

# A Bijection to Count (1-23-4)-Avoiding Permutations

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## Abstract

A permutation is (1-23-4)-avoiding if it contains no four entries, increasing left to right, with the middle two adjacent in the permutation. Here we give a 2-variable recurrence for the number of such permutations, improving on the previously known 4-variable recurrence. At the heart of the proof is a bijection from (1-23-4)-avoiding permutations to increasing ordered trees whose leaves, taken in preorder, are also increasing.

## 1 Introduction

There is a large literature on pattern avoidance in permutations; Miklós Bóna's book [2] contains a good bibliography. Babson and Steingrímsson [1] introduced the notion of “dashed” pattern: the absence of a dash indicates the corresponding letters in the permutation must be adjacent in the permutation. Thus a permutation avoids the dashed pattern 1-23-4 if it contains no four entries, increasing left to right, with the middle two adjacent. Dashed patterns are also known as vincular patterns and are subsumed by the notion of bivincular pattern [3] where permutation entries may be required to be consecutive in value and/or adjacent in position. Sergi Elizalde [5] obtained asymptotic bounds for the number  $u(n)$  of permutations of  $[n] := \{1, 2, \dots, n\}$  that avoid 1-23-4. A complicated recurrence for  $u(n)$  is posted to the [On-Line Encyclopedia of Integer Sequences](#) at [A113227](#). The purpose of this paper is to establish a much simpler recurrence:  $u(n) = \sum_{k=1}^n u(n, k)$  where  $u(n, k)$  is defined by the recurrence

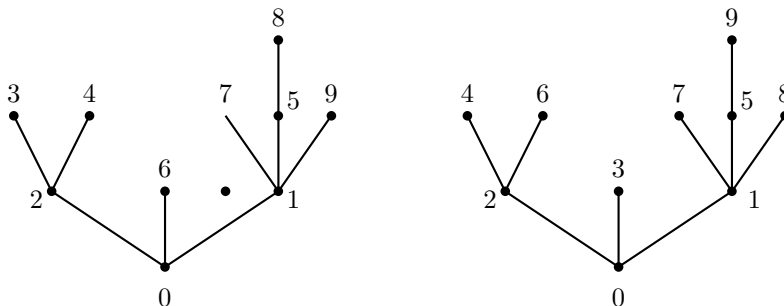
$$u(n, k) = u(n-1, k-1) + k \sum_{j=k}^{n-1} u(n-1, j) \quad 1 \leq k \leq n \quad (1)$$

with initial conditions  $u(0, 0) = 1$  and  $u(n, 0) = 0$  for  $n \geq 1$ .

In Section 2, we show that this recurrence counts  $n$ -edge increasing ordered trees with increasing leaves, by outdegree  $k$  of the root. This reduces the problem to identification of these trees and (1-23-4)-avoiding permutations, our main bijection. In Section 3, we review Stirling and Gessel permutations and give a baby version of the main bijection. In Section 4, we introduce what we call 2-configurations, a generalization of the notion of set partition in which some entries are allowed to appear twice. Sections 5 and 6 present the main bijection, between (1-23-4)-avoiding permutations of  $[n]$  and  $n$ -edge increasing ordered trees with increasing leaves, in the process representing both the permutations and the trees as 2-configurations. In Section 7, we observe that (1-23-4)-avoiding permutations are also equinumerous with certain marked-up Dyck paths. Section 8 gives a 4-variable generating function for statistics on permutations that arise naturally earlier in the paper. Finally, Section 9 briefly raises some further questions

## 2 Increasing ordered trees with increasing leaves

An *increasing ordered tree* of size  $n$  is an ordered tree (sometimes called a plane tree) with  $n + 1$  labeled vertices, the standard label set being  $0, 1, 2, \dots, n$ , such that each child exceeds its parent. Thus size measures the number of edges. Standard increasing ordered trees of size  $n$  are counted by the odd double factorial  $(2n - 1)!!$  (see, e.g., [6, 4]). An increasing ordered tree has *increasing leaves* if its leaves, taken in preorder (or “walkaround” order) are increasing. The figure below shows two increasing ordered trees, the first has increasing leaves while the second does not.



Now let  $\mathcal{I}_{n,k}$  denote the set of size- $n$  increasing ordered trees with increasing leaves in which the root has  $k$  children and let  $u(n, k) = |\mathcal{I}_{n,k}|$ . Consider a tree in  $\mathcal{I}_{n,k}$ . If vertex 1 is a leaf, then it must be the leftmost child of the root (else the leaves would not be increasing). Clearly, there are  $u(n - 1, k - 1)$  such trees. On the other hand, if vertex 1 is not a leaf, amalgamate it with the root, erasing its parent edge and re-planting

its child edges at the root. The pruned tree still has increasing leaves and its root has  $j \geq k$  children. Also, the original tree can be recovered from the pruned tree provided you know the position among the root edges of the leftmost re-planted edge (since the re-planted edges form a contiguous bunch of edges). There are  $k$  choices for this position, independent of  $j$ , and so  $k \sum_{j=k}^{n-1} u(n-1, j)$  choices altogether. Thus  $|\mathcal{I}_{n,k}|$  does indeed satisfy recurrence (1).

