

CHEBYSHEV SERIES REPRESENTATION OF FEIGENBAUM'S PERIOD-DOUBLING FUNCTION

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ABSTRACT. The Feigenbaum-Cvitanović equation $-\lambda g(x) = g(g(\lambda x))$ is solved over the interval $0 \leq x \leq 1$ with a Chebyshev series representation of $g(x)$. Accurate expansion coefficients are tabulated for solutions $g(x) = 1 + O(x^z)$ with even exponents from $z = 2$ up to $z = 14$.

1. INTRODUCTION

This article provides high precision approximations of functions $g(x)$ which solve the Feigenbaum–Cvitanović equation [11, 12, 8, 18, 9, 10, 6]

$$(1) \quad -\lambda g(x) = g(g(\lambda x)),$$

scaled such that

$$(2) \quad g(0) = 1.$$

The parameter λ plays the role of an eigenvalue bound to the solutions via

$$(3) \quad g(1) = -\lambda.$$

Further below we shall refer to $1/\lambda$ as *the* Feigenbaum constant(s)—although in a broader context a larger variety of numbers carries that name.

Only even functions $g(x) = g(-x)$ are discussed, so the standard representation is the Taylor series [12, 3, 2].

Definition 1. (*Taylor series coefficients b_n*)

$$(4) \quad g(x) = 1 + \sum_{n=z,2z,3z,\dots} b_n x^n; \quad z = 2, 4, 6, \dots$$

In this manuscript, the function $g(x)$ is expanded in a series of Chebyshev Polynomials $T(x)$ [1, §22][14, §18], which—for well understood reasons—supplies a more stable basis than the bare powers [7].

Definition 2. (*Chebyshev series expansion coefficients t_n*).

$$(5) \quad g(x) = \sum'_{n \geq 0} t_n T_n(x^d); \quad d = 1, 2, 3, \dots$$

The prime at the sum symbol indicates the term at $n = 0$ is halved.

Date: August 30, 2010.

2010 Mathematics Subject Classification. Primary 37G15, 37-04; Secondary 37M20.

Key words and phrases. Feigenbaum constant, period doubling, Chebyshev series.

We are considering even exponents z , so $g(x)$ is even and $t_n = 0$ whenever n is odd. The integer variable d plays the following role: Linear coupling equations between the t_n would have to be set up to remove the contributions by powers x^n ($z \nmid n$) from the solutions if $z > 2$ and $d = 1$. (The origin of this constraint that all exponents of x be multiples of the same z is not elaborated here.) Explicitly, equating equal powers of x in (4) and (5), the non-zero values of b_n are

$$(6) \quad 2^{n/d} \sum'_{\substack{s \geq n/d \\ s+n/d \equiv 0 \pmod{2}}} t_s \frac{s}{s+n/d} (-1)^{(s-n/d)/2} \binom{(s+n/d)/2}{n/d} = b_n, \quad d \mid n.$$

In the algorithm described further down, these would multiply the order of the linear system of equations by an approximate factor of $z/2$. For enhanced efficiency, d will be taken as

$$(7) \quad d = z/2,$$

which ensures that only powers of x with exponents divisible by z enter the calculation.

Equation (2) and the special values of the Chebyshev polynomials induce a sum rule and a coupling with λ :

$$(8) \quad g(0) = 1 \therefore 1 = \sum'_n (-1)^{n/2} t_n;$$

$$(9) \quad g(1) = -\lambda = \sum'_n t_n.$$

2. NUMERICAL ALGORITHM

2.1. Multivariate Newton Iteration. The right hand side of (1) is

$$(10) \quad g(g(\lambda x)) = \sum'_m t_m T_m(g^d[\lambda x]) = \sum'_m t_m T_m \left(\left\{ \sum'_n t_n T_n(\lambda^d x^d) \right\}^d \right).$$

Moving all terms to one side of the equation, the aim is to obtain a zero of the function f ,

$$(11) \quad f_j(\{t_m\}) \equiv \sum'_m t_m T_m \left(\left\{ \sum'_n t_n T_n(\lambda^d x_j^d) \right\}^d \right) + \lambda \sum'_n t_n T_n(x_j^d) = 0; \quad j = 1, 2, \dots$$

We solve for g with a finite, iteratively enlarged set of expansion coefficients t_n which are obtained by fitting at a finite set of the standard Chebyshev abscissa points $x_j = \cos \theta_j$, $\theta_j = j\pi/N$.

