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ON DELANNOY NUMBERS AND SCHRÖDER NUMBERS

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Abstract. The nth Delannoy number and the nth Schröder number given by

$$D_n=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\ \text{ and }\ S_n=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}$$

respectively arise naturally from enumerative combinatorics. Let p be an odd prime. We mainly show that

$$\sum_{k=1}^{p-1}\frac{D_k}{k^2}\equiv 2\left(\frac{-1}{p}\right)E_{p-3}\pmod p$$

and

$$\sum_{k=1}^{p-1}\frac{S_k}{m^k}\equiv\frac{m^2-6m+1}{2m}\left(1-\left(\frac{m^2-6m+1}{p}\right)\right)\pmod p,$$

where $(-)$ is the Legendre symbol, $E_0,E_1,E_2,\ldots$ are Euler numbers, and m is any integer not divisible by p. We also conjecture that

$$\sum_{k=1}^{p-1}\frac{D_k^2}{k^2}\equiv -2q_p(2)^2\pmod p$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

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1. Introduction

For $n\in\mathbb{N}=\{0,1,2,\ldots\}$, the (central) Delannoy number D_n denotes the number of lattice paths from the point $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$, while the Schröder number S_n represents the number of such paths that never rise above the line $y=x$. It is known that

$$D_n=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}$$

and

$$S_n=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}\frac{1}{k+1}=\sum_{k=0}^n\binom{n+k}{2k}C_k,$$

where C_k stands for the Catalan number $\binom{2k}{k}/(k+1)=\binom{2k}{k}-\binom{2k}{k+1}$. For information on D_n and S_n, the reader may consult [CHV], [S], and p. 178 and p. 185 of [St].

Despite their combinatorial backgrounds, surprisingly Delannoy numbers and Schröder numbers have some nice number-theoretic properties.

As usual, for an odd prime p we let $(\frac{\cdot}{p})$ denote the Legendre symbol. Recall that Euler numbers $E_0,E_1,E_2,\ldots$ are integers defined by $E_0=1$ and the recursion:

$$\sum_{\substack{k=0\\ 2\mid k}}^n\binom{n}{k}E_{n-k}=0\quad\text{for } n=1,2,3,\ldots.$$

Our first theorem is concerned with Delannoy numbers and their generalization.

Theorem 1.1. *Let p be an odd prime. Then*

$$\sum_{k=1}^{p-1}\frac{D_k}{k^2}\equiv 2\left(\frac{-1}{p}\right)E_{p-3}\pmod p \tag{1.1}$$

and

$$\sum_{k=1}^{p-1}\frac{D_k}{k}\equiv -q_p(2)\pmod p, \tag{1.2}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$. If we set

$$D_n(x)=\sum_{k=0}^n\binom{n}{k}\binom{n+k}{k}x^k\quad(n\in\mathbb{N}),$$

then for any p-adic integer x we have

$$\sum_{k=1}^{p-1}\frac{D_k(x)}{k}\equiv\frac{(-1+\sqrt{-x})^p+(-1-\sqrt{-x})^p+2}{p}\pmod p. \tag{1.3}$$

Corollary 1.1. *Let p be an odd prime. We have*

$$\sum_{k=1}^{p-1}\frac{D_k(3)}{k}\equiv-2q_p(2)\pmod p\quad\text{provided } p\neq 3,\tag{1.4}$$

$$\sum_{k=1}^{p-1}\frac{D_k(-4)}{k}\equiv\frac{3-3^p}{p}\pmod p,\tag{1.5}$$

$$\sum_{k=1}^{p-1}\frac{D_k(-9)}{k}\equiv-6q_p(2)\pmod p,\tag{1.6}$$

and also

$$\sum_{k=1}^{p-1}\frac{D_k(-2)}{k}\equiv-\frac{4}{p}P_{p-(\frac{2}{p})}\pmod p,\tag{1.7}$$

where the Pell sequence $\{P_n\}_{n\geqslant 0}$ is given by

$$P_0=0,\ P_1=1,\ \text{and } P_{n+1}=2P_n+P_{n-1}\ (n=1,2,3,\ldots).$$

If $p\neq 5$, then

$$\sum_{k=1}^{p-1}\frac{D_k(-5)}{k}\equiv-2q_p(2)-\frac{5}{p}F_{p-(\frac{p}{5})}\pmod p,\tag{1.8}$$

where the Fibonacci sequence $\{F_n\}_{n\geqslant 0}$ is defined by

$$F_0=0,\ F_1=1,\ \text{and } F_{n+1}=F_n+F_{n-1}\ (n=1,2,3,\ldots).$$

Now we propose two conjectures which seem challenging in the author's opinion.

