TABLE OF DIRICHLET L-SERIES AND PRIME ZETA MODULO FUNCTIONS FOR SMALL MODULI

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Abstract. The Dirichlet characters of reduced residue systems modulo m are tabulated for moduli $m \le 22$. The associated L-series are tabulated for $m \le 14$ and small positive integer argument s accurate to 10^{-50}, their first derivatives for $m \le 6$. Restricted summation over primes only defines Dirichlet Prime L-functions which lead to Euler products (Prime Zeta Modulo functions). Both are materialized over similar ranges of moduli and arguments. Formulas and numerical techniques are well known; the aim is to provide direct access to reference values.

1. Introduction

1.1. **Scope.** Dirichlet L-series are a standard modification of the Riemann series of the ζ-function with the intent to distribute the sum over classes of a modulo system. Section 1 enumerates the group representations; a first use of this classification is a table of L-series and some of their first derivatives at small integer arguments in Section 2.

In the same spirit as L-series generalize the ζ-series, multiplying the terms of the Prime Zeta Function by the characters imprints a periodic texture on these, which can be synthesized (in the Fourier sense) to distil the Prime Zeta Modulo Functions. Section 3 recalls numerical techniques and tabulates these for small moduli and small integer arguments.

If some of the Euler products that arise in growth rate estimators of number densities are to be factorized over the residue classes of the primes, taking logarithms proposes to use the Prime Zeta Modulo Function as a basis. The fundamental numerical examples are worked out in Section 4.

1.2. **Dirichlet Characters.** The Dirichlet characters $\chi_r(n)$ (mod m) are shown in Table 1–21 for small modulus m. The purpose of rolling out such basic information is that it tags each representation with a unique r for use in all tables further down.

Each table shows one character per line, lines enumerated by representation r, $1 \le r \le \varphi(m)$, where φ is Euler's totient function. The principal character χ_1 is the top line. The residues from 1 to m are indicated in the header row. The entries are either zero or roots of unity. As a shortcut to the notation,

$$u_j \equiv e^{2\pi i j/\varphi(m)}; \quad \bar{u}_j \equiv e^{-2\pi i j/\varphi(m)}, \tag{1}$$

denote a root of unity and its complex conjugation. The conductor f (smallest induced modulus) is another column; it also indicates whether the character is

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TABLE 1. $\chi_r(n)$, $m = 2$, $\varphi(m) = 1$.

r	1	2	f	
1	1	0	1	A000035

TABLE 2. $\chi_r(n)$, $m = 3$, $\varphi(m) = 2$.

r	1	2	3	f	
1	1	1	0	1	A011655
2	1	-1	0	3	A102283

TABLE 3. $\chi_r(n)$, $m = 4$, $\varphi(m) = 2$.

r	1	2	3	4	f	
1	1	0	1	0	1	A000035
2	1	0	-1	0	4	A101455

TABLE 4. $\chi_r(n)$, $m = 5$, $\varphi(m) = 4$.

r	1	2	3	4	5	f	
1	1	1	1	1	0	1	A011558
2	1	i	$-i$	-1	0	5	
3	1	-1	-1	1	0	5	A080891
4	1	$-i$	i	-1	0	5	

(im)primitive. Six-digit sequence numbers to matching periodic sequences in the Online Encyclopedia of Integer Sequences (OEIS) are included [27].

The cases $m \leq 7$ have been tabulated by Apostol [4], some of the real characters by Davies and Haselgrove [8], all up to $m = 10$ by Zucker and McPhedran [32].

The calculations are based on standard algorithms: m is decomposed into its prime number factorization, the character table of each factor is constructed and these representations are multiplied following the group property (multiplication rule) for each pair of representations, looping over all products of representations [4, 28, 33].

TABLE 5. $\chi_r(n)$, $m=6$, $\varphi(m)=2$.

r	1	2	3	4	5	6	f	
1	1	0	0	0	1	0	1	A120325
2	1	0	0	0	-1	0	3	A134667

TABLE 6. $\chi_r(n)$, $m=7$, $\varphi(m)=6$.

r	1	2	3	4	5	6	7	f	
1	1	1	1	1	1	1	0	1	A109720
2	1	u_2	u_1	$\bar{u}_2$	$\bar{u}_1$	-1	0	7	
3	1	$\bar{u}_2$	u_2	u_2	$\bar{u}_2$	1	0	7	
4	1	1	-1	1	-1	-1	0	7	A175629
5	1	u_2	$\bar{u}_2$	$\bar{u}_2$	u_2	1	0	7	
6	1	$\bar{u}_2$	$\bar{u}_1$	u_2	u_1	-1	0	7	

TABLE 7. $\chi_r(n)$, $m=8$, $\varphi(m)=4$.

r	1	2	3	4	5	6	7	8	f	
1	1	0	1	0	1	0	1	0	1	A000035
2	1	0	-1	0	-1	0	1	0	8	A091337
3	1	0	-1	0	1	0	-1	0	4	A101455
4	1	0	1	0	-1	0	-1	0	8	

TABLE 8. $\chi_r(n)$, $m=9$, $\varphi(m)=6$.

r	1	2	3	4	5	6	7	8	9	f	
1	1	1	0	1	1	0	1	1	0	1	A011655
2	1	u_1	0	u_2	$\bar{u}_1$	0	$\bar{u}_2$	-1	0	9	
3	1	u_2	0	$\bar{u}_2$	$\bar{u}_2$	0	u_2	1	0	9	
4	1	-1	0	1	-1	0	1	-1	0	3	A102283
5	1	$\bar{u}_2$	0	u_2	u_2	0	$\bar{u}_2$	1	0	9	
6	1	$\bar{u}_1$	0	$\bar{u}_2$	u_1	0	u_2	-1	0	9	

Remark 1. *Because the $\chi_r(n)$ are completely multiplicative, the sequences of Dirichlet inverses are*

$$\chi_r^{(-1)}(n) = \chi_r(n)\mu(n), \tag{2}$$

where μ is the Möbius function.

2. Dirichlet L-functions

2.1. **Symmetries.**

The L-series are defined via the character table [26, 3, 6].

Definition 1. *(Dirichlet L-functions)*

$$L(s,\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = \prod_p \frac{1}{1-\chi(p)p^{-s}} = \frac{1}{m^s}\sum_{n=1}^{m} \chi(n)\zeta(s,n/m), \tag{3}$$

where $\zeta(.,.)$ is the Hurwitz zeta-function [30, 2] *and the product extends over all primes p.*

TABLE 9. $\chi_r(n)$, $m = 10$, $\varphi(m) = 4$.

r	1	2	3	4	5	6	7	8	9	10	f	
1	1	0	1	0	0	0	1	0	1	0	1	
2	1	0	$-i$	0	0	0	i	0	-1	0	5	
3	1	0	-1	0	0	0	-1	0	1	0	5	
4	1	0	i	0	0	0	$-i$	0	-1	0	5	

TABLE 10. $\chi_r(n)$, $m = 11$, $\varphi(m) = 10$.

r	1	2	3	4	5	6	7	8	9	10	11	f	
1	1	1	1	1	1	1	1	1	1	1	0	1	A145568
2	1	u_1	$\bar{u}_2$	u_2	u_4	$\bar{u}_1$	$\bar{u}_3$	u_3	$\bar{u}_4$	-1	0	11	
3	1	u_2	$\bar{u}_4$	u_4	$\bar{u}_2$	$\bar{u}_2$	u_4	$\bar{u}_4$	u_2	1	0	11	
4	1	u_3	u_4	$\bar{u}_4$	u_2	$\bar{u}_3$	u_1	$\bar{u}_1$	$\bar{u}_2$	-1	0	11	
5	1	u_4	u_2	$\bar{u}_2$	$\bar{u}_4$	$\bar{u}_4$	$\bar{u}_2$	u_2	u_4	1	0	11	
6	1	-1	1	1	1	-1	-1	-1	1	-1	0	11	A011582
7	1	$\bar{u}_4$	$\bar{u}_2$	u_2	u_4	u_4	u_2	$\bar{u}_2$	$\bar{u}_4$	1	0	11	
8	1	$\bar{u}_3$	$\bar{u}_4$	u_4	$\bar{u}_2$	u_3	$\bar{u}_1$	u_1	u_2	-1	0	11	
9	1	$\bar{u}_2$	u_4	$\bar{u}_4$	u_2	u_2	$\bar{u}_4$	u_4	$\bar{u}_2$	1	0	11	
10	1	$\bar{u}_1$	u_2	$\bar{u}_2$	$\bar{u}_4$	u_1	u_3	$\bar{u}_3$	u_4	-1	0	11	

TABLE 11. $\chi_r(n)$, $m = 12$, $\varphi(m) = 4$.

r	1	2	3	4	5	6	7	8	9	10	11	12	f	
1	1	0	0	0	1	0	1	0	0	0	1	0	1	A120325
2	1	0	0	0	-1	0	1	0	0	0	-1	0	3	A134667
3	1	0	0	0	1	0	-1	0	0	0	-1	0	4	
4	1	0	0	0	-1	0	-1	0	0	0	1	0	12	A110161

TABLE 12. $\chi_r(n)$, $m = 13$, $\varphi(m) = 12$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	f	
1	1	1	1	1	1	1	1	1	1	1	1	1	0	1	
2	1	u_1	u_4	u_2	$-i$	u_5	$\bar{u}_1$	i	$\bar{u}_4$	$\bar{u}_2$	$\bar{u}_5$	-1	0	13	
3	1	u_2	$\bar{u}_4$	u_4	-1	$\bar{u}_2$	$\bar{u}_2$	-1	u_4	$\bar{u}_4$	u_2	1	0	13	
4	1	i	1	-1	i	i	$-i$	$-i$	1	-1	$-i$	-1	0	13	
5	1	u_4	u_4	$\bar{u}_4$	1	$\bar{u}_4$	$\bar{u}_4$	1	$\bar{u}_4$	u_4	u_4	1	0	13	
6	1	u_5	$\bar{u}_4$	$\bar{u}_2$	$-i$	u_1	$\bar{u}_5$	i	u_4	u_2	$\bar{u}_1$	-1	0	13	
7	1	-1	1	1	-1	-1	-1	-1	1	1	-1	1	0	13	A011583
8	1	$\bar{u}_5$	u_4	u_2	i	$\bar{u}_1$	u_5	$-i$	$\bar{u}_4$	$\bar{u}_2$	u_1	-1	0	13	
9	1	$\bar{u}_4$	$\bar{u}_4$	u_4	1	u_4	u_4	1	u_4	$\bar{u}_4$	$\bar{u}_4$	1	0	13	
10	1	$-i$	1	-1	$-i$	$-i$	i	i	1	-1	i	-1	0	13	
11	1	$\bar{u}_2$	u_4	$\bar{u}_4$	-1	u_2	u_2	-1	$\bar{u}_4$	u_4	$\bar{u}_2$	1	0	13	
12	1	$\bar{u}_1$	$\bar{u}_4$	$\bar{u}_2$	i	$\bar{u}_5$	u_1	$-i$	u_4	u_2	u_5	-1	0	13	

TABLE 13. $\chi_r(n)$, $m = 14$, $\varphi(m) = 6$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	f
1	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1
2	1	0	u_1	0	$\bar{u}_1$	0	0	0	u_2	0	$\bar{u}_2$	0	-1	0	7
3	1	0	u_2	0	$\bar{u}_2$	0	0	0	$\bar{u}_2$	0	u_2	0	1	0	7
4	1	0	-1	0	-1	0	0	0	1	0	1	0	-1	0	7
5	1	0	$\bar{u}_2$	0	u_2	0	0	0	u_2	0	$\bar{u}_2$	0	1	0	7
6	1	0	$\bar{u}_1$	0	u_1	0	0	0	$\bar{u}_2$	0	u_2	0	-1	0	7

TABLE 14. $\chi_r(n)$, $m = 15$, $\varphi(m) = 8$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	f
1	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0	1
2	1	i	0	-1	0	0	i	$-i$	0	0	1	0	$-i$	-1	0	5
3	1	-1	0	1	0	0	-1	-1	0	0	1	0	-1	1	0	5
4	1	$-i$	0	-1	0	0	$-i$	i	0	0	1	0	i	-1	0	5
5	1	-1	0	1	0	0	1	-1	0	0	-1	0	1	-1	0	3
6	1	$-i$	0	-1	0	0	i	i	0	0	-1	0	$-i$	1	0	15
7	1	1	0	1	0	0	-1	1	0	0	-1	0	-1	-1	0	15
8	1	i	0	-1	0	0	$-i$	$-i$	0	0	-1	0	i	1	0	15

TABLE 15. $\chi_r(n)$, $m = 16$, $\varphi(m) = 8$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	f
1	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1	0	1
2	1	0	i	0	$-i$	0	-1	0	-1	0	$-i$	0	i	0	1	0	16
3	1	0	-1	0	-1	0	1	0	1	0	-1	0	-1	0	1	0	8
4	1	0	$-i$	0	i	0	-1	0	-1	0	i	0	$-i$	0	1	0	16
5	1	0	-1	0	1	0	-1	0	1	0	-1	0	1	0	-1	0	4
6	1	0	$-i$	0	$-i$	0	1	0	-1	0	i	0	i	0	-1	0	16
7	1	0	1	0	-1	0	-1	0	1	0	1	0	-1	0	-1	0	8
8	1	0	i	0	i	0	1	0	-1	0	$-i$	0	$-i$	0	-1	0	16

Remark 2. *The Dirichlet inverse collects inverse powers of square-free n—rephrasing (2):*

$$\frac{1}{L(s,\chi)} = \prod_p [1 - \chi(p)p^{-s}] = \sum_{n=1}^{\infty} \frac{\mu(n)\chi(n)}{n^s}. \tag{4}$$

We provide an explicit table in Section 2.2, where the first column is the modulus m, the second column the representation r as in chapter 1.2, the third column s, and the final columns real and imaginary part of $L(s,\chi_r)$.

Cases where the values are complex conjugates of earlier values have been replaced by fillers. This happens whenever $\chi_r(n) = \bar{\chi}_{r'}(n)$ for all n.

Another symmetry stems from [4]

$$L(s,\chi) = L(s,\psi) \prod_{p|m} \left(1 - \frac{\psi(p)}{p^s}\right), \tag{5}$$

Table 16. $\chi_r(n)$, $m = 17$, $\varphi(m) = 16$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	f
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1
2	1	$\bar u_2$	u_1	$-i$	u_5	$\bar u_1$	$\bar u_5$	$\bar u_6$	u_2	u_3	u_7	$\bar u_3$	i	$\bar u_7$	u_6	-1	0	17
3	1	$-i$	u_2	-1	$\bar u_6$	$\bar u_2$	u_6	i	i	u_6	$\bar u_2$	$\bar u_6$	-1	u_2	$-i$	1	0	17
4	1	$\bar u_6$	u_3	i	$\bar u_1$	$\bar u_3$	u_1	$\bar u_2$	u_6	$\bar u_7$	u_5	u_7	$-i$	$\bar u_5$	u_2	-1	0	17
5	1	-1	i	1	i	$-i$	$-i$	-1	-1	$-i$	$-i$	i	1	i	-1	1	0	17
6	1	u_6	u_5	$-i$	$\bar u_7$	$\bar u_5$	u_7	u_2	$\bar u_6$	$\bar u_1$	u_3	u_1	i	$\bar u_3$	$\bar u_2$	-1	0	17
7	1	i	u_6	-1	$\bar u_2$	$\bar u_6$	u_2	$-i$	$-i$	u_2	$\bar u_6$	$\bar u_2$	-1	u_6	i	1	0	17
8	1	u_2	u_7	i	u_3	$\bar u_7$	$\bar u_3$	u_6	$\bar u_2$	u_5	u_1	$\bar u_5$	$-i$	$\bar u_1$	$\bar u_6$	-1	0	17
9	1	1	-1	1	-1	-1	-1	1	1	-1	-1	-1	1	-1	1	1	0	17
10	1	$\bar u_2$	$\bar u_7$	$-i$	$\bar u_3$	u_7	u_3	$\bar u_6$	u_2	$\bar u_5$	$\bar u_1$	u_5	i	u_1	u_6	-1	0	17
11	1	$-i$	$\bar u_6$	-1	u_2	u_6	$\bar u_2$	i	i	$\bar u_2$	u_6	u_2	-1	$\bar u_6$	$-i$	1	0	17
12	1	$\bar u_6$	$\bar u_5$	i	u_7	u_5	$\bar u_7$	$\bar u_2$	u_6	u_1	$\bar u_3$	$\bar u_1$	$-i$	u_3	u_2	-1	0	17
13	1	-1	$-i$	1	$-i$	i	i	-1	-1	i	i	$-i$	1	$-i$	-1	1	0	17
14	1	u_6	$\bar u_3$	$-i$	u_1	u_3	$\bar u_1$	u_2	$\bar u_6$	u_7	$\bar u_5$	$\bar u_7$	i	u_5	$\bar u_2$	-1	0	17
15	1	i	$\bar u_2$	-1	u_6	u_2	$\bar u_6$	$-i$	$-i$	$\bar u_6$	u_2	u_6	-1	$\bar u_2$	i	1	0	17
16	1	u_2	$\bar u_1$	i	$\bar u_5$	u_1	u_5	u_6	$\bar u_2$	$\bar u_3$	$\bar u_7$	u_3	$-i$	u_7	$\bar u_6$	-1	0	17

Table 17. $\chi_r(n)$, $m = 18$, $\varphi(m) = 6$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	f
1	1	0	0	0	1	0	1	0	0	0	1	0	1	0	0	0	1	0	1
2	1	0	0	0	$\bar u_1$	0	$\bar u_2$	0	0	0	u_1	0	u_2	0	0	0	-1	0	9
3	1	0	0	0	$\bar u_2$	0	u_2	0	0	0	u_2	0	$\bar u_2$	0	0	0	1	0	9
4	1	0	0	0	-1	0	1	0	0	0	-1	0	1	0	0	0	-1	0	3
5	1	0	0	0	u_2	0	$\bar u_2$	0	0	0	$\bar u_2$	0	u_2	0	0	0	1	0	9
6	1	0	0	0	u_1	0	u_2	0	0	0	$\bar u_1$	0	$\bar u_2$	0	0	0	-1	0	9

where d is an induced modulus, and $\psi(n)$ the character modulo d identified by

$$\chi(n) = \psi(n)\chi_1(n). \tag{6}$$

Example 1. *Examples are related to conductors $f < m$ in the tables of Section 1.2:*

- $L(s, \chi_1)$ *at* $m = 4$ *equals* $L(s, \chi_1)$ *at* $m = 2$, *relating Tables 3 and 1.*
- $L(s, \chi_2)$ *at* $m = 6$ *equals* $(1 + 2^{-s})L(s, \chi_2)$ *at* $m = 3$, *relating Tables 5 and 2.*
- $L(s, \chi_1)$ *at* $m = 8$ *equals* $L(s, \chi_1)$ *at* $m = 2$ *and* $L(s, \chi_3)$ *at* $m = 8$ *equals* $L(s, \chi_2)$ *at* $m = 4$, *relating Tables 7, 3 and 1.*
- $L(s, \chi_1)$ *at* $m = 9$ *equals* $L(s, \chi_1)$ *at* $m = 3$ *and* $L(s, \chi_4)$ *at* $m = 9$ *equals* $L(s, \chi_2)$ *at* $m = 3$, *relating Tables 8 and 2.*
- $L(s, \chi_2)$ *at* $m = 10$ *equals* $(1 - i/2^s)L(s, \chi_2)$ *at* $m = 5$, *relating Tables 9 and 4.*