### 3 Stirling and Gessel permutations

A size- $n$  Stirling permutation is a permutation of the multiset  $\{1, 1, 2, 2, \dots, n, n\}$  in which, for each  $i \in [n]$ , all entries between the two occurrences of  $i$  exceed  $i$  (the Stirling property for  $i$ ) [6, 4]. Stirling permutations are counted by the odd double factorial, and there is a simple bijection from increasing ordered trees to Stirling permutations due to Svante Janson [7]: given an increasing ordered tree, delete the root label and transfer the remaining labels from vertices to parent edges. Walk clockwise around the tree thereby traversing each edge twice and record labels in the order encountered; thus each label is recorded twice.  $\square$

As noted by Ira Gessel, permutations of the multiset  $\{1, 1, 2, 2, \dots, n, n\}$  in which the second occurrences of  $1, 2, 3, \dots, n$  occur in that order, equivalently the second occurrence of each  $i \in [n]$  is a right-to-left minimum (the Gessel property), are also counted by  $(2n-1)!!$ . Indeed, a Gessel permutation of size  $n$  is obtained from one of size  $n-1$  by inserting the first  $n$  in any one of the  $2n-1$  possible slots and the second  $n$  at the end. Similarly, a Stirling permutation of size  $n$  is obtained from one of size  $n-1$  by inserting the first  $n$  in any one of the  $2n-1$  possible slots and the second  $n$  immediately after it. These observations lead immediately to a recursively-defined bijection  $\phi$  from Gessel to Stirling permutations:

**Recursive  $\phi$ .** Given a Gessel permutation  $p$  of size  $n$ , let  $j$  denote the position of the first  $n$  in  $p$ , form  $p'$  by deleting the two  $n$ 's from  $p$ ; then  $\phi(p)$  is obtained by inserting two  $n$ 's into the recursively defined  $\phi(p')$  so that they occupy positions  $j$  and  $j+1$ . The base case is  $\phi((1, 1)) = (1, 1)$ .  $\square$

The bijection  $\phi$  is worth a closer look because it involves, in a simpler setting, ideas used in our main bijection. It has two non-recursive descriptions.

**Direct algorithmic  $\phi$ .** Suppose given a size- $n$  Gessel permutation. Find the largest

$i$  whose two occurrences bracket an entry  $< i$  (if there is no such such  $i$ , stop: the permutation is already Stirling). Place a divider just after each occurrence of  $i$  and circle (or “ensquare”) each entry  $\leq i$  between the two dividers. Then cyclically shift right the contents of the squares. This step ensures the Stirling property holds for  $i$ , but at the expense of the Gessel property: the second occurrence of  $i$  now initiates an inversion in the permutation. Do likewise with the next largest  $i$  whose two occurrences bracket an entry  $< i$ , and so on, until a Stirling permutation is obtained. For example,  $p = 2\ 3\ 1\ 5\ 4\ 1\ 2\ 3\ 4\ 6\ 5\ 6$  is processed as follows:

$$\begin{array}{l}
2\ 3\ 1\ 5\ 4\ 1\ 2\ 3\ 4\ 6\ \boxed{5}\ \boxed{6} \quad \rightarrow \quad 2\ 3\ 1\ 5\ 4\ 1\ 2\ 3\ 4\ 6\ \boxed{6}\ \boxed{5} \quad = \quad 2\ 3\ 1\ 5\ \boxed{4}\ \boxed{1}\ \boxed{2}\ \boxed{3}\ \boxed{4}\ 6\ 6\ \boxed{5} \quad \rightarrow \\
2\ 3\ 1\ 5\ \boxed{5}\ \boxed{4}\ \boxed{1}\ \boxed{2}\ \boxed{3}\ 6\ 6\ \boxed{4} \quad = \quad 2\ 3\ 1\ 5\ 5\ 4\ \boxed{1}\ \boxed{2}\ \boxed{3}\ 6\ 6\ \boxed{4} \quad \rightarrow \quad 2\ 3\ 1\ 5\ 5\ 4\ \boxed{4}\ \boxed{1}\ \boxed{2}\ 6\ 6\ \boxed{3} \quad = \\
2\ 3\ \boxed{1}\ 5\ 5\ 4\ 4\ \boxed{1}\ \boxed{2}\ 6\ 6\ \boxed{3} \quad \rightarrow \quad 2\ 3\ \boxed{3}\ 5\ 5\ 4\ 4\ \boxed{1}\ \boxed{1}\ 6\ 6\ \boxed{2} \quad = \quad 2\ \boxed{3}\ 3\ 5\ 5\ 4\ 4\ \boxed{1}\ \boxed{1}\ 6\ 6\ \boxed{2} \quad \rightarrow \\
2\ 3\ 3\ 5\ 5\ 4\ 4\ \boxed{2}\ \boxed{1}\ 6\ 6\ \boxed{1} \quad = \quad 2\ 3\ 3\ 5\ 5\ 4\ 4\ 2\ 1\ 6\ 6\ 1
\end{array}$$

To reverse the map, start with the smallest  $i$  whose second occurrence initiates an inversion. Take this second occurrence and all its inversion terminators, and rotate them left, and so on.  $\square$

**Constructive  $\phi$  via trapezoidal words.** Following Riordan, a *trapezoidal word*  $(w_i)_{i=1}^n$  is a member of the Cartesian product  $[1] \times [3] \times \dots \times [2n - 1]$ . Now  $\phi$  is the composition of the following bijections.