Conjecture 1.1. *Let $p>3$ be a prime. We have*

$$\sum_{k=1}^{p-1}\frac{D_k^2}{k^2}\equiv-2q_p(2)^2\pmod p,\tag{1.9}$$

$$\sum_{k=1}^{p-1}\frac{D_k}{k}\equiv-q_p(2)+p\,q_p(2)^2\pmod{p^2},\tag{1.10}$$

$$\sum_{k=1}^{p-1}D_kS_k\equiv-2p\sum_{k=1}^{p-1}\frac{(-1)^k+3}{k}\pmod{p^4},$$

and

$$\sum_{k=1}^{(p-1)/2}D_kS_k\equiv\begin{cases}4x^2\pmod p & \text{if } p\equiv 1\pmod 4\ \&\ p=x^2+y^2\ (2\nmid x,\ 2\mid y),\\ 0\pmod p & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Also, $\sum_{n=1}^{p-1} s_n^2/n \equiv -6 \pmod p$, *where*

$$s_n := \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k+1} = D_n - S_n.$$

Remark 1.1. Let p be an odd prime. Though there are many congruences for $q_p(2)$ mod p, (1.9) is curious since its left-hand side is a sum of squares. It is known that $\sum_{k=1}^{p-1} 1/k \equiv -p^2 B_{p-3}/3 \pmod{p^3}$ if $p > 3$, where $B_0, B_1, B_2, \ldots$ are Bernoulli numbers. In addition, we can prove that $\sum_{k=0}^{p-1} D_k \equiv (\frac{-1}{p}) - p^2 E_{p-3} \pmod{p^3}$ and $\sum_{k=0}^{p-1} D_k^2 \equiv (\frac{2}{p}) \pmod p$.

Conjecture 1.2. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}(-1)^k D_k(2)^3 \equiv \sum_{k=0}^{p-1}(-1)^k D_k\left(-\frac{1}{4}\right)^3 \equiv \left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}(-1)^k D_k\left(\frac{1}{8}\right)^3$$

$$\equiv \begin{cases} (\frac{-1}{p})(4x^2-2p) \pmod{p^2} & \text{if } p \equiv 1 \pmod 3 \ \&\ p = x^2+3y^2\ (x,y\in\mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } p \equiv 2 \pmod 3. \end{cases}$$

Also,

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}(-1)^k D_k\left(\frac{1}{2}\right)^3$$

$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p \equiv 1,7 \pmod{24} \text{ and } p = x^2+6y^2\ (x,y\in\mathbb{Z}), \\ 8x^2-2p \pmod{p^2} & \text{if } p \equiv 5,11 \pmod{24} \text{ and } p = 2x^2+3y^2\ (x,y\in\mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{-6}{p}) = -1. \end{cases}$$

And

$$\sum_{k=0}^{p-1}(-1)^k D_k(-4)^3 \equiv \left(\frac{-5}{p}\right)\sum_{k=0}^{p-1}(-1)^k D_k\left(-\frac{1}{16}\right)^3$$

$$\equiv \begin{cases} 4x^2-2p \pmod{p^2} & \text{if } p \equiv 1,4 \pmod{15} \text{ and } p = x^2+15y^2\ (x,y\in\mathbb{Z}), \\ 12x^2-2p \pmod{p^2} & \text{if } p \equiv 2,8 \pmod{15} \text{ and } p = 3x^2+5y^2\ (x,y\in\mathbb{Z}), \\ 0 \pmod{p^2} & \text{if } (\frac{-15}{p}) = -1. \end{cases}$$

Remark 1.2. Note that $(-1)^n D_n(x) = D_n(-x-1)$ for any $n \in \mathbb{N}$, since

$$\begin{aligned} D_n(-x-1) &= \sum_{k=0}^{n}\binom{n}{k}\binom{-n-1}{k}\sum_{j=0}^{k}\binom{k}{j}x^j \\ &= \sum_{j=0}^{n}\binom{n}{j}x^j\sum_{k=0}^{n}\binom{-n-1}{k}\binom{n-j}{n-k} \\ &= \sum_{j=0}^{n}\binom{n}{j}x^j\binom{-j-1}{n} = (-1)^n D_n(x). \end{aligned}$$

Concerning Schröder numbers we establish the following result.

