Preprint, arXiv:1008.3887

CONGRUENCES INVOLVING GENERALIZED CENTRAL TRINOMIAL COEFFICIENTS

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Abstract. For integers b and c the generalized trinomial coefficient $T_n(b,c)$ denotes the coefficient of x^n in the expansion of $(x^2+bx+c)^n$. Those $T_n=T_n(1,1)$ $(n=0,1,2,\ldots)$ are the usual central trinomial coefficients, and $T_n(3,2)$ coincides with the Delannoy number $D_n=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}$ in combinatorics. In this paper we investigate congruences involving generalized central trinomial coefficients systematically. Here are some typical results: For each $n=1,2,3,\ldots$ we have

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(b^2-4c)^{n-1-k}\equiv 0 \pmod{n^2}$$

and in particular $n^2\mid\sum_{k=0}^{n-1}(2k+1)D_k^2$; if p is an odd prime then

$$\sum_{k=0}^{p-1}T_k^2\equiv\left(\frac{-1}{p}\right)\pmod p\quad\text{and}\quad\sum_{k=0}^{p-1}D_k^2\equiv\left(\frac{2}{p}\right)\pmod p,$$

where $(-)$ denotes the Jacobi symbol. We also raise several conjectures some of which involve parameters in the representations of primes by certain binary quadratic forms.

1. Introduction

For $n\in\mathbb{N}=\{0,1,2,\ldots\}$, the nth central trinomial coefficient

$$T_n=[x^n](1+x+x^2)^n$$

2010 *Mathematics Subject Classification.* Primary 11A07, 11B75; Secondary 05A10, 05A15, 11B65, 11E25.

Keywords. Congruences, central trinomial coefficients, Motzkin numbers, central Delannoy numbers.

Supported by the National Natural Science Foundation (grant 10871087) and the Overseas Cooperation Fund (grant 10928101) of China.

is the coefficient of x^n in the expansion of $(1+x+x^2)^n$. Since T_n is the constant term of $(1+x+x^{-1})^n$, by the multi-nomial theorem we see that

$$T_n=\sum_{k=0}^{\lfloor n/2\rfloor}\frac{n!}{k!k!(n-2k)!}=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k}.$$

Central trinomial coefficients arise naturally in enumerative combinatorics (cf. Sloane [Sl]), e.g., T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$. As G. E. Andrews [A] pointed out, central trinomial coefficients were first studied by L. Euler. In 1987, Andrews and R. J. Baxter [AB] found that the q-analogues of central trinomial coefficients have applications in the hard hexagon model.

For $n\in\mathbb{N}$ the nth Motzkin number is defined by

$$M_n=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_k,$$

where C_k denotes the kth Catalan number $\frac{1}{k+1}\binom{2k}{k}=\binom{2k}{k}-\binom{2k}{k+1}$. It is known that M_n equals the number of paths from $(0,0)$ to $(n,0)$ in an $n\times n$ grid using only steps $(1,1)$, $(1,0)$ and $(1,-1)$ (cf. [Sl]).

Surprisingly we find that central trinomial coefficients and Motzkin numbers have nice congruence properties despite their combinatorial backgrounds. For example, we have the following conjecture.

Conjecture 1.1. (i) *For any $n\in\mathbb{Z}^+=\{1,2,3,\ldots\}$ we have*

$$\sum_{k=0}^{n-1}(8k+5)T_k^2\equiv 0 \pmod{n}.$$

If p is a prime, then

$$\sum_{k=0}^{p-1}(8k+5)T_k^2\equiv 3p\left(\frac{p}{3}\right) \pmod{p^2}.$$

(ii) *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}M_k^2\equiv(2-6p)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}kM_k^2\equiv(9p-1)\left(\frac{p}{3}\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}M_kT_k\equiv\frac{4}{3}\left(\frac{p}{3}\right)+\frac{p}{6}\left(1-9\left(\frac{p}{3}\right)\right) \pmod{p^2},$$

$$\sum_{k=0}^{p-1}\frac{M_kT_k}{(-3)^k}\equiv\frac{p}{2}\left(\left(\frac{p}{3}\right)-1\right) \pmod{p^2},$$

and

$$\sum_{k=0}^{p-1}\frac{T_kH_k}{3^k}\equiv\frac{3+(\frac{p}{3})}{2}-p\left(1+\left(\frac{p}{3}\right)\right)\pmod{p^2},$$

where H_k *denotes the harmonic number* $\sum_{0<j\leqslant k}1/j$.

Given $b,c\in\mathbb{Z}$, we define the *generalized central trinomial coefficients*

$$\begin{aligned}T_n(b,c):=&[x^n](x^2+bx+c)^n=[x^0](b+x+cx^{-1})^n\\ =&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\binom{n}{k}b^{n-2k}c^k\end{aligned}$$

and introduce the *generalized Motzkin numbers*

$$M_n(b,c):=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}C_kb^{n-2k}c^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n-k}{k}\binom{n}{k}\frac{b^{n-2k}c^k}{k+1}$$

$(n=0,1,2,\ldots)$. Note that

$$T_n=T_n(1,1),\ M_n=M_n(1,1),$$

$$T_n(2,1)=[x^n](x+1)^{2n}=\binom{2n}{n},$$

and

$$M_n(2,1)=\sum_{k=0}^{n}\binom{n}{2k}C_k2^{n-2k}=C_{n+1}.$$

Thus $T_n(b,c)$ can be viewed a natural common extension of central binomial coefficients and central trinomial coefficients, while $M_n(b,c)$ can be viewed as a natural common extension of Catalan numbers and Motzkin numbers. Let $d=b^2-4c$. H. S. Wilf [W, p. 159] observed that

$$\sum_{n=0}^{\infty}T_n(b,c)x^n=\frac{1}{\sqrt{1-2bx+dx^2}}$$

which implies the recursion

$$(n+1)T_{n+1}(b,c)=(2n+1)bT_n(b,c)-dnT_{n-1}(b,c)\quad(n\in\mathbb{Z}^+).$$

(See also T. D. Noe [N].) Also, the Zeilberger algorithm (cf. [PWZ]) yields the recursion

$$(n+3)M_{n+1}(b,c)=b(2n+3)M_n(b,c)-dnM_{n-1}(b,c)\ (n=1,2,3,\ldots)$$

which implies that

$$2cx^2\sum_{n=0}^{\infty}M_n(b,c)x^n=1-bx-\sqrt{1-2bx+dx^2}.$$

The central Delannoy numbers (see [CHV]) are defined by

$$D_n=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\ (n\in\mathbb{N}).$$

Such numbers also arise in many enumeration problems in combinatorics (cf. [Sl]); for example, D_n is the number of lattice paths from the point $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$. For $n\in\mathbb{N}$ we define the polynomial

$$D_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k.$$

Note that $D_n((x-1)/2)$ coincides with the well-known Legendre polynomial $P_n(x)$ of degree n. It is known that

$$\sum_{n=0}^{\infty}P_n(t)x^n=\frac{1}{\sqrt{1-2tx+x^2}}.$$

Thus, if $b,c\in\mathbb{Z}$ and $d=b^2-4c\neq 0$ then

$$\sum_{n=0}^{\infty}T_n(b,c)\left(\frac{x}{\sqrt{d}}\right)^n=\frac{1}{\sqrt{1-2bx/\sqrt{d}+d(x/\sqrt{d})^2}}=\sum_{n=0}^{\infty}P_n(b)x^n$$

and hence

$$T_n(b,c)=(\sqrt{d})^nP_n\left(\frac{b}{\sqrt{d}}\right).$$

It follows that

$$T_n(2x+1,x^2+x)=P_n(2x+1)=D_n(x)\ \text{ for all } x\in\mathbb{Z};$$

in particular, $D_n=T_n(3,2)$.

Motivated by Conjecture 1.1 we investigate congruences involving generalized central trinomial coefficients as well as generalized Motzkin numbers.

Now we state the main results of this paper.

Theorem 1.1. *Let p be an odd prime and let $b, c \in \mathbb{Z}$.*

(i) *For any integer $m \not\equiv 0 \pmod p$, we have*

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)}{m^k} \equiv \left(\frac{(m-b)^2-4c}{p}\right) \pmod p \tag{1.1}$$

and

$$2c \sum_{k=0}^{p-1} \frac{M_k(b,c)}{m^k} \equiv (m-b)^2 - ((m-b)^2-4c)\left(\frac{(m-b)^2-4c}{p}\right) \pmod p. \tag{1.2}$$

(ii) *If p does not divide $d = b^2 - 4c$, then we have*

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)^2}{d^k} \equiv \left(\frac{cd}{p}\right) \pmod p. \tag{1.3}$$

If $b \not\equiv 2c \pmod p$, then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c^2)^2}{(b-2c)^{2k}} \equiv \left(\frac{-c^2}{p}\right) \pmod p. \tag{1.4}$$

(iii) *Assume that $p \nmid c$. If $d = b^2 - 4c \not\equiv 0 \pmod p$, then*

$$\sum_{k=0}^{p-1} \frac{T_k(b,c)M_k(b,c)}{d^k} \equiv 0 \pmod p. \tag{1.5}$$

If $D = b^2 - 4c^2 \not\equiv 0 \pmod p$, then

$$\sum_{k=0}^{p-1} \frac{T_k(b,c^2)M_k(b,c^2)}{(b-2c)^{2k}} \equiv \frac{4b}{b+2c}\left(\frac{D}{p}\right) \pmod p. \tag{1.6}$$

Example 1.1. Let $p > 3$ be a prime. Applying Theorem 1.1(ii)-(iii) with $b = c = 1$ we get

$$\sum_{k=0}^{p-1} \frac{T_k^2}{(-3)^k} \equiv \left(\frac{p}{3}\right) \pmod p, \quad \sum_{k=0}^{p-1} \frac{T_k M_k}{(-3)^k} \equiv 0 \pmod p, \tag{1.7}$$

$$\sum_{k=0}^{p-1} T_k^2 \equiv \left(\frac{-1}{p}\right) \pmod p, \quad \sum_{k=0}^{p-1} T_k M_k \equiv \frac{4}{3}\left(\frac{p}{3}\right) \pmod p. \tag{1.8}$$