Remark 1. *With the presence of d , which effectively replaces the Chebyshev weights $1/\sqrt{1-x^2}$ by $x^{d-1}/\sqrt{1-x^{2d}}$, these abscissae may not be the optimum choice if $d > 1$.*

The common multivariate Newton algorithm with a $N \times N$ matrix of first derivatives is employed: we start with a set of approximations t_k , and calculate a vector

of corrections Δt_k which are the solutions to the linear system of equations

$$(12) \quad f_j(\{t_m\}) + \sum_k \frac{\partial f_j}{\partial t_k} \Delta t_k = 0; \quad j = 1, 2, \dots$$

This is actually done on $N - 1$ abscissa points x_j , because one row of the system of equations is reserved to accommodate (8):

$$(13) \quad \frac{1}{2}t_0 - t_2 + t_4 - \dots - 1 = 0.$$

$$(14) \quad \begin{pmatrix} \frac{1}{2} & -1 & 1 & -1 & \dots \\ \partial f_1/\partial t_0 & \partial f_1/\partial t_2 & \dots & & \\ \partial f_2/\partial t_0 & \partial f_2/\partial t_2 & \dots & & \\ \vdots & & & & \\ \partial f_{N-1}/\partial t_0 & \partial f_{N-1}/\partial t_2 & \dots & & \end{pmatrix} \cdot \begin{pmatrix} \Delta t_0 \\ \Delta t_2 \\ \Delta t_4 \\ \vdots \\ \Delta t_{2N-2} \end{pmatrix} = \begin{pmatrix} 1 - \sum'_n (-)^{n/2} t_n \\ -f_1 \\ \vdots \\ -f_{N-1} \end{pmatrix}.$$

To keep track of the prime at the sum symbols, a binary symbol which attains values of 2 or 1 is helpful:

Definition 3. (Neumann symbol ϵ)

$$(15) \quad \epsilon_0 = 2; \quad \epsilon_{>0} = 1.$$

The two terms in (11) are denoted $f^{(a)}$ and $f^{(b)}$. The second and higher rows in the matrix (14) are derivatives of $f = f^{(a)} + f^{(b)}$, which are computed with the chain and multiplication rules. The variable λ is eliminated with the aid of the derivative of (9),

$$(16) \quad \frac{\partial \lambda}{\partial t_k} = -\frac{1}{\epsilon_k}.$$

Remark 2. This implementation is one variant out of many. The simplicity of the previous formula means that the elimination of λ and $\Delta \lambda$ from the pool of unknowns produces no computational load.

The derivatives of (5) are

$$(17) \quad \frac{\partial g(\lambda x_j)}{\partial t_k} = \frac{1}{\epsilon_k} \left(T_k(\lambda^d x_j^d) - x_j^d d \lambda^{d-1} \sum_{l \geq 1} t_l T_l'(\lambda^d x_j^d) \right),$$

using the chain rule with the previous equation. The first term in (11) is

$$(18) \quad f_j^{(a)} = \frac{t_0}{2} T_0 \left[\left(\frac{t_0}{2} T_0(\lambda^d x_j^d) + t_1 T_1(\lambda^d x_j^d) + \dots \right)^d \right] \\ + t_1 T_1 \left[\left(\frac{t_0}{2} T_0(\lambda^d x_j^d) + t_1 T_1(\lambda^d x_j^d) + \dots \right)^d \right] \\ + t_2 T_2 \left[\left(\frac{t_0}{2} T_0(\lambda^d x_j^d) + t_1 T_1(\lambda^d x_j^d) + \dots \right)^d \right] + \dots$$

with derivatives

$$(19) \quad \frac{\partial f_j^{(a)}}{\partial t_k} = \frac{1}{\epsilon_k} T_k[g^d(\lambda x_j)] + dg^{d-1}(\lambda x_j) \frac{\partial g(\lambda x_j)}{\partial t_k} \sum_{l \geq 1} t_l T_l'[g^d(\lambda x_j)].$$

The term at $l = 0$ in the l -sum is skipped since $T_0'(\cdot) = 0$. (17) is inserted in front of the l -sum.