Theorem 1.2. *Let p be an odd prime and let m be an integer not divisible by p. Then*

$$\sum_{k=1}^{p-1}\frac{S_k}{m^k}\equiv\frac{m^2-6m+1}{2m}\left(1-\left(\frac{m^2-6m+1}{p}\right)\right)\pmod p. \tag{1.11}$$

Example 1.1. Theorem 1.2 in the case $m=6$ gives that

$$\sum_{k=1}^{p-1}\frac{S_k}{6^k}\equiv 0\pmod p\qquad\text{for any prime } p>3. \tag{1.12}$$

For technical reasons, we will prove Theorem 1.2 in the next section and show Theorem 1.1 and Corollary 1.1 in Section 3.

2. Proof of Theorem 1.2

Lemma 2.1. *Let p be an odd prime and let m be any integer not divisible by p. Then*

$$\sum_{k=1}^{p-1}\frac{C_k}{m^k}\equiv\frac{m-4}{2}\left(1-\left(\frac{m(m-4)}{p}\right)\right)\pmod p. \tag{2.1}$$

Proof. This follows from [Su10, Theorem 1.1] in which the author even determined $\sum_{k=1}^{p-1}C_k/m^k \bmod p^2$. However, we will give here a simple proof of (2.1).

For each $k=1,\ldots,p-1$, we clearly have

$$\binom{(p-1)/2}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod p.$$

Note also that

$$C_{p-1}=\frac{1}{2p-1}\prod_{k=1}^{p-1}\frac{p+k}{k}\equiv -1\pmod p.$$

Therefore

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{C_k}{m^k}\equiv&\sum_{0<k<p-1}\binom{(p-1)/2}{k}\frac{1}{k+1}\left(-\frac{4}{m}\right)^k+\frac{C_{p-1}}{m^{p-1}}\\ \equiv&-\frac{m}{4}\times\frac{2}{p+1}\sum_{k=1}^{(p-1)/2}\binom{(p+1)/2}{k+1}\left(-\frac{4}{m}\right)^{k+1}-1\\ \equiv&-\frac{m}{2}\left(\left(1-\frac{4}{m}\right)^{(p+1)/2}-1-\frac{p+1}{2}\left(-\frac{4}{m}\right)\right)-1\\ \equiv&-\frac{m}{2}\left(\frac{m-4}{m}\times\frac{(m(m-4))^{(p-1)/2}}{m^{p-1}}-1+\frac{2}{m}\right)-1\\ \equiv&-\frac{m-4}{2}\left(\frac{m(m-4)}{p}\right)+\frac{m}{2}-2 \pmod p\end{aligned}$$

and hence (2.1) follows. □

Lemma 2.2. *For any odd prime p we have*

$$\sum_{k=1}^{p-1}S_k\equiv 2\left(\frac{-1}{p}\right)-2^p \pmod{p^2}. \tag{2.2}$$

Proof. Recall the known identity (cf. (5.26) of [GKP, p. 169])

$$\sum_{n=0}^{m}\binom{n}{k}=\binom{m+1}{k+1}\quad (k,m\in\mathbb{N}).$$

Then

$$\begin{aligned}\sum_{n=0}^{p-1}S_n=&\sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n+k}{2k}C_k=\sum_{k=0}^{p-1}C_k\sum_{n=k}^{p-1}\binom{n+k}{2k}\\ =&\sum_{k=0}^{p-1}C_k\binom{p+k}{2k+1}=\sum_{k=0}^{p-1}\frac{p}{k!(k+1)!(2k+1)}\prod_{0<j\leqslant k}(p^2-j^2)\\ \equiv&\sum_{k=0}^{p-1}\frac{p(-1)^k(k!)^2}{k!(k+1)!(2k+1)}=p\sum_{k=0}^{p-1}(-1)^k\left(\frac{2}{2k+1}-\frac{1}{k+1}\right) \pmod{p^2}.\end{aligned}$$

Observe that

$$\begin{aligned}2p\sum_{k=0}^{p-1}\frac{(-1)^k}{2k+1}=&p\sum_{k=0}^{p-1}\left(\frac{(-1)^k}{2k+1}+\frac{(-1)^{p-1-k}}{2(p-1-k)+1}\right)\\ =&p\sum_{k=0}^{p-1}(-1)^k\left(\frac{1}{2k+1}+\frac{1}{2p-(2k+1)}\right)\\ \equiv&p(-1)^{(p-1)/2}\left(\frac{1}{p}+\frac{1}{2p-p}\right)=2\left(\frac{-1}{p}\right) \pmod{p^2}.\end{aligned}$$