Corollary 1.1. *Let p be an odd prime. For any integer x we have*

$$\sum_{k=0}^{p-1} D_k(x)^2 \equiv \left(\frac{x(x+1)}{p}\right) \pmod p. \tag{1.9}$$

In particular,

$$\sum_{k=0}^{p-1} D_k^2 \equiv \left(\frac{2}{p}\right) \pmod p. \tag{1.10}$$

Proof. It suffices to recall that $D_k(x) = T_k(2x+1, x^2+x)$ and apply Theorem 1.1(ii). $\square$

Theorem 1.2. *Let $b, c \in \mathbb{Z}$ and $d = b^2 - 4c$.*

(i) *For any $n \in \mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1} T_k(b, c^2)(b-2c)^{n-1-k} \equiv 0 \pmod n \tag{1.11}$$

and

$$6\sum_{k=0}^{n-1} kT_k(b, c^2)(b-2c)^{n-1-k} \equiv 0 \pmod n. \tag{1.12}$$

If p is an odd prime not dividing $b-2c$, then

$$\frac{2c}{p}\sum_{k=0}^{p-1} \frac{T_k(b, c^2)}{(b-2c)^k} \equiv -b + (b+2c)\left(\frac{b^2-4c^2}{p}\right) \pmod p \tag{1.13}$$

and

$$\frac{12c^2}{p}\sum_{k=0}^{p-1} \frac{kT_k(b, c^2)}{(b-2c)^k} \equiv (b+2c)^2\left(1 - \left(\frac{b^2-4c^2}{p}\right)\right) - 4c^2 \pmod p. \tag{1.14}$$

(ii) *Suppose that $d = 1$, i.e., there is an $m \in \mathbb{Z}$ such that $b = 2m+1$, $c = m^2 + m$, and hence $T_k(b, c) = D_k(m)$. Then*

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k(b, c) = \sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k \in \mathbb{Z} \tag{1.15}$$

for all $n \in \mathbb{Z}^+$. If p is a prime not dividing $b-1 = 2m$, then

$$\sum_{k=0}^{p-1}(2k+1)T_k(b, c) \equiv p + \frac{b+1}{b-1}p\left(\left(\frac{b+1}{2}\right)^{p-1} - 1\right) \pmod{p^3} \tag{1.16}$$

and

$$\sum_{k=0}^{p-1}(2k+1)^2T_k(b,c)\equiv\frac{2}{b-1}\left(\frac{(1-b)/2}{p}\right)=\frac{1}{m}\left(\frac{-m}{p}\right)\pmod p. \tag{1.17}$$

Example 1.2. Putting $b=1$ and $c=\pm1$ in (1.11) we get

$$\sum_{k=0}^{n-1}(-1)^kT_k\equiv0\pmod n\quad\text{and}\quad\sum_{k=0}^{n-1}3^{n-1-k}T_k\equiv0\pmod n,$$

where n is any positive integer. Also, for a prime $p>3$, (1.13) with $b=1$ and $c=\pm1$ yields $\sum_{k=0}^{p-1}(-1)^kT_k$ and $\sum_{k=0}^{p-1}T_k/3^k$ modulo p^2 given by H. Q. Cao and H. Pan [CP].

Remark 1.1. The author notes that for any $n\in\mathbb{Z}^+$ we have

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)T_k3^{n-1-k}=\sum_{k=0}^{n-1}\binom{n-1}{k}(-1)^{n-1-k}(k+1)\binom{2k}{k}.$$

If $b,c\in\mathbb{Z}$ with $b^2-4c=1$, then for any prime $p\nmid c$ by (1.16) we have

$$\sum_{k=0}^{p-1}(2k+1)T_k(b,c)\equiv p\pmod{p^2}.$$

Theorem 1.3. *Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$.*

(i) *For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k}\equiv0\pmod n, \tag{1.18}$$

and furthermore

$$b\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k}=nT_n(b,c)T_{n-1}(b,c). \tag{1.19}$$

(ii) *For any $n\in\mathbb{Z}^+$ we have*

$$\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2d^{n-1-k}=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}C_kc^kd^{n-1-k}\in\mathbb{Z}. \tag{1.20}$$

If c is nonzero and p is an odd prime not dividing d, then

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k}\equiv1+\frac{b^2}{c}\cdot\frac{(\frac{d}{p})-1}{2}\pmod p. \tag{1.21}$$

Now we give one more theorem.

Theorem 1.4. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(6,-3)^2}{48^k}\equiv\left(\frac{-1}{p}\right)+\frac{p^2}{3}E_{p-3}\pmod{p^3},\tag{1.22}$$

$$\sum_{k=0}^{p-1}\frac{T_k(2,-1)^2}{8^k}\equiv\left(\frac{-2}{p}\right)\pmod{p^2},\tag{1.23}$$

$$\sum_{k=0}^{p-1}\frac{T_k(2,-3)^2}{16^k}\equiv\left(\frac{p}{3}\right)\pmod{p^2},\tag{1.24}$$

$$\sum_{k=1}^{p-1}\frac{D_k^2}{k^2}\equiv-2q_p(2)^2\pmod{p},\tag{1.25}$$

where $E_0,E_1,E_2,\ldots$ are Euler numbers, and $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

Remark 1.2. (1.25) was conjectured by the author in [Su3].

We will show Theorems 1.1 and 1.2 in Sections 2 and 3 respectively. Section 4 is devoted to our proofs of Theorems 1.3 and 1.4. In Section 5 we are going to raise more conjectures for further research.

2. Proof of Theorem 1.1

The following lemma essentially follows from [ST, (1.5)], but we will give a direct proof.

Lemma 2.1. *Let p be an odd prime and let $m\in\mathbb{Z}$ with $m\not\equiv0\pmod{p}$. Then*

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}\equiv\left(\frac{m(m-4)}{p}\right)\pmod{p}\tag{2.1}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{C_k}{m^k}\equiv\frac{m}{2}-\frac{m-4}{2}\left(\frac{m(m-4)}{p}\right)\pmod{p}.\tag{2.2}$$

Proof. Clearly

$$\binom{2k}{k}=\binom{-1/2}{k}(-4)^k\equiv\binom{(p-1)/2}{k}(-4)^k$$

for all $k\in\mathbb{N}$. Thus

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}&\equiv\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}\frac{(-4)^k}{m^k}=\left(1-\frac{4}{m}\right)^{(p-1)/2}\\&=\frac{(m(m-4))^{(p-1)/2}}{m^{p-1}}\equiv\left(\frac{m(m-4)}{p}\right)\pmod{p}.\end{aligned}$$

This proves (2.1).

Observe that

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k+1}{k}}{m^k}=&\frac{\binom{p}{(p-1)/2}}{m^{(p-1)/2}}+\frac{1}{2}\sum_{k=0}^{(p-3)/2}\frac{\binom{2k+2}{k+1}}{m^k}\\ \equiv&\frac{m}{2}\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}-\frac{m}{2}\pmod p.\end{aligned}$$

Hence

$$\begin{aligned}\sum_{k=0}^{(p-1)/2}\frac{C_k}{m^k}=&\sum_{k=0}^{(p-1)/2}\frac{2\binom{2k}{k}-\binom{2k+1}{k}}{m^k}\\ \equiv&\left(2-\frac{m}{2}\right)\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{m^k}+\frac{m}{2}\\ \equiv&\frac{m}{2}-\frac{m-4}{2}\left(\frac{m(m-4)}{p}\right)\pmod p.\end{aligned}$$

So (2.2) also holds. We are done. □

Proof of Theorem 1.1(i). In the case $c\equiv 0\pmod p$, as $T_k(b,c)\equiv b^k\pmod c$ for all $k\in\mathbb{N}$, we have

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)}{m^k}\equiv\sum_{k=0}^{p-1}\frac{b^k}{m^k}\equiv\left(\frac{(m-b)^2}{p}\right)\pmod p.$$

So (1.1) holds if $p\mid c$. Note that (1.2) is trivial when $p\mid c$.

Suppose that $c\not\equiv 0\pmod p$. Note that for any $n\in\mathbb{N}$ we have

$$T_n(b,c)=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k\equiv\begin{cases}\binom{n}{n/2}c^{n/2}\pmod b & \text{if } 2\mid n,\\ 0\pmod b & \text{if } 2\nmid n.\end{cases}$$

In the case $b\equiv 0\pmod p$, by applying Lemma 2.1 we obtain

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)}{m^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}c^k}{m^{2k}}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}}{(m^2c^{p-2})^k}\equiv\left(\frac{m^2-4c}{p}\right)\pmod p$$

and

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{M_k(b,c)}{m^k}\equiv&\sum_{k=0}^{(p-1)/2}\frac{C_kc^k}{m^{2k}}\equiv\sum_{k=0}^{(p-1)/2}\frac{C_k}{(m^2c^{p-2})^k}\\ \equiv&\frac{m^2}{2c}-\frac{m^2-4c}{2c}\left(\frac{m^2-4c}{p}\right)\pmod p.\end{aligned}$$

So (1.1) and (1.2) hold when $p \mid b$.

Below we assume that $p \nmid bc$. Observe that

$$\sum_{n=0}^{p-1}\frac{T_n(b,c)}{m^n} =\sum_{n=0}^{p-1}\frac{1}{m^n}\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k$$
$$=\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{c^k}{b^{2k}}\sum_{n=0}^{p-1}\frac{b^n}{m^n}\binom{n}{2k}$$

and

$$\sum_{n=0}^{p-1}\frac{M_n(b,c)}{m^n} = \sum_{k=0}^{(p-1)/2}C_k\frac{c^k}{b^{2k}}\sum_{n=0}^{p-1}\frac{b^n}{m^n}\binom{n}{2k}$$

in a similar way.