The second term in (11) is

$$(20) \quad f_j^{(b)} = \lambda \left(\frac{t_0}{2} T_0(x_j^d) + t_1 T_1(x_j^d) + t_2 T_2(x_j^d) + \dots \right),$$

with derivatives

$$(21) \quad \frac{\partial f_j^{(b)}}{\partial t_k} = \frac{\partial \lambda}{\partial t_k} g(x_j) + \lambda \frac{\partial g(x_j)}{\partial t_k} = -\frac{g(x_j)}{\epsilon_k} + \lambda \frac{1}{\epsilon_k} T_k(x_j^d).$$

The sums of (19) and (21) fill all but the first row of the matrix (14), and the negated sums of (18) and (20) fill the right hand side.

2.2. Convergence. All digits of t_n and $1/\lambda$ are considered stable which remain the same if the order of the basis set $\{t_n\}$ is increased from N to $N + 4$, and these stable digits will be shown in Section 3. The two (truncated) Chebyshev series are translated into the two equivalent polynomials, and the stable (common) digits of b_n are also reported. [This is an application of (6) with error propagation.] In this sense, the floating point representations of t_n are b_n are rounded towards zero.

3. RESULTS

3.1. $z=2$. The most prominent solution is characterized by a leading order $z = 2$ with

$$(22) \quad \lambda \approx 0.39953.$$

Broadhurst's 1018 most-significant digits of $1/\lambda \approx 2.5029078$ are available via a reference in the Online Encyclopedia of Integer Sequences [15, A006891], updating an earlier value by Briggs [2].

The equivalent table of n and t_n at $d = 1$ follows. Long lines of digits are wrapped around:

```

0  0.5657908632724943155053234479753520221074493015133421406289563842809270230277394544246259
   011330003848845022918090496111898399
2  -0.700391573973713787145980440332471261711550900968326558936525103514630069672843746266611
   2605202911962834490844304668109574045
4  0.0173621867222441842092787099400084179514093184582945769724809006730767362263833677577478
   919456847233605679815143774768249724
6  0.0006236559136940908585298153810290290296475164268085752829679471895290731944075421401943
   614274816952820691319610401477096491
8  -0.00025266414376208310293595863274053777442956794506726672370626681912006533496105357064
   29129678242706445734230384800866127081
10 0.000002781260604292479399098215157081791712604067313923235379235837579995393225358243627
   3878484563185046160651019299165207065
12 0.000000077936819963404276117862621362965789982955679476321280713649199426441939964528134
   40475515594882948978268818766695877953
14 -0.00000000327785722586730123873708759372184079817288268443710493022130004551717855508077
   34091127699425296427128013117312173163
16 0.00000000006384210078705431885183925429981891093422337470885619930147560493556137215402
   444436471651246493296658874924317760
18 0.000000000001778107114594587345498329317747650578350774121312001420048384738123759595693
   3133389845995766737758138453317984213
20 -0.00000000000002799043155015243405083045051874952896840805205291021004938704453334036778
   277145181036915561331924809893972849572
22 -0.0000000000000065987765706423742082420617124884422275834957432503771525513188833732520

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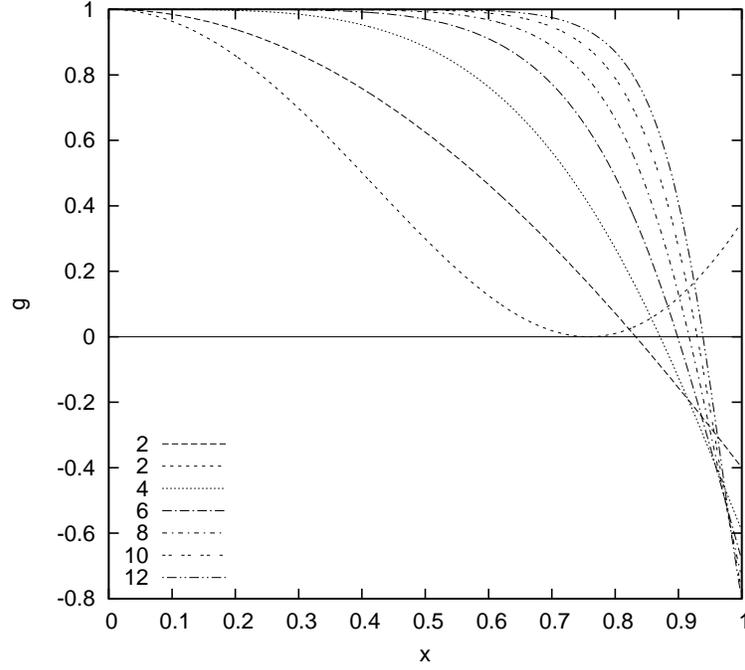



FIGURE 1. Solutions $g(x)$ discussed in this work, marked with their z -parameters. The three curves at $z = 2$ or $z = 4$ have been plotted by Campanino et. al. [5].

This corrects up to five digits in some rows of Lanford's table, where error bars are of the order 10^{-30} [13].

3.2. Extra $z=2$. The solution recorded by Stephenson and Wang at $z = 2$ which returns to a positive value $g(1) > 0$ (see Figure 1), hence assumes a negative value of λ , is also refined [17, 16].

Its table of n and t_n , value of $1/\lambda$, and table of n and b_n are:

```

0  0.695239872293346037618864099266551483193696717452792702
2  -0.2941925542257057866921887876571014540207923552631288627
4  0.33028072470630568964000834403239424732161343709604632350
6  -0.0315368919233901472203850867002152965050184294468376104
8  -0.0028714832609075151818101565163240137621094990590925332
10 0.00073699063183551803763665511752866415092050599857134560
12 -0.0000289143566935215723183757294107103328257404482568374
14 -0.00000639269493477664368809268119272428349626603668195306
16 0.00000089250734493829604749070500496361151531437475099399
18 -0.000000005841530325386329625967647262112183068522122098459
20 -0.00000000904338569609666460923438628990512000571101803815
22 0.000000000794308476080916066911148179642789312890062120789
24 0.00000000003025917767192722188179590965141382923724510004
26 -0.000000000105467686794658509993062892575928743129899731754
28 0.00000000000005178028438818918034562105519025020884646961949
30 0.0000000000000067093995175319688853016991708074433815242496
32 -0.00000000000010559545218086633148414702577521791670273535
34 0.000000000000000154347918061355616738318944738955043641955
36 0.000000000000000094274699599167949022401989148613318504987
38 -0.00000000000000009033368564852075748116667512849206793724
40 -0.00000000000000000000022125271892120189926978367638849260329
42 0.00000000000000000106886427363051040686657229276492155630
44 -0.00000000000000000000062817362134522292817164837731424606819

```


3.3. $\mathbf{z}=4$. A known solution with smoothness $z = 4$ is accurately described by the following table of n and t_n at $d = 2$ [17, 16]:

```

0 0.3259810020643102220511536428215443201428992452133
2 -0.800208447142190906917818576407006363536002907994
4 0.0418898841904523139724543647479542658619459241985
6 0.0043894976258619771863959729197546595065618948810
8 -0.00069010337244814876166850830661956401216977657241
10 0.0000144814075746787149177686108721351459836488957
12 0.00000476300216712418243311831014922536070727006694
14 -0.000000494555259455886663361734329319366255446136691
16 -0.00000002894498873151192531198855036171513344547245
18 0.000000043659575703295075770575545118494173777559517
20 -0.00000000278893954012460398480375551751458327306229
22 -0.0000000001905793842739537879484162909974046955218144
24 0.0000000000387515529016371762223417515171078981195
26 -0.0000000000010811170201073782340784343123593525691
28 -0.0000000000002781706617592988574104718860817240835
30 0.000000000000003140943030258792107754156277223196029
32 0.000000000000000113037783599017151502130939738831523
34 -0.0000000000000000294068162559927660239320286755977003
36 0.00000000000000000217380400301654878938939813471539029
38 0.00000000000000000000009589354775830741278121051671184423
40 -0.00000000000000000026674587926268032143341253347671646
42 0.000000000000000000000011684247116706485719678443835458607
44 0.0000000000000000000000014912430927612388099838950079617660
46 -0.000000000000000000000002158773018436811136342959944383313
48 0.00000000000000000000000028451466280962684221628155260692
50 0.0000000000000000000000000171842585248494306526822030351909
52 -0.0000000000000000000000001539313006678710338179573132491
54 -0.000000000000000000000000039226023013920647068619398834
56 0.000000000000000000000000001677001372390369884707804569
58 -0.0000000000000000000000000090135091823733200907687260
60 -0.00000000000000000000000000008349267645798270024985074
62 0.0000000000000000000000000001434920035967166259561752
64 -0.000000000000000000000000000319516654856190813929
66 -0.0000000000000000000000000000105418754132973448968
68 0.00000000000000000000000000000107373483256963049299
70 0.00000000000000000000000000000151525141050310900
72 -0.0000000000000000000000000000010833036852367435
74 0.000000000000000000000000000000067025066870316
76 0.00000000000000000000000000000048204263164179
78 -0.00000000000000000000000000000009684033207264
80 0.0000000000000000000000000000000028596298782
82 0.0000000000000000000000000000000066708949798
84 -0.00000000000000000000000000000000076077044388
86 -0.0000000000000000000000000000000000000036628899
88 0.0000000000000000000000000000000000721368704
90 -0.0000000000000000000000000000000000000005098973
92 -0.00000000000000000000000000000000000000000273795
94 0.00000000000000000000000000000000000000000672075
96 -0.000000000000000000000000000000000000000000257959
98 -0.0000000000000000000000000000000000000000000418300
100 0.000000000000000000000000000000000000000000005520

```

The associated Feigenbaum constant $1/\lambda$ (updating the value of 1.690302 [4]) and equivalent table of n and Taylor coefficients b_n are:

```

F = 1.690302971405244853343780150324161348228278059709
4 -1.83410790700941066477722032786167658753580656671
8 0.012962226191371748194249954526500692423364159
12 0.3119017366428453740938210685407183598371022
16 -0.0620146232838494154168020915170780000963
20 -0.037539476018044801855283971780562014903
24 0.0176482141699045721668837916436744792
28 0.00193502990250861524539045696343944
32 -0.002811394115124136184245956801096
36 0.00009519227150370368772733484141
40 0.000435491310822346046233354127
44 -0.000075173146500397955569561474
48 -0.00006736728882241037232621045
52 0.0000269344724659457260827140
56 0.000006286772295217247380353

```

60 -0.0000059502454648295960937
 64 0.000002101623794439990936
 68 0.000009189230440398699384
 72 -0.0000021302788365308075
 76 -0.0000010529385239013139
 80 0.000000526461946904
 84 0.000000076135782693113
 88 -0.00000009844458690011
 92 0.0000000047080672420
 96 0.0000000154345662656
 100 -0.0000000035684794770
 104 -0.0000000018996907837
 108 0.0000000009357108066
 112 0.00000000013961433
 116 -0.00000000017731035
 120 0.0000000000087943
 124 0.0000000000272384
 128 -0.000000000006104

3.4. $z=6$. The Chebyshev coefficients t_n for $z = 6$ with $d = 3$ are:

0 0.21021341309304017470311495545314716311007687224938007663559
 2 -0.85061644791828709112050514504136245282682454755905389538579
 4 0.05628161905542202323276948575124712239104414802830965331397
 6 0.009935058137903504135993082540678516096531085856931659844078
 8 -0.0020855090028753720240399596899815942535546644082405772236
 10 0.00002667282124614707868884031418105463714552279637627290637
 12 0.00003829372515583810559086517586800603514191708130147980083
 14 -0.000004625699664422667849249798381290140479802593287643188046
 16 -0.000002831151594281039534174488537456346214130161032444781955
 18 0.00000012338683107538209784038150537174370653614397236744514266
 20 -0.000000068174045281617081780530926809083421690600186685820884
 22 -0.00000001899873267576372584804450734908402877469178801808811196
 24 0.00000000346392675995598419474794310398532136946869032638576
 26 0.0000000000390068105896541570853177397127706284297914293009653
 28 -0.000000000079149906752964255297343374974010828294089885968213
 30 0.0000000000077931694780771578707264938426098858651135113840855
 32 0.000000000000852773790825838513873257851533380830018759450045
 34 -0.000000000000260383569822231813226762772106465631988119103374
 36 0.00000000000010025050056296133091452636511043678546651959869
 38 0.0000000000000449476081511898302858473633434434053219462678
 40 -0.00000000000000700314136799831204483340156830840996622492831
 42 -0.000000000000000211427500105270276952688421271708844343409553
 44 0.000000000000000017355760875149987461939747544759876131044824
 46 -0.0000000000000000141052513634382469967859280854950424746015099
 48 -0.0000000000000000022360690323524459052845995488637640910740059
 50 0.0000000000000000005463391250275824699752989313602943626365874
 52 -0.0000000000000000001032678941939516175678341770001697926196876
 54 -0.00000000000000000010759324343630651925920681050043268884835996
 56 0.00000000000000000013852158702642782628823590465480410418581454
 58 0.00000000000000000008484209644875932810101724300513674300141197
 60 -0.0000000000000000000039364298268040750575103277016650317751108005
 62 0.0000000000000000000024020661589849142360039060864756583132179518
 64 0.0000000000000000000006001745276715558023520845370409650872133890
 66 -0.000000000000000000000011746748308382063766464342078813559899538285
 68 -0.0000000000000000000000041146309997350933302276992059245312709099
 70 0.00000000000000000000000261692711387768568430763903240667516315367
 72 -0.0000000000000000000000027486585506723445468000794116756014447496
 74 -0.00000000000000000000000275308623842612943860805940900180953217
 76 0.00000000000000000000000090196778810492963298456791888344590739
 78 -0.000000000000000000000000376607611498847643758409590315035925
 80 -0.00000000000000000000000015696826935725241130869385256205437
 82 0.00000000000000000000000025350782115307816392785281062157
 84 0.00000000000000000000000006853080930004380405185843345
 86 -0.00000000000000000000000000631173846425375219150669720605
 88 0.0000000000000000000000000537223333770510172444422828
 90 0.00000000000000000000000008002743541502796444109333872
 92 -0.00000000000000000000000002051613050532287398283675699
 94 0.00000000000000000000000004791508300478340494151358
 96 0.0000000000000000000000000398185100966144593387684
 98 -0.0000000000000000000000000053917069856953084333496
 100 -0.000000000000000000000000028656289415773532604205
 102 0.00000000000000000000000001495957563527673051760