2.2. **Table.** The list of $L(s, \chi_r)$ starts as follows:

```
m  r  s  Re(L)                                                 Im(L)
2  1  2  1.23370055013616982735431137498451889191421242590510  0.00000000000000000000000000000000000000000000000000
2  1  3  1.05179979026464499972477089132251874191936300579794  0.00000000000000000000000000000000000000000000000000
2  1  4  1.01467803160419205454625346550734490885132901742381  0.00000000000000000000000000000000000000000000000000
2  1  5  1.00452376279513961613351031500525185030529714076748  0.00000000000000000000000000000000000000000000000000
2  1  6  1.00144707664094212190647858713793739465335159175109  0.00000000000000000000000000000000000000000000000000
2  1  7  1.00047154865237655475511163149159522241548962649832  0.00000000000000000000000000000000000000000000000000
2  1  8  1.00015517902529611930298724929572804156654297506137  0.00000000000000000000000000000000000000000000000000
2  1  9  1.00005134518384377259281790054250500567996990246638  0.00000000000000000000000000000000000000000000000000
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TABLE 18. $\chi_r(n)$, $m = 19$, $\varphi(m) = 18$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	f
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	0	1
2	1	u_1	$\bar{u}_5$	u_2	$\bar{u}_2$	$\bar{u}_4$	u_6	u_3	u_8	$\bar{u}_1$	$\bar{u}_6$	$\bar{u}_3$	u_5	u_7	$\bar{u}_7$	u_4	$\bar{u}_8$	-1	0	19
3	1	u_2	u_8	u_4	$\bar{u}_4$	$\bar{u}_8$	$\bar{u}_6$	u_6	$\bar{u}_2$	$\bar{u}_2$	u_6	$\bar{u}_6$	$\bar{u}_8$	$\bar{u}_4$	u_4	u_8	u_2	1	0	19
4	1	u_3	u_3	u_6	$\bar{u}_6$	u_6	1	-1	u_6	$\bar{u}_3$	1	-1	$\bar{u}_3$	u_3	$\bar{u}_3$	$\bar{u}_6$	$\bar{u}_6$	-1	0	19
5	1	u_4	$\bar{u}_2$	u_8	$\bar{u}_8$	u_2	u_6	$\bar{u}_6$	$\bar{u}_4$	$\bar{u}_4$	$\bar{u}_6$	u_6	u_2	$\bar{u}_8$	u_8	$\bar{u}_2$	u_4	1	0	19
6	1	u_5	$\bar{u}_7$	$\bar{u}_8$	u_8	$\bar{u}_2$	$\bar{u}_6$	$\bar{u}_3$	u_4	$\bar{u}_5$	u_6	u_3	u_7	$\bar{u}_1$	u_1	u_2	$\bar{u}_4$	-1	0	19
7	1	u_6	u_6	$\bar{u}_6$	u_6	$\bar{u}_6$	1	1	$\bar{u}_6$	$\bar{u}_6$	1	1	$\bar{u}_6$	u_6	$\bar{u}_6$	u_6	u_6	1	0	19
8	1	u_7	u_1	$\bar{u}_4$	u_4	u_8	u_6	u_3	u_2	$\bar{u}_7$	$\bar{u}_6$	$\bar{u}_3$	$\bar{u}_1$	$\bar{u}_5$	u_5	$\bar{u}_8$	$\bar{u}_2$	-1	0	19
9	1	u_8	$\bar{u}_4$	$\bar{u}_2$	u_2	u_4	$\bar{u}_6$	u_6	$\bar{u}_8$	$\bar{u}_8$	u_6	$\bar{u}_6$	u_4	u_2	$\bar{u}_2$	$\bar{u}_4$	u_8	1	0	19
10	1	-1	-1	1	1	1	1	-1	1	-1	1	-1	-1	-1	-1	1	1	-1	0	19
11	1	$\bar{u}_8$	u_4	u_2	$\bar{u}_2$	$\bar{u}_4$	u_6	$\bar{u}_6$	u_8	u_8	$\bar{u}_6$	u_6	$\bar{u}_4$	$\bar{u}_2$	u_2	u_4	$\bar{u}_8$	1	0	19
12	1	$\bar{u}_7$	$\bar{u}_1$	u_4	$\bar{u}_4$	$\bar{u}_8$	$\bar{u}_6$	$\bar{u}_3$	$\bar{u}_2$	u_7	u_6	u_3	u_1	u_5	$\bar{u}_5$	u_8	u_2	-1	0	19
13	1	$\bar{u}_6$	$\bar{u}_6$	u_6	$\bar{u}_6$	u_6	1	1	u_6	u_6	1	1	u_6	$\bar{u}_6$	u_6	$\bar{u}_6$	$\bar{u}_6$	1	0	19
14	1	$\bar{u}_5$	u_7	u_8	$\bar{u}_8$	u_2	u_6	u_3	$\bar{u}_4$	u_5	$\bar{u}_6$	$\bar{u}_3$	$\bar{u}_7$	u_1	$\bar{u}_1$	$\bar{u}_2$	u_4	-1	0	19
15	1	$\bar{u}_4$	u_2	$\bar{u}_8$	u_8	$\bar{u}_2$	$\bar{u}_6$	u_6	u_4	u_4	u_6	$\bar{u}_6$	$\bar{u}_2$	u_8	$\bar{u}_8$	u_2	$\bar{u}_4$	1	0	19
16	1	$\bar{u}_3$	$\bar{u}_3$	$\bar{u}_6$	u_6	$\bar{u}_6$	1	-1	$\bar{u}_6$	u_3	1	-1	u_3	$\bar{u}_3$	u_3	u_6	u_6	-1	0	19
17	1	$\bar{u}_2$	$\bar{u}_8$	$\bar{u}_4$	u_4	u_8	u_6	$\bar{u}_6$	u_2	u_2	$\bar{u}_6$	u_6	u_8	u_4	$\bar{u}_4$	$\bar{u}_8$	$\bar{u}_2$	1	0	19
18	1	$\bar{u}_1$	u_5	$\bar{u}_2$	u_2	u_4	$\bar{u}_6$	$\bar{u}_3$	$\bar{u}_8$	u_1	u_6	u_3	$\bar{u}_5$	$\bar{u}_7$	u_7	$\bar{u}_4$	u_8	-1	0	19

TABLE 19. $\chi_r(n)$, $m = 20$, $\varphi(m) = 8$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	f
1	1	0	1	0	0	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1
2	1	0	$-i$	0	0	0	i	0	-1	0	1	0	$-i$	0	0	0	i	0	-1	0	5
3	1	0	-1	0	0	0	-1	0	1	0	1	0	-1	0	0	0	-1	0	1	0	5
4	1	0	i	0	0	0	$-i$	0	-1	0	1	0	i	0	0	0	$-i$	0	-1	0	5
5	1	0	-1	0	0	0	-1	0	1	0	-1	0	1	0	0	0	1	0	-1	0	4
6	1	0	i	0	0	0	$-i$	0	-1	0	-1	0	$-i$	0	0	0	i	0	1	0	20
7	1	0	1	0	0	0	1	0	1	0	-1	0	-1	0	0	0	-1	0	-1	0	20
8	1	0	$-i$	0	0	0	i	0	-1	0	-1	0	i	0	0	0	$-i$	0	1	0	20

TABLE 20. $\chi_r(n)$, $m = 21$, $\varphi(m) = 12$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	f
1	1	1	0	1	1	0	0	1	0	1	1	0	1	0	0	1	1	0	1	1	0	1
2	1	u_4	0	$\bar{u}_4$	$\bar{u}_2$	0	0	1	0	u_2	$\bar{u}_4$	0	-1	0	0	u_4	u_2	0	$\bar{u}_2$	-1	0	7
3	1	$\bar{u}_4$	0	u_4	$\bar{u}_4$	0	0	1	0	u_4	u_4	0	1	0	0	$\bar{u}_4$	u_4	0	$\bar{u}_4$	1	0	7
4	1	1	0	1	-1	0	0	1	0	-1	1	0	-1	0	0	1	-1	0	-1	-1	0	7
5	1	u_4	0	$\bar{u}_4$	u_4	0	0	1	0	$\bar{u}_4$	$\bar{u}_4$	0	1	0	0	u_4	$\bar{u}_4$	0	u_4	1	0	7
6	1	$\bar{u}_4$	0	u_4	u_2	0	0	1	0	$\bar{u}_2$	u_4	0	-1	0	0	$\bar{u}_4$	$\bar{u}_2$	0	u_2	-1	0	7
7	1	-1	0	1	-1	0	0	-1	0	1	-1	0	1	0	0	1	-1	0	1	-1	0	3
8	1	$\bar{u}_2$	0	$\bar{u}_4$	u_4	0	0	-1	0	u_2	u_2	0	-1	0	0	u_4	$\bar{u}_4$	0	$\bar{u}_2$	1	0	21
9	1	u_2	0	u_4	u_2	0	0	-1	0	u_4	$\bar{u}_2$	0	1	0	0	$\bar{u}_4$	$\bar{u}_2$	0	$\bar{u}_4$	-1	0	21
10	1	-1	0	1	1	0	0	-1	0	-1	-1	0	-1	0	0	1	1	0	-1	1	0	21
11	1	$\bar{u}_2$	0	$\bar{u}_4$	$\bar{u}_2$	0	0	-1	0	$\bar{u}_4$	u_2	0	1	0	0	u_4	u_2	0	u_4	-1	0	21
12	1	u_2	0	u_4	$\bar{u}_4$	0	0	-1	0	$\bar{u}_2$	$\bar{u}_2$	0	-1	0	0	$\bar{u}_4$	u_4	0	u_2	1	0	21

TABLE 21. $\chi_r(n)$, $m = 22$, $\varphi(m) = 10$.

r	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	f
1	1	0	1	0	1	0	1	0	1	0	0	0	1	0	1	0	1	0	1	0	1	0	1
2	1	0	$\bar{u}_2$	0	u_4	0	$\bar{u}_3$	0	$\bar{u}_4$	0	0	0	u_1	0	u_2	0	$\bar{u}_1$	0	u_3	0	-1	0	11
3	1	0	$\bar{u}_4$	0	$\bar{u}_2$	0	u_4	0	u_2	0	0	0	u_2	0	u_4	0	$\bar{u}_2$	0	$\bar{u}_4$	0	1	0	11
4	1	0	u_4	0	u_2	0	u_1	0	$\bar{u}_2$	0	0	0	u_3	0	$\bar{u}_4$	0	$\bar{u}_3$	0	$\bar{u}_1$	0	-1	0	11
5	1	0	u_2	0	$\bar{u}_4$	0	$\bar{u}_2$	0	u_4	0	0	0	u_4	0	$\bar{u}_2$	0	$\bar{u}_4$	0	u_2	0	1	0	11
6	1	0	1	0	1	0	-1	0	1	0	0	0	-1	0	1	0	-1	0	-1	0	-1	0	11
7	1	0	$\bar{u}_2$	0	u_4	0	u_2	0	$\bar{u}_4$	0	0	0	$\bar{u}_4$	0	u_2	0	u_4	0	$\bar{u}_2$	0	1	0	11
8	1	0	$\bar{u}_4$	0	$\bar{u}_2$	0	$\bar{u}_1$	0	u_2	0	0	0	$\bar{u}_3$	0	u_4	0	u_3	0	u_1	0	-1	0	11
9	1	0	u_4	0	u_2	0	$\bar{u}_4$	0	$\bar{u}_2$	0	0	0	$\bar{u}_2$	0	$\bar{u}_4$	0	u_2	0	u_4	0	1	0	11
10	1	0	u_2	0	$\bar{u}_4$	0	u_3	0	u_4	0	0	0	$\bar{u}_1$	0	$\bar{u}_2$	0	u_1	0	$\bar{u}_3$	0	-1	0	11

```
2  1 10  1.00001704136304482548818390229983042421597459061568  0.00000000000000000000000000000000000000000000000000

3  1  2  1.46216361497620127686436903701868905708351102329493  0.00000000000000000000000000000000000000000000000000
3  1  3  1.15753627711664634890345156293695184295887568892048  0.00000000000000000000000000000000000000000000000000
3  1  4  1.06896121848013648544790488547275842249358118708022  0.00000000000000000000000000000000000000000000000000
3  1  5  1.03266056273537251922712118404363073526672256180849  0.00000000000000000000000000000000000000000000000000
3  1  6  1.01594752966348281716346920835087811291155436590386  0.00000000000000000000000000000000000000000000000000
3  1  7  1.00788821232596401439039663647537984292880738152520  0.00000000000000000000000000000000000000000000000000
3  1  8  1.00392431895420132088464794461465632862349989127618  0.00000000000000000000000000000000000000000000000000
3  1  9  1.00195748552573033298644404836825352712887742555272  0.00000000000000000000000000000000000000000000000000
3  1 10  1.00097762319679253337038382667184943548868636725125  0.00000000000000000000000000000000000000000000000000
3  2  1  0.60459978807807261686469275254738524409468874936425  0.00000000000000000000000000000000000000000000000000
3  2  2  0.78130241289648629686718742962409235636513433654529  0.00000000000000000000000000000000000000000000000000
3  2  3  0.88402381175007985674305791687101180774794618611766  0.00000000000000000000000000000000000000000000000000
3  2  4  0.94002568087712376869106944507088599164380309660335  0.00000000000000000000000000000000000000000000000000
3  2  5  0.96944058923515320144972479494643094087890176806038  0.00000000000000000000000000000000000000000000000000
3  2  6  0.98456036325367773106535026689304675871720067993823  0.00000000000000000000000000000000000000000000000000
3  2  7  0.99223652952251116935163174535130606577181948276643  0.00000000000000000000000000000000000000000000000000
3  2  8  0.99610656865181666898456918235051500602422797894078  0.00000000000000000000000000000000000000000000000000
3  2  9  0.99805019565707723722786382273031372573915214445692  0.00000000000000000000000000000000000000000000000000
3  2 10  0.99902429144886669520119689634565715196137413679602  0.00000000000000000000000000000000000000000000000000

4  1  *  same block as m=2, r=1 above
4  2  1  0.78539816339744830961566084581987572104929234984378  0.00000000000000000000000000000000000000000000000000
4  2  2  0.91596559417721901505460351493238411077414937428167  0.00000000000000000000000000000000000000000000000000
4  2  3  0.96894614625936938048363484584691860006954026768391  0.00000000000000000000000000000000000000000000000000
4  2  4  0.98894455174110533610842263322837782131586088706273  0.00000000000000000000000000000000000000000000000000
4  2  5  0.99615782807708806400631936863097528151139552938826  0.00000000000000000000000000000000000000000000000000
4  2  6  0.99868522221843813544160078786020654967836454612651  0.00000000000000000000000000000000000000000000000000
4  2  7  0.99955450789053990949634654989905898300218848194998  0.00000000000000000000000000000000000000000000000000
4  2  8  0.99984999024682965633806705924046378147600743300743  0.00000000000000000000000000000000000000000000000000
4  2  9  0.99994968418722008982135887329384752737274799691796  0.00000000000000000000000000000000000000000000000000
4  2 10  0.99998316402619687740554072995833414145685781649717  0.00000000000000000000000000000000000000000000000000

5  1  2  1.57913670417429737901351855998018418165019190515853  0.00000000000000000000000000000000000000000000000000
5  1  3  1.19244044793431753111654025621935839083886640200177  0.00000000000000000000000000000000000000000000000000
5  1  4  1.08059151653720037040957809062670203413031135039566  0.00000000000000000000000000000000000000000000000000
5  1  5  1.03659593826172404795493944950136791712330265360767  0.00000000000000000000000000000000000000000000000000
5  1  6  1.01727795202848213496957620064341390898803177371349  0.00000000000000000000000000000000000000000000000000
5  1  7  1.00833637051117233822761400044115868220134068231166  0.00000000000000000000000000000000000000000000000000
5  1  8  1.00407478575991247264117642907444188310864972771023  0.00000000000000000000000000000000000000000000000000
5  1  9  1.00200787979778508746375898729179421349063088276469  0.00000000000000000000000000000000000000000000000000
5  1 10  1.00099447262597359224857402037657282561335219014808  0.00000000000000000000000000000000000000000000000000
5  2  1  0.86480626597720996723118206585862333703828555690228  0.20415306613838514619400230664825930286316536836424
5  2  2  0.95871612271688315539193642933117852641597153075830  0.14556587678508959046170451181198645372080514688910
5  2  3  0.98819168162405719379746955041525485252124079065866  0.08910518834573959516382505339268229125210873564650
5  2  4  0.99676642741179814430615792373611283384761870873403  0.05034352479696035175591682828563520045910217644709
5  2  5  0.99914018175644358436955939861888522369387124002994  0.02716531388744971066264188817280532978424685390574
5  2  6  0.99977588996291134934990409095613448933356116531308  0.01425808563279482709723352766645291152583934704345
5  2  7  0.99994237287968123476301040409163638068961222509105  0.00735600297014121924386073641658070427527662713217
5  2  8  0.99998531753697854369258968020511839099444946801053  0.00375394923062927545514778625267954489947759787928
5  2  9  0.99999628233876434975046131554704541409373548102876  0.00190233717014360773086006880755397496523277821154
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5	2	10	0.99999906261295146952490527282124301526331935241782	0.00095963003013016502329918933981800880975393386893
5	3	1	0.43040894096400403888943323295060542542457068254029	0.00000000000000000000000000000000000000000000000000
5	3	2	0.70621140325974096993100317576256402766024647185295	0.00000000000000000000000000000000000000000000000000
5	3	3	0.85482476664854301023569008353813769713839646493701	0.00000000000000000000000000000000000000000000000000
5	3	4	0.92933695649491083980686610906564713050614170392099	0.00000000000000000000000000000000000000000000000000
5	3	5	0.96566836478393066124475433079984156013740840823482	0.00000000000000000000000000000000000000000000000000
5	3	6	0.98325856602631974769624638452177106667659152743151	0.00000000000000000000000000000000000000000000000000
5	3	7	0.99179339582931741534142186179786624204285859466114	0.00000000000000000000000000000000000000000000000000
5	3	8	0.99595698038636700210147238379794357697757501738067	0.00000000000000000000000000000000000000000000000000
5	3	9	0.99799995420138797524314109340968004179585079910036	0.00000000000000000000000000000000000000000000000000
5	3	10	0.99900746845894008421535713241894908695450211641298	0.00000000000000000000000000000000000000000000000000
5	4	*	complex conjugate of block m=5, r=2 above	
6	1	2	1.09662271123215095764827677776401679281263326747120	0.00000000000000000000000000000000000000000000000000
6	1	3	1.01284424247706555290520117569832862589016227805542	0.00000000000000000000000000000000000000000000000000
6	1	4	1.00215114232512795510741083013071102108773236288771	0.00000000000000000000000000000000000000000000000000
6	1	5	1.00038992014989212800127364704226727478963748175197	0.00000000000000000000000000000000000000000000000000
6	1	6	1.00007334951249089814529000197039564239731132893662	0.00000000000000000000000000000000000000000000000000
6	1	7	1.00001408566716742052797166275291593790592607385703	0.00000000000000000000000000000000000000000000000000
6	1	8	1.00000273958328647197494228858100532733981434482588	0.00000000000000000000000000000000000000000000000000
6	1	9	1.00000053731181289092982989983628428195870383683094	0.00000000000000000000000000000000000000000000000000
6	1	10	1.00000010598663941566201431121611520752434194697073	0.00000000000000000000000000000000000000000000000000
6	2	1	0.90689968211710892529703912882107786614203312404637	0.00000000000000000000000000000000000000000000000000
6	2	2	0.97662801612060787108398428703011544545641792068161	0.00000000000000000000000000000000000000000000000000
6	2	3	0.99452678821883983883594015647988828371643945938237	0.00000000000000000000000000000000000000000000000000
6	2	4	0.99877728593194400423426128538781636612154079014106	0.00000000000000000000000000000000000000000000000000
6	2	5	0.99973560764875173899502869478850690778136744831227	0.00000000000000000000000000000000000000000000000000
6	2	6	0.99994411892951644561324636481325061432215694056227	0.00000000000000000000000000000000000000000000000000
6	2	7	0.99998837740940578786219136836186314441066182247554	0.00000000000000000000000000000000000000000000000000
6	2	8	0.99999760993561282784779015571907170526651011948352	0.00000000000000000000000000000000000000000000000000
6	2	9	0.99999951244546996620682449425908386973473642598906	0.00000000000000000000000000000000000000000000000000
6	2	10	0.99999990110848472908322931518974470777383641622648	0.00000000000000000000000000000000000000000000000000
7	1	2	1.61136398385132385613624342855120834862264480118217	0.00000000000000000000000000000000000000000000000000
7	1	3	1.19855236408332724666679431847497346017966563259607	0.00000000000000000000000000000000000000000000000000
7	1	4	1.08187245352217062042416029641766054421466150962305	0.00000000000000000000000000000000000000000000000000
7	1	5	1.03686605895992592264680956538328769134094733939127	0.00000000000000000000000000000000000000000000000000
7	1	6	1.01733441471110228211998066625336567473240761984705	0.00000000000000000000000000000000000000000000000000
7	1	7	1.00834805297800295632565701341448026756877398799458	0.00000000000000000000000000000000000000000000000000
7	1	8	1.00407718202413396883088326257136711774176717738188	0.00000000000000000000000000000000000000000000000000
7	1	9	1.00200836799538014542852513817695847408347678061304	0.00000000000000000000000000000000000000000000000000
7	1	10	1.00099457158416398229104833306144079762503929918306	0.00000000000000000000000000000000000000000000000000
7	2	1	0.80420572938678064453602680236665980362176053423821	0.39866669881887309012652808390944874325573190865891
7	2	2	0.90224702530125686149636233530770009958391222926250	0.23254898127789499532985204147113474313969887412392
7	2	3	0.94900065744085650257652307364117767792866836617339	0.12093454100557844947665555914591001331455774253448
7	2	4	0.97318475700336659629588874918483336356693336159582	0.06017844317177939388236048602145251758906971756407
7	2	5	0.98599994212736087600607728025035165453963558799187	0.02951953363442772099408023091395505613560264336107
7	2	6	0.99276498435721643043064120863320290148811587034042	0.01445451816744452098610529504022403602429932612933
7	2	7	0.99629508297914797867297189660573928226356347275487	0.00709807042141154175731259463319416403598393430266
7	2	8	0.99811618902400718178473542229094126524402635499584	0.00349949908664805504650399864559900345320076012546
7	2	9	0.99904709602524537604651839190722720158169721059753	0.00173171006537630576137313987221381988394773503229
7	2	10	0.99951974493944251917233025937674571422645249684763	0.00085947985090885868750565281722856171336856369371
7	3	1	0.53774738050490415906598555390898822375399700059565	-0.10529754563079555866579778611255217411793239854732
7	3	2	0.79921874363828629008917129529621684333258829115112	-0.10184729711946994015173540667599101706782444023755
7	3	3	0.91240522085515469791868540419778964075978702679431	-0.06976362901709455799415333135521407886901102521939
7	3	4	0.96069475033602908351906123463215608905238699078834	-0.04146710952018380372504169202059970019181384483466
7	3	5	0.98181270644919194883806240753332647240769039896362	-0.02293503224533897104297842285180493097499824290892
7	3	6	0.99137115305174119351788903189418321893876028456642	-0.01218825115375538918117788900886114648523257085628
7	3	7	0.99583208189077681485907422279505804906229884302486	-0.00632813765305032572275376476423896476601761030970
7	3	8	0.99796239397636132381861446725992048655200552288598	-0.00323992779704102114804391413080566545590637392988
7	3	9	0.99899597620671592382274543827658769855584304634032	-0.00164459817186975288583457255998149374143657356800
7	3	10	0.99950274042585653742901397144414398508837913176813	-0.00083032463457240362968451390003482633212237222935
7	4	1	1.18741041172372594878462529794936302999233468616504	0.00000000000000000000000000000000000000000000000000
7	4	2	1.15192547054449104710169239732054996479782140468657	0.00000000000000000000000000000000000000000000000000
7	4	3	1.09334306942953357197657981500770023478019258483238	0.00000000000000000000000000000000000000000000000000
7	4	4	1.05199412759596611776500555032380144198785360478349	0.00000000000000000000000000000000000000000000000000
7	4	5	1.02770129237545454551940843149715298941542199657669	0.00000000000000000000000000000000000000000000000000
7	4	6	1.01441679288251066345375357363759477537167189691006	0.00000000000000000000000000000000000000000000000000
7	4	7	1.00740051628718717867663635839026011802055297911677	0.00000000000000000000000000000000000000000000000000
7	4	8	1.00376601206832360100225310394601858773949662862450	0.00000000000000000000000000000000000000000000000000
7	4	9	1.00190553240571369595395042661328947819347674390036	0.00000000000000000000000000000000000000000000000000
7	4	10	1.00096046328382044118026384519742117406407857792938	0.00000000000000000000000000000000000000000000000000