Now we consider the case $m \equiv b \pmod p$. For $k \in \{0,1,\ldots,(p-1)/2\}$ we have

$$\sum_{k=0}^{p-1}\frac{b^n}{m^n}\binom{n}{2k} \equiv \sum_{n=2k}^{p-1}\binom{n}{2k} = \binom{p}{2k+1} \pmod p$$

with the help of a well-known identity of Chu (see, (1.52) of H. Gould [G, p, 7] or (5.26) of [GKP, p. 169]). Thus, by the above,

$$\sum_{n=0}^{p-1}\frac{T_n(b,c)}{m^n} \equiv \binom{p-1}{(p-1)/2}\frac{c^{(p-1)/2}}{b^{p-1}} \equiv \left(\frac{-c}{p}\right) = \left(\frac{(m-b)^2-4c}{p}\right) \pmod p$$

and

$$\sum_{n=0}^{p-1}\frac{M_n(b,c)}{m^n} \equiv C_{(p-1)/2}\frac{c^{(p-1)/2}}{b^{p-1}} \equiv 2\left(\frac{-c}{p}\right) = 2\left(\frac{(m-b)^2-4c}{p}\right) \pmod p.$$

So (1.1) and (1.2) are true.

Below we consider the remaining case $m \not\equiv b \pmod p$. Observe that

$$\sum_{n=0}^{p-1}\frac{b^n}{m^n}\binom{n}{2k} = [x^{2k}]\sum_{n=0}^{p-1}\frac{b^n}{m^n}(1+x)^n$$
$$\equiv [x^{2k}]\sum_{n=0}^{p-1}(b+bx)^n m^{p-1-n} = [x^{2k}]\frac{(b+bx)^p-m^p}{b+bx-m}$$
$$=[x^{2k}]\frac{(b+bx)^p-m^p}{-(m-b)^p}\cdot\frac{(bx)^p-(m-b)^p}{bx-(m-b)}$$
$$\equiv[x^{2k}]\frac{b^p+b^px^p-m^p}{-(m-b)^p}\sum_{j=0}^{p-1}(bx)^j(m-b)^{p-1-j} \equiv \frac{b^{2k}}{(m-b)^{2k}} \pmod p.$$

Therefore, with the help of Lemma 2.1,

$$\sum_{k=0}^{p-1}\frac{T_n(b,c)}{m^n}\equiv\sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{c^k}{b^{2k}}\cdot\frac{b^{2k}}{(m-b)^{2k}}$$
$$\equiv\left(1-\frac{4c}{(m-b)^2}\right)^{(p-1)/2}\equiv\left(\frac{(m-b)^2-4c}{p}\right)\pmod p.$$

This proves (1.1)

In a similar way,

$$\sum_{n=0}^{p-1}\frac{M_n(b,c)}{m^n}\equiv\sum_{k=0}^{(p-1)/2}C_k\frac{c^k}{(m-b)^{2k}}\equiv\sum_{k=0}^{(p-1)/2}\frac{C_k}{M^k}\pmod p,$$

where $M:=(m-b)^2c^{p-2}$. Applying Lemma 2.1 we get the desired (1.2). □

Lemma 2.2. *Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. Let p be any odd prime and let $n\in\{0,\ldots,p-1\}$. If $p\nmid d$ or $p/2<n<p$, then*

$$T_n(b,c)\equiv\left(\frac{d}{p}\right)d^nT_{p-1-n}(b,c)\pmod p. \tag{2.3}$$

Proof. If $p\mid d$, then

$$T_n(b,c)\equiv[x^n]\left(x^2+bx+\frac{b^2}{4}\right)^n=[x^n]\left(x+\frac{b}{2}\right)^{2n}=\binom{2n}{n}\frac{b^n}{2^n}\pmod p.$$

Note that for $n=(p+1)/2,\ldots,p-1$ we have

$$\binom{2n}{n}=\frac{(2n)!}{(n!)^2}\equiv 0\pmod p.$$

Now assume that $p\nmid d$. Then

$$d^nT_{p-1-n}(b,c)=d^n(\sqrt{d})^{p-1-n}P_{p-1-n}\left(\frac{b}{\sqrt{d}}\right)$$
$$=d^{(p-1)/2}\sum_{k=0}^{p-1-n}\binom{p-1-n+k}{2k}\binom{2k}{k}\left(\frac{b/\sqrt{d}-1}{2}\right)^k(\sqrt{d})^n$$
$$=d^{(p-1)/2}\sum_{k=0}^{p-1}\binom{n+k-p}{2k}\binom{2k}{k}\left(\frac{b-\sqrt{d}}{2\sqrt{d}}\right)^k(\sqrt{d})^n$$
$$=d^{(p-1)/2}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\left(\frac{b-\sqrt{d}}{2\sqrt{d}}\right)^k(\sqrt{d})^n$$
$$\equiv\left(\frac{d}{p}\right)(\sqrt{d})^nP_n\left(\frac{b}{\sqrt{d}}\right)=\left(\frac{d}{p}\right)T_n(b,c)\pmod p.$$

This concludes the proof. □

Remark 2.1. Lemma 2.2 in the case $p\nmid d$ is essentially known (see, e.g., [N, (14)]), but our proof is simple and direct. By Lemma 2.2, for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{T_k^2}{9^k}=\sum_{k=0}^{p-1}\left(\frac{T_k}{(-3)^k}\right)^2\equiv\sum_{k=0}^{p-1}\left(\left(\frac{-3}{p}\right)T_{p-1-k}\right)^2=\sum_{j=0}^{p-1}T_j^2 \pmod p$$

and hence $\sum_{k=0}^{p-1}T_k^2/9^k\equiv(\frac{-1}{p}) \pmod p$ in light of Example 1.1.

Let A and B be integers. The Lucas sequence $u_n=u_n(A,B)$ ($n=0,1,2,\ldots$) is defined by

$$u_0=0,\ u_1=1,\ \text{and}\ u_{n+1}=Au_n-Bu_{n-1}\ (n=1,2,3\ldots).$$

Let α and β be the two roots of the equation $x^2-Ax+B=0$. It is well-known that if $\Delta=A^2-4B\ne0$ then

$$u_n=\frac{\alpha^n-\beta^n}{\alpha-\beta}\quad \text{for all } n=0,1,2,\ldots.$$

Lemma 2.3. *Let A and B be integers. For any odd prime p we have*

$$u_p(A,B)\equiv\left(\frac{A^2-4B}{p}\right) \pmod p.$$

Proof. Though this is a known result, here we provide a simple proof.

If $\Delta=A^2-4B\equiv0 \pmod p$, then

$$u_n(A,B)\equiv u_n\left(A,\frac{A^2}{4}\right)=n\left(\frac{A}{2}\right)^{n-1} \pmod p\quad \text{for } n=1,2,3,\ldots$$

and in particular $u_p(A,B)\equiv0 \pmod p$.

When $\Delta\not\equiv0 \pmod p$, we have

$$\Delta u_p(A,B)=(\alpha-\beta)(\alpha^p-\beta^p)\equiv(\alpha-\beta)(\alpha-\beta)^p=\Delta^{(p+1)/2} \pmod p$$

with α and β the two roots of the equation $x^2-Ax+B=0$, hence $u_p(A,B)\equiv(\frac{\Delta}{p}) \pmod p$ as desired. □

Proof of Theorem 1.1(ii). Suppose that $d=b^2-4c\not\equiv0 \pmod p$. By Lemma 2.2,

$$\begin{aligned}&\left(\frac{d}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(b,c)^2}{d^k}\\ \equiv&\sum_{k=0}^{p-1}T_k(b,c)T_{p-1-k}(b,c)=[x^{p-1}]\bigg(\sum_{n=0}^{\infty}T_n(b,c)x^n\bigg)^2\\ =&[x^{p-1}]\frac{1}{1-2bx+dx^2}=[x^p]\frac{x}{1-2bx+dx^2} \pmod p.\end{aligned}$$

Write

$$\frac{x}{1-2bx+dx^2}=\sum_{n=0}^{\infty}u_nx^n.$$

Then $u_0=0$ and $u_1=1$. Since $(1-2bx+dx^2)\sum_{n=0}^{\infty}u_nx^n=x$, we have $u_n-2bu_{n-1}+du_{n-2}=0$ for $n=2,3,\ldots$, hence $u_n=u_n(2b,d)$ for all $n\in\mathbb{N}$. Thus, with the help of Lemma 2.3, from the above we obtain

$$\left(\frac{d}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(b,c)^2}{d^k}\equiv u_p(2b,d)\equiv\left(\frac{4b^2-4d}{p}\right)=\left(\frac{c}{p}\right)\ (\mathrm{mod}\ p).$$

This proves (1.3).

