Noncrossing Linked Partitions and Large $(3, 2)$-Motzkin Paths

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Abstract. Noncrossing linked partitions arise in the study of certain transforms in free probability theory. We explore the connection between noncrossing linked partitions and colored Motzkin paths. A $(3, 2)$-Motzkin path can be viewed as a colored Motzkin path in the sense that there are three types of level steps and two types of down steps. A large $(3, 2)$-Motzkin path is defined to be a $(3, 2)$-Motzkin path for which there are only two types of level steps on the x-axis. We establish a one-to-one correspondence between the set of noncrossing linked partitions of $[n + 1]$ and the set of large $(3, 2)$-Motzkin paths of length n. In this setting, we get a simple explanation of the well-known relation between the large and the little Schröder numbers.

Keywords: Noncrossing linked partition, Schröder path, large $(3, 2)$-Motzkin path, $(3, 2)$-Motzkin path, Schröder number

AMS Classifications: 05A15, 05A18.

1 Introduction

The notion of noncrossing linked partitions was introduced by Dykema [4] in the study of free probabilities. He showed that the generating function of the number of noncrossing linked partitions of $[n + 1] = \{1, 2, \ldots, n + 1\}$ equals

$$F(x) = \sum_{n=0}^{\infty} f_{n+1} x^n = \frac{1 - x - \sqrt{1 - 6x + x^2}}{2x}. \tag{1.1}$$

This implies that the number of noncrossing linked partitions of $[n+1]$ is equal to the number of large Schröder paths of length $2n$, namely, the n-th large Schröder number S_n. A large Schröder path of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ consisting of up steps $(1,1)$, level steps $(2,0)$ and down steps $(1,-1)$ and never lying below the x-axis. The first few values of S_n's are $1, 2, 6, 22, 90, 394, 1806, \ldots$. The sequence of the large Schröder numbers is listed as entry A006318 in OEIS [8]. Chen, Wu and Yan [2] established a bijection between the set of noncrossing linked partitions of $[n+1]$ and the set of large Schröder paths of length $2n$.

Motivated by the correspondence between noncrossing partitions and 2-Motzkin paths, we are led to the question whether there is any connection between noncrossing linked partitions and colored Motzkin paths. It is known that little Schröder paths of length $2n$ equals the number of $(3,2)$-Motzkin paths of length $n-1$. Recall that a little Schröder path of length $2n$ is a lattice path from $(0,0)$ to $(2n,0)$ consisting of up steps $(1,1)$, down steps $(1,-1)$ and level steps $(2,0)$ not lying below the x-axis with the additional condition that there are no level steps on the x-axis. The number of such paths of length $2n$ is referred to as the little Schröder number s_n. Yan [9] found a bijective proof of this fact. Since the large Schröder numbers and the little Schröder numbers are related by a factor of two, we see that the number of noncrossing linked partitions of $[n+1]$ is numerically related to the number of $(3,2)$-Motzkin paths of length n.

Indeed, the main result of this paper is to introduce a class of Motzkin paths, which we call the large $(3,2)$-Motzkin paths, such that noncrossing linked partitions of $[n+1]$ are in one-to-one correspondence with large $(3,2)$-Motzkin paths of length n. By examining the connection between large $(3,2)$-Motzkin paths and the ordinary $(3,2)$-Motzkin paths, we immediately get the relation between the large and the little Schröder numbers.

Let us recall some terminology. A $(3,2)$-Motzkin path of length n is a lattice path from $(0,0)$ to $(n,0)$ consisting of up steps $u=(1,1)$, level steps $(1,0)$ and down steps $(1,-1)$ with each down step receiving one of the two colors d_1, d_2, and each level step receiving one of the three colors l_1, l_2, l_3. Let m_n denote the n-th $(3,2)$-Motzkin number, that is, the number of $(3,2)$-Motzkin paths consisting of n steps, or of length n. A large $(3,2)$-Motzkin path is a $(3,2)$-Motzkin path for which each level step at the x-axis receives only one of the two colors l_1 or l_2. An elevated large $(3,2)$-Motzkin path is defined as a large $(3,2)$-Motzkin path that does not touch the x-axis except for the origin and the destination. Denote the set of large $(3,2)$-Motzkin paths by L and the set of large $(3,2)$-Motzkin paths of length n by $L(n)$. Meanwhile, we use L_n to denote the number of paths in $L(n)$.