7	5	*	complex conjugate of block m=7, r=3 above	
7	6	*	complex conjugate of block m=7, r=2 above	
8	1	*	same block as m=2, r=1 above	
8	2	1	0.62322524014023051339402008025056800265069531234657	0.00000000000000000000000000000000000000000000000000
8	2	2	0.87235802495485994176969511702117566123998328386851	0.00000000000000000000000000000000000000000000000000
8	2	3	0.95838045456309456205166940286157781882489531793978	0.00000000000000000000000000000000000000000000000000
8	2	4	0.98654286069397050390153449061672691096683375790950	0.00000000000000000000000000000000000000000000000000
8	2	5	0.99563385631296745634221771712751768218319824059632	0.00000000000000000000000000000000000000000000000000
8	2	6	0.99857397195353054767027051610666807303195493006369	0.00000000000000000000000000000000000000000000000000
8	2	7	0.99953131567937557755040925824403013635232953894177	0.00000000000000000000000000000000000000000000000000
8	2	8	0.99984521547922560046287988947718520778494833398948	0.00000000000000000000000000000000000000000000000000
8	2	9	0.99994870961103321295581212285432008166030642503296	0.00000000000000000000000000000000000000000000000000
8	2	10	0.99998296629534976661522190750326347218135136023530	0.00000000000000000000000000000000000000000000000000
8	3	1	0.78539816339744830961566084581987572104929234984378	0.00000000000000000000000000000000000000000000000000
8	3	*	same block as m=4, r=2 above	
8	4	1	1.11072073453959156175397024751517342465365542234392	0.00000000000000000000000000000000000000000000000000
8	4	2	1.06473417104350337039282745146166888948309915177449	0.00000000000000000000000000000000000000000000000000
8	4	3	1.02772258593685856787925661800225767210100318537 00	0.00000000000000000000000000000000000000000000000000
8	4	4	1.01050894057394275298789820730233717945908389474823	0.00000000000000000000000000000000000000000000000000
8	4	5	1.00375568639565501098499975447095047854454794291814	0.00000000000000000000000000000000000000000000000000
8	4	6	1.00130144245443407196300635551705830033223670716771	0.00000000000000000000000000000000000000000000000000
8	4	7	1.00044347460565500357302750536448229851749828880011	0.00000000000000000000000000000000000000000000000000
8	4	8	1.00014970877409314113080112352920534314965829703675	0.00000000000000000000000000000000000000000000000000
8	4	9	1.00005027137748272819466305550760489152952812322802	0.00000000000000000000000000000000000000000000000000
8	4	10	1.00001682946462637816456731816360613057968348822610	0.00000000000000000000000000000000000000000000000000
9	1	*	same block as m=3, r=1 above	
9	2	1	1.13627591877137772867946858378927249748292799678081	0.41357061234626267713312691756971607765808508039554
9	2	2	1.09924073772918719207477594230466608269357483133795	0.22450633533683796656069968736383136018364034275073
9	2	3	1.05644449535924978290277143269472486228673687642286	0.11294076651112729133580834935100749139342128637150
9	2	4	1.02976333851242225322436999956769517083783845938162	0.05582169736672454866622487819092111703070011747053
9	2	5	1.01524842945055492022455823053287317840736438491878	0.02758589174346302566892049676562875529278796209783
9	2	6	1.00771556402305063959727586103539595578221438570588	0.01368080691260537708503578099549129148480927514897
9	2	7	1.00388116792079898276400053235729817594406041714603	0.00680659222762275777787704551341070970120168960300
9	2	8	1.00194664113692266435177100893395270139351033405522	0.00339376332884039838632460202507686338961059006694
9	2	9	1.00097489248744716761105512405887248102838173225098	0.00169429502176905993749996807191595460263374951666
9	2	10	1.00048785302807757983798450597516719533385277052014	0.00084646212870020778828343357630122085664973565229
9	3	1	0.58948981737626090425789465558502894133017371989959	0.17309778850634728994376104426232298389452252600035
9	3	2	0.83135555718602012627281200677167522445209838588321	0.14659041529381451985174091305925190715916525545672
9	3	3	0.92650439456797372688556298132104800713011600712802	0.09048467928895974683351272231483601266444775135675
9	3	4	0.96607423281837262129851555797296592839794969214625	0.04973921168882777734030587162752648598784346041292
9	3	5	0.98373245077565709342207654853061929308559707199022	0.02599428917231768623889775238588695118066265318375
9	3	6	0.99203356119260609235275930580372496257016032556757	0.01327240031455404057094387497709283474997898039490
9	3	7	0.99605676477514825718742088008902620249197459955994	0.00670295780912172161830983153354202667426192647657
9	3	8	0.99803794524570708401315427951067202120174444304323	0.00336763304230532769162487521510116471102553158885
9	3	9	0.99902126993075679376403294767989311310894881052356	0.00168773055497479996899505548340412101810174871071
9	3	10	0.99951118994959295567625570024765603468142402120408	0.00084481643698373544858470467901024631695040377649
9	4	1	0.60459978807807261686469275254738524409468874936425	0.00000000000000000000000000000000000000000000000000
9	4	*	same block as m=3, r=2 above	
9	5	*	complex conjugate of block m=9, r=3 above	
9	6	*	complex conjugate of block m=9, r=2 above	
10	1	2	1.18435252813072303426013891998513813623764392886889	0.00000000000000000000000000000000000000000000000000
10	1	3	1.04338539194252783972697272419193859198400810175155	0.00000000000000000000000000000000000000000000000000
10	1	4	1.01305454675362534725897945996253315699716689099593	0.00000000000000000000000000000000000000000000000000
10	1	5	1.00420231519104517145634759170445016971319944568243	0.00000000000000000000000000000000000000000000000000
10	1	6	1.00138298402803710161067657250836056666009377724922	0.00000000000000000000000000000000000000000000000000
10	1	7	1.00045874261655380433521076606271212999664270823110	0.00000000000000000000000000000000000000000000000000
10	1	8	1.00015261862803781454492183364836984450275656471136	0.00000000000000000000000000000000000000000000000000
10	1	9	1.00005083315755503846480633301973992791740699432179	0.00000000000000000000000000000000000000000000000000
10	1	10	1.00001693896129978991239377230979882871333915089989	0.00000000000000000000000000000000000000000000000000
10	2	1	0.96688279904640254032818321918275298846986824108440	-0.22825006685021983742158872628105236565597741008690
10	2	2	0.99510759191315555300736255728417513984617281748057	-0.09411315389413119838627959552080817788318773580048
10	2	3	0.99932983016727464319294768208934013892775438261447	-0.03441877185726755406085864040922456531304636318583
10	2	4	0.99991289771160816629090272550396503387631259476197	-0.01195437691627703226321804194787185165637399284878
10	2	5	0.99998909781542638782776695762428539024962895421450	-0.00405781679243915134890684303403483345618662234520
10	2	6	0.99999867255092376852329836482592281607615240511064	-0.00136341264787566273635872375473668986999754616456
10	2	7	0.99999984165288546303835306609489091744176282374052	-0.00045604681798129040272528236538520494886096837636
10	2	8	0.99999998140116068929983635124516792046671305237725	-0.00015224341599954698115139218562169881534447035514
10	2	9	0.99999999784104978648431065161893516795108945129870	-0.00005078056879929138974630094937384810916904889984
10	2	10	0.99999999975165276851418583843583268128754762774386	-0.00001693155445279539615872596567148632233915143617

10	3	1	0.64561341144600605833414984942590813813685602381043	0.00000000000000000000000000000000000000000000000000
10	3	2	0.88276425407467621241375396970320503457530808981618	0.00000000000000000000000000000000000000000000000000
10	3	3	0.96167786247961088651515134398040490928069602305413	0.00000000000000000000000000000000000000000000000000
10	3	4	0.98742051627584276729479524088225007616277556041605	0.00000000000000000000000000000000000000000000000000
10	3	5	0.99584550118342849440865290363733660889170242099216	0.00000000000000000000000000000000000000000000000000
10	3	6	0.99862198112048099375400023427992373959341327004763	0.00000000000000000000000000000000000000000000000000
10	3	7	0.99954178173423395764877672009316207205881842743193	0.00000000000000000000000000000000000000000000000000
10	3	8	0.99984743734100124820343126029715429407514366979232	0.00000000000000000000000000000000000000000000000000
10	3	9	0.99994917286193756113228785335774582312748332019235	0.00000000000000000000000000000000000000000000000000
10	3	10	0.99998306168985701789134869211857696692223112238604	0.00000000000000000000000000000000000000000000000000
10	4	*	complex conjugate of block m=10, r=2 above	
11	1	2	1.63133957042799316013793239667374398930804948880013	0.00000000000000000000000000000000000000000000000000
11	1	3	1.20115378001672456767967825305050975786433641533649	0.00000000000000000000000000000000000000000000000000
11	1	4	1.08224930957797029737000847738287672267074338747969	0.00000000000000000000000000000000000000000000000000
11	1	5	1.03692131663783352252184967242615912205197659179877	0.00000000000000000000000000000000000000000000000000
11	1	6	1.01734248772081272840881650911281246900882543284855	0.00000000000000000000000000000000000000000000000000
11	1	7	1.00834922563766105677975000165526663258467189421903	0.00000000000000000000000000000000000000000000000000
11	1	8	1.00407735151384936970050176553798866286370353392364	0.00000000000000000000000000000000000000000000000000
11	1	9	1.00200839240113284143106999567942030786558894683600	0.00000000000000000000000000000000000000000000000000
11	1	10	1.00099457508922541121755640600316837214195047622188	0.00000000000000000000000000000000000000000000000000
11	2	2	1.22609858506160642391001983506463949221811785315475	0.09794327555369501394097411130959497326271012469243
11	2	3	1.11200312562997040762761378707362780308244657899633	0.05351953374912626513736120163760186428914262286987
11	2	4	1.05456639992479817674326471300831366133980808034624	0.02895568665712734966196812767526044939559327036457
11	2	5	1.02665531433440702827369828184213318207843338337000	0.01545917919763627249206480072823363078172793482591
11	2	6	1.01309998813959902215311978062402231567771501603310	0.00813126441419070477579769023695571751362271001027
11	2	7	1.00647238918578388093790524153326577874468856777068	0.00421985758129577050915637523096468392077797060677
11	2	8	1.00321035018788822666677942270252303991349474335377	0.00216662536615362777277923474707040200172461491000
11	2	9	1.00159664310988233426039306737538580935022201717142	0.00110355204297348660578435175464430036409015500531
11	2	10	1.00079551243566790687944919944582706686825124692209	0.00055885762932065370498370574457586668246765487261
11	3	2	0.94431158005757224770616334312144439381298674100888	0.16022433204748269183651810235473620070102575804964
11	3	3	0.99764167226514886313525892608064983797317170219478	0.09626536408815421043574063325934752361804478702326
11	3	4	1.00653559723554287486030666062600169231616142172155	0.05248245827568837498974930835340433530986684249913
11	3	5	1.00562090730480311717673889685461030789568232806235	0.02748321650869558387348574287182200128000884002782
11	3	6	1.00353936317584301845813553655183151962394673641266	0.01412085746162437829303805722308760097398177298271
11	3	7	1.00199877271588461989409444471113309646740588946963	0.00718231097210474209723469088738949659940707654800
11	3	8	1.00107225161572362359218592617603514712778003017359	0.00363153443324445518593363779531783535320296490708
11	3	9	1.00055952510810713429664750729861937431710716452283	0.00182934328243390595880534478583866216418620160250
11	3	10	1.00028733543543352864425447597650040704794846067613	0.00091926118400837517652620786250658618762035007966
11	4	2	0.80997029123410977860339310872223766187006139099475	0.27486838125694933769780812234474840636695659565884
11	4	3	0.92328470914052625275117582902280310876002396791016	0.13441181614988013985259386974519745611945012301922
11	4	4	0.96828846351526629411364015260839054117537940616578	0.06518410415533524864831005594058082923244740731744
11	4	5	0.98635240499503562655754783891200742723935089736535	0.03175136175249363945963636575923915276163112452059
11	4	6	0.99388716431253145497703961666700124707639452049900	0.01556472939348223520589730291425351175037762285819
11	4	7	0.99717070258417439142533816937119439868165347007748	0.00767204577077528290336709691434819129324638046729
11	4	8	0.99865804484225259029879220470741196924717761387430	0.00379759809395063287770150763867824705097853033002
11	4	9	0.99935241624560192314746224676721443591299541467456	0.00188555302875380543887722662059913858961177528122
11	4	10	0.99968378342791048214051567636306964597827292671812	0.00093824269676595313342801537897979869664935415480
11	5	2	0.80911450111572235388643957604269295305215469043651	0.15715853047123663431985584014732642985019601555132
11	5	3	0.90649414971814574906988952365424002810061454063682	0.08643394617385632924894074325592189980695271949808
11	5	4	0.95273891801542594763371493123296640870408313289878	0.04330770409137898568852195427307140126243840860926
11	5	5	0.97595466021079727388051971956293077452468156058584	0.02107326767532597831079332426638447570307517966006
11	5	6	0.98779278117878633495123974451978679262375105329273	0.01020309922378050209627283071736588299124894404395
11	5	7	0.99382693649902727969603146998036665960763557171318	0.00495869993312218882134368634364095342737760497705
11	5	8	0.99688910354369141764185196425431376066398802810374	0.00242453165382455947091211943142506291227594321255
11	5	9	0.99843628168050059791839550265017895345389879175117	0.00119233458134123577898674459674696726630800382042
11	5	10	0.99921537731054403531421880724460718416866400275270	0.00058913604525888928900462727385608660068375627264
11	6	1	0.94722582509948293642963438181697406661998807266176	0.00000000000000000000000000000000000000000000000000
11	6	2	0.90953910532388370153208593905860526925918567918243	0.00000000000000000000000000000000000000000000000000
11	6	3	0.92714818236965512499162713935981177313229130635256	0.00000000000000000000000000000000000000000000000000
11	6	4	0.95402214180179009475618991411135239034848107657977	0.00000000000000000000000000000000000000000000000000
11	6	5	0.97395415902841414785573191135465686119873226035666	0.00000000000000000000000000000000000000000000000000
11	6	6	0.98602234339724985055556436152196199817189922454767	0.00000000000000000000000000000000000000000000000000
11	6	7	0.99271345463685035096616776875938210825305162058405	0.00000000000000000000000000000000000000000000000000
11	6	8	0.99626317149787303867008700913208576905050130544380	0.00000000000000000000000000000000000000000000000000
11	6	9	0.99810187725448244155848978629295478260842294969019	0.00000000000000000000000000000000000000000000000000
11	6	10	0.99904140785341483492129776179859194407299390623981	0.00000000000000000000000000000000000000000000000000
11	7	*	complex conjugate of block m=11, r=5 above	
11	8	*	complex conjugate of block m=11, r=4 above	
11	9	*	complex conjugate of block m=11, r=3 above	
11	10	*	complex conjugate of block m=11, r=2 above	

12	1	*	same block as m=6, r=1 above	
12	2	1	0.90689968211710892529703912882107786614203312404637	0.00000000000000000000000000000000000000000000000000
12	2	*	same block as m=6, r=2 above	
12	3	1	1.04719755119659774615421446109316762806572313312504	0.00000000000000000000000000000000000000000000000000
12	3	2	1.01773954908579890561622612770264901197127708253519	0.00000000000000000000000000000000000000000000000000
12	3	3	1.00483304056527195013117687717458225192396768500554	0.00000000000000000000000000000000000000000000000000
12	3	4	1.00115374373790910569000809783613557219630361406351	0.00000000000000000000000000000000000000000000000000
12	3	5	1.00025724300744645110099558002451838966576341222525	0.00000000000000000000000000000000000000000000000000
12	3	6	1.00005516079486946347375661884492562587819769365207	0.00000000000000000000000000000000000000000000000000
12	3	7	1.00001155156127175216186842760820350014119268335919	0.00000000000000000000000000000000000000000000000000
12	3	8	1.00000238317325045037195489143970786984385928599219	0.00000000000000000000000000000000000000000000000000
12	3	9	1.00000048689433725794053894538007898840650162939253	0.00000000000000000000000000000000000000000000000000
12	3	10	1.00000009882888661299593862900370253608066951284794	0.00000000000000000000000000000000000000000000000000
12	4	1	0.76034599630094634753109425488040582420162773094718	0.00000000000000000000000000000000000000000000000000
12	4	2	0.94970312629400939526349849174574151587365195090969	0.00000000000000000000000000000000000000000000000000
12	4	3	0.99004001943815994979181677686330405085688506765724	0.00000000000000000000000000000000000000000000000000
12	4	4	0.99807159983792868732970969812023495572578667744209	0.00000000000000000000000000000000000000000000000000
12	4	5	0.99962849256584767855073228195424063872164784181061	0.00000000000000000000000000000000000000000000000000
12	4	6	0.99992821782510442180143305623089402637092535781348	0.00000000000000000000000000000000000000000000000000
12	4	7	0.99998604981392933627635184591264701868765588376300	0.00000000000000000000000000000000000000000000000000
12	4	8	0.99999727223893094714678187496378392724448624970057	0.00000000000000000000000000000000000000000000000000
12	4	9	0.99999946372666051276785050408735003688442321757795	0.00000000000000000000000000000000000000000000000000
12	4	10	0.99999989410504733421178089865193068769238548049671	0.00000000000000000000000000000000000000000000000000
13	1	2	1.63520072917456829187790383429900728869102712072628	0.00000000000000000000000000000000000000000000000000
13	1	3	1.20150976756416433806910559976292406905776508783784	0.00000000000000000000000000000000000000000000000000
13	1	4	1.08228533856623041033917109251131806670798946769367	0.00000000000000000000000000000000000000000000000000
13	1	5	1.03692496239544539403496794767907051931022046945055	0.00000000000000000000000000000000000000000000000000
13	1	6	1.01734285121516823706244702445409659496381892787357	0.00000000000000000000000000000000000000000000000000
13	1	7	1.00834926131223185100489731085613974855374179166569	0.00000000000000000000000000000000000000000000000000
13	1	8	1.00407735496705119002291353234695055908709250660742	0.00000000000000000000000000000000000000000000000000
13	1	9	1.00200839273159322841467709490462663234152781784097	0.00000000000000000000000000000000000000000000000000
13	1	10	1.00099457512055705584449610793047616209254354410815	0.00000000000000000000000000000000000000000000000000
13	2	1	1.30081758341281856338867798716541347186150765955546	0.53393691918336212379364825856659605266462296463270
13	2	2	1.17773426763753060339662545725312650970661930668303	0.23615979034382488214395198527865525468640024549471
13	2	3	1.09516884356510246849501336001225792089000938151393	0.10083179696687069225215841125169409578248320972117
13	2	4	1.04950317167496335188448321308099988272436883913607	0.04391687781063347384439576632425313554285612418092
13	2	5	1.02542391983051724136779469108722770753708796726788	0.01975481392589917432822365987509252065831021085435
13	2	6	1.01295556970941051686937178678187970606934381682593	0.00915550198251887478676620887609526889218133725965
13	2	7	1.00656556175387919604012573992844299566116277505645	0.00434366506848755514769005370872779009459153594044
13	2	8	1.00331395581072587637447684503699825195618752057447	0.00209601533096467408680315738771531637868800327996
13	2	9	1.00166789482440760532997572787305933162120148595672	0.00102339416313220076991299212719902314215356449487
13	2	10	1.00083772583173915192646750130114778225492857787652	0.00050367805037036407745328901855207450212209942018
13	3	1	0.82312737623913308981206491555803810496010468988604	0.18592807779632539444088402690104272501249542791817
13	3	2	1.00821607982301287963449907798249864953028098132743	0.14251842359221182291918142532605901474943524577644
13	3	3	1.02992688544187646816970759536133245487424052322266	0.08421325565302656149088225957417066323141288555471
13	3	4	1.02184548874845619907891506107678209778491573416576	0.04589912318494090325056372212806462060433021109831
13	3	5	1.01281758659198567353627343361808678242152575374031	0.02419407326744560721664960640791121114557453324000
13	3	6	1.00695099067130217172979779232242222680927771350728	0.01253047994581218081478305359069497956747663861362
13	3	7	1.00363613225581503988013824462322506499824824403761	0.00641868989645515782902851364082701019231842127282
13	3	8	1.00186704118318459566710943631544309097588187537163	0.00326348013769578098288477748854041148806718487896
13	3	9	1.00094879373531253503890842325040103531699299287265	0.00165065519702279616885920426450447761179283177110
13	3	10	1.00047924342219943849717941855500040010074058736910	0.00083187043109339965876411311719644120346010534149
13	4	1	1.08642943411465667904756436036751417209703758075237	0.58143938788146905079695262401134406190499575662569
13	4	2	1.04959036238141747190460523780820534267456555014343	0.27870238977660638674590011796870117134402086477344
13	4	3	1.02161286435740978687559566727975746365214033121502	0.13237089541536577070832078716201641673186793893552
13	4	4	1.00847314520198350349922480129469737630854819582831	0.06416716326588355324950361988206301509326366811176
13	4	5	1.00314370299058370520412014510289530971146717884202	0.03160401308851732225945916899216647812080907612716
13	4	6	1.00112830013343918849316462642070135733324000247791	0.01569766294629419913640664978654554197046013093130
13	4	7	1.00039630420836347672206002822445391728154725405798	0.00782713696964016962588218428469605169271479412179
13	4	8	1.00013716867613330440200760101054444792311483368130	0.00390916810648802106004934990726362771730372556913
13	4	9	1.00004699200820870915118866359823328247641825620953	0.00195370360410053999292267211009099222485418389537
13	4	10	1.00001598158801789203116573480033226168802498666845	0.00097667693016979388884900619399224695691674265855
13	5	1	0.56632991654275189572266519728012770768142013470892	0.31509644476552622235125814130206432577181155325467
13	5	2	0.81214349264552248429294160055006741911845042046775	0.22320407708461733377325043897093491668848811024702
13	5	3	0.91644908310944324620646826767013349212899809941778	0.12071069032415912357929469919910320265228233452964
13	5	4	0.96176967074941481843252969985588891492196271859704	0.06043056253156463306672186827369244029424462142815
13	5	5	0.98207368646999041146259796068109810089935992685554	0.02961876135964329139693765329424121106754176036827
13	5	6	0.99143107380989776273173987436162205847800884219531	0.01448204422764191193781938594489768750501628347459
13	5	7	0.99584534448225859812516353079186704701859594494415	0.00710476272187907509308658848812715358037729376154
13	5	8	0.99796525662272479753482088371753651431622900641601	0.00350102026261014081882865486222077539479501488788
13	5	9	0.99899658318352693340871032766918931410978198558733	0.00173204251656577521384335328130019916498829126307