Now suppose that $b\not\equiv 2c\ (\mathrm{mod}\ p)$ and set $D=b^2-4c^2=(b-2c)(b+2c)$. If $p\mid D$, then $b\equiv-2c\not\equiv 0\ (\mathrm{mod}\ p)$ and hence

$$\sum_{k=0}^{p-1}\frac{T_k(b,c^2)^2}{(b-2c)^{2k}}\equiv\sum_{k=0}^{p-1}\frac{(\binom{2k}{k}(b/2)^k)^2}{(2b)^{2k}}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv\left(\frac{-1}{p}\right)\ (\mathrm{mod}\ p).$$

The last step can be easily explained as follows:

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}\equiv&\sum_{k=0}^{(p-1)/2}\binom{-1/2}{k}^2\\ \equiv&\sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}\binom{(p-1)/2}{(p-1)/2-k}\\ =&[x^{(p-1)/2}](1+x)^{(p-1)/2+(p-1)/2}\\ =&\binom{p-1}{(p-1)/2}\equiv\left(\frac{-1}{p}\right)\ (\mathrm{mod}\ p).\end{aligned}$$

Below we assume that $p\nmid D$. By Lemma 2.2 and Fermat's little theorem,

$$\left(\frac{D}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(b,c^2)^2}{(b-2c)^{2k}}\equiv C\ (\mathrm{mod}\ p)$$

where

$$\begin{aligned}C=&\sum_{k=0}^{p-1}D^kT_k(b,c^2)(b-2c)^{2(p-1-k)}T_{p-1-k}(b,c^2)\\ =&[x^{p-1}]\left(\sum_{k=0}^{\infty}T_k(b,c^2)(Dx)^k\right)\sum_{l=0}^{\infty}T_l(b,c^2)(b-2c)^{2l}x^l\\ =&[x^{p-1}]\frac{1}{\sqrt{1-2b(Dx)+D(Dx)^2}}\cdot\frac{1}{\sqrt{1-2b(b-2c)^2x+D(b-2c)^4x^2}}\\ =&[y^{p-1}]\frac{(b-2c)^{p-1}}{\sqrt{(1-2b(b+2c)y+(b+2c)^2Dy^2)(1-2b(b-2c)y+D(b-2c)^2y^2)}}\end{aligned}$$

(Note that y corresponds to $(b-2c)x$.) Therefore

$$\begin{aligned}C\equiv&[y^{p-1}]\frac{1}{1-Dy}\cdot\frac{1}{\sqrt{(1-(b+2c)^2y)(1-(b-2c)^2y)}}\\ \equiv&[y^{p-1}]\sum_{n=0}^{\infty}(Dy)^n\frac{1}{\sqrt{1-2(b^2+4c^2)y+D^2y^2}}\pmod p.\end{aligned}$$

Observe that $(b^2+4c^2)^2-4(4b^2c^2)=(b^2-4c^2)^2=D^2$ and hence

$$\frac{1}{\sqrt{1-2(b^2+4c^2)y+D^2y^2}}=\sum_{k=0}^{\infty}T_k(b^2+4c^2,4b^2c^2)y^k.$$

So we have

$$\begin{aligned}C\equiv&\sum_{k=0}^{p-1}T_k(b^2+4c^2,4b^2c^2)D^{p-1-k}\equiv\sum_{k=0}^{p-1}\frac{T_k(b^2+4c^2,4b^2c^2)}{D^k}\\ \equiv&\left(\frac{(D-(b^2+4c^2))^2-4(4b^2c^2)}{p}\right)=\left(\frac{-16c^2D}{p}\right)\pmod p\end{aligned}$$

with the help of the first part of Theorem 1.1.

Combining the above, we finally obtain (1.4). We are done. □

Lemma 2.4. *Let b and c be integers. For any odd prime p, we have*

$$T_p(b,c)\equiv b\pmod p,\quad T_{p+1}(b,c)\equiv b^2\pmod p,\tag{2.4}$$

and

$$T_{p-1}(b,c)\equiv\left(\frac{b^2-4c}{p}\right)\pmod p.\tag{2.5}$$

Proof. Since $\binom{p}{k}\equiv 0\pmod p$ for all $k=1,\ldots,p-1$, we have

$$T_p(b,c)=\sum_{k=0}^{p}\binom{p}{2k}\binom{2k}{k}b^{p-2k}c^k\equiv\binom{p}{0}b^p\equiv b\pmod p$$

with the help of Fermat's little theorem. If $1<k<p$, then

$$\binom{p+1}{k}=\frac{p(p+1)}{k(k-1)}\binom{p-1}{k-2}\equiv 0\pmod p.$$

Thus

$$\begin{aligned}T_{p+1}(b,c)=&\sum_{k=0}^{(p+1)/2}\binom{p+1}{k}\binom{p+1-k}{k}b^{p+1-2k}c^k\\ \equiv&b^{p+1}+\binom{p+1}{1}\binom{p}{1}b^{p-1}c\equiv b^2\pmod p.\end{aligned}$$

If $p \mid b$, then (2.5) is valid since

$$\begin{aligned} T_{p-1}(b,c) &= \sum_{k=0}^{(p-1)/2}\binom{p-1}{2k}\binom{2k}{k}b^{p-1-2k}c^k \\ &\equiv \binom{p-1}{(p-1)/2}c^{(p-1)/2} \equiv \left(\frac{-c}{p}\right) = \left(\frac{b^2-4c}{p}\right) \pmod p. \end{aligned}$$

When $p \nmid b$, we have

$$\begin{aligned} T_{p-1}(b,c) &\equiv \sum_{k=0}^{(p-1)/2}\binom{2k}{k}\frac{c^k}{b^{2k}} = \sum_{k=0}^{(p-1)/2}\binom{-1/2}{k}(-4)^k\frac{c^k}{b^{2k}} \\ &\equiv \sum_{k=0}^{(p-1)/2}\binom{(p-1)/2}{k}\left(-\frac{4c}{b^2}\right)^k = \left(1-\frac{4c}{b^2}\right)^{(p-1)/2} \\ &\equiv \left(\frac{b^2-4c}{p}\right) \pmod p. \end{aligned}$$

This concludes the proof. □

Proof of Theorem 1.1(iii). Suppose that $d = b^2 - 4c \not\equiv 0 \pmod p$. By Lemma 2.2,

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)M_k(b,c)}{d^k} \equiv \left(\frac{d}{p}\right)S_1 \pmod p$$

where

$$\begin{aligned} S_1 =& \sum_{k=0}^{p-1}T_{p-1-k}(b,c)M_k(b,c) = [x^{p-1}]\sum_{j=0}^{\infty}T_j(b,c)x^j\sum_{k=0}^{\infty}M_k(b,c)x^k \\ =& [x^{p-1}]\frac{1}{\sqrt{1-2bx+dx^2}} \times \frac{1-bx-\sqrt{1-2bx+dx^2}}{2cx^2} \\ =& \frac{1}{2c}[x^{p+1}]\left(\frac{1-bx}{\sqrt{1-2bx+dx^2}}-1\right) = \frac{T_{p+1}(b,c)-bT_p(b,c)}{2c}. \end{aligned}$$

In light of Lemma 2.4, $S_1 \equiv 0 \pmod p$ and hence (1.5) follows.

Now suppose that $D = b^2 - 4c^2 \not\equiv 0 \pmod p$. In view of Lemma 2.2 and Fermat's little theorem,

$$\begin{aligned} &\sum_{k=0}^{p-1}\frac{T_k(b,c^2)M_k(b,c^2)}{(b-2c)^{2k}} \\ \equiv& \left(\frac{D}{p}\right)\sum_{k=0}^{p-1}\frac{D^kT_{p-1-k}(b,c^2)}{(b-2c)^{2k}}M_k(b,c^2) \equiv \left(\frac{D}{p}\right)S_2 \pmod p, \end{aligned}$$

where

$$\begin{aligned}S_2 =&\sum_{k=0}^{p-1}(b-2c)^{p-1-k}T_{p-1-k}(b,c^2)M_k(b,c^2)(b+2c)^k\\ =&[x^{p-1}]\sum_{j=0}^{\infty}T_j(b,c^2)((b-2c)x)^j\sum_{k=0}^{\infty}M_k(b,c^2)((b+2c)x)^k\\ =&[x^{p-1}]\frac{1-b(b+2c)x-\sqrt{1-2b(b+2c)x+D(b+2c)^2x^2}}{2c^2((b+2c)x)^2\sqrt{1-2b(b-2c)x+D(b-2c)^2x^2}}\\ =&\frac{1}{2c^2(b+2c)^2}[x^{p+1}]\frac{1-b(b+2c)x}{\sqrt{1-2b(b-2c)x+D(b-2c)^2x^2}}\\ &-\frac{1}{2c^2(b+2c)^2}[x^{p+1}]\frac{\sqrt{(1-Dx)(1-(b+2c)^2x)}}{\sqrt{(1-Dx)(1-(b-2c)^2x)}}.\end{aligned}$$

Recall the identity $(b^2+4c^2)^2-4(4b^2c^2)=D^2$ and observe that

$$\begin{aligned}2c^2(b+2c)^2S_2 =&[y^{p+1}]\frac{(b-2c)^{p+1}}{\sqrt{1-2by+Dy^2}}-b(b+2c)[y^p]\frac{(b-2c)^p}{\sqrt{1-2by+Dy^2}}\\ &-[x^{p+1}]\frac{1-(b+2c)^2x}{\sqrt{1-2(b^2+4c^2)x+D^2x^2}}\\ \equiv&(b-2c)^2T_{p+1}(b,c^2)-b(b+2c)(b-2c)T_p(b,c^2)\\ &-T_{p+1}(b^2+4c^2,4b^2c^2)+(b+2c)^2T_p(b^2+4c^2,4b^2c^2)\ (\mathrm{mod}\ p).\end{aligned}$$

Applying Lemma 2.4 we get

$$\begin{aligned}2c^2(b+2c)^2S_2 \equiv&(b-2c)^2b^2-b^2D-(b^2+4c^2)^2+(b+2c)^2(b^2+4c^2)\\ \equiv&8bc^2(b+2c)\ (\mathrm{mod}\ p).\end{aligned}$$

Thus $S_2\equiv 4b/(b+2c)\ (\mathrm{mod}\ p)$ and this concludes the proof of (1.6). □

3. Proof of Theorem 1.2

Lemma 3.1. *Let b and c be integers. For all $n=1,2,3,\ldots$ we have*

$$2c\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k}=-nT_n(b,c^2)+(b+2c)nT_{n-1}(b,c^2)\quad(3.1)$$

Proof. In the case $n=1$ both sides of (3.1) coincide with $2c$. Denote by $f(n)$ the right-hand side of (3.1). Clearly it suffices to show that for any positive integer n we have

$$\begin{aligned}&f(n+1)-(b-2c)f(n)\\ =&2c\sum_{k=0}^{n}T_k(b,c^2)(b-2c)^{n-k}-2c\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-k}=2cT_n(b,c^2).\end{aligned}$$

Observe that

$$\begin{aligned}
&f(n+1)-(b-2c)f(n)\\
=&-(n+1)T_{n+1}(b,c^2)+(b+2c)(n+1)T_n(b,c^2)\\
&-(b-2c)\left(-nT_n(b,c^2)+(b+2c)nT_{n-1}(b,c^2)\right)\\
=&-(n+1)T_{n+1}(b,c^2)+(4c^2-b^2)nT_{n-1}(b,c)\\
&+\left(n(b-2c)+(n+1)(b+2c)\right)T_n(b,c^2)\\
=&-(2n+1)bT_n(b,c^2)+\left(n(b-2c)+(n+1)(b+2c)\right)T_n(b,c^2)=2cT_n(b,c^2)
\end{aligned}$$

with the help of the recursion for $T_n(b,c^2)$.