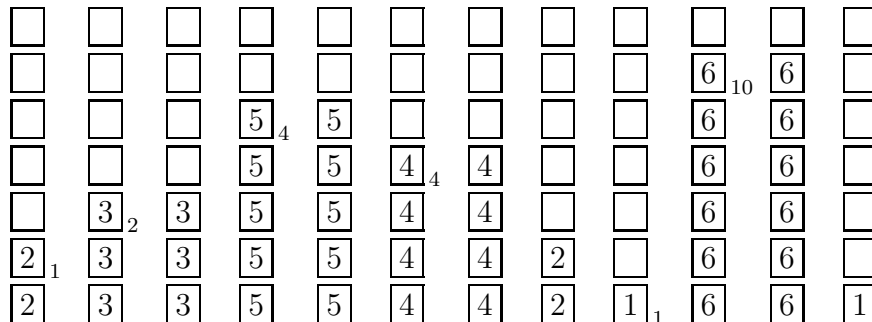
**Gessel permutation  $\rightarrow$  trapezoidal word:** Given a Gessel permutation  $\sigma$ , for  $1 \leq i \leq n$  set  $w_i =$  number of entries weakly preceding the first  $i$  in  $\sigma$  and  $\leq i$ .

**Trapezoidal word  $\rightarrow$  Gessel permutation:** Given a trapezoidal word  $(w_i)_{i=1}^n$ , start with a row of  $2n$  empty squares to be filled with the entries of the permutation. Place the first  $n$  in the  $w_n$ -th square and the second  $n$  in the last square. Then for  $i = n - 1, n - 2, \dots, 1$  in turn, place the first  $i$  in the  $w_i$ -th unoccupied square and the second  $i$  in the last unoccupied square.

**Stirling permutation  $\rightarrow$  trapezoidal word:** Given a Stirling permutation  $\sigma$ , for  $1 \leq i \leq n$  set  $w_i =$  number of entries weakly preceding the first  $i$  in  $\sigma$  and  $\leq i$ .

**Trapezoidal word  $\rightarrow$  Stirling permutation:** Given a trapezoidal word  $(w_i)_{i=1}^n$ , start with a row of  $2n$  empty squares to be filled with the entries of the permutation. Place the first  $n$  in the  $w_n$ -th square and the second  $n$  in the next square. Then for  $i = n - 1, n - 2, \dots, 1$  in turn, place the first  $i$  in the  $w_i$ -th unoccupied square and the second  $i$  in the next unoccupied square.

For example, the Gessel permutation  $p$  above gives the trapezoidal word 1 1 2 4 4 10. Then  $\phi(p)$  is constructed as follows (subscripts indicate placement positions among unoccupied squares).

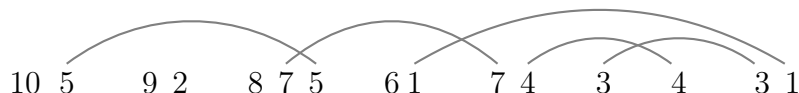


## 4 2-Configurations

Fix a positive integer  $n$  and take a multiset  $M$  consisting of the union of  $[n]$  and an arbitrary subset of  $n$ . An element of  $[n]$  that occurs twice in  $M$  is called a *repeater*. Consider a partition of  $M$  into a list of decreasing lists (blocks) such that

1. The entries in each block are distinct.
2. Each block contains at most one first occurrence of a repeater.
3. If a block contains a first occurrence of a repeater, then all later entries in the block are second occurrences of other repeaters.

Call such a partition a *2-configuration* of size  $n$ . Draw an arc connecting the two occurrences of each repeater. We can then view the 2-configuration (i) as an arc diagram, and speak of noncrossing arcs, (ii) as a graph with the blocks as vertices, and speak of its (connected) components. By directing each arc from left to right we may also speak of incoming and outgoing arcs. In these terms, conditions 2 and 3 say that each block has at most one outgoing arc and all entries in a block after an outgoing arc have incoming arcs.



The example illustrated for  $n = 10$  has repeaters 1, 3, 4, 5, 7; it has 8 blocks and 5 arcs. It has 3 components, namely,  $C_1 = (6\ 1, 3, 3\ 1)$ ,  $C_2 = (9\ 2)$ ,  $C_3 = (10\ 5, 8\ 7\ 5, 7\ 4, 4)$ .

As here, the blocks in a component will always be listed in left-to-right order and the components themselves will always be ordered by increasing minimum entry. The first of these components has noncrossing arcs, the second is a singleton with one block and no arcs, and the third has some crossing arcs. For each nonsingleton component, its *critical* block is the second block if the minimum entry in the component occurs in the first block, otherwise it is the first block. (Note this definition would not make sense for a singleton component.) Thus, the critical block for  $C_1$  is the second one, (3), and for  $C_3$  is the first one, (10 5).