Also,

$$\begin{aligned}-p\sum_{k=0}^{p-1}\frac{(-1)^k}{k+1}&=p\sum_{k=1}^{p}\frac{(-1)^k}{k}\\ \equiv-\sum_{k=1}^{p-1}\frac{p}{k}\binom{p-1}{k-1}-1&=-\sum_{k=0}^{p-1}\binom{p}{k}=1-2^p \pmod{p^2}.\end{aligned}$$

Combining the above, we obtain

$$\sum_{n=0}^{p-1}S_n\equiv 2\left(\frac{-1}{p}\right)+1-2^p \pmod{p^2}$$

and hence (2.2) holds. □

Proof of Theorem 1.2. In the case $m\equiv 1 \pmod p$, (1.11) reduces to the congruence

$$\sum_{k=1}^{p-1}S_k\equiv -2\left(1-\left(\frac{-1}{p}\right)\right) \pmod p$$

which follows from (2.2) in view of Fermat's little theorem.

Below we assume that $m\not\equiv 1 \pmod p$. Then

$$\sum_{n=1}^{p-1}\frac{1}{m^n}\equiv\sum_{n=1}^{p-1}m^{p-1-n}=\frac{m^{p-1}-1}{m-1}\equiv 0 \pmod p$$

and hence

$$\sum_{n=1}^{p-1}\frac{S_n}{m^n}\equiv\sum_{n=1}^{p-1}\frac{S_n-1}{m^n}=\sum_{n=1}^{p-1}\frac{\sum_{k=1}^{n}\binom{n+k}{2k}C_k}{m^n}=\sum_{k=1}^{p-1}\frac{C_k}{m^k}\sum_{n=k}^{p-1}\frac{\binom{n+k}{2k}}{m^{n-k}} \pmod p.$$

Given $k\in\{1,\ldots,p-1\}$, we have

$$\sum_{n=k}^{p-1}\frac{\binom{n+k}{2k}}{m^{n-k}}=\sum_{r=0}^{p-1-k}\frac{\binom{2k+r}{r}}{m^r}=\sum_{r=0}^{p-1-k}\frac{\binom{-2k-1}{r}}{(-m)^r}\equiv\sum_{r=0}^{p-1-k}\frac{\binom{p-1-2k}{r}}{(-m)^r} \pmod p.$$

If $(p-1)/2<k<p-1$, then

$$C_k=\frac{(2k)!}{k!(k+1)!}\equiv 0 \pmod p.$$

Therefore

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{S_n}{m^n}\equiv&\sum_{k=1}^{(p-1)/2}\frac{C_k}{m^k}\left(1-\frac{1}{m}\right)^{p-1-2k}+\frac{C_{p-1}}{m^{p-1}}\\ \equiv&\sum_{k=1}^{p-1}\frac{C_k}{m^k}\left(\frac{m}{m-1}\right)^{2k}\equiv\sum_{k=1}^{p-1}\frac{C_k}{m_0^k}\pmod p,\end{aligned}$$

where m_0 is an integer with $m_0\equiv(m-1)^2/m \pmod p$. By Lemma 2.1,

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{C_k}{m_0^k}\equiv&\frac{m_0-4}{2}\left(1-\left(\frac{m_0(m_0-4)}{p}\right)\right)\\ =&\frac{mm_0-4m}{2m}\left(1-\left(\frac{mm_0(mm_0-4m)}{p}\right)\right)\\ \equiv&\frac{(m-1)^2-4m}{2m}\left(1-\left(\frac{(m-1)^2-4m}{p}\right)\right)\pmod p.\end{aligned}$$

So (1.11) follows. We are done. □

3. Proofs of Theorem 1.1 and Corollary 1.1

We need some combinatorial identities.

Lemma 3.1. *For any $n\in\mathbb{N}$, we have*

$$\sum_{r=0}^{2n}\frac{(-1)^r\binom{2n}{r}}{2n+1-2r}=\frac{(-16)^n}{(2n+1)\binom{2n}{n}} \tag{3.1}$$

and

$$\sum_{r=0}^{2n}\frac{(-1)^r\binom{2n}{r}}{(2n+1-2r)^2}=\frac{(-16)^n}{(2n+1)^2\binom{2n}{n}}, \tag{3.2}$$

that is,

$$\sum_{k=-n}^{n}\frac{(-1)^k}{(2k+1)^s}\binom{2n}{n-k}=\frac{16^n}{(2n+1)^s\binom{2n}{n}}\quad \text{for } s=1,2. \tag{3.3}$$