It can be shown that the generating function

$$L(x) = \sum_{n=0}^{\infty} L_n x^n$$

satisfies the functional equation

$$L(x) = 1 + 2xL(x) + 2x^2M(x)L(x), \tag{1.2}$$

where

$$M(x) = \sum_{n=0}^{\infty} m_n x^n = \frac{1 - 3x - \sqrt{1 - 6x + x^2}}{4x^2} \tag{1.3}$$

is the generating function of the $(3, 2)$-Motzkin numbers. From (1.2) and (1.3), it follows that $L(x) = F(x)$. This yields that

$$L_n = f_{n+1}. \tag{1.4}$$

Examining the connection between the large $(3, 2)$-Motzkin paths and ordinary $(3, 2)$-Motzkin paths, we are led to a simple explanation of the following relation:

$$L_n = 2m_{n-1}. \tag{1.5}$$

In fact, the argument for the above relation is essentially the same argument for the relation (1.2) between $L(x)$ and $M(x)$. Since the little Schröder number is equal to the $(3, 2)$-Motzkin number, see Chen, Li, Shapiro and Yan [1] and Yan [9], (1.5) is equivalent to the well-known relation $S_n = 2s_n$, which has been proved combinatorially by Shapiro and Sulanke [7], Deutsch [3], Gu, Li and Mansour [5] and Huq [6].

2 Noncrossing Linked Partitions

We begin with an overview of noncrossing linked partitions. In particular, we shall give a description of the linear representation of a linked partition. A linked partition $\pi = \{B_1, B_2, \ldots, B_k\}$ of $[n]$ is a collection of nonempty subsets, called blocks, of $[n]$, such that the union of $B_1, B_2, \ldots, B_k$ is $[n]$ and any two distinct blocks of π are nearly disjoint. Two blocks B_i and B_j are said to be nearly disjoint if for every $k \in B_i \cap B_j$, one of the following conditions holds:

(a) $k = \min(B_i), |B_i| > 1$ and $k \neq \min(B_j)$, or

(b) $k = \min(B_j), |B_j| > 1$ and $k \neq \min(B_i)$.

We say that π is a noncrossing linked partition if in addition, for any two distinct blocks $B_i, B_j \in \pi$, there does not exist $i_1, j_1 \in B_i$ and $i_2, j_2 \in B_j$ such that $i_1 < i_2 < j_1 < j_2$. Let $NCL(n)$ denote the set of noncrossing linked partitions of $[n]$.

In this paper, we shall adopt the linear representation of noncrossing linked partitions, see Chen, Wu and Yan [2]. For a noncrossing linked partition π of $[n]$, first we draw n vertices on a horizontal line with points or vertices $1, 2, \ldots, n$ arranged in increasing order. For each block $B = \{i_1, i_2, \ldots, i_k\}$ with $i_1 = \min(B)$ and $k \geq 2$, draw an arc between i_1 and any other vertex in B. We shall use a pair (i, j) to denote an arc between i and j, where we assume that $i < j$. For example, the linear representation of $\pi = \{1, 4, 8\}\{2, 3\}\{5, 6\}\{6, 7\}\{8, 9\} \in NCL(9)$ is shown in Figure 1.

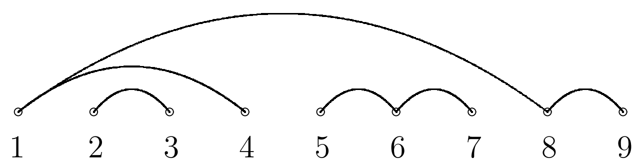

Figure 1: The linear representation of $\pi = \{1, 4, 8\}\{2, 3\}\{5, 6\}\{6, 7\}\{8, 9\}$

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Here is the main result of this paper.

Theorem 2.1 *There is a bijection between the set of large* $(3, 2)$*-Motzkin paths of length* n *and the set of noncrossing linked partition of* $[n + 1]$*.*

Proof. We describe a map φ from $L(n)$ to $NCL(n+1)$ in terms of a recursive procedure. Let $P \in L(n)$. We wish to construct a noncrossing linked partition $\pi = \varphi(P)$.

If $P = \emptyset$, then set $\varphi(P) = \{1\}$. For $n \geq 1$, write $P = P_1 P_2 \cdots P_k$, where each P_i is a nonempty elevated large $(3, 2)$-Motzkin paths of length p_i. We consider the following cases.

Case 1. (i) If $P_i = l_1$, then set $\varphi(P_i) = \{1, 2\}$.

(ii) If $P_i = l_2$, then set $\varphi(P_i) = \{1\}\{2\}$.