A *bad* repeater is a repeater whose two occurrences enclose a smaller entry in the 2-configuration. Any such smaller entry is a *delinquent* for the bad repeater. In other words, a delinquent is an entry  $i$  below an arc whose (equal) endpoints are  $> i$ . In the example, the bad repeaters are 5, 7, 4 with delinquent entries  $\{2\}$ ,  $\{5, 6, 1\}$ ,  $\{3\}$  respectively.

Now we list some properties a 2-configuration may (or may not) have.

- The *good-component* property: For each nonsingleton component  $C_i$ , all blocks in the components  $C_1, C_2, \dots, C_{i-1}$  (including, of course, singleton components) precede the critical block of  $C_i$  in the 2-configuration. In the example,  $C_3$  badly fails this condition.
- The *single-incoming-arc* property: No block has more than one incoming arc. In the example, the last block fails this condition because it has 2 incoming arcs.
- The *no-crossing-in-component* property: The arcs within each component, considered as connecting entries rather than blocks, are noncrossing. In the example,  $C_3$  fails this condition because the arcs incident with the block 8 7 5 do cross, while  $C_1$  and  $C_2$  satisfy it.
- The *Gessel* property: The second occurrence of each bad repeater is the last entry in its block, and repeaters that end their second block are right-to-left minima in the flattened configuration (to flatten means to concatenate all the blocks). In the example, the bad repeater 7 fails the first part of this condition, and the repeaters 4, 5 fail the second part.
- The *Stirling* property: No bad repeaters.
- The *restricted-first-entry* property: The first entry of a block is never a repeater and these first entries increase left to right.

## 5 First steps in the main bijection

The identification of (1-23-4)-avoiding permutations and increasing ordered trees with increasing leaves can be described as a composition of four bijections:

$$\mathcal{I}_n \longleftrightarrow \mathcal{J}_n \longleftrightarrow \mathcal{S}_n \longleftrightarrow \mathcal{A}_n \longleftrightarrow \mathcal{P}_n,$$

where  $\mathcal{I}_n$  is the set of all size- $n$  increasing ordered trees with increasing leaves,  $\mathcal{J}_n$  is a certain subset of the size- $n$  Stirling permutations,  $\mathcal{S}_n$  and  $\mathcal{A}_n$  are subsets of the 2-configurations of size  $n$ , and  $\mathcal{P}_n$  is the set of (1-23-4)-avoiding permutations of  $[n]$ .

### 5.1 $\mathcal{I}_n \rightarrow \mathcal{J}_n$

A *plateau* in a Stirling permutation is a pair of adjacent entries that are equal. Thus Janson's bijection of Section 3 identifies leaves in the tree with plateaus in the permutation, and sends  $\mathcal{I}_n$ , the size- $n$  increasing ordered trees with increasing leaves, onto  $\mathcal{J}_n$ , defined as the set of size- $n$  Stirling permutations whose plateaus increase left to right. This is the first bijection.

### 5.2 $\mathcal{J}_n \rightarrow \mathcal{S}_n$

Given a permutation in  $\mathcal{J}_n$ , underline the longest decreasing run starting at the second entry of each plateau and extract these runs to form a set of decreasing lists (blocks) with increasing first entries:

$$1\ 3\ 5\ \underline{5}\ \underline{3}\ \underline{1}\ 2\ 6\ 7\ \underline{7}\ \underline{6}\ \underline{4}\ 8\ 9\ \underline{9}\ \underline{8}\ \underline{4}\ \underline{2}\ 10\ \underline{10} \rightarrow 5\ 3\ 1\quad 7\ 6\ 4\quad 9\ 8\ 4\ 2\quad 10$$

We call the result  $\rho$  a *Stirling configuration*. The original permutation can be recovered from  $\rho$ . First, flatten  $\rho$ . Then, for each  $i$  that occurs only once, insert a second  $i$  as far to the left of the existing  $i$  as possible so that the two  $i$ 's don't enclose a smaller entry. (The order in which the missing  $i$ 's are inserted is immaterial.)

The first occurrence of a repeater in a Stirling configuration (4 is the only repeater in the example) is always the last entry in its block for otherwise a bad repeater would be present. The set of Stirling configurations of size  $n$ , denoted  $\mathcal{S}_n$ , can be characterized as the set of 2-configurations of size  $n$  that have the restricted-first-entry and Stirling properties of the previous section.

Extraction of underlined runs is thus a bijection from  $\mathcal{J}_n$ , the increasing-plateau Stirling permutations, to  $\mathcal{S}_n$ . This is the second bijection.

