Partitions and Partial Matchings Avoiding Neighbor Patterns

William Y. C. Chen[1], Neil J. Y. Fan[2] and Alina F. Y. Zhao[3]

Center for Combinatorics, LPMC-TJKLC
Nankai University, Tianjin 300071, P. R. China

[1]chen@nankai.edu.cn, [2]fjy@cfc.nankai.edu.cn, [3]zfeiyan@cfc.nankai.edu.cn

Abstract. We obtain the generating functions for partial matchings avoiding neighbor alignments and for partial matchings avoiding neighbor alignments and left nestings. We show that there is a bijection between partial matchings avoiding three neighbor patterns (neighbor alignments, left nestings and right nestings) and set partitions avoiding right nestings via an intermediate structure of integer compositions. Such integer compositions are known to be in one-to-one correspondence with self-modified ascent sequences or $3\bar{1}52\bar{4}$-avoiding permutations, as shown by Bousquet-Mélou, Claesson, Dukes and Kitaev.

Keywords: set partition, partial matching, neighbor alignment, left nesting, right nesting.

AMS Subject Classification: 05A15, 05A19

1 Introduction

This paper is concerned with the enumeration of partial matchings and set partitions that avoid certain neighbor patterns. Recall that a partition π of $[n] = \{1, 2, \ldots, n\}$ can be represented as a diagram with vertices drawn on a horizontal line in increasing order. For a block B of π, we write the elements of B in increasing order. Suppose that $B = \{i_1, i_2, \ldots, i_k\}$. Then we draw an arc from i_1 to i_2, an arc from i_2 to i_3, and so on. Such a diagram is called the linear representation of π. If (i, j) is an arc in the diagram of π, we call i a left-hand endpoint, and call j a right-hand endpoint.

A partial matching is a partition for which each block contains at most two elements. A partial matching is also called a poor partition by Klazar [10], see also [2], and it can be viewed as an involution on a set. A partition for which each block contains exactly two elements is called a perfect matching.

Perfect matchings avoiding certain patterns have been studied in [3, 4, 5, 7, 8, 9, 11, 12, 16]. Bousquet-Mélou, Claesson, Dukes and Kitaev [1] investigated perfect matchings avoiding left nestings and right nestings, and found bijections with other combinatorial objects such as $(2 + 2)$-free posets. Claesson and Linusson [5] established a correspondence between permutations and perfect matchings avoiding left nestings.

A nesting of a partition π is formed by two arcs (i_1, j_1) and (i_2, j_2) in the linear representation such that $i_1 < i_2 < j_2 < j_1$. If we further require that $i_1 + 1 = i_2$, then this nesting is called a left nesting. Similarly, one can define right nestings, as well as left crossings and right crossings. We say that k arcs $(i_1, j_1), (i_2, j_2), \ldots, (i_k, j_k)$ form a k-crossing if

$i_1 < i_2 < \cdots < i_k < j_1 < j_2 < \cdots < j_k$. An alignment of a partition π is formed by two arcs (i_1, j_1) and (i_2, j_2) such that $i_1 < j_1 < i_2 < j_2$.

In this paper, we define a neighbor alignment as an alignment consisting of two arcs (i_1, j_1) and (i_2, j_2) such that $j_1 + 1 = i_2$. The aforementioned patterns with neighbor constraints are called neighbor patterns. Left nestings and right nestings were introduced by Stoimenow [15] in the study of regular linearized chord diagrams, and were further explored in [1, 5, 6]. An illustration of neighbor patterns is given in Figure 1.

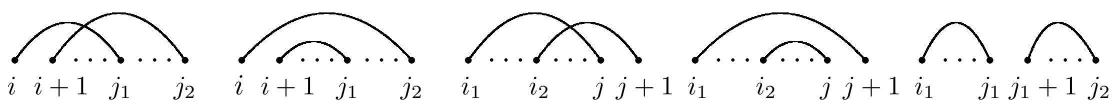

Figure 1: Left crossing, left nesting, right crossing, right nesting and neighbor alignment.

