$, $0 \leq k \leq 5$), and $\Delta \in \mathcal{W}_6$, where

$$\Phi_j = \Phi_j(q) = \sum_{n=1}^{\infty} \frac{n^j q^n}{1 - q^n} = \sum_{m, n \geq 1} n^j q^{nm} = \sum_{n=1}^{\infty} \sigma_j(n) q^n,$$

and

$$\Delta = \Delta(q) = \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1 - q^n)^{24}.$$

Since $\dim \mathcal{W}_6 = 22$ by [6, Cor.3.6], there is a linear relation between these functions and $S_4^*(q)/P$. In fact, we can write the function $S_4^*(q)/P$ as a linear combination of the 22 functions $\delta_q^j(\Phi_{2k+1})$ ($0 \leq j \leq 5 - k$, $0 \leq k \leq 5$), and $\Delta \in \mathcal{W}_6$. The coefficients in this linear combination are rational numbers, and we find that we need to multiply each coefficient by 3^5 to obtain 3-integral rationals. The congruence (6.18) then implies a congruence mod 3^6 between the arithmetic functions $n^j(\sigma_{2k+1}(n))$ ($0 \leq j \leq 5 - k$, $0 \leq k \leq 5$), and $\tau(n)$. Solving this congruence for $\tau(n)$ gives the result (6.16). \square

Ashworth [5] (see also [18]) has also obtained congruences for $\tau(n)$ mod powers of 3. Ashworth's congruences have a different form and depend on the residue of n mod 3.

7. CONCLUDING REMARKS

It should be pointed out that Bringmann, Mahlburg and Rhoades [10] have proved that there are positive constants α_k and β_k such that

$$(7.1) \quad M_{2k}(n) \sim N_{2k}(n) \sim \alpha_k n^k p(n)$$

$$(7.2) \quad M_{2k}(n) - N_{2k}(n) \sim \beta_k n^{k-\frac{1}{2}} p(n),$$

as $n \rightarrow \infty$ when k is fixed. This implies that

$$(7.3) \quad \text{spt}_k(n) \sim \frac{\beta_k}{(2k)!} n^{k-\frac{1}{2}} p(n),$$

as $n \rightarrow \infty$ when k is fixed. It would be interesting to consider whether the new identity (1.3) could lead to an elementary upper bound for $\text{spt}_k(n)$.

Folsom and Ono [14] found nontrivial congruences for Andrews' spt -function mod 2 and 3. Ono [21] also found simple explicit congruences for Andrews' spt -function modulo every prime > 3 . These congruences are related to the action of a weight $\frac{3}{2}$ Hecke operator. It would be interesting to determine whether such behavior continues for the higher degree spt -functions and higher weight Hecke operators.

The function

$$(7.4) \quad A_k(q) = \sum_{n_k \geq n_{k-1} \geq \dots \geq n_1 \geq 1} \frac{q^{n_1+n_2+\dots+n_k}}{(1-q^{n_k})^2(1-q^{n_{k-1}})^2 \dots (1-q^{n_1})^2}$$

occurs in equation (1.4) so that

$$(7.5) \quad \sum_{n=1}^{\infty} \mu_{2k}(n) q^n = \frac{1}{(q)_{\infty}} A_k(q).$$

The function $A_k(q)$ was first studied by MacMahon [20] as a generalization of

$$(7.6) \quad A_1(q) = \sum_{n=1}^{\infty} \sigma_1(n) q^n = \sum_{n=1}^{\infty} \frac{q^n}{(1-q^n)^2}.$$

He conjectured that the coefficients of $A_k(q)$ could be expressed in terms of divisor functions. This conjecture was recently proved by Andrews and Rose [4] by showing that in general $A_k(q)$ is a quasimodular form. The result also follows from (7.5), (4.4) and the fact that the generating function for $M_{2k}(n)$ is $P(q)$ times a quasimodular form, which was proved by Atkin and the author [6, Theorem 4.2]. Andrews and Rose's proof is more direct. Andrews and Rose's were motivated by a certain curve-counting problem on Abelian surfaces.

8. TABLE

For reference we include a table of $\text{spt}_k(n)$ for $1 \leq k \leq 6$, $1 \leq n \leq 29$.

$n \setminus k$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	3	1	0	0	0	0
3	5	5	1	0	0	0
4	10	15	7	1	0	0
5	14	35	28	9	1	0
6	26	75	85	45	11	1
7	35	140	217	166	66	13
8	57	259	497	505	287	91
9	80	435	1036	1341	1013	456
10	119	735	2044	3223	3081	1834
11	161	1155	3787	7149	8372	6293
12	238	1841	6797	14916	20824	19125
13	315	2765	11648	29480	48192	52781
14	440	4200	19558	55902	105117	134643
15	589	6125	31703	101892	217945	321622
16	801	8975	50645	180245	433017	726650
17	1048	12731	78674	309297	828346	1564696
18	1407	18179	120932	518859	1534271	3231635
19	1820	25235	181664	849563	2759132	6432859
20	2399	35180	270600	1366441	4837638	12395504
21	3087	48055	395682	2154789	8283014	23195905
22	3998	65681	574329	3348972	13894554	42287433
23	5092	88299	820834	5119981	22856717	75274166
24	6545	118895	1166109	7733835	36968045	131143033
25	8263	157690	1634668	11520100	58818578	223982780
26	10486	209230	2279242	16985374	92258215	375713010
27	13165	274510	3142903	24746334	142699970	619712403
28	16562	359779	4312063	35735413	218041302	1006599177
29	20630	466970	5859616	51073008	329162610	1611563058

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