### 5.3 $\mathcal{P}_n \rightarrow \mathcal{A}_n$

Now we turn to the pattern-avoiding permutations. An *ascent* in a permutation is a pair of adjacent entries  $a, b$  with  $a < b$ ;  $a$  is the *initiator*,  $b$  the *terminator*. Clearly, a permutation is (1-23-4)-avoiding if and only if, for each ascent, either its initiator is a left-to-right minimum or its terminator is a right-to-left maximum. But an asymmetrical viewpoint is more convenient. Any permutation can be split into *LRMin segments* starting at its left-to-right minima. The first two entries of a nonsingleton segment necessarily form an ascent, an *LRMin ascent* (for otherwise the second entry would start a new segment); other ascents are *free*. Then a permutation is

- (1-23)-avoiding iff it has no free ascents
- (1-23-4)-avoiding iff each free ascent terminates at a right-to-left maximum.

Suppose given a (1-23-4)-avoiding permutation. We will perform a series of six reversible steps, using

$$23 \ 4 \ 21 \ 6 \ 25 \ 24 \ 14 \ 22 \ 18 \ 20 \ 16 \ 13 \ 11 \ 19 \ 7 \ 5 \ 2 \ 8 \ 17 \ 12 \ 10 \ 1 \ 15 \ 3 \ 9$$

as a working example. The resulting configurations will form  $\mathcal{A}_n$ .

Step 1. Split the permutation into its LRMin segments. Working left to right, for each ascent initiator  $i$ , place an overline starting at  $i$  and extending as far right as possible so that it does not cover an entry  $< i$ . Each new overline is placed below any existing overlines. Also, if a segment has just one entry and thus no ascents, overline that entry. Now every entry is covered by a unique overline (“cover” means no intervening overlines). Clearly, no overline will straddle two segments and each pair of overlines is either disjoint or nested (no overlaps). Also, an overline starting at a free ascent initiator will cover both entries of the ascent.

$$\overline{23} \ / \ \overline{\overline{4 \ 21 \ 6 \ 25 \ 24 \ 14 \ 22 \ 18 \ 20 \ 16 \ 13 \ 11 \ 19 \ 7 \ 5}} \ / \ \overline{\overline{2 \ 8 \ 17 \ 12 \ 10}} \ / \ \overline{\overline{1 \ 15 \ 3 \ 9}}$$

Step 2. For each LRMin segment, extract the entries covered by an overline to obtain a list of blocks. (With a slight abuse of words, we will refer to this list as a segment of blocks.)

$$23 \ / \ 4 \ 21 \ 5 \ 6 \ 25 \ 24 \ 13 \ 7 \ 14 \ 22 \ 16 \ 18 \ 20 \ 11 \ 19 \ / \ 2 \ 8 \ 17 \ 12 \ 10 \ / \ 1 \ 15 \ 3 \ 9$$



To recapture the original LRMin segments, “coalesce” the blocks in each segment, inserting each one (starting at the second block) into its current predecessor just far enough from the end so that its first entry does not terminate a free ascent.

Step 3. Only the first block in a segment can be a singleton block. Furthermore, if the first block is a singleton and the segment has more than one block, then the second entry of the second block, being a free ascent terminator, is a right-to-left maximum and, in particular, the largest entry in the segment. In this case, interchange the first and third (smallest and largest) entries in the segment. In the example step 3 alters the third segment.

23 / 4 21 5 6 25 24 13 7 14 22 16 18 20 11 19 / 17 8 2 12 10 / 1 15 3 9

Step 4. Sort each block into decreasing order.

23 / 21 5 4 25 24 13 7 6 22 16 14 20 18 19 11 / 12 10 8 2 17 , / 15 1 9 3

This step amounts to transferring the first entry of each block to the end of the block except, in case the first block in a segment is a singleton, transferring the first two entries of the second block (if there is one) to the end of the block. Thus the step can be reversed.

Step 5. Sort blocks in each segment in order of increasing first entries.

23 / 19 11 20 18 21 5 4 22 16 14 25 24 13 7 6 / 12 10 8 2 17 / 9 3 15 1

Originally, the block containing the smallest entry of a segment was the first block in the segment and the remaining blocks were in order of decreasing first entries, both statements holding unless the telltale singleton block is present in which case *all* blocks were in order of decreasing first entries. So this step is reversible.

Step 6. Establish links between the blocks in a segment by inserting a second copy of the last entry of each block (save for the last one) into the next block, maintaining monotonicity.

23 / 19 11 20 18 11 21 18 5 4 22 16 14 4 25 24 14 13 7 6 /  
12 10 8 2 17 2 / 9 3 15 3 1

This step is obviously reversible.

Step 7. Erase dividers and sort the entire collection of blocks in order of increasing first entry.

9 3 12 10 8 2 15 3 1 17 2 19 11 20 18 11 21 18 5 4 22 16 14 4 23 25 24 14 13 7 6

To reverse Step 7, follow the trail of the repeaters to determine the blocks in each segment. The order of blocks within a segment is determined by their first entries, which must be increasing, and the order of the segments themselves is determined by their smallest entries, which must be decreasing.

We call the result of Step 7 an *avoider configuration*. Avoider configurations form the set  $\mathcal{A}_n$  and they are characterized as 2-configurations that have the restricted-first-entry and single-arc properties (both by construction) and also the good-component property (reflecting the fact that each free ascent in the permutation terminates at a right-to-left maximum).