13	5	10	0.99950286746665302319953194073487874968360615077598	0.00085955072838989276653174705464419976732718508704
13	6	1	0.55412431351625958564168009971360119455679936292757	-0.22744728590401645018371809581699718116388636327401
13	6	2	0.76604213380762222889024455717590483297307051661167	-0.03400940750026205350901745993054346319167227528268
13	6	3	0.88252337663708677299734636895977797472695450058645	0.01317252810194530072026962706552613176965464113892
13	6	4	0.94195469665529784867844259712599681903658969146081	0.01616827560489748421494090262044999350808977032876
13	6	5	0.97142579003746381249347520409409215202693613736932	0.01097985321945528710150035577093728589761425650067
13	6	6	0.98591551903995513958080163724190651988357235848082	0.00636155022565435307206622206620088093934740585744
13	6	7	0.99303807855901627673007272448415536100186250568051	0.00344649912209668096329417511731335142917289713286
13	6	8	0.99654887055961925663262755812518536429782794620881	0.00180565139460541026009997704023742975391149916117
13	6	9	0.99828511273076420289073286135728805537147799829150	0.00092879578129663112272835656162870045311612770528
13	6	10	0.99914629254214567802043168041926482818084543456451	0.00047269447310452088093533938739278522419385736520
13	7	1	0.66273539107184558971369651547685026175894744301379	0.00000000000000000000000000000000000000000000000000
13	7	2	0.84225715353071571897038449041624170195601708736066	0.00000000000000000000000000000000000000000000000000
13	7	3	0.91250792492569485803584849386454803179537451045450	0.00000000000000000000000000000000000000000000000000
13	7	4	0.95096490087075612249552605859398871256659929791460	0.00000000000000000000000000000000000000000000000000
13	7	5	0.97332897779749143326952338841277998135817609137862	0.00000000000000000000000000000000000000000000000000
13	7	6	0.98589587045954845873708424364378207292612934682994	0.00000000000000000000000000000000000000000000000000
13	7	7	0.99268801124035986319128350227204058492126380541325	0.00000000000000000000000000000000000000000000000000
13	7	8	0.99625806750397379033803269427689447511687378287258	0.00000000000000000000000000000000000000000000000000
13	7	9	0.99810085488639360199402958145847387201610416507261	0.00000000000000000000000000000000000000000000000000
13	7	10	0.99904120321947801800537256197782075518612943677070	0.00000000000000000000000000000000000000000000000000
13	8	*	complex conjugate of block m=13, r=6 above	
13	9	*	complex conjugate of block m=13, r=5 above	
13	10	*	complex conjugate of block m=13, r=4 above	
13	11	*	complex conjugate of block m=13, r=3 above	
13	12	*	complex conjugate of block m=13, r=2 above	
14	1	2	1.20852298788849289210218257141340626146698360088663	0.00000000000000000000000000000000000000000000000000
14	1	3	1.04873331857291134083344502866560177765720742852156	0.00000000000000000000000000000000000000000000000000
14	1	4	1.01425542517703495664765027789155676020124516527161	0.00000000000000000000000000000000000000000000000000
14	1	5	1.00446399461742823756409676646505995098654273503530	0.00000000000000000000000000000000000000000000000000
14	1	6	1.00143856448124130896185596834315683606471375078694	0.00000000000000000000000000000000000000000000000000
14	1	7	1.00047033381411230822936281799717964047839294121337	0.00000000000000000000000000000000000000000000000000
14	1	8	1.00015500553185219551513762482694771493808839934523	0.00000000000000000000000000000000000000000000000000
14	1	9	1.00005132040163916858198505001645660206378249002591	0.00000000000000000000000000000000000000000000000000
14	1	10	1.00001703782285132215209223117368548434610859674245	0.00000000000000000000000000000000000000000000000000
14	2	1	1.17788490614348768447699320606769646308407281193961	0.15010207776461851033980434077571796116283286112483
14	2	2	1.06537623481662618830894077155842681122710193447386	0.06627539283767446838162979466383442628104244574261
14	2	3	1.02140474661914026269374873051451447072169365236250	0.02576061512443766596772240355532003964848650767777
14	2	4	1.00685403444390674895139059878910381659680840753372	0.00938384938710366476939118746943705405785634017687
14	2	5	1.00220508703670350171759588103112234105973206171447	0.00329637015958187787582859594794636928487486207286
14	2	6	1.00071655423395203762834974350231619907118814733170	0.00113369883373689574372328679933946706222992900288
14	2	7	1.00023488493846334820712595595214868897567045344624	0.00038504060645246625248639368748072560670613729402
14	2	8	1.00007747320221640540587238950121998354190105128767	0.00012979507852314535358319993524809105520859968740
14	2	9	1.00002565706610714905407146710081843293764762169984	0.00004355711673910688609353645878829900017096627327
14	2	10	1.00000851857601932502492640185271188410286271875238	0.00001457775054143153413483470835462096600604730809
14	3	1	0.71777940036734023738564210708568325350212511906916	0.10122951412939744931550286351780535716253528144991
14	3	2	0.92117167324613273884912580158198653238569223110527	0.05845772453345597741221510776937535472244381944376
14	3	3	0.97698268153222650344053943899937160887812023275785	0.02464690664510144071123037419002416912942701310806
14	3	4	0.99296093442565441228729866181152704443450830322583	0.00923617199939483130735845535679633289861872866350
14	3	5	0.99777422750499428563012953192523023256375356523300	0.00327769367405741963882726090422089427880013682168
14	3	6	0.99928116729631506473475566005687455151534586920616	0.00113141255788525141441494915111192104550918174619
14	3	7	0.99976486602289849318469890691560813380720109445886	0.00038476712815618209702991788635673042953956187720
14	3	8	0.99992249966685681420546328177707980019313011208046	0.00012976291118766184887604513688128441182910013415
14	3	9	0.99997433997995657736626545706526628855334068233991	0.00004355338005898880262283752278818241891512512635
14	3	10	0.99999148110206733430502305517352886191394494402336	0.00001457732054836704880044305237520882203858279786
14	4	1	0.59370520586186297439231264897468151499616734308252	0.00000000000000000000000000000000000000000000000000
14	4	2	0.86394410290836828532626929799041247359836605351493	0.00000000000000000000000000000000000000000000000000
14	4	3	0.95667518575084187547950733813173770543266851172834	0.00000000000000000000000000000000000000000000000000
14	4	4	0.98624449462121823540469270342856385186361275448452	0.00000000000000000000000000000000000000000000000000
14	4	5	0.99558562698872159097192691801286695849619005918367	0.00000000000000000000000000000000000000000000000000
14	4	6	0.99856653049372143433728867404950735700648952352084	0.00000000000000000000000000000000000000000000000000
14	4	7	0.99953019975369352884322513684033621084851740896742	0.00000000000000000000000000000000000000000000000000
14	4	8	0.99984505108368171193583805275872945263113921991893	0.00000000000000000000000000000000000000000000000000
14	4	9	0.99994868566273378639154036718631039718138010963493	0.00000000000000000000000000000000000000000000000000
14	4	10	0.99998296283139483528067374378609556744878162619312	0.00000000000000000000000000000000000000000000000000
14	5	*	complex conjugate of block m=14, r=3 above	
14	6	*	complex conjugate of block m=14, r=2 above	

Table 22. Closed forms of Dirichlet series.

m	r	s	$L(s,\chi_r)$	$L(s,\chi_r)$ [27]
2	1	s	$(1-2^{-s})\zeta(s)$	
2	1	2	$\pi^2/8$	A111003
2	1	3	$7\zeta(3)/8=-\psi''(1/4)/64-\pi^3/32$	
2	1	4	$\pi^4/96$	
3	1	s	$(1-3^{-s})\zeta(s)$	
3	1	2	$4\pi^2/27$	
3	1	3	$26\zeta(3)/27=-\psi''(1/3)/27-4\pi^3/(81\sqrt{3})$	
3	1	4	$8\pi^4/729$	
3	1	5	$242\zeta(5)/243$	
3	2	1	$\pi/(3\sqrt{3})$	A073010
3	2	2	$2\psi'(1/3)/9-4\pi^2/27$	A086724
3	2	3	$4\pi^3/(81\sqrt{3})$	A129404
3	2	4	$\psi'''(1/3)/243-8\pi^4/729$	
4	2	1	$\pi/4$	A003881
4	2	2	$\psi'(1/4)/8-\pi^2/8$	A006752
4	2	3	$\pi^3/32$	A153071
4	2	4	$\psi'''(1/4)/768-\pi^4/96$	A175572
5	1	s	$(1-5^{-s})\zeta(s)$	
5	1	2	$4\pi^2/25$	
5	1	3	$124\zeta(3)/125$	
5	2	1	$\frac{\pi}{5}[1/\sqrt{5-2\sqrt{5}}+i/\sqrt{5+2\sqrt{5}}]$	
5	2	2	$\frac{2}{25}[\psi'(1/5)-4\pi^2/(5-\sqrt{5})-i\psi'(2/5)+4i\pi^2/(5+\sqrt{5})]$	
5	3	1	$2\log[(1+\sqrt{5})/2]/\sqrt{5}$	A086466
5	3	2	$4\pi^2/(25\sqrt{5})$	
6	1	s	$(1+6^{-s}-2^{-s}-3^{-s})\zeta(s)$	
6	1	2	$\pi^2/9$	A100044
6	2	1	$\pi/(2\sqrt{3})$	A093766
6	2	3	$\pi^3/(18\sqrt{3})$	

The case $s=1$ has been included for the non-principal characters, where

$$\sum_{n=1}^{m}\chi_r(n)=0,\quad r>1 \tag{7}$$

leads to cancellations which ensure convergence [13]. The inner sum in

$$L(s,\chi)=\sum_{k\ge 0}\sum_{n=1}^{m}\frac{\chi(n)}{(km+n)^s} \tag{8}$$

is converted to a rational polynomial in k with denominator degree at least 2 larger than numerator degree, and the k-sum becomes essentially an overlay of Harmonic sums. Table 22 shows the basic examples. ψ are the polygamma functions [1, §6.4].

Remark 3. *For the principal character* χ_1, [4]

$$L(s,\chi_1)=\zeta(s)\prod_{p|m}\left(1-\frac{1}{p^s}\right), \tag{9}$$

so $L(s,\chi_1)$ is a rational multiple of π^s if s is even [5, 29, 14].

2.3. **First Derivative.** The derivative of (3) with respect to s is

$$L'(s,\chi) = -\sum_{n\ge 2}\chi(n)\frac{\log n}{n^s}. \tag{10}$$

For $s>1$, values are calculated by direct application of (3),

$$L'(s,\chi) = \frac{1}{m^s}\left(-\log m\sum_{n=1}^{m}\chi(n)\zeta(s,n/m) + \sum_{n=1}^{m}\chi(n)\zeta'(s,n/m)\right). \tag{11}$$

For $s=1$, each period of the character $\chi(n)=\chi(n+m)$ in

$$\begin{aligned} L'(1,\chi) &= -\sum_{k\ge 0}\sum_{n=1}^{m}\chi(n)\frac{\log(km+n)}{km+n} \\ &= -\sum_{n=1}^{\lfloor (m-1)/2\rfloor}\chi(n)\frac{\log(n)}{n} - \sum_{k=1}^{\infty}\sum_{n=-\lfloor (m-1)/2\rfloor}^{\lfloor (m-1)/2\rfloor}\chi(n)\frac{\log(km+n)}{km+n} \end{aligned} \tag{12}$$

is expanded in a series around a center $X\equiv km$ to which the indices have distances $\epsilon=-n$,

(13)

$$\begin{aligned} \frac{\log(X-\epsilon)}{X-\epsilon} &= \frac{\log X}{X} + (-1+\log X)\frac{\epsilon}{X^2} + (-\frac{3}{2}+\log X)\frac{\epsilon^2}{X^3} + (-\frac{11}{6}+\log X)\frac{\epsilon^3}{X^4}+\cdots \\ &= \frac{\log X}{X} + \sum_{j\ge 1}(-\sum_{u=1}^{j}\frac{1}{u}+\log X)\frac{\epsilon^j}{X^{j+1}}. \end{aligned}$$

Remark 4. *The generalization to other s is* [23]

$$\frac{\log(X-\epsilon)}{(X-\epsilon)^s} = \frac{\log X}{X} + \sum_{j=1}^{\infty}\left[-\sum_{u=0}^{j-1}\frac{(s)_u}{(j-u)u!} + \frac{(s)_j}{j!}\log X\right]\frac{\epsilon^j}{X^{s+j}}, \tag{14}$$

where $(s)_j\equiv s(s+1)\cdots(s+j-1)$, $(s)_0=1$, is Pochhammer's symbol.

Multiplication with $\chi(n)$ and summation over the classes n for any non-principal character cancels the leading $\log X/X$ term—see (7)—and generates series [18, 15]

$$L'(1,\chi) = -\sum_{n=1}^{\lfloor (m-1)/2\rfloor}\chi(n)\frac{\log(n)}{n} - \sum_{k=1}^{\infty}\sum_{j\ge 2}\frac{\alpha_j+\beta_j\log(km)}{(km)^j} \tag{15}$$

with constants α_j and β_j depending on χ and on the order j.

Remark 5. *Depending on the parity of the character, either the terms with even or those with odd j vanish. This reduction of terms has been driving the choice of X.*

Summations

$$\alpha_j\sum_{k\ge 1}\frac{1}{(km)^j} = \frac{\alpha_j}{m^j}\zeta(j), \tag{16}$$

and

$$\beta_j \sum_{k\ge 1} \frac{\log(km)}{(km)^j} = \frac{\beta_j}{m^j}[\log m\, \zeta(j) - \zeta'(j)] \tag{17}$$

rewrite L' as a series over Zeta Functions. The k-terms up to some freely chosen $M \approx 40$ are summed directly to accelerate convergence:

Algorithm 1.

$$L'(1,\chi) = -\sum_{n=2}^{Mm+\lfloor (m-1)/2\rfloor} \chi(n)\frac{\log n}{n} \tag{18}$$
$$-\sum_{j\ge 2}\left[\frac{\alpha_j}{m^j}\zeta(j) + \frac{\beta_j}{m^j}[\zeta(j)\log m - \zeta'(j)] - \sum_{1\le k\le M}\frac{\alpha_j + \beta_j \log(km)}{(km)^j}\right].$$

Remark 6. *This can probably be formulated much more efficiently in terms of the Gauss sums* [16].

The results for $L'(s,\chi_r)$ start as follows:

m	r	s	Re(L')	Im(L')
2	1	2	-0.41811583807616962590828560617781139711923775611996	0.00000000000000000000000000000000000000000000000000
2	1	3	-0.06921016836272074748362627760768005927930796015366	0.00000000000000000000000000000000000000000000000000
2	1	4	-0.01771623065878015673143625539935151371812832223282	0.00000000000000000000000000000000000000000000000000
2	1	5	-0.00522011393348035186442563804164248718723344171968	0.00000000000000000000000000000000000000000000000000
2	1	6	-0.00163309262853731373611967895541271800687857377431	0.00000000000000000000000000000000000000000000000000
2	1	7	-0.00052595465151771844771643934267364011741770800196	0.00000000000000000000000000000000000000000000000000
2	1	8	-0.00017197075267004922546732124391835435192533514509	0.00000000000000000000000000000000000000000000000000
2	1	9	-0.00005669458151516296038034817251083265989289171083	0.00000000000000000000000000000000000000000000000000
2	1	10	-0.00001877753891961746692900793448874128383181319749	0.00000000000000000000000000000000000000000000000000
3	1	2	-0.63258236162649762708200575756500570930611425093721	0.00000000000000000000000000000000000000000000000000
3	1	3	-0.14187732701989244492420427021033210559381975265229	0.00000000000000000000000000000000000000000000000000
3	1	4	-0.05338083539288848748859404708635490112000898019216	0.00000000000000000000000000000000000000000000000000
3	1	5	-0.02376820291781272030522239322060071116333138263190	0.00000000000000000000000000000000000000000000000000
3	1	6	-0.01130138631860072648326657049664519466207000887400	0.00000000000000000000000000000000000000000000000000
3	1	7	-0.00552422641475663255895936723479880503970450093810	0.00000000000000000000000000000000000000000000000000
3	1	8	-0.00273338165371887933413456355075883756420758755113	0.00000000000000000000000000000000000000000000000000
3	1	9	-0.00135998290214225322961491777490967865422189100126	0.00000000000000000000000000000000000000000000000000
3	1	10	-0.00067839760411486935474878475316385767840903369166	0.00000000000000000000000000000000000000000000000000
3	2	1	0.22266298696860150948666026276474436188657161605715	0.00000000000000000000000000000000000000000000000000
3	2	2	0.13489092252345807409064587918196369202754729945570	0.00000000000000000000000000000000000000000000000000
3	2	3	0.07527569062722762962331216028061202962234911131947	0.00000000000000000000000000000000000000000000000000
3	2	4	0.04007380990190515691818068756656827444733394284007	0.00000000000000000000000000000000000000000000000000
3	2	5	0.02075846279913862669323460796527515934928909079509	0.00000000000000000000000000000000000000000000000000
3	2	6	0.01058516532249050924386643900827692965142388036618	0.00000000000000000000000000000000000000000000000000
3	2	7	0.00534970119320939519232648443149160751998413890293	0.00000000000000000000000000000000000000000000000000
3	2	8	0.00269034603907081869458536474978701888606095226299	0.00000000000000000000000000000000000000000000000000
3	2	9	0.00134930467679590672574612091099997528531317044446	0.00000000000000000000000000000000000000000000000000
3	2	10	0.00067573917582867185510738563249105312628698915310	0.00000000000000000000000000000000000000000000000000
4	1	*	same block as m=2, r=1 above	
4	2	1	0.19290131679691242936318976402803278524509686762001	0.00000000000000000000000000000000000000000000000000
4	2	2	0.08158073611659279510291216978594115145773887519886	0.00000000000000000000000000000000000000000000000000
4	2	3	0.03157707945712738788724616744286091406753844890852	0.00000000000000000000000000000000000000000000000000
4	2	4	0.01157054792451164165145272825991055112687726366004	0.00000000000000000000000000000000000000000000000000
4	2	5	0.00409487498794859210069139346473920353166828166325	0.00000000000000000000000000000000000000000000000000
4	2	6	0.00141739694744032189042296189559147079679885558589	0.00000000000000000000000000000000000000000000000000
4	2	7	0.00048373355305791317333505432592425184709053568205	0.00000000000000000000000000000000000000000000000000
4	2	8	0.00016362101836044233132872653651523994833718831247	0.00000000000000000000000000000000000000000000000000
4	2	9	0.00005503463265438472183757623135734483380574563555	0.00000000000000000000000000000000000000000000000000
4	2	10	0.00001844662524176733922410472153863013759986794730	0.00000000000000000000000000000000000000000000000000
5	1	2	-0.79414955411761222469720192670858817205698401344150	0.00000000000000000000000000000000000000000000000000
5	1	3	-0.18106414531976634265272520503820279224780330702467	0.00000000000000000000000000000000000000000000000000
5	1	4	-0.06601391659734281254780710518639245085828845039340	0.00000000000000000000000000000000000000000000000000