Case 2. (i) If $P_i = uP_c d_1$, where $P_c \in L$, that is, P_c is a large $(3, 2)$-Motzkin path, then set

$$\varphi(P_i) = \{1, p_i, p_i + 1\} \cup \varphi(P_c),$$

see Figure 2 for an illustration of this operation.

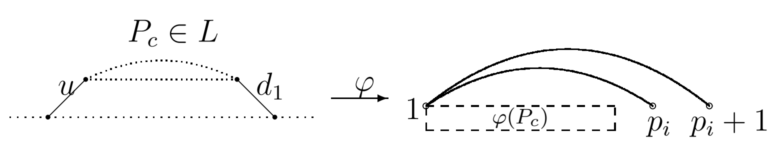

Figure 2: Case 2 (i).

(ii) If $P_i = uP_cd_1$ and $P_c \in \overline{L}$, that is, P_c is a $(3,2)$-Motzkin path with at least one l_3 step on the x-axis. Then set

$$P_c = P_c^{(1)} l_3 P_c^{(2)} l_3 \cdots l_3 P_c^{(k)},\ k \geq 2,$$

where $P_c^{(i)} \in L$ is of length $t_i \geq 0$. We proceed to construct $\varphi(P_i)$ via the following steps. Let

$$\tau(P_c^{(i)}) = \{1, t_i + 2\} \cup \varphi(P_c^{(i)}),\ i = 1, 2, \ldots, k.$$

For $i = 1, 2, \ldots, k-1$, merge the last vertex $t_i + 2$ of $\tau(P_c^{(i)})$ and the first vertex 1 of $\tau(P_c^{(i+1)})$ and relabel the vertices by $\{1, 2, \ldots, p_i\}$ in increasing order. Denote the resulting noncrossing linked partition by $\omega(P_c)$. Then set

$$\varphi(P_i) = \{1, p_i + 1\} \cup \omega(P_c).$$

An illustration of the above construction is given in Figure 3.

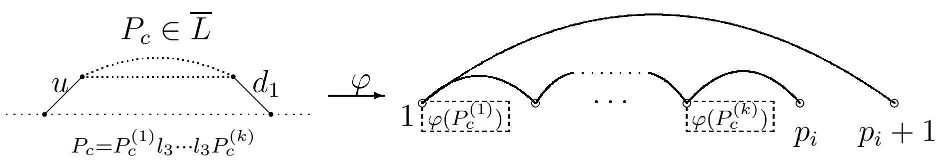

Figure 3: Case 2 (ii).

Case 3. (i) If $P_i = uP_cd_2$, where $P_c \in L$, then set

$$\varphi(P_i) = (\{1, p_i + 1\}\{p_i\}) \cup \varphi(P_c).$$

Figure 4 is an illustration of this operation.

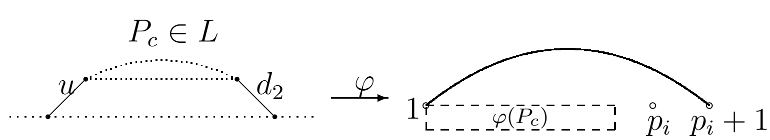

Figure 4: Case 3 (i).

(ii) If $P_i = uP_cd_2$ and $P_c \in \overline{L}$, then set

$$P_c = P_c^{(1)} l_3 P_c^{(2)} l_3 \cdots l_3 P_c^{(k)},\ k \geq 2,$$

where $P_c^{(i)} \in L$ is of length $t_i \geq 0$. Denote by $\tau(P_c^{(i)})$ the noncrossing linked partition $\{1, t_i + 2\} \cup \varphi(P_c^{(i)}),\ i = 2, 3, \ldots, k$. For each $i = 2, 3, \ldots, k-1$, we merge the last vertex $t_i + 2$ of $\tau(P_c^{(i)})$ and the first vertex 1 of $\tau(P_c^{(i+1)})$ and relabel the vertices by $\{t_1 + 2, t_1 + 3, \ldots, p_i\}$ in increasing order. Let the resulting noncrossing linked partition be denoted by $\nu(P_c)$. Then set

$$\varphi(P_i) = \{1, p_i + 1\} \cup \varphi(P_c^{(1)}) \cup \nu(P_c).$$

This operation is illustrated by Figure 5.

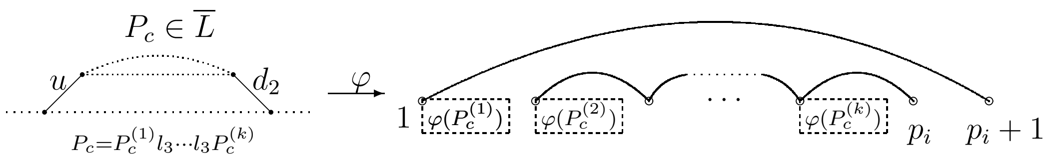

Figure 5: Case 3 (ii).