The above proof of (3.1) is simple. However, the reader might wonder how (3.1) was found. Set $D=b^2-4c^2$. Then

$$\begin{aligned}
\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k}=&[x^{n-1}]\frac{1}{\sqrt{1-2bx+Dx^2}}\cdot\frac{1}{1-(b-2c)x}\\
=&[x^{n-1}](1-(b-2c)x)^{-3/2}(1-(b+2c)x)^{-1/2}
\end{aligned}$$

and hence

$$-2c\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k}=[x^{n-1}]\frac{d}{dx}\sqrt{\frac{1-(b+2c)x}{1-(b-2c)x}}.$$

Observe that

$$\begin{aligned}
\sqrt{\frac{1-(b+2c)x}{1-(b-2c)x}}=&\frac{1-(b+2c)x}{\sqrt{1-2bx+Dx^2}}=(1-(b+2c)x)\sum_{k=0}^{\infty}T_k(b,c^2)x^k\\
=&1+\sum_{k=1}^{\infty}(T_k(b,c)-(b+2c)T_k(b,c))x^k
\end{aligned}$$

and thus

$$[x^{n-1}]\frac{d}{dx}\sqrt{\frac{1-(b+2c)x}{1-(b-2c)x}}=n\left(T_n(b,c^2)-(b+2c)T_{n-1}(b,c^2)\right).$$

Therefore (3.1) follows. □

Lemma 3.2. *Let $b\in\mathbb{Z}$, $c\in\mathbb{Z}\setminus\{0\}$ and $n\in\mathbb{Z}^+$. Then*

$$\begin{aligned}
&\frac{3}{n}\sum_{k=0}^{n-1}kT_k(b,c^2)(b-2c)^{n-1-k}-\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k}\\
&\quad=\frac{(b+4c)T_n(b,c^2)-(b+2c)^2T_{n-1}(b,c^2)}{4c^2}.
\end{aligned}\tag{3.2}$$

Proof. Note that for any $k\in\mathbb{N}$ we have

$$T_k(2c,c^2)=[x^k](x^2+2cx+c^2)^k=[x^k](x+c)^{2k}=\binom{2k}{k}c^k.$$

In the case $b=2c$, we can easily verify that both sides of (3.2) coincide with $(2-3/n)\binom{2n-2}{n-1}c^{n-1}$.

Below we assume $b\ne 2c$ and define

$$\sigma_n:=\sum_{k=0}^{n-1}(n-k)T_k(b,c^2)(b-2c)^{n-1-k}.$$

Clearly

$$\sigma_n=[x^{n-1}]\bigg(\sum_{k=0}^\infty T_k(b,c^2)x^k\bigg)\sum_{l=0}^\infty(l+1)(b-2c)^lx^l.$$

For $|z|<1$ we have

$$\frac{1}{(1-z)^2}=\sum_{l=0}^\infty\binom{-2}{l}(-z)^l=\sum_{l=0}^\infty\binom{l+1}{l}z^l.$$

Thus

$$\begin{aligned}\sigma_n=&[x^{n-1}]\frac{1}{\sqrt{1-2bx+(b^2-4c^2)x^2}}\times\frac{1}{(1-(b-2c)x)^2}\\=&[x^{n-1}](1-(b+2c)x)^{-1/2}(1-(b-2c)x)^{-5/2}=[x^{n-1}]\frac{d}{dx}f(x),\end{aligned}$$

where

$$\begin{aligned}f(x)=&\bigg(-\frac{b(b+2c)}{12c^2(b-2c)}+\frac{(b+2c)^2}{12c^2}x+\frac{2}{3(b-2c)(1-(b-2c)x)}\bigg)\\&\times\frac{1}{\sqrt{1-2bx+(b^2-4c^2)x^2}}\\=&\bigg(-\frac{b(b+2c)}{12c^2(b-2c)}+\frac{(b+2c)^2}{12c^2}x+\frac{2}{3(b-2c)}\sum_{j=0}^\infty(b-2c)^jx^j\bigg)\\&\times\sum_{k=0}^\infty T_k(b,c^2)x^k.\end{aligned}$$

Therefore

$$\begin{aligned}\frac{\sigma_n}{n}=[x^n]f(x)=&-\frac{b(b+2c)}{12c^2(b-2c)}+\frac{(b+2c)^2}{12c^2}T_{n-1}(b,c^2)\\&+\frac{2}{3(b-2c)}\sum_{k=0}^nT_k(b,c^2)(b-2c)^{n-k},\end{aligned}$$

i.e.,

$$\begin{aligned}&\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k}-\frac{1}{n}\sum_{k=0}^{n-1}kT_k(b,c^2)(b-2c)^{n-1-k}\\ =&\frac{2}{3}\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k}+\frac{2}{3}\cdot\frac{T_n(b,c^2)}{b-2c}\\ &+\frac{b+2c}{12c^2(b-2c)}\left((b^2-4c^2)T_{n-1}(b,c^2)-bT_n(b,c^2)\right).\end{aligned}$$

This yields the desired (3.2). □

Proof of Theorem 1.2(i). Let n be any positive integer. Since $T_k(b,0)=[x^k]x^k(x+b)^k=b^k$ for all $k\in\mathbb{N}$, (1.11) and (1.12) hold. trivially when $c=0$.

Now assume that $c\neq 0$. By Lemma 3.1 we have

$$\frac{1}{n}\sum_{k=0}^{n-1}T_k(b,c^2)(b-2c)^{n-1-k}=\frac{bT_{n-1}(b,c^2)-T_n(b,c^2)}{2c}+T_{n-1}(b,c^2).$$

Observe that

$$\begin{aligned}&T_n(b,c^2)-bT_{n-1}(b,c^2)\\ =&\sum_{k\in\mathbb{N}}\binom{n}{2k}\binom{2k}{k}b^{n-2k}(c^2)^k-\sum_{k\in\mathbb{N}}\binom{n-1}{2k}\binom{2k}{k}b^{n-2k}(c^2)^k\\ =&\sum_{k=1}^{n}\binom{n-1}{2k-1}\binom{2k}{k}b^{n-2k}c^{2k}=2c\sum_{k=1}^{n}\binom{n-1}{2k-1}\binom{2k-1}{k-1}b^{n-2k}c^{2k-1}\\ =&2c\sum_{0<k\leqslant\lfloor n/2\rfloor}\binom{n-1}{k-1}\binom{n-k}{k}b^{n-2k}c^{2k-1}\equiv 0\pmod{2c}.\end{aligned}$$

Therefore (1.11) holds. In light of Lemma 3.2, (1.12) is reduced to the congruence

$$(b+4c)T_n(b,c^2)\equiv(b+2c)^2T_{n-1}(b,c^2)\pmod{2c^2}.$$

In fact, as $\binom{2k}{k}=2\binom{2k-1}{k-1}$ for all $k\in\mathbb{Z}^+$, we have

$$\begin{aligned}&(b+4c)T_n(b,c^2)-(b+2c)^2T_{n-1}(b,c^2)\\ =&(b+4c)\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^{2k}\\ &-(b+2c)^2\sum_{k=0}^{\lfloor (n-1)/2\rfloor}\binom{n-1}{2k}\binom{2k}{k}b^{n-1-2k}c^{2k}\\ \equiv&(b+4c)b^n-(b+2c)^2b^{n-1}\equiv 0\pmod{2c^2}.\end{aligned}$$

So (1.12) is valid.

Now write $D=b^2-4c^2$ and suppose that p is an odd prime not dividing $b-2c$. In view of Lemmas 2.4 and 3.1 and Fermat's little theorem, we have

$$\begin{aligned}\frac{2c}{p}\sum_{k=0}^{p-1}\frac{T_k(b,c^2)}{(b-2c)^k}=&\frac{(b+2c)T_{p-1}(b,c^2)-T_p(b,c^2)}{(b-2c)^{p-1}}\\ \equiv&(b+2c)\left(\frac{D}{p}\right)-b \pmod p.\end{aligned}$$

This proves (1.13). If $p\mid c$, then $(\frac{D}{p})=(\frac{b^2}{p})=1$ and hence (1.14) becomes obvious. When $p\nmid c$, by (3.2), (1.13) and Lemma 2.4 we get

$$\begin{aligned}\frac{3}{p}\sum_{k=0}^{p-1}\frac{kT_k(b,c^2)}{(b-2c)^k}\equiv&\frac{(b+4c)T_p(b,c^2)-(b+2c)^2T_{p-1}(b,c^2)}{4c^2}\\ \equiv&\frac{(b+4c)b-(b+2c)^2(\frac{D}{p})}{4c^2} \pmod p\end{aligned}$$

and hence (1.14) follows. □

Lemma 3.3. *For $k\in\mathbb{N}$ and $n\in\mathbb{Z}^+$ we have*

$$\sum_{m=0}^{n-1}(2m+1)^2\binom{m+k}{2k}=(4n^2-1)\frac{n-k}{2k+3}\binom{n+k}{2k}. \tag{3.3}$$

Proof. Observe that

$$\begin{aligned}&(4n^2-1)\frac{n-k}{2k+3}\binom{n+k}{2k}+(2n+1)^2\binom{n+k}{2k}\\ =&(4n^2+8n+3)\frac{n+1-k}{2k+3}\binom{n+k}{2k}\\ =&(4(n+1)^2-1)\frac{n+1-k}{2k+3}\binom{n+1+k}{2k}.\end{aligned}$$

So we can easily prove (3.3) by induction on n. □

Proof of Theorem 1.2(ii). We prove (1.15) by induction. (1.15) is obvious when $n=1$.