This is the fourth of the four bijections. In the next section we present a bijection from the 2-configurations of size  $n$  with the single-arc and good-component properties to the 2-configurations of size  $n$  with the Stirling property. The restriction of this bijection to 2-configurations with the restricted-first-entry property then completes the proof, by providing the third of the desired bijections, from  $\mathcal{A}_n$  to  $\mathcal{S}_n$ .

## 6 Bijections between sets of 2-configurations

The following four subsets of the 2-configurations of size  $n$  turn out to be equinumerous.

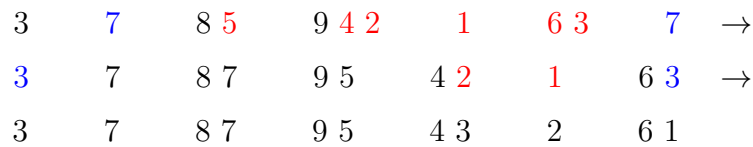
1.  $\mathcal{W}_n$  : The 2-configurations that have both the single-incoming-arc and the good-component property.
2.  $\mathcal{X}_n$  : The 2-configurations that have both the good-component and the no-crossing-in-component property.
3.  $\mathcal{Y}_n$  : The 2-configurations that have both the no-crossing-in-component and the Gessel property.
4.  $\mathcal{Z}_n$  : The 2-configurations that have the Stirling property.

We will give bijections  $\mathcal{W}_n \rightarrow \mathcal{X}_n \rightarrow \mathcal{Y}_n \rightarrow \mathcal{Z}_n$ . A 2-configuration in  $\mathcal{Z}_n$  has no bad repeaters, so we must turn bad repeaters in  $\sigma \in \mathcal{W}_n$  into good ones. Say a bad repeater is of Type 1 if its first occurrence is not the last entry in its block, of Type 2 if its second occurrence is not the last entry in its block, and of Type 3 if it is neither of Type 1 or 2. (It may be both Type 1 and 2). The three bijections eliminate the bad repeaters of Type 1, 2, and 3 in turn. All three bijections preserve the statistic “number of components”.

The first also preserves the support multiset of each component while the other two do not.

**Iterative descriptions:**

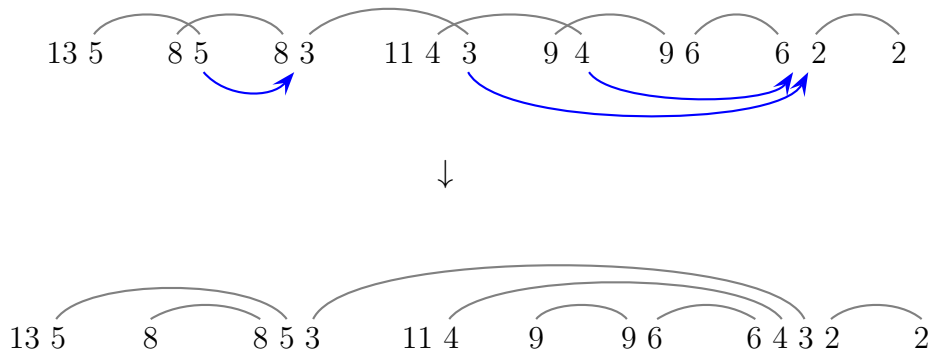
1.  $\mathcal{W}_n \rightarrow \mathcal{X}_n$  As long as there is a bad repeater of Type 1 in the current 2-configuration, find the largest one and transfer the last entry of its first block to its second block.
2.  $\mathcal{X}_n \rightarrow \mathcal{Y}_n$  As long as there is a bad repeater of Type 2 in the current 2-configuration, find the largest one and transfer its second occurrence to the block of its leftmost delinquent.
3.  $\mathcal{Y}_n \rightarrow \mathcal{Z}_n$  As long as there is a bad repeater of Type 3 in the current 2-configuration, find the largest one. Let  $A_1, A_2, \dots, A_k$  ( $k \geq 1$ ) denote the blocks containing its delinquents and let  $A_{k+1}$  denote the second block of the bad repeater. Transfer the bad repeater from  $A_{k+1}$  to  $A_1$  and transfer all delinquents in  $A_i$  to  $A_{i+1}$ ,  $1 \leq i \leq k$ . The illustration shows bad repeaters in blue, delinquents in red.