5	1	5	-0.02803059823039338648618168977463624264981206724792	0.00000000000000000000000000000000000000000000000000
5	1	6	-0.01274655216161707783567191612822547752813257089256	0.00000000000000000000000000000000000000000000000000
5	1	7	-0.00601266692474181544340142251056728527848680260418	0.00000000000000000000000000000000000000000000000000
5	1	8	-0.00289780816429208681148594110523507826723163904303	0.00000000000000000000000000000000000000000000000000
5	1	9	-0.00141515581506736113080511793801987035047486225809	0.00000000000000000000000000000000000000000000000000
5	1	10	-0.00069686796644059205567056482789101956496211513721	0.00000000000000000000000000000000000000000000000000
5	2	1	0.15455633174545896654216402988318223319631324458045	-0.04416511200957409045660097689544036344145251055763
5	2	2	0.05050979313230396347463529389188839699537071768701	-0.06288371253648251957890661793245158835052207797768
5	2	3	0.01507105948497285624292789584118242153531510296102	-0.04799063018512213874476530058592708250322082165337
5	2	4	0.00424184607216138083790041032739339236339500612291	-0.03010137917480593897728507435161327269043871511918
5	2	5	0.00114861565079184996476050087704459470274830844873	-0.01719585118829897996253763843920629701525900442239
5	2	6	0.00030304940840236474636155422803615095905036896641	-0.00933236123888727278715072907924912021531877618753
5	2	7	0.00007856512689696300450206984503991405022425728234	-0.00491427713632333387301179780383192804031806346576
5	2	8	0.00002012751493845638650556527725292650875212564245	-0.00254037671848717154548440828698513402355480977756
5	2	9	0.00000511524664696431121123658694448119840671360542	-0.00129802077872988575355809063127949978984266825936
5	2	10	0.00000129298554161385699494858629450007762837834559	-0.00065830142225906966333365915041268268627304973615
5	3	1	0.35624064703076149886468458637127319729600101706517	0.00000000000000000000000000000000000000000000000000
5	3	2	0.20266211487080801527488733720785334085091900565061	0.00000000000000000000000000000000000000000000000000
5	3	3	0.10388625869208176709713993389724045176780995165888	0.00000000000000000000000000000000000000000000000000
5	3	4	0.05104273008990150830689821112259545031075078923703	0.00000000000000000000000000000000000000000000000000
5	3	5	0.02473640090329101340316683838755544717970256852729	0.00000000000000000000000000000000000000000000000000
5	3	6	0.01198055022616085816574664564068743667497734955752	0.00000000000000000000000000000000000000000000000000
5	3	7	0.00582939152208435310011173652194507030486352146162	0.00000000000000000000000000000000000000000000000000
5	3	8	0.00285323832859547683771660707120286577974119330454	0.00000000000000000000000000000000000000000000000000
5	3	9	0.00140420989593554401404925170274317235732888543766	0.00000000000000000000000000000000000000000000000000
5	3	10	0.00069416308492606798469708268390211537633278700735	0.00000000000000000000000000000000000000000000000000
5	4	*	complex conjugate of block m=5, r=2 above	
6	1	2	-0.22106312441035038531715323839767664970922350123298	0.00000000000000000000000000000000000000000000000000
6	1	3	-0.02384978528249851972724435786366761692582963218901	0.00000000000000000000000000000000000000000000000000
6	1	4	-0.00373531726099356257429393164827326376542863699524	0.00000000000000000000000000000000000000000000000000
6	1	5	-0.00065714165364763572457150480485955255715025474529	0.00000000000000000000000000000000000000000000000000
6	1	6	-0.00012165894201189454533419008342071367659157573371	0.00000000000000000000000000000000000000000000000000
6	1	7	-0.00002313970297458484939698379205383906267748210396	0.00000000000000000000000000000000000000000000000000
6	1	8	-0.00000447269734236844559590073628663864560648820581	0.00000000000000000000000000000000000000000000000000
6	1	9	-0.00000087354855781095376482688783796727343060881158	0.00000000000000000000000000000000000000000000000000
6	1	10	-0.00000017180828814384318028884506434742083274351198	0.00000000000000000000000000000000000000000000000000
6	2	1	0.12445616121617402941801461275592899266970070044133	0.00000000000000000000000000000000000000000000000000
6	2	2	0.03322426198835216013333563169691854919063967469441	0.00000000000000000000000000000000000000000000000000
6	2	3	0.00809032537282812029136942953937439659172001832609	0.00000000000000000000000000000000000000000000000000
6	2	4	0.00185491362365415188068745267800728147981196264481	0.00000000000000000000000000000000000000000000000000
6	2	5	0.00040825816321425536634349133557907194970873493671	0.00000000000000000000000000000000000000000000000000
6	2	6	0.00008735165752376249519608704307916158530192375585	0.00000000000000000000000000000000000000000000000000
6	2	7	0.00001832422685076895091380802807933370090496864408	0.00000000000000000000000000000000000000000000000000
6	2	8	0.00000379090798028155362700114637172277474692570545	0.00000000000000000000000000000000000000000000000000
6	2	9	0.00000077660160018611457020318183843244621675802787	0.00000000000000000000000000000000000000000000000000
6	2	10	0.00000015799247627892726540465636999955610941416444	0.00000000000000000000000000000000000000000000000000
7	1	2	-0.85309025126436133004946970313518822354904214298483	0.00000000000000000000000000000000000000000000000000
7	1	3	-0.19072909719908906534161215116264945239873942439401	0.00000000000000000000000000000000000000000000000000
7	1	4	-0.06800538708274220825519746102881664932138469374313	0.00000000000000000000000000000000000000000000000000
7	1	5	-0.02845202516805281803177421465758424399556399041637	0.00000000000000000000000000000000000000000000000000
7	1	6	-0.01283522907322728553125394548131939197516809981662	0.00000000000000000000000000000000000000000000000000
7	1	7	-0.00603112705456884998744292659522213499402046341079	0.00000000000000000000000000000000000000000000000000
7	1	8	-0.00290161312373374662114970629774909396918409617538	0.00000000000000000000000000000000000000000000000000
7	1	9	-0.00141593387383728528835417591601881706412707398657	0.00000000000000000000000000000000000000000000000000
7	1	10	-0.00069702611007132015872795136262345138110120062692	0.00000000000000000000000000000000000000000000000000
7	2	1	0.14361034321910282672247043842346402006888045267441	-0.17369230259554111555029265006994697564320400789026

3. Prime Zeta Modulo Functions

3.1. **Dirichlet Prime L-functions.** Restricting the summation of the Dirichlet series to primes defines functions $S(s, \chi)$ that mediate between Dirichlet series and the Prime Zeta Function. We recall an algorithmic standard of numerical computation [17, 7, 11]. The constituents are the numbers

$$l_p \equiv \frac{\chi(p)}{p^s} \tag{19}$$

with Mercator series

$$-\log(1-\chi(p)/p^s) = -\log(1-l_p) = \sum_{t\ge 1}\frac{1}{t}l_p^t. \tag{20}$$

Because the Dirichlet inverse of $1/t$ is $\mu(t)/t$, the Möbius inverse is [19, 9, 31]

$$l_p = -\sum_{t\ge 1}\frac{\mu(t)}{t}\log(1-[\chi(p)/p^s]^t). \tag{21}$$

Definition 2. *(Dirichlet Prime L-series)*

$$S(s,\chi) \equiv \sum_p \frac{\chi(p)}{p^s}. \tag{22}$$

To accelerate convergence, the sum is split at some threshold value M,

$$S(s,\chi) = \sum_{p\le M}\frac{\chi(p)}{p^s} + \sum_{p>M}\frac{\chi(p)}{p^s}, \quad S(M,s,\chi) \equiv \sum_{p>M}\frac{\chi(p)}{p^s}, \tag{23}$$

in which (21) is inserted at the right hand side:

$$S(s,\chi) = \sum_{p\le M}\frac{\chi(p)}{p^s} - \sum_{t\ge 1}\frac{\mu(t)}{t}\sum_{p>M}\log[1-\chi^t(p)p^{-st}]. \tag{24}$$

Cutting also through the Euler products (3)

$$L(s,\chi) = \prod_{p\le M}\frac{1}{1-\chi(p)p^{-s}}\prod_{p>M}\frac{1}{1-\chi(p)p^{-s}} \tag{25}$$

proposes

Definition 3. *(Incomplete Dirichlet Euler Product)*

$$L(M,s,\chi) \equiv L(s,\chi)\prod_{p\le M}(1-\frac{\chi(p)}{p^s}) = \prod_{p>M}\frac{1}{1-\chi(p)p^{-s}}. \tag{26}$$

The logarithm

$$\log L(M,s,\chi) = -\sum_{p>M}\log(1-\chi(p)p^{-s}) \tag{27}$$

is inserted into the right hand side of (24).

Algorithm 2.

$$S(s,\chi) = \sum_{p\le M}\frac{\chi(p)}{p^s} + \sum_{t\ge 1}\frac{\mu(t)}{t}\log L(M,st,\chi^t). \tag{28}$$

This generates a table of $S(s,\chi_r)$:

m	r	s	Re(S)	Im(S)
2	1	2	0.20224742004106549850654336483224793417323134323989	0.00000000000000000000000000000000000000000000000000
2	1	3	0.04976263929944353642311331466570670097541212192615	0.00000000000000000000000000000000000000000000000000
2	1	4	0.01449313976424684494261929593315787016204105971484	0.00000000000000000000000000000000000000000000000000
2	1	5	0.00450501748392425713281824253885571113169727672665	0.00000000000000000000000000000000000000000000000000
2	1	6	0.00144508685063651295413367326605939920958594187454	0.00000000000000000000000000000000000000000000000000
2	1	7	0.00047133285613359253512413872944872308918332888531	0.00000000000000000000000000000000000000000000000000
2	1	8	0.00015515536651783056052343914268308052297714451207	0.00000000000000000000000000000000000000000000000000
2	1	9	0.00005134257496245066307358514078311753682292034974	0.00000000000000000000000000000000000000000000000000
3	1	2	0.34113630892995438739543225372113682306212023212878	0.00000000000000000000000000000000000000000000000000
3	1	3	0.13772560226240649938607627762866966393837508488911	0.00000000000000000000000000000000000000000000000000
3	1	4	0.06464746075190116593027361692081219114969538070250	0.00000000000000000000000000000000000000000000000000

3	1	5	0.03163979114647569746203634953474048479424871705587	0.00000000000000000000000000000000000000000000000000
3	1	6	0.01569834473815365973053970893135432376376975531762	0.00000000000000000000000000000000000000000000000000
3	1	7	0.00782658548530597479392615061788036460724460003300	0.00000000000000000000000000000000000000000000000000
3	1	8	0.00390898957624195798012410977216029436233090156130	0.00000000000000000000000000000000000000000000000000
3	1	9	0.00195366231153715980294047535060885548327417269948	0.00000000000000000000000000000000000000000000000000
3	2	1	-0.64194483853319570866139263972173431667541104401589	0.00000000000000000000000000000000000000000000000000
3	2	2	-0.27470520828551848628957789816086063009994509882225	0.00000000000000000000000000000000000000000000000000
3	2	3	-0.13052475556852058147112238237019273405338475461522	0.00000000000000000000000000000000000000000000000000
3	2	4	-0.06372483064121439204992455625399328246981734399486	0.00000000000000000000000000000000000000000000000000
3	2	5	-0.03151448029153218645003121115269230790129727571063	0.00000000000000000000000000000000000000000000000000
3	2	6	-0.01568088471610726339651562337306385986464759578854	0.00000000000000000000000000000000000000000000000000
3	2	7	-0.00782412274079501095457725272363147659937126191727	0.00000000000000000000000000000000000000000000000000
3	2	8	-0.00390864007053523196268812155398435360012142057735	0.00000000000000000000000000000000000000000000000000
3	2	9	-0.00195361255478026763866872767417697973059282808296	0.00000000000000000000000000000000000000000000000000
4	1	*	same block as m=2, r=1 above	
4	2	1	-0.33498132529999318106331712148754357377997538075508	0.00000000000000000000000000000000000000000000000000
4	2	2	-0.09461989289295015794518679014917480960188034024972	0.00000000000000000000000000000000000000000000000000
4	2	3	-0.03225247383350252743465978303821335087318799948530	0.00000000000000000000000000000000000000000000000000
4	2	4	-0.01119397145618826174462577330588016650706523982717	0.00000000000000000000000000000000000000000000000000
4	2	5	-0.00385806941548066209579442616677354317681180931505	0.00000000000000000000000000000000000000000000000000
4	2	6	-0.00131658492330696537193003498997224601161685078059	0.00000000000000000000000000000000000000000000000000
4	2	7	-0.00044569595893400199862208556511570721754211538618	0.00000000000000000000000000000000000000000000000000
4	2	8	-0.00015003262315703762090694227805813265008262330846	0.00000000000000000000000000000000000000000000000000
4	2	9	-0.00005031836933794511539693316106370484242184103002	0.00000000000000000000000000000000000000000000000000
5	1	2	0.41224742004106549850654336483224793417323134323989	0.00000000000000000000000000000000000000000000000000
5	1	3	0.16676263929944353642311331466570670097541212192615	0.00000000000000000000000000000000000000000000000000
5	1	4	0.07539313976424684494261929593315787016204105971484	0.00000000000000000000000000000000000000000000000000
5	1	5	0.03543501748392425713281824253885571113169727672665	0.00000000000000000000000000000000000000000000000000
5	1	6	0.01700608685063651295413367326605939920958594187454	0.00000000000000000000000000000000000000000000000000
5	1	7	0.00827103285613359253512413872944872308918332888531	0.00000000000000000000000000000000000000000000000000
5	1	8	0.00405884536651783056052343914268308052297714451207	0.00000000000000000000000000000000000000000000000000
5	1	9	0.00200395557496245066307358514078311753682292034974	0.00000000000000000000000000000000000000000000000000
5	2	1	0.05558076757159122797438373110392029200397141552077	0.25744607231019845831553919953785168450933618650489
5	2	2	0.00587745504741124399339243532481344811691373319858	0.15525350477709758586525784166633774678670632020307
5	2	3	0.00061185910379617932279038303371260872339769646209	0.09055484241783400177149050749754610178341378435528
5	2	4	0.00006064197279815635344055135957389107001014491352	0.05054452415058793994664192927792361208763038172014
5	2	5	0.00000580003308992913574711743668771979341610600613	0.02719213801008999533003275544381628939733258482756
5	2	6	0.00000054287198362827368021161276466778954793060719	0.01426158552668629463074677498250709051205907255207
5	2	7	0.00000005018054931017382182274452449237991840398809	0.00735645310955466879930470745507895671903321075689
5	2	8	0.00000000460549113371719109246140782054160828361855	0.00375400658116490237713537870580380742684581756294
5	2	9	0.00000000042097057300122419349132192179194835304856	0.00190234443109101218028442898933440462363147712115
5	3	1	-1.00799654793986117226166607551267856699903195664933	0.00000000000000000000000000000000000000000000000000
5	3	2	-0.38071874461129711635917015923497145487567001845229	0.00000000000000000000000000000000000000000000000000
5	3	3	-0.16474980378367446941888405281224241856194292947043	0.00000000000000000000000000000000000000000000000000
5	3	4	-0.07523495147825830029432718746871438806203449993040	0.00000000000000000000000000000000000000000000000000
5	3	5	-0.03542159890156654964644026157148756761652565948421	0.00000000000000000000000000000000000000000000000000
5	3	6	-0.01700490923570389119218181500524906761727983414130	0.00000000000000000000000000000000000000000000000000
5	3	7	-0.00827092778629868551875580553631885600410697969667	0.00000000000000000000000000000000000000000000000000
5	3	8	-0.00405883591198651733558468768304964106526021571057	0.00000000000000000000000000000000000000000000000000
5	3	9	-0.00200395472034918349581165226497195360585348682960	0.00000000000000000000000000000000000000000000000000
6	1	2	0.09113630892995438739543225372113682306212023212878	0.00000000000000000000000000000000000000000000000000
6	1	3	0.01272560226240649938607627762866966393837508488911	0.00000000000000000000000000000000000000000000000000
6	1	4	0.00214746075190116593027361692081219114969538070250	0.00000000000000000000000000000000000000000000000000
6	1	5	0.00038979114647569746203634953474048479424871705587	0.00000000000000000000000000000000000000000000000000
6	1	6	0.00007334473815365973053970893135432376376975531762	0.00000000000000000000000000000000000000000000000000
6	1	7	0.00001408548530597479392615061788036460724460003300	0.00000000000000000000000000000000000000000000000000
6	1	8	0.00000273957624195798012410977216029436233090156130	0.00000000000000000000000000000000000000000000000000
6	1	9	0.00000053731153715980294047535060885548327417269948	0.00000000000000000000000000000000000000000000000000
6	2	1	-0.14194483853319570866139263972173431667541104401589	0.00000000000000000000000000000000000000000000000000
6	2	2	-0.02470520828551848628957789816086063009994509882225	0.00000000000000000000000000000000000000000000000000
6	2	3	-0.00552475556852058147112238237019273405338475461522	0.00000000000000000000000000000000000000000000000000
6	2	4	-0.00122483064121439204992455625399328246981734399486	0.00000000000000000000000000000000000000000000000000
6	2	5	-0.00026448029153218645003121115269230790129727571063	0.00000000000000000000000000000000000000000000000000
6	2	6	-0.00005588471610726339651562337306385986464759578854	0.00000000000000000000000000000000000000000000000000
6	2	7	-0.00001162274079501095457725272363147659937126191727	0.00000000000000000000000000000000000000000000000000
6	2	8	-0.00000239007053523196268812155398435360012142057735	0.00000000000000000000000000000000000000000000000000
6	2	9	-0.00000048755478026763866872767417697973059282808296	0.00000000000000000000000000000000000000000000000000

Remark 7. *For the principal character* χ_1*,*

$$S(s,\chi_1) = P(s) - 2^{-s}, \quad m=2, \tag{29}$$

$$S(s,\chi_1) = P(s) - 3^{-s}, \quad m=3, \tag{30}$$

$$S(s,\chi_1) = P(s) - 2^{-s} - 3^{-s}, \quad m=6, \tag{31}$$

$$S(s,\chi_1) = P(s) - \sum_{\substack{p\le m\\ (p,m)>1}} p^{-s}, \tag{32}$$

where $P(s)$ *is the Prime Zeta Function* [12, 22, 20].

3.2. **Prime Zeta Modulo Functions.** The sum (22) over all primes may be divided into distinct residue classes

$$S(s,\chi) = \sum_{n=1}^{m} \sum_{p\equiv n \pmod m} \frac{\chi(p)}{p^s} = \sum_{n=1}^{m} \chi(n) \sum_{p\equiv n \pmod m} \frac{1}{p^s}. \tag{33}$$

This defines non-overlapping subseries of the Prime Zeta Function.

Definition 4. *(Prime Zeta Modulo Functions)*

$$P_{m,n}(s) \equiv \sum_{p\equiv n \pmod m} \frac{1}{p^s}; \quad P_{m,n}(M,s) \equiv \sum_{\substack{p\equiv n \pmod m\\ p>M}} \frac{1}{p^s}. \tag{34}$$

Remark 8. *The Prime Zeta-function* $P(s)$ *is recovered if the filtering with the modular combs is undone:*

$$\sum_{n=1}^{m} P_{m,n}(s) = P(s). \tag{35}$$

The matrix of the characters is orthogonal [13], so the inversion of (33) is

$$P_{m,n}(s) = \frac{1}{\varphi(m)} \sum_{r=1}^{\varphi(m)} \bar\chi_r(n) S(s,\chi_r) + \sum_{\substack{p=n\le m\\ (p,m)>1}} \frac{1}{p^s}, \tag{36}$$

which acts on the individual terms and therefore remains valid for the incomplete sums,

$$P_{m,n}(M,s) = \frac{1}{\varphi(m)} \sum_{r=1}^{\varphi(m)} \bar\chi_r(n) S(M,s,\chi_r) + \sum_{\substack{M<p=n\le m\\ (p,m)>1}} \frac{1}{p^s}. \tag{37}$$

The second terms take care of small primes for which $\chi(p)=0$.