```

2 -0.8842303740823175512755608808
4 0.0665209634842108260820275058
6 0.0160708249063224794092104569
8 -0.0038907994146791521600746136
10 -0.000097551920092753105368942
12 0.000123849745247201425628817
14 -0.00001527270699745541145444107
16 -0.0000021352008988295402412375489
18 0.000000772398623891616273984173
20 -0.000000024219287650319566438404
22 -0.000000023416621864635256604937
24 0.000000003826707490135532768257
26 0.00000000032042363484105018598
28 -0.00000000017467929304630940850
30 0.00000000001152216008656868781
32 0.000000000004587462209785440857
34 -0.000000000000995958581659698440
36 -0.0000000000000312775335815343648
38 0.0000000000000038873506228181227
40 -0.000000000000000392306669959245007
42 -0.000000000000000085951077958354364
44 0.0000000000000000249251998355112348
46 -0.0000000000000000018980520249321427
48 -0.000000000000000000856801506577853604
50 0.00000000000000000001177536979181658135
52 0.000000000000000000001511172911297034026
54 -0.0000000000000000000006142892999320489
56 0.0000000000000000000000261014286449232
58 0.000000000000000000000018461512874019
60 -0.0000000000000000000000333176674195
62 -0.0000000000000000000000022406130989
64 0.0000000000000000000000014800851697
66 -0.000000000000000000000000113012074
68 -0.000000000000000000000000038099450
70 0.00000000000000000000000000908411
72 0.0000000000000000000000000018534
74 -0.00000000000000000000000000346207
76 0.0000000000000000000000000003859
78 0.000000000000000000000000000073

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The corresponding Feigenbaum constant (updating 1.35798 [4]) and the associated Taylor coefficients b_n are:

```

F = 1.3580172791380503454873763331
8 -1.89735300467491340532977247
16 -0.73884380388552253531567
24 0.989774470130239041388
32 0.445857788637156558
40 -0.5879109210650583
48 -0.268029707317435
56 0.326144611238
64 0.20652701084
72 -0.201257882
80 -0.1608771
88 0.1402826
96 0.1105
104 -0.094

```

3.6. $z=10$. The Chebyshev coefficients t_n for $z = 10$ with $d = 5$ are:

```

0 0.087844965370486707
2 -0.9093012275250430520
4 0.074588019670768413
6 0.022306567620926918
8 -0.00591629904025435163
10 -0.00011666771033891637
12 0.000271811762145798921
14 -0.0000328109239789392
16 -0.00000770036081694509
18 0.00000258451857056376
20 -0.00000003534676935799
22 -0.000000121746886087767
24 0.0000000176227062237214
26 0.00000000328463293938500