Now suppose the validity of (1.15) for a fixed $n\in\mathbb{Z}^+$. Observe that

$$\begin{aligned}&(n+1)\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n+1+k}{k}\left(\frac{b-1}{2}\right)^k-n\sum_{k=0}^{n-1}\binom{n}{k+1}\binom{n+k}{k}b^k\\ =&\sum_{k=0}^{n}\left((n+1+k)\binom{n+1}{k+1}-n\binom{n}{k+1}\right)\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k\\ =&(2n+1)\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{b-1}{2}\right)^k=(2n+1)D_n(m)=(2n+1)T_n(b,c).\end{aligned}$$

Therefore, by the induction hypothesis, we have

$$\begin{aligned}&(n+1)\sum_{k=0}^{n}\binom{n+1}{k+1}\binom{n+1+k}{k}\left(\frac{b-1}{2}\right)^k\\&=\sum_{k=0}^{n-1}(2k+1)T_k(b,c)+(2n+1)T_n(b,c)=\sum_{k=0}^{n}(2k+1)T_k(b,c).\end{aligned}$$

This proves (1.15) with n replaced by $n+1$.

Let p be a prime not dividing $b-1=2m$. In light of (1.15),

$$\begin{aligned}&\frac{1}{p}\sum_{k=0}^{p-1}(2k+1)T_k(b,c)=\sum_{k=0}^{p-1}\binom{p}{k+1}\binom{p+k}{k}m^k\\&=\binom{2p-1}{p-1}m^{p-1}+\sum_{k=0}^{p-2}\binom{p}{k+1}\binom{p+k}{k}m^k\\&\equiv m^{p-1}+\sum_{k=0}^{p-2}\binom{p}{k+1}m^k=m^{p-1}+\frac{(m+1)^p-m^p-1}{m}\\&\equiv 1+\frac{(m+1)^p-(m+1)}{m}=1+\frac{b+1}{b-1}\left(\left(\frac{b+1}{2}\right)^{p-1}-1\right)\pmod{p^2}\end{aligned}$$

and hence (1.16) follows.

Now we show (1.17). In view of Lemma 3.3,

$$\begin{aligned}\sum_{n=0}^{p-1}(2n+1)^2T_n(b,c)&=\sum_{n=0}^{p-1}(2n+1)^2\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}m^k\\&=\sum_{k=0}^{p-1}\binom{2k}{k}m^k\sum_{n=0}^{p-1}(2n+1)^2\binom{n+k}{2k}\\&=(4p^2-1)\sum_{k=0}^{p-1}\frac{p-k}{2k+3}\binom{p+k}{2k}\binom{2k}{k}m^k\\&=(4p^2-1)\sum_{k=0}^{p-1}\frac{pm^k}{2k+3}\prod_{0<j\leqslant k}\left(\frac{p^2}{j^2}-1\right)\\&\equiv-\sum_{k=0}^{p-1}\frac{p(-m)^k}{2k+3}\pmod{p^2}\\&\equiv-(-m)^{(p-3)/2}\equiv\frac{1}{m}\left(\frac{-m}{p}\right)\pmod{p}.\end{aligned}$$

This proves (1.17). □

4. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3(i). We first prove (1.19) by induction.

When $n=1$, both sides of (1.19) are equal to b.

Now assume that (1.19) holds for a fixed integer $n\geqslant 1$. Then

$$\begin{aligned}&b\sum_{k=0}^{(n+1)-1}(2k+1)T_k(b,c)^2(-d)^{(n+1)-1-k}\\=&b(2n+1)T_n(b,c)^2-bd\sum_{k=0}^{n-1}(2k+1)T_k(b,c)^2(-d)^{n-1-k}\\=&b(2n+1)T_n(b,c)^2-dnT_n(b,c)T_{n-1}(b,c)\\=&(n+1)T_n(b,c)T_{n+1}(b,c).\end{aligned}$$

This concludes the induction step.

Now we fix a positive integer n and want to show (1.18). Recall that

$$T_n(b,c)\equiv\begin{cases}\binom{n}{n/2}c^{n/2}\pmod b&\text{if }2\mid n,\\0\pmod b&\text{if }2\nmid n.\end{cases}$$

When $b\neq 0$, b divides $T_n(b,c)$ or $T_{n-1}(b,c)$ since n or $n-1$ is odd, therefore (1.18) follows from (1.19).

Now it remains to consider the case $b=0$. Note that $T_k(0,c)=0$ for $k=1,3,5,\ldots$, and $T_k(0,c)=\binom{k}{k/2}c^{k/2}$ for $k=0,2,4,\ldots$. Thus

$$\begin{aligned}&\sum_{k=0}^{n-1}(2k+1)T_k(0,c)^2(4c-0^2)^{n-1-k}\\=&\sum_{k=0}^{\lfloor(n-1)/2\rfloor}(4k+1)\left(\binom{2k}{k}c^k\right)^2(4c)^{n-1-2k}\\=&(4c)^{n-1}\sum_{k=0}^{\lfloor(n-1)/2\rfloor}(4k+1)\frac{\binom{2k}{k}^2}{16^k}.\end{aligned}$$

By induction, for any $m\in\mathbb{N}$ we have the identity

$$\sum_{k=0}^{m}(4k+1)\frac{\binom{2k}{k}^2}{16^k}=\frac{(m+1)^2}{16^m}\binom{2m+1}{m}^2=\frac{(2m+1)^2}{16^m}\binom{2m}{m}^2,$$

which was pointed out to the author by R. Tauraso. It follows that

$$4^{n-1}\sum_{k=0}^{\lfloor(n-1)/2\rfloor}(4k+1)\frac{\binom{2k}{k}^2}{16^k}=n^2\binom{n-1}{\lfloor n/2\rfloor}^2.$$

Therefore

$$\sum_{k=0}^{n-1}(2k+1)T_k(0,c)^2(4c-0^2)^{n-1-k}\equiv 0 \pmod{n^2}$$

and hence (1.18) holds when $b=0$. We are done. $\square$

Lemma 4.1. *Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. For any $n\in\mathbb{N}$ we have*

$$T_n(b,c)^2=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2c^kd^{n-k}. \tag{4.1}$$

Proof. If $d=0$ (i.e., $b^2=4c$), then

$$T_n(b,c)=[x^n]\left(x^2+bx+\frac{b^2}{4}\right)^n=[x^n]\left(x+\frac{b}{2}\right)^{2n}=\binom{2n}{n}\frac{b^n}{2^n}$$

and hence (4.1) holds.

Now assume that $d\neq 0$. By [S2, Theorem 3.1],

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2x^k(x+1)^k=\left(\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k\right)^2$$

which follows from comparing coefficients of powers of x and verifying the corresponding identities via the Zeilberger algorithm. Therefore

$$\begin{aligned}T_n(b,c)^2=&\left((\sqrt{d})^nP_n\left(\frac{b}{\sqrt{d}}\right)\right)^2=d^nD_n\left(\frac{b/\sqrt{d}-1}{2}\right)^2\\=&d^n\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\left(\frac{b/\sqrt{d}-1}{2}\right)^k\left(\frac{b/\sqrt{d}+1}{2}\right)^k\\=&d^n\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\left(\frac{b^2/d-1}{4}\right)^k=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2c^kd^{n-k}.\end{aligned}$$

This completes the proof. $\square$

Lemma 4.2. *For any $k\in\mathbb{N}$ and $n\in\mathbb{Z}^+$ we have*

$$\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}=\frac{n(n-k)}{k+1}\binom{n+k}{2k}. \tag{4.2}$$

Proof. (4.2) can be easily proved by induction on n. $\square$

Proof of Theorem 1.3(ii). Let $n\in\mathbb{Z}^+$. In view of Lemmas 4.1 and 4.2, we have

$$\begin{aligned}&\sum_{m=0}^{n-1}(2m+1)T_m(b,c)^2d^{n-1-m}\\ =&\sum_{m=0}^{n-1}(2m+1)d^{n-1-m}\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}^2c^kd^{m-k}\\ =&\sum_{k=0}^{n-1}\binom{2k}{k}^2c^kd^{n-1-k}\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}\\ =&\sum_{k=0}^{n-1}\binom{2k}{k}^2c^kd^{n-1-k}\frac{n(n-k)}{k+1}\binom{n+k}{2k}\\ =&n\sum_{k=0}^{n-1}(n-k)\binom{n}{k}\binom{n+k}{k}C_kc^kd^{n-1-k}\\ =&n^2\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}C_kc^kd^{n-1-k}.\end{aligned}$$

This proves (1.20).