The case $m=2$ is simply $P_{2,1}(s) = S(s,\chi_1)$ because there is only one even prime, repeating values provided above. The block for $m=4$, $s\le 8$ has been tabulated earlier [25]. Starting at $m\ge 3$, the $P_{m,n}(s)$ are:

m	n	s	P
3	1	2	0.033215550322217950552927177780138096481087566665327
3	2	2	0.307920758607736436842505075940998726581032665475551
3	1	3	0.003600423346942958957476947629238464942495165136940
3	2	3	0.134125178915463540428599329999431198995879919752170
3	1	4	0.000461315055343386940174530333409454339939018353820
3	2	4	0.064186145696557778990099086587402736809756362348680
3	1	5	0.000062655427471755506002569191024088446475720672620
3	2	5	0.031577135719003941956033780343716396347772996383250
3	1	6	0.000008730011023198167012042779145231949561079764540

3	2	6	0.01568961472713046156352766615220909181420867555308
3	1	7	0.00000123137225548191967444894712444400393666905787
3	2	7	0.00782535411305049287425170167075592060330793097513
3	1	8	0.00000017475285336300871799410908797038110474049198
3	2	8	0.00390881482338859497140611566307232398122616106932
3	1	9	0.00000002487837844608213587383821593787634067230826
3	2	9	0.00195363743315871372080460151239291760693350039122
3	1	10	0.00000000354755157005073612886511663730357082627733
3	2	10	0.00097666493907697988040868523830634809488146371787
4	1	2	0.05381376357405767028067828734153656228567550149509
4	3	2	0.14843365646700782822586507749071137188755584174481
4	1	3	0.00875508273297050449422676581374667505111206122043
4	3	3	0.04100755656647303192888654885196002592430006070572
4	1	4	0.00164958415402929159899676131363885182748790994383
4	3	4	0.01284355561021755334362253461951901833455314977101
4	1	5	0.00032347403422179751851190818604108397744273370580
4	3	5	0.00418154344970245961430633435281462715425454302085
4	1	6	0.00006425096366477379110181913804357659898454554698
4	3	6	0.00138083588697173916303185412801582261060139632757
4	1	7	0.00001281844859979526825102658216650793582060674956
4	3	7	0.00045851440753379726687311214728221515336272213574
4	1	8	0.00000256137168039646980824843231247393644726060181
4	3	8	0.00015259399483743409071519071037060658652988391026
4	1	9	0.00000051210281225277383832598985970634720053965986
4	3	9	0.00005083047215019788923525915092341118962238068988
4	1	10	0.00000010240775251510279580486929749957117091037360
4	3	10	0.00001693866668446511506004583085023984513096217169
5	1	2	0.01082089638114771753353951906172584388284719779619
5	2	2	0.27586829355163944664905730184997372065557850052458
5	3	2	0.12061478877454186078379946018363597386887218032151
5	4	2	0.00494344133373647354014708373691239576593346459761
5	1	3	0.00080913843084035641245250698022237496506614634497
5	2	3	0.12815553197969650234624459561826033077604565502678
5	3	3	0.03760068956186250057475408812071422899263187067150
5	4	3	0.00019727932704417708966212394650976624166844988289
5	1	4	0.00006986805789621433879330279589781606000671240287
5	2	4	0.06292928488592025628255758548942987059983408077138
5	3	4	0.01238476073533231633591565621150625851220369905124
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5	1	5	0.00000625466213439143946805396018589577550095731367
5	2	5	0.03131022310141769935983100374949396438572202646650
5	3	5	0.00411808509132770402979824830567767498838944163893
5	4	5	0.00000045462904446230372093652349817598208485130755
5	1	6	0.00000056583972496957732807037158491679285049223691
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5	4	6	0.00000002296774134130364785875882024900330256162971
5	1	7	0.00000005135773338184100299467054471296122828929121
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5	3	8	0.00015241702904363578545934235353127668363643127419
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5	1	9	0.00000000042413860329242757996461375187871653490432
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6	1	2	0.03321555032221795055292717778013809648108756665327
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6	1	3	0.00360042334694295895747694762923846494249516513694
6	5	3	0.00912517891546354042859932999943119899587991975217
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6	5	9	0.00000051243315871372080460151239291760693350039122
6	1	10	0.00000000354755157005073612886511663730357082627733
6	5	10	0.00000010243907697988040868523830634809488146371787
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7	2	2	0.25309105534677419585838789294127430386526757184310
7	3	2	0.11639715869790706431310433422397739230981168440572
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7	6	2	0.00700951134571340516000169799675822107497051929142
7	1	3	0.00005804318475905574364933421290638511768635690102
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7	1	7	0.00000000006176660482334123360593904477690007671484
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7	1	8	0.00000000000208617300170230400726962584682245349808
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7	5	8	0.00000256005892801890273373942320287012333528915764
7	6	8	0.00000000122602055737564094051849217646524322084646
7	1	9	0.00000000000007094349319404656244040521756222506490
7	2	9	0.00195312500056290352684233687342252458639878703099
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8	3	3	0.03795923737874621050097172756821732453455944389710
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8	3	6	0.00137232803938853551236876963794880625582124542544
8	5	6	0.00006420931345861947918404525870715385796928283867
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8	1	7	0.00000000244227705387080249482274869134009349823858
8	3	7	0.00045729980966012391873727903917860227855869182633
8	5	7	0.00001281600632274139744853175941781659572710851098
8	7	7	0.00000121459787367334813583310810361287480403030941
8	1	8	0.00000000014348051242565858216560023784749195377568
8	3	8	0.00015242051432548838564315112854960660265969307627
8	5	8	0.00000256122819988404414966626671223608895530682613
8	7	8	0.00000017348051194570507203958182099998387019083399
8	1	9	0.00000000000843564128353088757035062026925167636757
8	3	9	0.00005080569062402872344247706183871749797040231687
8	5	9	0.00000051209437661149030743841950908607794886329229
8	7	9	0.00000002478152616916579278208908469369165197837301
8	1	10	0.00000000000049610802941223825264777808779980414291
8	3	10	0.00001693512652591184376250291459388467539088986097
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8	7	10	0.00000000354015855327129754291625635516974007231072
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9	2	2	0.26039173226766023741732954512096620015351462748034
9	4	2	0.00752255314281775931487450953309022672123401907520
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9	7	2	0.02167141174185584311623524345298505307951813341174
9	8	2	0.00440747289217139603232051681702349713391944691181
9	1	3	0.00017008424576062139060622195078389739472568490955
9	2	3	0.12580572311897707843586565043069790850018156355436
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9	4	5	0.00000272908573930689364034153551975616239291768163
9	5	5	0.00032016550299262414302048445792287319178479768673
9	7	5	0.00005950746362025771484480376104430641274401410669
9	8	5	0.00000070750459002147370230256118556571980152038669
9	1	6	0.00000002165314420330617700577923467863736900218271
9	2	6	0.01562556625209709052016651772875887563938535907812
9	4	6	0.00000020831510574072208249530862192022798415406849
9	5	6	0.00006400699017674680116881995009262009505162631770
9	7	6	0.00000850004277325413875254169128863308420792351334
9	8	6	0.00000004148485662424219232847335759607977169015726
9	1	7	0.00000000112936105133831008373964509933527361911788
9	2	7	0.00781255137580477014545459089248034091018305787202
9	4	7	0.00000001597315332810527813272756669841821856004569
9	5	7	0.00001280029924401125735339208130937537206109959683
9	7	7	0.00000121426974110247608623247991264625044448989429
9	8	7	0.00000000243800171147144371869696620432106377350629
9	1	8	0.00000000005916646720564647646539923300104797195776
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9	4	8	0.00000000122706977496722943559128332841057231961576

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9	1	9	0.00000000000310668372015285362010410932189277922708
9	2	9	0.00195312542416744980289165249739994733511945265134
9	4	9	0.00000000009433745499118703336858038475534132792061
9	5	9	0.00000051200055837045246513736210734988438231726862
9	7	9	0.00000002478093430737079598684953144379910656516057
9	8	9	0.00000000000843289346544781165288562038743173047126
9	1	10	0.00000000000016331196596546999409540120445937620322
9	2	10	0.00097656253855672497852795390313933029665018147175
9	4	10	0.00000000000725503565044170240033226878114294608674
9	5	10	0.00000010240002421559270781200281133981669058688493
9	7	10	0.00000000354013322243432895647068896731796850398736
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10	1	2	0.01082089638114771753353951906172584388284719779619
10	3	2	0.12061478877454186078379946018363597386887218032151
10	7	2	0.02586829355163944664905730184997372065557850052458
10	9	2	0.00494344133373647354014708373691239576593346459761
10	1	3	0.00080913843084035641245250698022237496506614634497
10	3	3	0.03760068956186250057475408812071422899263187067150
10	7	3	0.00315553197969650234624459561826033077604565502678
10	9	3	0.00019727932704417708966212394650976624166844988289
10	1	4	0.00006986805789621433879330279589781606000671240287
10	3	4	0.01238476073533231633591565621150625851220369905124
10	7	4	0.00042928488592025628255758548942987059983408077138
10	9	4	0.00000922608509805798535275143632392498999656748935
10	1	5	0.00000625466213439143946805396018589577550095731367
10	3	5	0.00411808509132770402979824830567767498838944163893
10	7	5	0.00006022310141769935983100374949396438572202646650
10	9	5	0.00000045462904446230372093652349817598208485130755
10	1	6	0.00000056583972496957732807037158491679285049223691
10	3	6	0.00137195625824195372120548457657357145068690772793
10	7	6	0.00000854178492824835195225955908066196274598028000
10	9	6	0.00000002296774134130364785875882024900330256162971
10	1	7	0.00000005135773338184100299467054471296122828929121
10	3	7	0.00045726360583073511381763233890241641380597176705
10	7	7	0.00000121671538540391312233979398137313283918252394
10	9	7	0.00000000117718407166718117192602022058130988530311
10	1	8	0.00000000466637839516483023409561227013523337400965
10	3	8	0.00015241702904363578545934235353127668363643127419
10	7	8	0.00000017361020853816259472105933508411048224883713
10	9	8	0.00000000006088726144763914163420444959362509039110
10	1	9	0.00000000042413860329242757996461375187871653490432
10	3	9	0.00005080535828240244957909485677156547385336323426
10	7	9	0.00000002478937341462986352384610597009748484035541
10	9	9	0.00000000000316803029120338647329183008676818185576
10	1	10	0.00000000003855562522808473135592118445688884204375
10	3	10	0.00001693509508643674413086026760791168357555923310
10	7	10	0.00000000354062943543015412120109166142062067593252
10	9	10	0.00000000000016548281548613787552698185521679533591

There are obvious refinements in the table whenever classes of a (non-prime) modulus are a union of classes of smaller modulus [32]:

Example 2. *The value* $P_{6,1}(s)$ *equals* $P_{3,1}(s)$. *The value* $P_{6,5}(s)$ *equals* $P_{3,2}(s) - 1/2^s$.

Example 3. *The sum* $P_{8,1}(s) + P_{8,5}(s)$ *equals* $P_{4,1}(s)$. *The sum* $P_{8,3}(s) + P_{8,7}(s)$ *equals* $P_{4,3}(s)$.

Example 4. *The sum* $P_{9,1}(s) + P_{9,4}(s) + P_{9,7}(s)$ *equals* $P_{3,1}(s)$.

Example 5. *The* $P_{m,n}(s)$ *for the four classes at* $m = 10$ *are essentially a permutation of the four classes at* $m = 5$, *but* $P_{5,2}(s)$ *for* $m = 5$ *is* $P_{10,7}(s) + 1/2^s$ *at* $m = 10$.

3.3. **Euler modulo product.** A further set of constants is obtained if the filtered modular values of (34) are re-organized into associated Euler products.

Definition 5. *(Euler modulo product)*

$$\zeta_{m,n}(s) \equiv \prod_{p=n \pmod m} \frac{1}{1-p^{-s}} = \sum_{q=1}^{\infty} \frac{c_{m,n}(q)}{q^s}. \tag{38}$$

The completely multiplicative arithmetic function c is defined as $c_{m,n}(q) = 0$ if q has at least one prime factor $\neq n \pmod m$, and $c_{m,n}(q) = 1$ if all prime factors of q are $\equiv n \pmod m$ or if $q = 1$.

Remark 9. *The characteristic function $c_{m,n}(q)$ in (38) selects integers q that have prime power signatures with $k = 1, 2, \ldots$ (not necessarily distinct) primes all in the same residuum class, which is isomorph to the combinatorics of almost-prime signatures* [20]*. Therefore*

$$\zeta_{m,n}(s) = \sum_{k=0}^{\infty} P_{m,n,k}(s), \tag{39}$$

where

$$P_{m,n,k}(s) \equiv \frac{1}{k!} \sum_{\substack{k_1+2k_2+\cdots kk_k=k \\ k_k \geq 0}} (k; k_1 k_2 \cdots k_k)^* P_{m,n}^{k_1}(s) P_{m,n}^{k_2}(2s) \cdots P_{m,n}^{k_k}(ks), \tag{40}$$

are multinomials in the $P_{m,n}(s)$.

The $\zeta_{m,n}$ are accessible via multiplicative mixing of the L-series if $\varphi(m) = 2$ [11, 10]. The simpler approach implemented here is to accumulate the $P_{m,n}(s)$ obtained in Section 3.2,

$$\log \zeta_{m,n}(s) = -\sum_{p=n \pmod m} \log(1-p^{-s}) = \sum_{t \geq 1} \frac{1}{t} P_{m,n}(st). \tag{41}$$

This evaluation may again be split at M using the same cut as in (34):

Definition 6. *(Incomplete Euler modulo product)*

$$\zeta_{m,n}(M, s) \equiv \prod_{\substack{p=n \pmod m \\ p>M}} \frac{1}{1-p^{-s}}. \tag{42}$$

The associates to Definition 3 are

$$\zeta_{m,n}(s) = \zeta_{m,n}(M, s) \prod_{\substack{p=n \pmod m \\ p \leq M}} \frac{1}{1-p^{-s}}; \tag{43}$$

$$\log \zeta_{m,n}(M, s) = \sum_{t \geq 1} \frac{1}{t} P_{m,n}(M, st). \tag{44}$$

Remark 10. *Reminiscent of (35), the Riemann zeta function collects all residue classes,*

$$\prod_{n=1}^{m} \zeta_{m,n}(s) = \zeta(s). \tag{45}$$

The product over the numbers listed times $\prod_{p\le m,(p,m)>1}(1-p^{-s})$ (that fall into the "trenches" where the $\chi=0$) compared with $\zeta(s)$ provides a check of numerical consistency.

Since $\zeta_{2,2}(s)=1/(1-2^{-s})$ and $\zeta_{2,1}(s)=(1-2^{-s})\zeta(s)$, the noteworthy values of $\zeta_{m,n}(s)$ start at $m=3$:

m	n	s	Zeta
3	1	2	1.03401487541434188053903064441304762857896542848910
3	2	2	1.41406439089214763756550181907982937990769506939316
3	1	3	1.00361130175701910913029426716259929490867629745799
3	2	3	1.15337110601499930871280075651004504066539534451062
3	1	4	1.00046150891836112253350741835804432442530982567393
3	2	4	1.06846811091796338426848094038546450399003516889694
3	1	5	1.00006265916432119872202518332049765156349326287357
3	2	5	1.03259586114163185601846011315870956950391798144999
3	1	6	1.00000873008527587107163357184986055948903446917689
3	2	6	1.01593866043234218859244626456559006285437559081239
3	1	7	1.00000123137375097073547807909042996909669398074936
3	2	7	1.00788697124040368356072041975798727403112540606682
3	1	8	1.00000017475288367836474357787477390452756306811768
3	2	8	1.00392414351556224720764569154793430922419486589601
3	1	9	1.00000002487837906260076012221313463298443805962493
3	2	9	1.00195746059865282361229356952338128232146781888209
3	1	10	1.00000000354755158260959501372671716742807152615965
3	2	10	1.00097761964577279463923730364876047710509000379085
4	1	2	1.05618212172681614173793076531621989058758042546071
4	3	2	1.16807558541051428866969673706404040136467902145555
4	1	3	1.00882610230910848889719384450675105022197334662081
4	3	3	1.04259771615462145839338168202497838298447961204548
4	1	4	1.00165222963665150519631577153104361173127254438959
4	3	4	1.01300431585148634665399481938602650341480916534996
4	1	5	1.00032357758896100656211660961706285853691779918112
4	3	5	1.00419882656000385808584623974706967482770644252094
4	1	6	1.00006425507604345265654515567536392156836383588910
4	3	6	1.00138273271730277119044572005520471705067106858197
4	1	7	1.00001281861267833768577063221287540357494369161606
4	3	7	1.00045872415949089035961518870218728065924793791041
4	1	8	1.00000256137823752646031567559721051916577582675427
4	3	8	1.00015261725614807427399274998465395446569675667036
4	1	9	1.00000051210307444955772344253708036843655634283193
4	3	9	1.00005083305473755942032762323454360451035601968338
4	1	10	1.00000010240776300165778365731512915793270702921448
4	3	10	1.00001693895354714349004828400848126437102468981399
5	1	2	1.01091516060101952260495658428951492098453862758174
5	2	2	1.36857205387664908586076389048310999017020782888590
5	3	2	1.13576487866892162686864300947208228951193641300547
5	4	2	1.00496032392229755899374962481025218479551029418802
5	1	3	1.00080974916213981237591658238294535965537553124834
5	2	3	1.14647406699308384207156816123876504746097490224826
5	3	3	1.03904714620903325172616881494173461251449412325900
5	4	3	1.00019731027508315895211726710719665423271453210039
5	1	4	1.00006987283218426141419635264600625153236814679615
5	2	4	1.06712476150223425563458216313613707388509171652801
5	3	4	1.01253957164493590352210027269115214047836280278775
5	4	4	1.00000922615810257514168804143847154751101040479537
5	1	5	1.00000625470097284441750106943552347775641813117440
5	2	5	1.03232023399778748951275181693907418137524756424676
5	3	5	1.00413510197962756396832622100191068943129241061254
5	4	5	1.00000045462923054757077178219810008398847605853520
5	1	6	1.00000056584004437312082346506704946507930051287745
5	2	6	1.01588169331559068093936585611050025186958374297954
5	3	6	1.00137384081358621066443457151047388312761077458853
5	4	6	1.00000002296774183238118385390343751552162917665031
5	1	7	1.00000005135773601730647261631573028226879596939639
5	2	7	1.00787524204534206227219710199330711789815839187329
5	3	7	1.00045747278405872997753571892593244385904755534268
5	4	7	1.00000000117718407298751975989140730455605047299986
5	1	8	1.00000000466637841693383148380229067863361198146146

5	2	8	1.00392174291851409944906566519007445056477589859023
5	3	8	1.00015244026334684517576289935485631243810502238780
5	4	8	1.000000000006088726145122822326132985063692994015709
5	1	9	1.00000000042413860347230375311685307458456339360109
5	2	9	1.00195697200031213857939523509681714132558926260870
5	3	9	1.00005080793959315758516083734272448531900142192340
5	4	9	1.00000000000316803029121320886942290191856601223145
5	1	10	1.00000000003855562522957121761521053110699996912154
5	2	10	1.00097752065063983843167256545404741314588424745646
5	3	10	1.00001693538188861609719720557413066198806885305437
5	4	10	1.000000000000016548281548616487205238135341187159743
6	1	2	1.03401487541434188053903064441304762857896542848910
6	5	2	1.06054829316911072817412636430987203493077130204487
6	1	3	1.00361130175701910913029426716259929490867629745799
6	5	3	1.00919971776312439512370066194628941058222092644679
6	1	4	1.00046150891836112253350741835804432442530982567393
6	5	4	1.00168885398559067275170088161137297249065797084088
6	1	5	1.00006265916432119872202518332049765156349326287357
6	5	5	1.00032724048095586051788323462249989545692054452968
6	1	6	1.00000873008527587107163357184986055948903446917689
6	5	6	1.00006461886308684189568929168175271812227597220594
6	1	7	1.00000123137375097073547807909042996909669398074936
6	5	7	1.00001285427758802978290229147862799845275723883193
6	1	8	1.00000017475288367836474357787477390452756306811768
6	5	8	1.00000256482995458217949082556532519082878785470111
6	1	9	1.00000002487837906260076012221313463298443805962493
6	5	9	1.00000051243342107981617580864540592825443370204834
6	1	10	1.00000000354755158260959501372671716742807152615965
6	5	10	1.00000010243908746964447242346941598445166706433402
7	1	2	1.00222953381974042627186415913822019244863756540129
7	2	2	1.33746389207369371366872360528892767208848738545304
7	3	2	1.13097002184898936694171439997659798834205226307093
7	4	2	1.00925809090246245773689160380680443030003452731001
7	5	2	1.04577714521003242013083380033831302979424845783143
7	6	2	1.00705203260307404805671935242888870692893671473688
7	1	3	1.00005804579333842582733346516941555537502172043921
7	2	3	1.14297797173549754260058384863294164013896229848439
7	3	3	1.03871737324706394518763779322963241056749953425477
7	4	3	1.00076390858271896863295594860561693331663187488964
7	5	3	1.00822899406805182364465607752987243099420200380309
7	6	3	1.00047382657441886608438150501246871993371039652912
7	1	4	1.00000175761304005094062160690765751034301172679602
7	2	4	1.06667108835339684152070322152775177935342710035520
7	3	4	1.01251335185061365241572541659058762432780978612662
7	4	4	1.00006849652871724122119914536012770276225904212045
7	5	4	1.00161055789897831028832952888681771283602863700909
7	6	4	1.00003540628815905289912251372448655311275078200634
7	1	5	1.00000005620337259746341656129434729738657491157251
7	2	5	1.03225824022068145735171855495352158074798557667844
7	3	5	1.00413297568911102402999888618334524759035772751544
7	4	5	1.00000621249325042367673128564710034860046130098397
7	5	5	1.00032051227106870211998736145861461994162073529346
7	6	5	1.00000270233597598941712969634399477195726107101327
7	1	6	1.00000000184792769925706246450107536978725984676770
7	2	6	1.01587302313633197293338202964424201385855538191516
7	3	6	1.001373669019435819499977843262717757828370222759547
7	4	6	1.00000056453130909717912341530968058196969919192491
7	5	6	1.00006402546875715285850304341261216795397018133427
7	6	6	1.00000020739126411912695649676728624986487607457979
7	1	7	1.00000000006176660482693590699613669924470496018582
7	2	7	1.00787401605472157878701446721875022221822450340248
7	3	7	1.00045745901662079449496711619068106442900549239719
7	4	7	1.00000005131683815237409132236443379307023790633963
7	5	7	1.00001280128490975474769220576907611675583462516742
7	6	7	1.00000001594181727051843251503394875825925790685075
7	1	8	1.00000000000208617300170648175746494841280261573005
7	2	8	1.00392156864055720045806690391550435549440658339755
7	3	8	1.00015243916894651396603872407269447926888124517854
7	4	8	1.00000000466509241175435577864199625679249785018148
7	5	8	1.00000256006548178653957699161193792689539712692820