Proof. If we denote by a_n the left-hand side of (3.1), then the well-known Zeilberger algorithm (cf. [PWZ]) yields the recursion

$$a_{n+1}=-\frac{8(n+1)}{2n+3}a_n\quad (n=0,1,2,\ldots).$$

So (3.1) can be easily proved by induction. (3.2) is equivalent to [Su11, (2.5)] which was shown by a similar method. Clearly (3.3) is just a combination of (3.1) and (3.2). We are done. □

Proof of Theorem 1.1. Let $s\in\{1,2\}$ and let x be any p-adic integer. We claim that

$$\delta_{s,2}\,\delta_{p,3}+\sum_{n=1}^{p-1}\frac{D_n(x)}{n^s}\equiv\sum_{k=1}^{(p-1)/2}\frac{(-x)^k}{k^s}\pmod p. \tag{3.4}$$

Clearly,

$$\sum_{n=1}^{p-1}\frac{D_n(x)-1}{n^s}=\sum_{n=1}^{p-1}\frac{\sum_{k=1}^n\binom{n+k}{2k}\binom{2k}{k}x^k}{n^s}=\sum_{k=1}^{p-1}\binom{2k}{k}x^k\sum_{n=k}^{p-1}\frac{\binom{n+k}{2k}}{n^s}.$$

Note that $\sum_{n=1}^{p-1}1/n^s\equiv-\delta_{s,2}\,\delta_{p,3}\pmod p$ since

$$\sum_{k=1}^{p-1}\frac{1}{(2k)^s}\equiv\sum_{n=1}^{p-1}\frac{1}{n^s}\pmod p.$$

As $p\mid\binom{2k}{k}$ for $k=(p+1)/2,\ldots,p-1$, and

$$\sum_{n=k}^{p-1}\frac{\binom{n+k}{2k}}{n^s}=\sum_{r=0}^{p-1-k}\frac{\binom{2k+r}{r}}{(k+r)^s}\equiv(-2)^s\sum_{r=0}^{p-1-k}\frac{(-1)^r\binom{p-1-2k}{r}}{(p-2k-2r)^s}\pmod p$$

for $k=1,\ldots,(p-1)/2$, by applying Lemma 3.1 we obtain from the above that

$$\begin{aligned}\delta_{s,2}\,\delta_{p,3}+\sum_{n=1}^{p-1}\frac{D_n(x)}{n^s}\equiv&(-2)^s\sum_{k=1}^{(p-1)/2}\binom{2k}{k}x^k\frac{(-16)^{(p-1)/2-k}}{(p-2k)^s\binom{p-1-2k}{(p-1)/2-k}}\\ \equiv&\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{x^k}{k^s}4^{(p-1)/2-k}\binom{-1/2}{(p-1)/2-k}^{-1}\\ \equiv&\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{x^k}{k^s4^k}\binom{(p-1)/2}{k}^{-1}\\ \equiv&\sum_{k=1}^{(p-1)/2}\binom{2k}{k}\frac{x^k}{k^s4^k}\binom{-1/2}{k}^{-1}=\sum_{k=1}^{(p-1)/2}\frac{(-x)^k}{k^s}\pmod p.\end{aligned}$$

In the case $s=2$ and $x=1$, (3.4) yields the congruence

$$\delta_{p,3}+\sum_{n=1}^{p-1}\frac{D_n}{n^2}\equiv\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}\pmod p.$$

By Lehmer [L, (20)],

$$\sum_{\substack{k=1\\2|k}}^{(p-1)/2}\frac{1}{k^2}\equiv\delta_{p,3}+\left(\frac{-1}{p}\right)E_{p-3}\pmod p$$

and hence

$$\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k^2}=2\sum_{\substack{k=1\\2|k}}^{(p-1)/2}\frac{1}{k^2}-\sum_{k=1}^{(p-1)/2}\frac{1}{k^2}\equiv\delta_{p,3}+2\left(\frac{-1}{p}\right)E_{p-3}\pmod p$$

since $\sum_{k=1}^{(p-1)/2}(1/k^2+1/(p-k)^2)=\sum_{k=1}^{p-1}1/k^2\equiv 0\pmod p$ if $p>3$. So (1.1) follows.