Now assume $c\neq 0$ and let p be an odd prime not dividing d. By (1.20),

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k}=\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{p+k}{k}C_k\frac{c^k}{d^k}.$$

For $k=0,1,\ldots,p-1$, clearly

$$\begin{aligned}\binom{p-1}{k}\binom{p+k}{k}=&\prod_{0<j\leqslant k}\left(\frac{p-j}{j}\cdot\frac{p+j}{j}\right)=(-1)^k\prod_{0<j\leqslant k}\left(1-\frac{p^2}{j^2}\right)\\ \equiv&(-1)^k\left(1-p^2H_k^{(2)}\right)\pmod{p^4},\end{aligned}$$

where $H_k^{(2)}=\sum_{0<j\leqslant k}1/j^2$. Thus

$$\begin{aligned}\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k}\equiv&\sum_{k=0}^{p-1}C_k\left(-\frac{c}{d}\right)^k(1-p^2H_k^{(2)})\pmod{p^4}\\ \equiv&\sum_{k=0}^{p-1}C_k\left(-\frac{c}{d}\right)^k\pmod{p^2}.\end{aligned}$$

If $p \mid c$, then $(\frac{d}{p}) = (\frac{b^2}{p}) = 1$ and hence (1.21) follows. In the case $p \nmid c$, we take an integer $m \equiv -d/c \pmod{p^2}$ and then get

$$\frac{1}{p^2}\sum_{k=0}^{p-1}(2k+1)\frac{T_k(b,c)^2}{d^k} \equiv \sum_{k=0}^{p-1}\frac{C_k}{m^k} \pmod{p^2}.$$

By [Su3, Lemma 2.1],

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{C_k}{m^k} \equiv& \frac{m-4}{2}\left(1-\left(\frac{m(m-4)}{p}\right)\right)\\ \equiv& -\frac{d+4c}{2c}\left(1-\left(\frac{d(d+4c)}{p}\right)\right) = \frac{b^2}{2c}\left(\left(\frac{d}{p}\right)-1\right) \pmod p.\end{aligned}$$

(Moreover, the author [Su1] determined $\sum_{k=1}^{p-1} C_k/m^k \bmod p^2$ in terms of Lucas sequences.) So (1.21) is valid. We are done. □

Remark 4.1. Let $p > 3$ be a prime. As $D_k = T_k(3,2)$, by refining the proof of Theorem 1.3(ii) and using two auxiliary congruences

$$\sum_{k=1}^{p-1}(-2)^k C_k \equiv -4p\, q_p(2) \pmod{p^3}$$

and

$$\sum_{k=1}^{p-1}(-2)^k C_k H_k^{(2)} \equiv 2q_p(2)^2 \pmod p$$

(the author has a proof of them), we get

$$\sum_{k=0}^{p-1}(2k+1)D_k^2 \equiv p^2 - 4p^3 q_p(2) - 2p^4 q_p(2)^2 \pmod{p^5}.$$

Lemma 4.3. *Let $b, c \in \mathbb{Z}$. Suppose that $p > 3$ is a prime not dividing $d = b^2 - 4c$. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)^2}{d^k} \equiv \left(\frac{16c}{d}\right)^{(p-1)/2} + p\sum_{\substack{k=0\\ k\neq (p-1)/2}}^{p-1}\frac{\binom{2k}{k}}{2k+1}\left(-\frac{c}{d}\right)^k \pmod{p^3}. \tag{4.3}$$

Proof. With the help of (4.1), we have

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n(b,c)^2}{d^n} =& \sum_{n=0}^{p-1}\frac{1}{d^n}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2 c^k d^{n-k}\\ =& \sum_{k=0}^{p-1}\binom{2k}{k}^2\frac{c^k}{d^k}\sum_{n=k}^{p-1}\binom{n+k}{2k} = \sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{p+k}{2k+1}\left(\frac{c}{d}\right)^k\\ =& \sum_{k=0}^{p-1}\frac{p}{2k+1}\binom{2k}{k}\left(\prod_{0<j\leqslant k}\frac{p^2-j^2}{j^2}\right)\left(\frac{c}{d}\right)^k\end{aligned}$$

and hence

$$\begin{aligned}\sum_{n=0}^{p-1}\frac{T_n(b,c)^2}{d^n}\equiv&\sum_{k=0}^{p-1}\frac{p(-1)^k}{2k+1}\binom{2k}{k}\left(1-p^2H_k^{(2)}\right)\left(\frac{c}{d}\right)^k \pmod{p^4}\\ \equiv&(-1)^{(p-1)/2}\binom{p-1}{(p-1)/2}\left(1-p^2H_{(p-1)/2}^{(2)}\right)\left(\frac{c}{d}\right)^{(p-1)/2}\\ &+p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{\binom{2k}{k}}{2k+1}\left(-\frac{c}{d}\right)^k \pmod{p^3}.\end{aligned}$$

As Wolstenholme observed, $H_{p-1}^{(2)}\equiv 0 \pmod p$ since $\sum_{j=1}^{p-1}1/(2j)^2\equiv\sum_{k=1}^{p-1}1/k^2 \pmod p$. Therefore

$$H_{(p-1)/2}^{(2)}\equiv\frac{1}{2}\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=\frac{H_{p-1}^{(2)}}{2}\equiv 0 \pmod p.$$

Recall Morley's congruence (cf. [M])

$$\binom{p-1}{(p-1)/2}\equiv(-1)^{(p-1)/2}4^{p-1} \pmod{p^3}.$$

So we have

$$(-1)^{(p-1)/2}\binom{p-1}{(p-1)/2}\left(1-p^2H_{(p-1)/2}^{(2)}\right)\equiv 4^{p-1} \pmod{p^3}$$

and hence (4.3) follows. □

Proof of Theorem 1.4. (i) Applying Lemma 4.3 with $b=6$ and $c=-3$ we get

$$\sum_{k=0}^{p-1}\frac{T_k(6,-3)^2}{48^k}\equiv\left(\frac{-1}{p}\right)+p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{\binom{2k}{k}}{(2k+1)16^k} \pmod{p^3}.$$

By [Su2, (1.4)-(1.5)],

$$\sum_{k=0}^{(p-3)/2}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv 0 \pmod{p^2}$$

and

$$\sum_{k=(p+1)/2}^{p-1}\frac{\binom{2k}{k}}{(2k+1)16^k}\equiv\frac{p}{3}E_{p-3} \pmod{p^2}.$$

So (1.22) follows.

(ii) Now we prove (1.23) and (1.24). Since $p \mid \binom{2k}{k}$ for every $k = (p+1)/2, \ldots, p-1$, by Lemma 4.3 with $b = 2$ and $c \in \{-1, -3\}$ we obtain

$$\sum_{k=0}^{p-1} \frac{T_k(2,-1)^2}{8^k} \equiv (-2)^{(p-1)/2} + p \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)8^k} \pmod{p^2} \tag{4.4}$$

and

$$\sum_{k=0}^{p-1} \frac{T_k(2,-3)^2}{16^k} \equiv (-3)^{(p-1)/2} + p \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)} \left(\frac{3}{16}\right)^k \pmod{p^2}. \tag{4.5}$$

For $n \in \mathbb{N}$ define

$$u_n = (2n+1) \sum_{k=0}^{n} \binom{n+k}{2k} \frac{(-2)^k}{2k+1} \quad \text{and} \quad v_n = (2n+1) \sum_{k=0}^{n} \binom{n+k}{2k} \frac{(-3)^k}{2k+1}.$$

Via the Zeilberger algorithm (cf. [PWZ]) we find the recurrence relations

$$u_n + u_{n+2} = 0 \quad \text{and} \quad v_n + v_{n+1} + v_{n+2} = 0.$$

So, by induction we have

$$u_n = (-1)^{n(n-1)/2} = \left(\frac{-2}{2n+1}\right) \quad \text{and} \quad v_n = \left(\frac{2n+1}{3}\right)$$

for every $n = 0, 1, 2, \ldots$. Taking $n = (p-1)/2$ and noting that $\binom{n+k}{2k} \equiv \binom{2k}{k}/(-16)^k \pmod{p^2}$ for $k = 0, \ldots, n$ (cf. [S1, Lemma 2.2]), we then obtain

$$(-2)^{(p-1)/2} + p \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)8^k} \equiv u_{(p-1)/2} = \left(\frac{-2}{p}\right) \pmod{p^3}$$

and

$$(-3)^{(p-1)/2} + p \sum_{k=0}^{(p-3)/2} \frac{\binom{2k}{k}}{(2k+1)} \left(\frac{3}{16}\right)^k \equiv v_{(p-1)/2} = \left(\frac{p}{3}\right) \pmod{p^3}.$$

Combining these with (4.4) and (4.5) we immediately get (1.23) and (1.24).

(iii) Finally we show (1.25). Applying (4.1) with $b = 3$ and $c = 2$ we obtain

$$\sum_{k=0}^{n} \binom{n+k}{2k} \binom{2k}{k}^2 2^k = D_n^2.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{D_n^2-1}{n^2}=&\sum_{n=1}^{p-1}\frac{1}{n^2}\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}^2 2^k\\ =&\sum_{k=1}^{p-1}2^k\binom{2k}{k}^2\sum_{n=k}^{p-1}\frac{\binom{n+k}{2k}}{n^2}=\sum_{k=1}^{p-1}2^k\binom{2k}{k}^2\sum_{r=0}^{p-1-k}\frac{\binom{2k+r}{r}}{(k+r)^2}.\end{aligned}$$

If $k\in\{(p+1)/2,\ldots,p-1\}$ then $p\mid\binom{2k}{k}$. For each $k=1,\ldots,(p-1)/2$, clearly

$$\sum_{r=0}^{p-1-k}\frac{\binom{2k+r}{r}}{(k+r)^2}=4\sum_{r=0}^{p-1-k}\frac{(-1)^r\binom{-2k-1}{r}}{(-2k-2r)^2}\equiv 4\sum_{r=0}^{p-1-2k}\frac{(-1)^r\binom{p-1-2k}{r}}{(p-2k-2r)^2}\pmod p.$$

By [Su2, (2.5)], we have the identity

$$\sum_{r=0}^{2n}\frac{(-1)^r\binom{2n}{r}}{(2n+1-2r)^2}=\frac{(-16)^n}{(2n+1)^2\binom{2n}{n}}.$$

Also, $\sum_{k=1}^{p-1}1/k^2\equiv 0 \pmod p$ since $\sum_{k=1}^{p-1}1/(2k)^2\equiv\sum_{k=1}^{p-1}1/k^2 \pmod p$. So, by the above, we have

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{D_k^2}{k^2}\equiv&\sum_{k=1}^{(p-1)/2}2^k\binom{2k}{k}^2\frac{4(-16)^{(p-1)/2-k}}{(p-2k)^2\binom{p-1-2k}{(p-1)/2-k}}\\ \equiv&\sum_{k=1}^{(p-1)/2}\frac{2^k\binom{2k}{k}^2 4^{(p-1)/2-k}}{k^2\binom{p-1-2k}{(p-1)/2-k}/(-4)^{(p-1)/2-k}}\pmod p\end{aligned}$$

For each $k\in\{1,\ldots,(p-1)/2\}$, obviously

$$\begin{aligned}\frac{\binom{2k}{k}}{(-4)^k}=\binom{-1/2}{k}\equiv&\binom{(p-1)/2}{k}=\binom{(p-1)/2}{(p-1)/2-k}\\ \equiv&\binom{-1/2}{(p-1)/2-k}=\frac{\binom{p-1-2k}{(p-1)/2-k}}{(-4)^{(p-1)/2-k}}\pmod p.\end{aligned}$$

Therefore

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{D_k^2}{k^2}\equiv&\sum_{k=1}^{(p-1)/2}\frac{2^k\binom{2k}{k}^2 2^{p-1}/4^k}{k^2\binom{2k}{k}/(-4)^k}\\ \equiv&\sum_{k=1}^{(p-1)/2}\frac{(-2)^k\binom{2k}{k}}{k^2}\equiv\sum_{k=1}^{p-1}\frac{(-2)^k}{k^2}\binom{2k}{k}\pmod p.\end{aligned}$$

The author [Su2] conjectured that

$$\sum_{k=1}^{p-1}\frac{(-2)^k}{k^2}\binom{2k}{k}\equiv -2q_p(2)^2 \pmod p,$$

which was later confirmed by S. Mattarei and R. Tauraso [MT]. So we finally get (1.25). This ends the proof. □

5. More conjectures for further research

Motivated by part (ii) of Theorem 1.2, we raise the following conjecture.