7	6	8	1.00000000122602055887861311034095486231862309598186
7	1	9	1.00000000000007094349319405145668869193592605078547
7	2	9	1.00195694716299061958071122607016739847773365273626
7	3	9	1.00005080785320217744383794322741759529927086244099
7	4	9	1.00000000042409795891226589464058012067287236620938
7	5	9	1.00000051200336210006761095863858791048582890631081
7	6	9	1.00000000009430265664963407876051382328271602225165
7	1	10	1.00000000000000242353263829862208174870945612586559
7	2	10	1.00097751710657373561753719161737821320340978142840
7	3	10	1.00001693537510775031560912509072853821083229508912
7	4	10	1.00000000003855433521585762081599098999990534328999
7	5	10	1.00000010240017360999818977149868475813442017230296
7	6	10	1.00000000000725388960805038399127728615979988697545
8	1	2	1.00483506503419118711835983116910411959790731754341
8	3	2	1.13941877710821151502705893077334020887255996109629
8	5	2	1.05109998494218330793687755600633505680120101845818
8	7	2	1.02515037392575917991619543556094158794331100276024
8	1	3	1.00022487189858708836232213399171649391737471516971
8	3	3	1.03941995442465269726466028414808844655561938824520
8	5	3	1.00859929667035262471282393658930645974303187198527
8	7	3	1.00305724526111078841419961903241251128776224554545
8	1	4	1.00001240072777353501966319275809128848338463112908
8	3	4	1.01257739917435104034672760553946857732303378692076
8	5	4	1.00163980857405824257174676874861412537808205141553
8	7	4	1.00042161387118003163153986625406555739471421431602
8	1	5	1.00000071379019830210721017813879624740460863399749
8	3	5	1.00413888121924847360311597885777832338928143876735
8	5	5	1.00032286356830585400934496169979204029466533001408
8	7	5	1.00005969825676165180885820298465127947408592737522
8	1	6	1.00000004165020787989301833933248046151599801990394
8	3	6	1.00137421310570835820170401275086689129181158794032
8	5	6	1.00006421342316107034018869960936872036224692472247
8	7	6	1.00000850791989938493702496782374314457377807985424
8	1	7	1.00000000244227705982268643969925689092778090852564
8	3	7	1.00045750900445223186038575648293787031867590037346
8	5	7	1.00001281617036997722419481790704275345557942138612
8	7	7	1.00000121459934851974962938194693153961770963877695
8	1	8	1.00000000014348051244622704724492022817584432875083
8	3	8	1.00015244374916001126716002732963454166550275486299
8	5	8	1.00000256123475664652681324962099734476444330055668
8	7	8	1.00000017348054203877213135423542699753177525705254
8	1	9	1.00000000000843564128360202167483991763433742998011
8	3	9	1.00005080827195166959301669724244142405059548197828
8	5	9	1.00000051209463880395427474465633795499257239686876
8	7	9	1.00000002478152678327512869947740743068742148552662
8	1	10	1.00000000000049610802941248433872995024850173405997
8	3	10	1.00001693541332862363755720116203998935696348266958
8	5	10	1.00000010240726689357756610560030809483919985171972
8	7	10	1.00000000354015856580393032582392645037510385491765
9	1	2	1.00403383505128879798247811992474748948252289579877
9	2	2	1.34730860006734125382172429232240042513609379247219
9	4	2	1.00756920580396055044764395809846530439886346418110
9	5	2	1.04492546987837193323296862915029330679603098038265
9	7	2	1.02212392135990189822906820506116992751646742163236
9	8	2	1.00442332356455015978660825839058205396611967230788
9	1	3	1.00017010954035563217936315831074138833385687151460
9	2	3	1.14377866433934793890564873335241094138548462302908
9	4	3	1.00049432514572171244268626193755112793108393453682
9	5	3	1.00816891576421280744766931461155671842390273719182
9	7	3	1.00294482608929152112978326220983595984160379842542
9	8	3	1.00021594427393638968283773920259822881957855052866
9	1	4	1.00000825662747111714397548870033416941196594578155
9	2	4	1.06674129214220456654470650822375143155155057667681
9	4	4	1.00003616169631672484677498115868698087952268890395
9	5	4	1.00160659432813253476579891611141515042772719305834
9	7	4	1.00041707177024616512635742252815870747696948360196
9	8	4	1.00001216547711220254005455135543391724191546972805
9	1	5	1.00000041887828157638534779627660537201000587317543
9	2	5	1.03226452929064327997355449602961167060717076318574
9	4	5	1.00000272909309079900485466890016088517679112737683

9	5	5	1.00032026798877483551103148925212733203033814567290
9	7	5	1.00005951100446665907486972236762505690900866452338
9	8	5	1.00000070750508832285169131456798807758159600734212
9	1	6	1.00000002165314466371698346720223021679754301660010
9	2	6	1.01587359111356620012257087979459886025025352202905
9	4	6	1.00000020831514889994872382675170867217993946412643
9	5	6	1.00006401108688635479741673945040757241833536646872
9	7	6	1.00000850011502303974258689264448187739466524133492
9	8	6	1.00000004148485834292898694068124853022614282928380
9	1	7	1.00000000112936105260187075408758426117342509455760
9	2	7	1.00787406752837281919012443939348536889214116945799
9	4	7	1.00000001597315358266487003837608080110404243661118
9	5	7	1.00001280046308993889648810309344249215545718583233
9	7	7	1.00000121427121555033802270555170630250282870733628
9	8	7	1.00000000243800171741288241395233016494508463776092
9	1	8	1.00000000005916646720913030668400492666448392595332
9	2	8	1.00392157331286877568744980969805132863571065112483
9	4	8	1.00000000122706977647148919769931136709832310681548
9	5	8	1.00000256001945535201357654534560922276642839788231
9	7	8	1.00000017346664721149243396402338564097435048672861
9	8	8	1.00000000014337153252426939215553645556341336026956
9	1	9	1.00000000000310668372016248120395836883309448482379
9	2	9	1.00195694758742413775182659896672656412321870400605
9	4	9	1.00000000009433745500008301863547212255982749672275
9	5	9	1.00000051200082051487256906213445307047779625726255
9	7	9	1.00000002478093492146546468562613434792098197445067
9	8	9	1.00000000000843289346551892257692342553991325843936
9	1	10	1.00000000000016331196596549663093548060562311709040
9	2	10	1.00097751714514377945213570884484620273462462324346
9	4	10	1.00000000000725503565049432908845638623592279214807
9	5	10	1.00000010240003470135626123146851051814671333124481
9	7	10	1.00000000354013323496687206422761916659309124111146
9	8	10	1.00000000000049603930917316538434479004525647605792
10	1	2	1.01091516060101952260495658428951492098453862758174
10	3	2	1.13576487866892162686864300947208228951193641300547
10	7	2	1.02642904040748681439557291786233249262765587166442
10	9	2	1.00496032392229755899374962481025218479551029418802
10	1	3	1.00080974916213981237591658238294535965537553124834
10	3	3	1.03904714620903325172616881494173461251449412325900
10	7	3	1.00316480861894836181262214108391941652835303946723
10	9	3	1.00019731027508315895211726710719665423271453210039
10	1	4	1.00006987283218426141419635264600625153236814679615
10	3	4	1.01253957164493590352210027269115214047836280278775
10	7	4	1.00042946390834461465742077794012850676727348424501
10	9	4	1.00000922615810257514168804143847154751101040479537
10	1	5	1.00000625470097284441750106943552347775641813117440
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14	9	4	1.00000414533130953892565927018226729314383790658300
14	11	4	1.00006849652871724122119914536012770276225904212045
14	13	4	1.00003540628815905289912251372448655311275078200634
14	1	5	1.00000005620337259746341656129434729738657491157251
14	3	5	1.00413297568911102402999888618334524759035772751544
14	5	5	1.00032051227106870211998736145861461994162073529346
14	9	5	1.00000017021378516180947735011122403134961102740724
14	11	5	1.00000621249325042367673128564710034860046130098397
14	13	5	1.00000270233597598941712969634399477195726107101327
14	1	6	1.00000000184792769925706246450107536978725984676770
14	3	6	1.00137366901943581949977843262717757828370222759547
14	5	6	1.00006402546875715285850304341261216795397018133427
14	9	6	1.00000000714982678585629793543105073239201545407273
14	11	6	1.00000056453130909717912341530968058196969919192491
14	13	6	1.00000020739126411912695649676728624986487607457979
14	1	7	1.00000000006176660482693590699613669924470496018582
14	3	7	1.00045745901662079449496711619068106442900549239719
14	5	7	1.00001280128490975474769220576907611675583462516742
14	9	7	1.00000000030429406645274091669360373610714462446965
14	11	7	1.00000005131683815237409132236443379307023790633963
14	13	7	1.00000001594181727051843251503394875825925790685075
14	1	8	1.00000000000208617300170648175746494841280261573005
14	3	8	1.00015243916894651396603872407269447926888124517854
14	5	8	1.00000256006548178653957699161193792689539712692820
14	9	8	1.00000000001305502389377758007208441660575655768115
14	11	8	1.00000000466509241175435577864199625679249785018148
14	13	8	1.00000000122602055887861311034095486231862309598186
14	1	9	1.00000000000007094349319405145668869193592605078547
14	3	9	1.00005080785320217744383794322741759529927086244099
14	5	9	1.00000051200336210006761095863858791048582890631081
14	9	9	1.00000000000056290352684264945674910277758182919576
14	11	9	1.00000000042409795891226589464058012067287236620938
14	13	9	1.00000000009430265664963407876051382328271602225165
14	1	10	1.00000000000000242353263829862208174870945612586559
14	3	10	1.00001693537510775031560912509072853821083229508912
14	5	10	1.00000010240017360999818977149868475813442017230296
14	9	10	1.00000000000002434720384819045368936729207832656373

14	11	10	1.00000000000385543352158576208159909899999905343289 99
14	13	10	1.0000000000072538896080503839912772861597998869754 5

Redundancies of $\zeta_{m,n}$ correlate with those of $P_{m,n}$ mentioned in Section 3.2—sums over P replaced by products over ζ:

Example 6. *$\zeta_{6,1}(s)$ equals $\zeta_{3,1}(s)$. $\zeta_{6,5}(s)$ equals $(1-2^{-s})\zeta_{3,2}(s)$.*

Example 7. *$\zeta_{8,1}(s)\zeta_{8,5}(s)$ equals $\zeta_{4,1}(s)$. $\zeta_{8,3}(s)\zeta_{8,7}(s)$ equals $\zeta_{4,3}(s)$.*

Example 8. *The $\zeta_{14,n}(s)$ are basically a permutation of the $\zeta_{7,n}(s)$, but $\zeta_{14,9}(s) = (1-2^{-s})\zeta_{7,2}(s)$.*

4. Zeta Expansions

Euler products over the unrestricted set of primes split in that modulo basis via exponential product expansions [24]. The exponents $\gamma_{s,j}^{(\cdot)}$ and coefficients in the logarithmic series of the Euler products $A_1^{(s)}$, $Q_1^{(s)}$, $F_1^{(s)}$ and $C_1^{(s)}$ are exactly those of my earlier work [21].

Definition 7. *(Artin's constants of order s)*

$$A_{m,n}^{(s)} \equiv \prod_{p=n \pmod m} \left(1-\frac{1}{p^s(p-1)}\right) = \prod_j \zeta_{m,n}(j)^{-\gamma_{s,j}^{(A)}}. \tag{46}$$

$$\log A_{m,n}^{(s)} = -\sum_{t=s+1}^{\infty} P_{m,n}(t) \sum_{j=1}^{\lfloor t/(1+s)\rfloor} \frac{1}{j}\binom{t-sj-1}{j-1}, \tag{47}$$

In numerical practise we employ the variant equivalent to (44) with a threshold parameter M defining $P_{m,n}(M,t)$ and defining an incomplete $A_{m,n}^{(s)}$. The $A_{m,n}^{(s)}$ are:

m	n	s	A
3	1	1	0.963036288983694666072706507298106468187694030872 95
3	2	1	0.465970993488325919476263392391075709286658514057 55
3	*	1	0.373955813619202288054728054346416415111629248606 15
3	1	2	0.995868068887020592961815716776269466497721801271 60
3	2	2	0.741595069915237630122679876024935721777881958387 23
3	*	2	0.697501358496365903284670350820922924073153946214 52
3	1	3	0.999465888208127617685732681666160311081302970596 16
3	2	3	0.873167954623326340277190173210771104227907528410 41
3	*	3	0.856540444853542174426167984135953882166572800317 65
3	1	4	0.999927179657643213523954512888535596067449083983 13
3	2	4	0.937117681192613430148599828174606269212661250541 19
3	*	4	0.931265184160004334389237205550676982558423734587 80
3	1	5	0.999989834849773360492278781984377557963648413160 94
3	2	5	0.968671846979450782457756062796318640338686633956 54
3	*	5	0.966668868596777512740328372930016264211423822118 19
4	1	1	0.936183046894868404853747352174836025595452955710 98
4	3	1	0.798894649629768017983389350720773587150408625441 74
4	*	1	0.373955813619202288054728054346416415111629248606 15
4	1	2	0.989199851033153625793494398103958380965290887507 35
4	3	2	0.940155632208357765945867587566663366149331650888 63
4	*	2	0.697501358496365903284670350820922924073153946214 52
4	1	3	0.997946778193945179713239341786829192384580404908 03
4	3	3	0.980917406555984831791289877675458524983774424869 71
4	*	3	0.856540444853542174426167984135953882166572800317 65
4	1	4	0.999596256571077900515145185468926978314743654455 95
4	3	4	0.993750750106013000122015019463779711582207538350 63
4	*	4	0.931265184160004334389237205550676982558423734587 80
4	1	5	0.999919729126536832841750978742816236467379827130 30
4	3	5	0.997931840186174269548611661909254262570249506762 03
4	*	5	0.966668868596777512740328372930016264211423822118 19

5	1	1	0.98831956638305746883087282795162021735104254639947
5	2	1	0.48531082947754999932252444786658812444646497522733
5	3	1	0.82493234111986491690153714571045421637133702809175
5	4	1	0.99485772777378307465830719513542008131033561396718
5	*	1	0.37395581361920228805472805434641641511162924860615
5	1	2	0.99911416667905889380339439800697914957598443799056
5	2	2	0.74725940266248257162661615183905048245682166762209
5	3	2	0.94387233155461696180996684651869263721075148724586
5	4	2	0.99979302441165734257764415786782157555012208036948
5	*	2	0.69750135849636590328467035082092292407315394621452
5	1	3	0.99992325507141931782483437831240399829723649073316
5	2	3	0.87456297038427688306528625223021965638998225414189
5	3	3	0.98144009040883263601170435021757121554704060929700
5	4	3	0.99999029508991648136798168771812469418297569009643
5	*	3	0.85654044485354217442616798413595388216657280031765
5	1	4	0.99999312300779961065284240241206938654053912931887
5	2	4	0.93743420242628703757024971525968083790937519251505
5	3	4	0.99382408909519320610177720172252856184425578311066
5	4	4	0.99999952116182270606071177223396763347910848780771
5	*	4	0.93126518416000433438923720555067698255842373458780
5	1	5	0.99999937766961442465172997736994048856195463128477
5	2	5	0.96874035025205510610892797235078265238823889538516
5	3	5	0.99794215558580146608046689108068784979274640836406
5	4	5	0.99999997579084470604995769973504106813807405835542
5	*	5	0.96666886859677751274032837293001626421142382211819
6	1	1	0.96303628898369466607270650729810646818769403087295
6	5	1	0.93194198697665183895252678478215141857331702811511
6	*	1	0.37395581361920228805472805434641641511162924860615
6	1	2	0.99586806888702059296181571677626946649772180127160
6	5	2	0.98879342655365017349690650136658096237050927784964
6	*	2	0.69750135849636590328467035082092292407315394621452
6	1	3	0.99946588820812761768573268166616031108130297059616
6	5	3	0.99790623385523010317393162652659554768903717532618
6	*	3	0.85654044485354217442616798413595388216657280031765
6	1	4	0.99992717965764321352395451288853559606744908398313
6	5	4	0.99959219327212099215850648338624668716017200057727
6	*	4	0.93126518416000433438923720555067698255842373458780
6	1	5	0.99998983484977336049227878198437755796364841316094
6	5	5	0.99991932591427177544026432288652246744638620279385
6	*	5	0.96666886859677751274032837293001626421142382211819
7	1	1	0.99771564294568838736155996492787546367559477468412
7	2	1	0.49840088692369542648778312810294075345529599481565
7	3	1	0.82871803666235390545933514079733883693844062023419
7	4	1	0.98998905141183604558273699292198729033368569094988
7	5	1	0.94610323757016626868761072363193190084000156782121
7	6	1	0.99248607820796097215463092567055639457362125429046
7	*	1	0.37395581361920228805472805434641641511162924860615
7	1	2	0.99994014189209871584452460006551005110995697243087
7	2	2	0.74991747228874878297031470134449669622114285848704
7	3	2	0.94419863275192592923695459672054797957947335493860
7	4	2	0.99916135030829557805642100162223289908196026590518
7	5	2	0.98983016697259050948663754611004452800101496357527
7	6	2	0.99948806669231373616034944437904869460306577636950
7	*	2	0.69750135849636590328467035082092292407315394621452
7	1	3	0.99999818427485449426581627589239112109518799672518
7	2	3	0.87499621737893938504603267043258995819277235099382
7	3	3	0.98146776697270810153477930104022054022783560088610
7	4	3	0.99992467473018342384264858111921255578780699414136
7	5	3	0.99799160362258728065847744039609703225891846735173
7	6	3	0.99996166797698051410762218911629473139618175755620
7	*	3	0.85654044485354217442616798413595388216657280031765
7	1	4	0.99999994188477665613746471458679347599374570540990
7	2	4	0.93749983342357481295568615381996549141887642266209
7	3	4	0.99382637891296178623822692780002794908822962790985
7	4	4	0.99999316656592074861469998930147905723527287783743
7	5	4	0.99959956791155045893890680467063608451168904075951
7	6	4	0.99999707301010952296050938802699541446883588698900
7	*	4	0.93126518416000433438923720555067698255842373458780
7	1	5	0.99999999808814607332653001930682318795744533803823

7	2	5	0.96874999276560345204224635032110372021185903647059
7	3	5	0.99794234170960789630101905258256187373104983998247
7	4	5	0.99999937902057396614860818298900423682749345076897
7	5	5	0.99991997744742418905121707179384935036968868082300
7	6	5	0.99999977533878029037234181445905570571069241006908
7	*	5	0.96666886859677751274032837293001626421142382211819

The lines with a star in the n-column are [21]

$$A_1^{(s)} = \prod_{n=1}^{m} A_{m,n}^{(s)}, \tag{48}$$

i.e., the noted $A_{m,n}^{(s)}$ multiplied by the finite product $\prod_p \{1 - 1/[p^s(p-1)]\}$ from primes up to and not coprime to m; see Remark 10. A005596 and A065414–A065416 in the OEIS display $A_1^{(s)}$ for $s \le 4$ [27].