```

```

28 -0.0000000132079164712664
30 0.00000000033312229401120
32 0.00000000058825803651164
34 -0.0000000001012188314110
36 -0.00000000013849849099
38 0.000000000069207426816
40 -0.00000000000327774239
42 -0.00000000000290338777
44 0.00000000000058170125
46 0.0000000000000577884
48 -0.0000000000000369611
50 0.0000000000000025399
52 0.000000000000001449192
54 -0.00000000000000033648
56 -0.0000000000000000223

```

This translates into $1/\lambda$ and the Taylor coefficients b_n as follows:

```

F = 1.2915168672623445696
10 -1.8517140134795485
20 -1.124743004799023
30 1.0746332407420
40 1.07554234877
50 -0.6752703771
60 -0.99566771
70 0.2886430
80 0.99702
90 -0.0150

```

3.7. **z=12.** The list of n and t_n for $z = 12$ with $d = 6$ starts:

```

0 0.050315274170474025
2 -0.9292027766462839346
4 0.081321549028015737
6 0.0284367418795728353
8 -0.00805828726265292218
10 -0.00029782425094054862
12 0.000483912632074647391
14 -0.00005617186728366212
16 -0.00001926578700820676
18 0.000006201794732562265
20 0.000000194426753767854
22 -0.0000004017148408827121
24 0.00000005067173288804275
26 0.00000001645633128370378
28 -0.00000005657763568442150
30 -0.000000001193549987650470
32 0.00000000365299595939128
34 -0.0000000049892905014794
36 -0.00000000014698792552832
38 0.0000000000539147386315
40 0.000000000006412134576
42 -0.0000000000034507823335
44 0.0000000000005035434125
46 0.000000000000135831268
48 -0.000000000000052890573
50 -0.00000000000000142316
52 0.0000000000000033350259
54 -0.000000000000000518608
56 -0.0000000000000001272412
58 0.000000000000000052730
60 -0.000000000000000003532
62 -0.000000000000000003262
64 0.000000000000000000539

```

This translates into $1/\lambda$ and Taylor coefficients b_n :

```

F = 1.2465277517207492954
12 -1.79116162311222203
24 -1.463168537263831
36 0.9856559520745
48 1.76384751144
60 -0.352844894
72 -1.91762338
84 -0.54023
96 2.0135

```

108 1.521
 120 -1.866
 132 -2.43
 144 1.

3.8. **z=14.** The list of n and t_n for $z = 14$ with $d = 7$ starts:

0 0.0210003942667285
 2 -0.945643213077312
 4 0.08714605274152282
 6 0.03437323735545195
 8 -0.01025712295409548
 10 -0.000548744504139
 12 0.000757490037990091
 14 -0.000083828165279389
 16 -0.000038842635955055
 18 0.000012160982312532
 20 0.0000007928961871139
 22 -0.000001006742945403
 24 0.00000010854938696682
 26 0.00000005550666127079
 28 -0.000000017168885225179
 30 -0.0000000012329704551329
 32 0.0000000014734835693269
 34 -0.0000000001538909310688
 36 -0.000000000083949894876
 38 0.000000000025491091508
 40 0.000000000002018718384
 42 -0.00000000000224862047
 44 0.00000000000022506633
 46 0.0000000000001314485
 48 -0.00000000000003902405
 50 -0.00000000000000336485
 52 0.00000000000000035144
 54 -0.000000000000000033657
 56 -0.0000000000000000209532
 58 0.000000000000000006079
 60 0.00000000000000000565

This translates into the Feigenbaum constant and Taylor coefficients b_n :

F = 1.21391238764424391
 14 -1.72516768360581
 28 -1.7485120998868
 42 0.76642269718
 56 2.385250550
 70 0.383241
 84 -2.65092
 98 -2.3006
 112 2.30
 126 4.6

4. SUMMARY

Three representations of the Feigenbaum Function $g(x^z)$ for orders $z = 2$ and $z = 4$ have been computed with higher precision than previously published. The principal solutions in the parameter range $z = 6-14$, some of which have been characterized in the literature by Feigenbaum constants with 5-digit accuracy, have been made explicit.

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