Conjecture 5.1. *Let x be any integer. Then*

$$\sum_{k=0}^{n-1}(2k+1)D_k(x)^m\equiv 0 \pmod n$$

for all $m,n\in\mathbb{Z}^+$. If p is a prime not dividing $x(x+1)$, then

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^3\equiv p\left(\frac{-4x-3}{p}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(2k+1)D_k(x)^4\equiv p \pmod{p^2}.$$

Now we propose the following conjecture related to Theorem 1.1(ii).

Conjecture 5.2. *Let $b,c\in\mathbb{Z}$. For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}(8ck+4c+b)T_k(b,c^2)^2(b-2c)^{2(n-1-k)}\equiv 0 \pmod n.$$

If p is an odd prime not dividing $b(b-2c)$, then

$$\sum_{k=0}^{p-1}(8ck+4c+b)\frac{T_k(b,c^2)^2}{(b-2c)^{2k}}\equiv p(b+2c)\left(\frac{b^2-4c^2}{p}\right) \pmod{p^2}.$$

Remark 5.1. Conjecture 5.2 in the case $b=c=1$ yields the first part of Conjecture 1.1.

By Theorem 1.1(ii), if p is an odd prime then

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{2^{2k}}\equiv\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{6^{2k}}\equiv\left(\frac{-1}{p}\right) \pmod p.$$

Motivated by this and (1.22)-(1.24), we now give a further conjecture.

Conjecture 5.3. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1}\frac{T_k(2,2)^2}{4^k}-\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{8^k}\equiv\begin{cases}0\ (\mathrm{mod}\ p^3)&\text{if}\ p\equiv 1\ (\mathrm{mod}\ 4),\\0\ (\mathrm{mod}\ p^2)&\text{if}\ p\equiv 3\ (\mathrm{mod}\ 4).\end{cases}$$

If $p>3$, then

$$\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{4^k}\equiv\sum_{k=0}^{p-1}\frac{T_k(4,1)^2}{36^k}\equiv\left(\frac{-1}{p}\right)\ (\mathrm{mod}\ p^2).$$

Now we raise a conjecture related to Theorem 1.1(iii).

Conjecture 5.4. *Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. For any $n\in\mathbb{Z}^+$ we have*

$$\sum_{k=0}^{n-1}T_k(b,c)M_k(b,c)d^{n-1-k}\equiv 0\ (\mathrm{mod}\ n).$$

If p is an odd prime not dividing cd, then

$$\sum_{k=0}^{p-1}\frac{T_k(b,c)M_k(b,c)}{d^k}\equiv\frac{pb^2}{2c}\left(\left(\frac{d}{p}\right)-1\right)\ (\mathrm{mod}\ p^2).$$

By Conjecture 5.4, for any prime $p>3$ we should have

$$\sum_{k=0}^{p-1}\frac{T_k(3,3)M_k(3,3)}{(-3)^k}\equiv\frac{3p}{2}\left(\left(\frac{p}{3}\right)-1\right)\ (\mathrm{mod}\ p^2).$$

This can be further strengthened.

Conjecture 5.5. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{T_k(3,3)M_k(3,3)}{(-3)^k}\equiv\begin{cases}2p^2\ (\mathrm{mod}\ p^3)&\text{if}\ p\equiv 1\ (\mathrm{mod}\ 3),\\p^3-p^2-3p\ (\mathrm{mod}\ p^4)&\text{if}\ p\equiv 2\ (\mathrm{mod}\ 3).\end{cases}$$

In view of Theorem 1.1(ii), for $b,c\in\mathbb{Z}$ and a prime $p\nmid(b-2c)$, it is natural to investigate whether the sum $\sum_{k=0}^{p-1}T_k(b,c^2)^3/(b-2c)^{3k}$ mod p has a pattern. This leads us to raise the following two conjectures.

Conjecture 5.6. *Let $p > 3$ be a prime. Then*

$$\left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{8^k} \equiv \sum_{k=0}^{p-1}\frac{T_k(2,3)^3}{(-64)^k}$$
$$\equiv \sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{(-64)^k} \equiv \left(\frac{3}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(2,9)^3}{512^k}$$
$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,7 \pmod{24} \text{ and } p=x^2+6y^2,\\ 2p-8x^2 \pmod{p^2} & \text{if } p\equiv 5,11 \pmod{24} \text{ and } p=2x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(3k+2)\frac{T_k(2,3)^3}{8^k} \equiv p\left(3\left(\frac{3}{p}\right)-1\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(3k+1)\frac{T_k(2,3)^3}{(-64)^k} \equiv p\left(\frac{-2}{p}\right) \pmod{p^3}.$$

When $(\frac{-6}{p})=1$ we have

$$\sum_{k=0}^{p-1}(72k+47)\frac{T_k(2,9)^3}{(-64)^k} \equiv 42p \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(72k+25)\frac{T_k(2,9)^3}{512^k} \equiv 12p\left(\frac{3}{p}\right) \pmod{p^2}.$$

Also,

$$\sum_{k=0}^{n-1}(3k+2)T_k(2,3)^3 8^{n-1-k} \equiv 0 \pmod{2n}$$

and

$$\sum_{k=0}^{n-1}(3k+1)T_k(2,3)^3(-64)^{n-1-k} \equiv 0 \pmod{n}$$

for every positive integer n.

Remark 5.2. Let $p > 3$ be a prime. If $p \equiv 1,7 \pmod{24}$ then $p = x^2+6y^2$ for some $x, y \in \mathbb{Z}$; if $p \equiv 5, 11 \pmod{24}$ then $p = 2x^2+3y^2$ for some $x, y \in \mathbb{Z}$. The reader may consult [BEW] and [Co] for such known facts.

Conjecture 5.7. *Let $p>3$ be a prime. Then*

$$\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{8^{3k}}\equiv\sum_{k=0}^{p-1}\frac{T_k(18,49)^3}{16^{3k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv1\pmod 4\ \&\ p=x^2+y^2\ (2\nmid x,\ 2\mid y),\\0\pmod{p^2}&\text{if } p\equiv3\pmod 4.\end{cases}$$

And

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{(-8)^{3k}}\equiv\left(\frac{6}{p}\right)\sum_{k=0}^{p-1}\frac{T_k(10,49)^3}{12^{3k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p\equiv1,3\pmod 8\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if } (\frac{-2}{p})=-1,\ i.e.,\ p\equiv5,7\pmod 8.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(7k+4)\frac{T_k(10,49)^3}{(-8)^{3k}}\equiv\frac{p}{14}\left(\frac{2}{p}\right)\left(65-9\left(\frac{p}{3}\right)\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(7k+3)\frac{T_k(10,49)^3}{12^{3k}}\equiv\frac{3p}{28}\left(13+15\left(\frac{p}{3}\right)\right)\pmod{p^2}.$$

For each $n=1,2,3,\ldots$ we have

$$\sum_{k=0}^{n-1}(7k+4)T_k(10,49)^3(-8^3)^{n-1-k}\equiv0\pmod{4n}$$

and

$$\sum_{k=0}^{n-1}(7k+3)T_k(10,49)^3(12^3)^{n-1-k}\equiv0\pmod{n}.$$

Since $T_n(2x+1,x(x+1))=D_n(x)$, and $T_n(-b,c)=(-1)^nT_n(b,c)$ for all $b,c\in\mathbb{Z}$, we see that $(-1)^nD_n(x)=D_n(-x-1)$.

Conjecture 5.8. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^kD_k(2)^3\equiv\sum_{k=0}^{p-1}(-1)^kD_k\left(-\frac14\right)^3\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}(-1)^kD_k\left(\frac18\right)^3$$
$$\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2}&\text{if } p\equiv1\pmod 3\ \&\ p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if } p\equiv2\pmod 3.\end{cases}$$

Conjecture 5.9. *Let $p > 3$ be a prime. Then*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}(-1)^k D_k\left(\frac{1}{2}\right)^3$$
$$\equiv\begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,7 \pmod{24} \text{ and } p=x^2+6y^2,\\ 8x^2-2p \pmod{p^2} & \text{if } p\equiv 5,11 \pmod{24} \text{ and } p=2x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p})=-1.\end{cases}$$

We also have

$$\sum_{k=0}^{p-1}(-1)^k D_k(-4)^3 \equiv \left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}(-1)^k D_k\left(-\frac{1}{16}\right)^3$$
$$\equiv\begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p\equiv 1,4 \pmod{15} \text{ and } p=x^2+15y^2,\\ 2p-20x^2 \pmod{p^2} & \text{if } p\equiv 2,8 \pmod{15} \text{ and } p=5x^2+3y^2,\\ 0 \pmod{p^2} & \text{if } (\frac{p}{15})=-1.\end{cases}$$

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