Finally, $\pi = \varphi(P)$ is constructed by merging the last vertex of $\varphi(P_i)$ and the first vertex of $\varphi(P_{i+1})$, for $i = 1, 2, \ldots, k-1$, and relabeling the vertices by $\{1, 2, \ldots, n+1\}$. It can be seen that π is a noncrossing linked partition of $[n+1]$.

To show that φ is a bijection, we give the inverse map of φ. Let $\pi \in NCL(n+1)$. We still work with the linear representation of π. First, we make use of the outer arc decomposition of π. Here an outer arc is an arc in the linear representation of π that is not covered by any other arc. To be more specific, the outer arc decomposition of π is given by

$$\pi = (\pi_1, \pi_2, \ldots, \pi_m),$$

where each π_i is a noncrossing linked partition of the set $\{s_i, s_i + 1, \ldots, s_{i+1} - 1, s_{i+1}\}$ with $s_1 = 1$, $s_i < s_{i+1}$ and $s_{m+1} = n+1$, such that $\pi_i = \{s_i\}\{s_{i+1}\}$, or s_i and s_{i+1} are contained in the same block of π_i. Next, consider $\varphi^{-1}(\pi_i)$, for $i = 1, 2, \ldots, m$. If $\pi_i = \{s_i, s_{i+1}\}$ (or $\{s_i\}\{s_{i+1}\}$), then set $\pi_i = l_1$ (or l_2). If $s_{i+1} \geq s_i + 2$ and there are arcs forming a path from the vertex s_i to the vertex $s_{i+1} - 1$, then we deduce that $\varphi^{-1}(\pi_i)$ starts with an up step u and ends with a down step d_1. If there are no paths from the vertex s_i to the vertex $s_{i+1} - 1$, then we have that $\varphi^{-1}(\pi_i)$ starts with an up step u and ends with a down step d_2. Hence $\varphi^{-1}(\pi_i)$ can be reconstructed recursively according to the cases for the map φ. Finally, putting all the pieces together, that is, setting

$$\varphi^{-1}(\pi) = \varphi^{-1}(\pi_1)\varphi^{-1}(\pi_2) \cdots \varphi^{-1}(\pi_m),$$

we are led to a large $(3,2)$-Motzkin path of length n. This completes the proof. ∎

An example of the above bijection is given in Figure 6.

The above bijection implies that the large Schröder number S_n equals the number L_n of large $(3,2)$-Motzkin paths of length n. On the other hand, there is a one-to-one

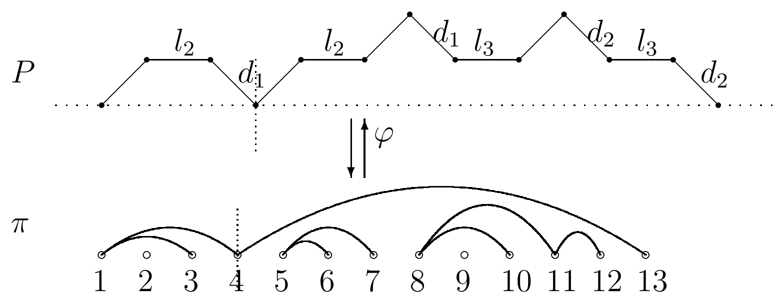

Figure 6: Bijection φ: $L(12) \to NCL(13)$.

correspondence between $(3,2)$-Motzkin paths of length $n-1$ and little Schröder paths of length $2n$. Therefore, the relation $S_n = 2s_n$ can be rewritten as

$$L_n = 2m_{n-1}, \tag{2.6}$$

that is, the number of large $(3,2)$-Motzkin paths of length n is twice the number of ordinary $(3,2)$-Motzkin paths of length $n-1$. Here we give a combinatorial interpretation of this fact.

Let P be a $(3,2)$-Motzkin path of length $n-1$. If P does not have any level step l_3 on the x-axis, then we get two large $(3,2)$-Motzkin paths by adding a level step l_1 or l_2 at the end of P. Otherwise, remove the first level step l_3 on the x-axis in P, and elevate the path after this l_3 level step. Concerning the elevated $(3,2)$-Motzkin path, there are two choices for the last down step. It is easy to see that the above construction is reversible. Hence we obtain (2.6).

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