The bijection  $\mathcal{Y}_n \rightarrow \mathcal{Z}_n$

**More direct descriptions:**

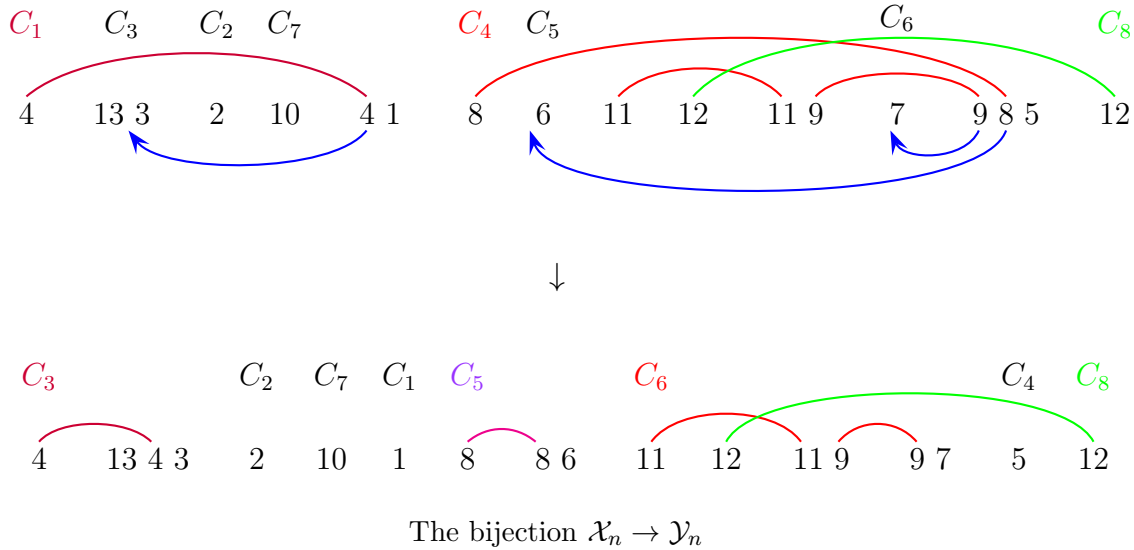
1.  $\mathcal{W}_n \rightarrow \mathcal{X}_n$  Given  $\sigma \in \mathcal{W}_n$ , transform each component as illustrated.



A component and its image under the bijection  $\mathcal{W}_n \rightarrow \mathcal{X}_n$

For each bad repeater  $i$  of Type 1, transfer its second occurrence rightward to the first later block in its component whose outgoing arc starts at an entry  $< i$ , or to the last block in the component if there is no such block. The single-incoming-arc property has been traded for the no-crossing-in-component property, and the resulting configuration has no bad repeaters of Type 1.

2.  $\mathcal{X}_n \rightarrow \mathcal{Y}_n$  Transfer the second occurrence of each bad repeater  $i$  of Type 2 to the block of its leftmost delinquent, as illustrated by the blue arrows. For the reader's convenience, the components are indicated:  $C_i$  is placed directly above the  $i$ -th component's unique block if the  $i$ -th component is a singleton, and above its critical block otherwise. (The color coding is merely for the reader's convenience.)



The good-component property has been traded for the Gessel property.

3.  $\mathcal{Y}_n \rightarrow \mathcal{Z}_n$  For each  $i \in [n]$ , record the number  $w_i$  of blocks that weakly precede the first appearance of  $i$  and contain an entry  $\leq i$ . Define an *ender* to be an element of  $[n]$  that appears as the last entry in a block. Start with the same number of blocks, initially empty, as in the given 2-configuration. For each ender  $i$  from largest to smallest in turn, place a copy of  $i$  in the  $w_i$ -th empty block and, if  $i$  is a repeater, place a second copy in the next empty block. When done, each block contains one entry. Next, for each non-ender  $i$ , place  $i$  in the  $w_i$ -th *available* block where the available blocks for  $i$  are those that don't contain an ender  $> i$ . (The order in which

the non-enders are placed is immaterial.) The resulting configuration is in  $\mathcal{Z}_n$ . For the example above, the enders are 1, 2, 3, 5, 7 of which 3, 7 are repeaters, and the  $w_i$  are as follows.

	enders					non-enders			
$i$	1	2	3	5	7	4	6	8	9
$w_i$	1	1	1	2	2	2	5	3	4

The enders are placed successively as illustrated in the following table.

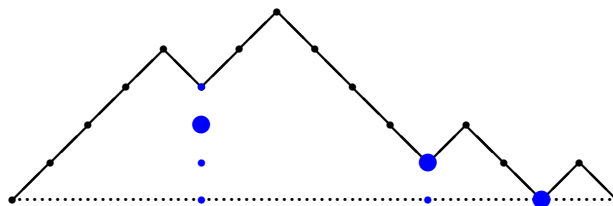
	7	7				
	7	7	5			
3	7	7	5	3		
3	7	7	5	3	2	
3	7	7	5	3	2	1

As for the non-enders, the available blocks for  $i = 4$  are the first, fourth, etc., and since  $w_i = 2$ , 4 goes in the fourth block. For  $i = 6$  only the second and third blocks are unavailable and  $w_i = 5$  puts 6 in the last block. All blocks are available for 8 and 9 and the final result is the same as in the first description.  $\square$

It is fairly easy to see that the steps are reversible for all three mappings and that they do map onto the claimed sets.

## 7 Another manifestation of the counting sequence

A *valley-marked* Dyck path is a Dyck path in which, for each valley ( $DU$ ), one of the lattice points between the valley vertex and the  $x$ -axis inclusive (in blue in the figure below) is marked.