Definition 8. *(Quadratic Class numbers of order s)*

$$Q_{m,n}^{(s)} \equiv \prod_{p=nl \pmod m} \left(1 - \frac{1}{p^s(p+1)}\right) = \prod_j \zeta_{m,n}(j)^{-\gamma_{s,j}^{(Q)}}. \tag{49}$$

Again, the algorithm is the $M-$deferred variant of a known formula [21],

$$\log Q_{m,n}^{(s)} = -\sum_{t=s+1}^{\infty} P_{m,n}(t) \sum_{j=1}^{\lfloor t/(1+s)\rfloor} \frac{(-1)^{t-(s+1)j}}{j} \binom{t-sj-1}{j-1}. \tag{50}$$

The table of $Q_{m,n}^{(s)}$ starts:

m	n	s	Q
3	1	1	0.97024910467115917306803567015977644528885523914057
3	2	1	0.79204649341104628208008707618094468295489090223018
3	*	1	0.70444220099916559273660335032663721018858643141710
3	1	2	0.99680763522216452381503254510333176771190098607201
3	2	2	0.90960373636135175343904777594792044585740214764741
3	*	2	0.88151383972517077692839182290322784712986925720808
3	1	3	0.99959370433770807882786299647928332089368010624696
3	2	3	0.95697939892931332309349292388556861626669448272441
3	*	3	0.94773326214367537593952153765418961303363163231741
3	1	4	0.99994499634458014625186893933369950019928710526941
3	2	4	0.97889912348656360383240568800722815385319039104803
3	*	4	0.97582415304766824167901143659479983197176497122921
3	1	5	0.99999234838384640058266382093557284740952517853584
3	2	5	0.98952999643109135827233602414981315609523742685183
3	*	5	0.98850439774124690875110662385118666440095808327535
4	1	1	0.95406039246755865186005688760123226145264727334098
4	3	1	0.88603472890500783721446805186195985509972671129727
4	*	1	0.70444220099916559273660335032663721018858643141710
4	1	2	0.99262946073126248692585121240213394187141980334665
4	3	2	0.96879198085650902105058006862070103808727094151243
4	*	2	0.88151383972517077692839182290322784712986925720808
4	1	3	0.99862038502725376088960590123790296746879346645464
4	3	3	0.99030529616410002251835282806899969957047647793415
4	*	3	0.94773326214367537593952153765418961303363163231741
4	1	4	0.99973009395621554677063646518040509444080859765319
4	3	4	0.99685542637431681147759855262917235702927491774292
4	*	4	0.97582415304766824167901143659479983197176497122921
4	1	5	0.99994643288800372098610092291896044680141586572085
4	3	5	0.99896321876577456056788702369534663838265580294255
4	*	5	0.98850439774124690875110662385118666440095808327535
5	1	1	0.98994541651532837528981290049449971750949436487113
5	2	1	0.81417661748357764967519511911124771377014221986475
5	3	1	0.90845960510191820259198957462683801148959562318257
5	4	1	0.99525215425415153950030018010344295622027317317902
5	*	1	0.70444220099916559273660335032663721018858643141710

5	1	2	0.9992550334407123366205807435320216678735160464283 9
5	2	2	0.91411979597676293619735190404927935915076781045200
5	3	2	0.97170968219087668252029279604183386856774563347235
5	4	2	0.99981152134879912586741302748663655518106124260658
5	*	2	0.88151383972517077692839182290322784712986925720808
5	1	3	0.99993586794053861211626882911611188040468932785472
5	2	3	0.95797248791315543969785734684555954155114677761234
5	3	3	0.99070465607889613208223242762827729704972901822545
5	4	3	0.99999120670666619470789435284342251314622703330412
5	*	3	0.94773326214367537593952153765418961303363163231741
5	1	4	0.99999426409769122005699815077922330151446689642523
5	2	4	0.97911501946597037170638532422142794193185794775165
5	3	4	0.99691092876378462606358844096686476050711455139439
5	4	4	0.99999956721940822422487479489052376349550108528067
5	*	4	0.97582415304766824167901143659479983197176497122921
5	1	5	0.99999948124042658176935512678249171240320614070572
5	2	5	0.98957593423045818490913570593415192280274622185561
5	3	5	0.99897099455900934654360997881391321362270950221429
5	4	5	0.99999997815156628432244203179623561003978532541884
5	*	5	0.98850439774124690875110662385118666440095808327535
6	1	1	0.97024910467115917306803567015977644528885523914057
6	5	1	0.95045579209325553849610449141713361954586908267621
6	*	1	0.70444220099916559273660335032663721018858643141710
6	1	2	0.99680763522216452381503254510333176771190098607201
6	5	2	0.99229498512147464011532484648864048638989325197899
6	*	2	0.88151383972517077692839182290322784712986925720808
6	1	3	0.99959370433770807882786299647928332089368010624696
6	5	3	0.99858719888276172844538392057624551262611598197330
6	*	3	0.94773326214367537593952153765418961303363163231741
6	1	4	0.99994499634458014625186893933369950019928710526941
6	5	4	0.99972676441180963795649942604993513585006678234693
6	*	4	0.97582415304766824167901143659479983197176497122921
6	1	5	0.99999234838384640058266382093557284740952517853584
6	5	5	0.99994610165668179362257114019349539984360834713448
6	*	5	0.98850439774124690875110662385118666440095808327535
7	1	1	0.99783169567561415395612278331682981321793848654811
7	2	1	0.83084436115736277012960608040205678524297640045329
7	3	1	0.91204173657631900150492713248862784862731325200951
7	4	1	0.99152664451130171888186444781116508688643378122073
7	5	1	0.96301728477470669833755668029320880900891574734534
7	6	1	0.99343749409772808721220289503432216846512348209610
7	*	1	0.70444220099916559273660335032663721018858643141710
7	1	2	0.99994366073240649656784328959429546613893530201614
7	2	2	0.91657341209621074525122836181093861113002239080188
7	3	2	0.97199490345740290514191139256429322419400292540344
7	4	2	0.99929947053146742408555681415248114978845320558949
7	5	2	0.99317882616912445691586599920122870731373256645422
7	6	2	0.99955929237466022465905605429286339331226987661148
7	*	2	0.88151383972517077692839182290322784712986925720808
7	1	3	0.99999829680521359669791936140827355119253342304436
7	2	3	0.95832951728972205581988628973822050125077925004215
7	3	3	0.99072837048111798598099398660127807953178764545713
7	4	3	0.99993720312563673896846596275692472208855362653198
7	5	3	0.99865908524683796954174391736600314663890463047532
7	6	3	0.99996710470497322840547153373281750802705003754425
7	*	3	0.94773326214367537593952153765418961303363163231741
7	1	4	0.99999994558480920546185156129258500018727263672918
7	2	4	0.97916650671420237250529489743286478175232028721336
7	3	4	0.99691288143967753375215937932656902663383764671792
7	4	4	0.99999430503583323315981125802879383765208262150784
7	5	4	0.99973294403625574083545067264002211232543320540325
7	6	4	0.99999749025918214169988795654177516084234849072519
7	*	4	0.97582415304766824167901143659479983197176497122921
7	1	5	0.99999999821182133621213755516477429295696450571644
7	2	5	0.98958332654672320978345933759407259691090563702888
7	3	5	0.99897115320706590516862955106329442434349627356586
7	4	5	0.99999948250950920843035994399632033679802298131632
7	5	5	0.99994664636169323397746114726903151081859982442482
7	6	5	0.99999980741214224538109413884142031379247371320710
7	*	5	0.98850439774124690875110662385118666440095808327535

Equivalent to Remark 10, lines with a star in the n-column are

$$Q_1^{(s)} = \prod_{n=1}^{m} Q_{m,n}^{(s)}. \tag{51}$$

The Niklasch values of $Q_1^{(s)}$, $s \le 5$ are A065463 and A065465–A065468 in the OEIS [27].

Definition 9. *(Feller-Tornier constants)*

$$F_{m,n}^{(s)} \equiv \prod_{p=n \pmod m} \left(1 - \frac{2}{p^s}\right) = \prod_j \zeta_{m,n}(j)^{-\gamma_{s,j}^{(F)}}. \tag{52}$$

Based on [21]

$$\log F_{m,n}^{(s)} = -\sum_{t=1}^{\infty} \frac{2^t}{t} P_{m,n}(st), \tag{53}$$

the $F_{m,n}^{(s)}$ become:

m	n	s	F
3	1	2	0.93484201367742708692711271624051049601103614106357
3	2	2	0.44372767162332273561525453355445652782381952938988
3	*	2	0.32263409893924467057953169254823706657095057966583
3	1	3	0.99280761653561283079975439424575351687703956375940
3	2	3	0.73634019702793989821414960486873306430411328482419
3	*	3	0.67689273700988199361023732672438921279767839745979
3	1	4	0.99907744600555800880829233535859351936265583252243
3	2	4	0.87204973197830029518258016493629726761761662731638
3	*	4	0.84973299138471876626505370362916043989282010424286
3	1	5	0.99987468990135792609587644151500113697139680938750
3	2	5	0.93688662911191901929922845845837418044555880822266
3	*	5	0.92905919295966281511524587198420062376637612342100
4	1	2	0.89484122456248817072566150690843732198754780892072
4	3	2	0.72109797824075241583243117750350641933238009488227
4	*	2	0.32263409893924467057953169254823706657095057966583
4	1	3	0.98251462525135333541051144592244831503480272183851
4	3	3	0.91858546035955424632246635972848794535685587324250
4	*	3	0.67689273700988199361023732672438921279767839745979
4	1	4	0.99670115119737736072418577538862522209341552226715
4	3	4	0.97433761118740866145644826461858163613349229991035
4	*	4	0.84973299138471876626505370362916043989282010424286
4	1	5	0.99935305638694748001375333214108685197931481807727
4	3	5	0.99163800636498496397333514320139495841461274016083
4	*	5	0.92905919295966281511524587198420062376637612342100
5	1	2	0.97845248073805828720735998931024731012225701761724
5	2	2	0.47437087172424840070720711055620424815486422859867
5	3	2	0.76307380442876918582110108389946175183811018177911
5	4	2	0.99014350052786791479556620332141840911985643518766
5	*	2	0.32263409893924467057953169254823706657095057966583
5	1	3	0.99838190086287504193401769542494480352829930716181
5	2	3	0.74526882528822037898411982038512181070115931122217
5	3	3	0.92488231672061579498342311673595319971735637472649
5	4	3	0.99960547324813363288137636241078869060175533783316
5	*	3	0.67689273700988199361023732672438921279767839745979
5	1	4	0.99986026431454149472891209002225661373275171174437
5	2	4	0.87424877013139375867977083071682392372665769680115
5	3	4	0.97523240905393793969475076372750984033219642558801
5	4	4	0.99998154787827064113659669538698369167247356564844
5	*	4	0.84973299138471876626505370362916043989282010424286
5	1	5	0.99998749067686156347656227622782664876124713464261
5	2	5	0.93738708184645276170019195939946207242296096965759
5	3	5	0.99176387687679582568564123108609354980158058605822
5	4	5	0.99999909074199348489740247879140513547360047968785

5	*	5	0.92905919295966281511524587198420062376637612342100
6	1	2	0.93484201367742708692711271624051049601103614106357
6	5	2	0.88745534324664547123050906710891305564763905877975
6	*	2	0.32263409893924467057953169254823706657095057966583
6	1	3	0.99280761653561283079975439424575351687703956375940
6	5	3	0.98178692937058653095219947315831075240548437976559
6	*	3	0.67689273700988199361023732672438921279767839745979
6	1	4	0.99907744600555800880829233535859351936265583252243
6	5	4	0.99662826511805748020866304564148259156299043121872
6	*	4	0.84973299138471876626505370362916043989282010424286
6	1	5	0.99987468990135792609587644151500113697139680938750
6	5	5	0.99934573771938028725251035568893245914192939543750
6	*	5	0.92905919295966281511524587198420062376637612342100
7	1	2	0.99555404713198948244184584044601450508178750229430
7	2	2	0.49691435036587215810973241763330728415652207058013
7	3	2	0.76957796537542257228306221725905383464002237343862
7	4	2	0.98166945377655916042561037674324801490546565135125
7	5	2	0.91277467867618400750325447073175642635692243929785
7	6	2	0.98600841416269679563485056385653389004971689718240
7	*	2	0.32263409893924467057953169254823706657095057966583
7	1	3	0.99988391667262821518818274735430677265688389455317
7	2	3	0.74984143188165689857204887699782457488954387149038
7	3	3	0.92546983387929396966566578391620995434498396242975
7	4	3	0.99847336720476224584831435899199360231273056600688
7	5	3	0.98367895449893787275102115297201007664596764353105
7	6	3	0.99905281257738254491306703845873477181422457136253
7	*	3	0.67689273700988199361023732672438921279767839745979
7	1	4	0.99999648478110131582478600860698357487906770714195
7	2	4	0.87499274570389232686770562590711676511699533369021
7	3	4	0.97528291952775784702284332764806085376404886702137
7	4	4	0.99986301635151287548319500298252048598086571436272
7	5	4	0.99678408919749714690368672145075838680245939386610
7	6	4	0.99992918995830049683594411988434204666642097898894
7	*	4	0.84973299138471876626505370362916043989282010424286
7	1	5	0.99999988759326185799713244165714207306583554795699
7	2	5	0.93749968084921148189646542628572899629520177899819
7	3	5	0.99176807708490809250200263138026260692965359369641
7	4	5	0.99998757509072907576280787952699016917459084613132
7	5	5	0.99935918111048145887987957173392322340983918846234
7	6	5	0.99999459534270191193438256531181007469985761332297
7	*	5	0.92905919295966281511524587198420062376637612342100

The redundancies are

$$F_{6,1}^{(s)} = F_{3,1}^{(s)}; \quad F_{6,5}^{(s)} = F_{3,2}^{(s)}/(1-2^{1-s}). \tag{54}$$

Equivalent to Remark 10, lines with a star in the n-column are

$$F_1^{(s)} = \prod_{n=1}^{m} F_{m,n}^{(s)} \tag{55}$$

as known [21]. $F_1^{(2)}$ is A065474 in the OEIS [27].

Definition 10. *(Hardy-Littlewood constants)*

$$C_{m,n}^{(s)} \equiv \prod_{\substack{p=n \pmod m \\ p>s}} \frac{p^{s-1}(p-s)}{(p-1)^s} = \prod_j \zeta_{m,n}(j)^{-\gamma_{s,j}^{(C)}}. \tag{56}$$

Based on [21]

$$\log C_{m,n}^{(s)} = -\sum_{t=2}^{\infty} \frac{s^t - s}{t} P_{m,n}(t), \tag{57}$$

the $C_{m,n}^{(s)}$ read:

m	n	s	C
3	1	2	0.95836482362808797487757575013966951613301317628078
3	2	2	0.91845582471428180359847772409497399151465223444510
3	*	2	0.66016181584686957392781211001455577843262336028473
3	1	3	0.86751218171239491908907658476288886972026952686301
3	2	3	0.73216995449044542201998526124057450315277708109614
3	*	3	0.63516635460427120720669659127252241734206568733237
3	1	4	0.72258658325534766847436578690250565003731801889725
3	2	4	0.4255474511760544876327159311166763672980634588268
3	*	4	0.30749487875832709312335448607107685302217851995066
3	1	5	0.52346534507199690512570203587861939776333944516009
3	2	5	0.78300290352910122201164489284179973132539677801906
3	*	5	0.40987488508823647447878121233795527789635801325495
3	1	6	0.27710605535004326975691888903900717559837928534397
3	2	6	0.67343998355656740459077411005798448137406580764280
3	*	6	0.18661429735835839665692484794418833784007394494559
4	1	2	0.92306113221757975924356514047403417887101279477641
4	3	2	0.71518753504535953339073535622822204345296583727742
4	*	2	0.66016181584686957392781211001455577843262336028473
4	1	3	0.74414958837024499127360452418064114273268267540782
4	3	3	0.85354660478324400008551727753202067459718543554908
4	*	3	0.63516635460427120720669659127252241734206568733237
4	1	4	0.44091192969092295875998558949907810730433587749524
4	3	4	0.69740657499079114022349189307772080129218829858526
4	*	4	0.30749487875832709312335448607107685302217851995066
4	1	5	0.83597682802260698833719482963834225780151289192106
4	3	5	0.49029455284991749943661473521891278555391304309671
4	*	5	0.40987488508823647447878121233795527789635801325495
4	1	6	0.75229690708395736448771308785547959989749911962742
4	3	6	0.24805937071004332290260100799138708454500603005911
4	*	6	0.18661429735835839665692484794418833784007394494559
5	1	2	0.98735247301453588313535598749142090390363595767527
5	2	2	0.96641612578310894608587557568989440501197703436705
5	3	2	0.74195287815639505497135252353507072030513398594048
5	4	2	0.99464113053183188600569420316192917886306904257169
5	*	2	0.66016181584686957392781211001455577843262336028473
5	1	3	0.96012864968975466210115275000677088726813802997313
5	2	3	0.89069345898428997391945247100168623128503858445870
5	3	3	0.96661807398837421625616971838928560460691085028529
5	4	3	0.98352374917967957309861313654634184087997254268265
5	*	3	0.63516635460427120720669659127252241734206568733237
5	1	4	0.91635769426465932563156516687748163064942656924781
5	2	4	0.76395458708732551236931737716038770467396880305946
5	3	4	0.93096047342619617143737099126866851277161150022652
5	4	4	0.96628229818335611095281967822459833058807820685263
5	*	4	0.30749487875832709312335448607107685302217851995066
5	1	5	0.85407396102427097380601359543999477960804600701910
5	2	5	0.57764567367339449979254761372677952826205357641920
5	3	5	0.88138997545439239487651612116012354934561245726479
5	4	5	0.94259730749543865248035195590178804888244553462061
5	*	5	0.40987488508823647447878121233795527789635801325495
5	1	6	0.77137630056008079159358507773745815248542947723585
5	2	6	0.32451219145047009948254991056074596924905555805738
5	3	6	0.81724710072367521785130485832746372286203725036689
5	4	6	0.91220871378010997522334196835556787989433736577703
5	*	6	0.18661429735835839665692484794418833784007394494559
6	1	2	0.95836482362808797487757575013966951613301317628078
6	5	2	0.91845582471428180359847772409497399151465223444510
6	*	2	0.66016181584686957392781211001455577843262336028473
6	1	3	0.86751218171239491908907658476288886972026952686301
6	5	3	0.73216995449044542201998526124057450315277708109614
6	*	3	0.63516635460427120720669659127252241734206568733237
6	1	4	0.72258658325534766847436578690250565003731801889725
6	5	4	0.4255474511760544876327159311166763672980634588268
6	*	4	0.30749487875832709312335448607107685302217851995066
6	1	5	0.52346534507199690512570203587861939776333944516009
6	5	5	0.78300290352910122201164489284179973132539677801906
6	*	5	0.40987488508823647447878121233795527789635801325495
6	1	6	0.27710605535004326975691888903900717559837928534397

6	5	6	0.67343998355656740459077411005798448137406580764280
6	*	6	0.18661429735835839665692484794418833784007394494559
7	1	2	0.99765398801824805745575199848586590104838977156875
7	2	2	0.99668740154736185119549542357183826086432853703231
7	3	2	0.74564040438722187807736297411080874332720005893822
7	4	2	0.98906852604128092494920011866206688759944804504343
7	5	2	0.93348561251953561042303182283359764377636008578062
7	6	2	0.99193337190767667267067400622701760558938762002681
7	*	2	0.66016181584686957392781211001455577843262336028473
7	1	3	0.99284563576602975595901031078857636226019241830154
7	2	3	0.98984306267452432665619052548686584276034684896281
7	3	3	0.98203592018701854285981806936191667723888824054206
7	4	3	0.96524528800661834895816280975502100995118818242669
7	5	3	0.77093874395874628323816728886762758318909948223296
7	6	3	0.97465778149976704833101678046677018357146007753252
7	*	3	0.63516635460427120720669659127252241734206568733237
7	1	4	0.98546514686054195689724549716779516136368771372367
7	2	4	0.97925593342382577716321982515617176017498868620909
7	3	4	0.96303959431570229715988416824794900218954678573364
7	4	4	0.92635560049467742885049474387330246912217450891583
7	5	4	0.47504942820632988734750925376986894672657306491884
7	6	4	0.94695503795677660236507040844485582354536105333782
7	*	4	0.30749487875832709312335448607107685302217851995066
7	1	5	0.97541078884978790157546934143855452639378834459773
7	2	5	0.96472640730111318180467286493968492162356472526847
7	3	5	0.93672333482275622639932836356657031526367035173684
7	4	5	0.86999884056567783769854473353741503126140009551379
7	5	5	0.95367427149821135161097352295533167315911015618065
7	6	5	0.90753546045655108160772300725983280806657638255495
7	*	5	0.40987488508823647447878121233795527789635801325495
7	1	6	0.96259053430958360955758267936940830333133617188597
7	2	6	0.94607005665812543459253596932520179647066529165643
7	3	6	0.90265831568425453492025154169848478262640590887244
7	4	6	0.79353818470465233918261062748102769445314469036766
7	5	6	0.92878975832084684493539660980925986457038262893686
7	6	6	0.85504445986218770161405444452761140400159991710681
7	*	6	0.18661429735835839665692484794418833784007394494559

For the standard reasons, $C_{6,1}^{(s)} = C_{3,1}^{(s)}$ and $C_{6,5}^{(s)} = C_{3,2}^{(s)}$. Lines with a star in the n-column are [21]

$$C_1^{(s)} = \prod_{n=1}^{m} C_{m,n}^{(s)}, \tag{58}$$

compatible with constants A005597, A065418 and A065419 in the OEIS [27].

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