With the help of (3.4) in the case $s=x=1$, we have

$$\begin{aligned}\sum_{n=1}^{p-1}\frac{D_n}{n}\equiv\sum_{k=1}^{(p-1)/2}\frac{(-1)^k}{k}\equiv\frac{1}{2}\sum_{k=1}^{(p-1)/2}\left(\frac{(-1)^k}{k}+\frac{(-1)^{p-k}}{p-k}\right)\\ \equiv-\frac{1}{2}\sum_{k=1}^{p-1}\frac{1}{k}\binom{p-1}{k-1}=-\frac{1}{2p}\sum_{k=1}^{p-1}\binom{p}{k}=-q_p(2)\pmod p.\end{aligned}$$

This proves (1.2).

Now fix a p-adic integer x. Observe that

$$\begin{aligned}p\sum_{k=1}^{(p-1)/2}\frac{(-x)^k}{k}\equiv&-2\sum_{k=1}^{(p-1)/2}\frac{p}{2k}\binom{p-1}{2k-1}(-x)^k\\ =&\sum_{\substack{j=1\\2|j}}^{p}\binom{p}{j}(-1)^{p-j}((\sqrt{-x})^j+(-\sqrt{-x})^j)\\ =&(-1+\sqrt{-x})^p+(-1-\sqrt{-x})^p+2\pmod{p^2}.\end{aligned}$$

Combining this with (3.4) in the case $s=1$ we immediately get (1.3).

The proof of Theorem 1.1 is now complete. □

Remark 3.1. By modifying our proof of (1.2) and using the new identity $\sum_{r=0}^{2n}\binom{2n}{r}/(2n+1-2r)=2^{2n}/(2n+1)$, we can prove the congruence $\sum_{k=1}^{p-1}(-1)^k s_k/k\equiv 4((\frac{2}{p})-1)\pmod p$ for any odd prime p. Combining this with $\sum_{k=1}^{p-1}(-1)^k D_k/k\equiv -4P_{p-(\frac{2}{p})}/p\pmod p$ (an equivalent form of (1.7)) we obtain that $\sum_{k=1}^{p-1}(-1)^k S_k/k\equiv 4(1-(\frac{2}{p})-P_{p-(\frac{2}{p})}/p)\pmod p$.

Proof of Corollary 1.1. Note that $\omega=(-1+\sqrt{-3})/2$ is a primitiv e cubic root of unity. If $p\neq 3$, then

$$(-1+\sqrt{-3})^p+(-1-\sqrt{-3})^p=(2\omega)^p+(2\omega^2)^p=-2^p$$

and hence (1.3) with $x=3$ yields the congruence in (1.4).

Clearly (1.5) follows from (1.3) with $x=-4$.

Since $2^p-4^p+2=(2-2^p)(2^p+1)\equiv 6(1-2^{p-1}) \pmod{p^2}$, (1.3) in the case $x=-9$ yields (1.6).

The companion sequence $\{Q_n\}_{n\geqslant 0}$ of the Pell sequence is defined by $Q_0=Q_1=2$ and $Q_{n+1}=2Q_n+Q_{n-1}$ $(n=1,2,3,\ldots)$. It is well known that

$$Q_n=(1+\sqrt{2})^n+(1-\sqrt{2})^n \quad \text{for all } n\in\mathbb{N}.$$

(1.3) with $x=-2$ yields the congruence

$$\sum_{k=1}^{p-1}\frac{D_k(-2)}{k}\equiv\frac{2-Q_p}{p} \pmod{p}.$$

Since $Q_p-2\equiv 4P_{p-(\frac{2}{p})} \pmod{p^2}$ by the proof of [ST, Corollary 1.3], (1.7) follows immediately.

Recall that the Lucas sequence $\{L_n\}_{n\geqslant 0}$ is given by

$$L_0=2,\ L_1=1,\ \text{and } L_{n+1}=L_n+L_{n-1}\ (n=1,2,3,\ldots).$$

It is well known that

$$L_n=\left(\frac{1+\sqrt{5}}{2}\right)^n+\left(\frac{1-\sqrt{5}}{2}\right)^n \quad \text{for all } n\in\mathbb{N}.$$

Putting $x=-5$ in (1.3) we get

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{D_k(-5)}{k}\equiv&\frac{2-2^pL_p}{p}=\frac{2^p(1-L_p)+2-2^p}{p}\\ \equiv&-\frac{2}{p}(L_p-1)-2q_p(2) \pmod{p}.\end{aligned}$$

It is known that $2(L_p-1)\equiv 5F_{p-(\frac{p}{5})} \pmod{p^2}$ provided $p\neq 5$ (see the proof of [ST, Corollary 1.3]). So (1.8) holds if $p\neq 5$. We are done. $\square$

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