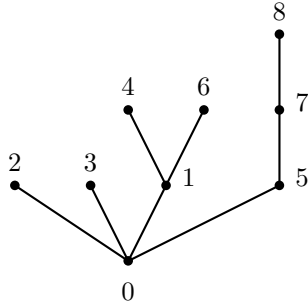
A valley-marked Dyck path

We have seen in Section 2 that recurrence (1) counts  $\mathcal{I}_{n,k}$ . It also counts  $\mathcal{D}_{n,k}$ , the valley-marked Dyck paths of semilength  $n$  with first ascent of length  $k$ . To see this,

observe that deleting the first peak produces a smaller-size valley-marked Dyck path. So every path in  $\mathcal{D}_{n,k}$  is obtained by inserting a new first peak at height  $k$  either into a path in  $\mathcal{D}_{n-1,k-1}$  (no new valley is created) or into a path in  $\mathcal{D}_{n-1,j}$  with  $j \geq k$  (a new valley is created with  $k$  choices for its marked lattice point), and the recurrence follows. From their common recurrence it is easy to construct a recursively-defined bijection between  $\mathcal{D}_{n,k}$  and  $\mathcal{I}_{n,k}$ . But we can also give a direct bijection by exhibiting identical codings for  $\mathcal{D}_{n,k}$  and  $\mathcal{I}_{n,k}$ . Given a marked path in  $\mathcal{D}_{n,k}$ , let  $a_i = \#U$ s between the  $i$ -th  $D$  and the  $(i+1)$ -st  $D$ ,  $0 \leq i \leq n-1$ . Let  $h_i$  denote the height above the line  $y = -1$  of the mark for the valley corresponding to each  $i \in [1, n-1]$  with  $a_i \geq 1$ . The  $a_i$ 's code the path and the  $h_i$ 's code the marks. The example above yields

$$\begin{array}{rcccccccc} i = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ a_i = & 4 & 2 & 0 & 0 & 0 & 1 & 0 & 1 \\ h_i = & & 3 & & & & 2 & & 1. \end{array}$$

For an ordered tree, to prune a non-root vertex means to delete the vertex and its parent edge and attach all its children to its parent, maintaining order. Now given a tree in  $\mathcal{I}_{n,k}$ , let  $a_i$  denote the outdegree ( $= \#$  children) of vertex  $i$ ,  $0 \leq i \leq n-1$  (vertex  $n$  is necessarily a leaf), and for each non-root non-leaf vertex  $i$ , let  $h_i$  denote the position of  $i$  among its siblings in the tree obtained by pruning vertices  $1, 2, \dots, i-1$ . For example, the tree below yields the same code as the path above.



We leave the reader to verify that both paths and trees can be uniquely retrieved from their codes, and that the codings are identical: pairs of sequences of nonnegative integers  $((a_i)_{i=0}^{n-1}, (h_i)_{i \geq 1, a_i \geq 1})$  such that  $a_0 \geq 1$ ,  $a_0 + a_1 \geq 2, \dots, a_0 + a_1 + \dots + a_{n-1} \geq n$  with equality in the last inequality, and  $1 \leq h_i \leq \sum_{j=0}^{i-1} a_j - (i-1)$  for each  $i \geq 1$  such that  $a_i \geq 1$ .

## 8 A generating function

In Section 5.3 we considered the LRMin segments of a permutation and distinguished between LRMin and free ascents. Call a (LRMin) segment *short* if it has length 1, otherwise *long*. There is an elegant generating function to count all of these notions for general permutations. Let  $u(n, i, j, k)$  denote the number of permutations of  $[n]$  with  $i$  short segments,  $j$  long segments (and hence  $j$  LRMin ascents), and  $k$  free ascents. It is straightforward to obtain the recurrence

$$u(n, i, j, k) = u(n-1, i-1, j, k) + (j+k)u(n-1, i, j, k) + (i+1)u(n-1, i+1, j-1, k) + (n-i-j-k)u(n-1, i, j, k-1),$$

valid for  $n \geq 2$ ,  $i \geq 1$ ,  $j \geq 0$ ,  $k \geq 0$ ,  $i+j \geq 1$ ,  $i+2j \leq n$ ,  $k \leq n-i-j$ , with initial condition  $u(1, 1, 0, 0) = 1$ .

Set  $F(x, y, z, w) := 1 + \sum_{n \geq 1, i \geq 1, j \geq 0, k \geq 0} u(n, i, j, k) x^n / n! y^i z^j w^k$ . The recurrence for  $u$  translates into a first-order linear partial differential equation for  $F$ :

$$F_x = yF + zF_y + zF_z + wF_w + xwF_x - wyF_y - wzF_z - w^2F_w.$$

The solution of this PDE is a nice application of the standard method of characteristic curves, which yields

$$F(x, y, z, w) = e^{x(y-z)} \left( \frac{1-w}{1-we^{x(1-w)}} \right)^{\frac{z}{w}}.$$

Putting  $y = z = 1$  in  $F$  yields the generating function for free ascents:

$$\left( \frac{1-w}{1-we^{(1-w)x}} \right)^{\frac{1}{w}}.$$

## 9 Further questions

The fast recurrence for (1-23-4)-avoiding permutations established in this paper raises the question whether there might be fast but nonobvious recurrences for other similar patterns. For example, there is a slow recurrence for (12-34)-avoiding permutations posted to OEIS at [A113226](#). Can you do better?

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