Our main results are the generating functions for three classes of partial matchings avoiding neighbor patterns, which are denoted by $\mathcal{P}(n), \mathcal{Q}(n), \mathcal{R}(n)$, respectively. Denote the set of partial matchings of $[n]$ by $\mathcal{M}(n)$. The set of partial matchings in $\mathcal{M}(n)$ with no neighbor alignments is denoted by $\mathcal{P}(n)$, and the set of partial matchings in $\mathcal{P}(n)$ with k arcs is denoted by $\mathcal{P}(n, k)$. The set of partial matchings in $\mathcal{P}(n)$ with no left nestings is denoted by $\mathcal{Q}(n)$, and the set of partial matchings in $\mathcal{Q}(n)$ with k arcs is denoted by $\mathcal{Q}(n, k)$. Moreover, the set of partial matchings in $\mathcal{Q}(n)$ with no right nestings is denoted by $\mathcal{R}(n)$, and the set of partial matchings in $\mathcal{R}(n)$ with k arcs is denoted by $\mathcal{R}(n, k)$. For $0 \leq k \leq \lfloor n/2 \rfloor$, we set

$$P(n, k) = |\mathcal{P}(n, k)|, \quad Q(n, k) = |\mathcal{Q}(n, k)|, \quad R(n, k) = |\mathcal{R}(n, k)|.$$

Denote the set of partitions of $[n]$ by $\mathcal{S}(n)$ and denote the set of partitions in $\mathcal{S}(n)$ with k blocks by $\mathcal{S}(n, k)$. The set of partitions in $\mathcal{S}(n)$ with no right nestings is denoted by $\mathcal{T}(n)$, and the set of partitions in $\mathcal{T}(n)$ with k arcs is denoted by $\mathcal{T}(n, k)$. For $0 \leq k \leq n - 1$, we set $T(n, k) = |\mathcal{T}(n, k)|$.

We obtain the following generating function formulas for the numbers $P(n, k)$ and $Q(n, k)$.

Theorem 1.1.

$$\sum_{n \geq 1} \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} P(n, k) x^n y^k = \sum_{n \geq 1} \prod_{k=1}^{n} (1 + kxy) x^n. \tag{1}$$

Theorem 1.2.

$$\sum_{n \geq 1} \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} Q(n - 1, k) x^n y^k = \sum_{n \geq 1} \frac{x^n}{\prod_{k=1}^{n} (1 - kx^2 y)}. \tag{2}$$

It is clear that when $y = 1$, the right-hand side of (1) reduces to

$$\sum_{n \geq 1} \prod_{k=1}^{n} (1 + kx) x^n$$

which is the generating function of the sequence $A124380$ in OEIS [13], whose first few entries are

$$1, 2, 4, 9, 22, 57, 157, 453, 1368, 4290, \ldots.$$

It seems that no combinatorial interpretations of this sequence are known. Thus Theorem 1.1 can be considered as a combinatorial interpretation of the above generating function.

Meanwhile, when $y = 1$ the right-hand side of (2) reduces to

$$\sum_{n\geq 1} \frac{x^n}{\prod_{k=1}^{n}(1 - kx^2)}$$

which is the generating function of the sequence $A024428$ in OEIS [13], whose first few entries are

$$1, 1, 2, 4, 8, 18, 42, 102, 260, 684, 1860, \ldots.$$

This sequence can be expressed in terms of Stirling numbers of the second kind. So Theorem 1.2 can be considered as another combinatorial interpretation of the above generating function.

We derive the generating function for the numbers $R(n, k)$ by establishing a connection with compositions of the integer $n-k$ into $\binom{k+1}{2}$ components. Moreover, we show that there is a correspondence between the set $\mathcal{R}(n, k)$ and the set $\mathcal{T}(n-k+1, k)$. Hence by Theorem 1.3 we obtain the generating function for $T(n, k)$ as stated in Theorem 1.4. Furthermore, it turns out that this generating function coincides with the generating function for the number of self-modified ascent sequences of length n with largest element $k-1$ or $3\bar{1}52\bar{4}$-avoiding permutations having k right-to-left minima, as derived by Bousquet-Mélou, Claesson, Dukes and Kitaev [1].

Theorem 1.3.

$$\sum_{n\geq 1}\sum_{k=0}^{n-1} R(n+k-1, k)x^n y^k = \sum_{n\geq 1} \frac{x^n}{(1-xy)^{\binom{n+1}{2}}}. \tag{3}$$

Theorem 1.4.

$$\sum_{n\geq 1}\sum_{k=0}^{n-1} T(n, k)x^n y^k = \sum_{n\geq 1} \frac{x^n}{(1-xy)^{\binom{n+1}{2}}}. \tag{4}$$

This paper is structured as follows. In Section 2, we give a proof of Theorem 1.1 by deriving a recurrence relation of $P(n, k)$. Section 3 gives a proof of Theorem 1.2 by establishing a correspondence between $\mathcal{S}(n-k, n-2k)$ and $\mathcal{Q}(n-1, k)$. In Section 4, we give a bijection between $\mathcal{R}(2n-k-1, n-k)$ and the set of compositions of $n-k$ into $\binom{k+1}{2}$ components, which leads to the generating function in Theorem 1.3. In Section 5 we present a proof of Theorem 1.4 by constructing a correspondence between the set $\mathcal{R}(n, k)$ and the set $\mathcal{T}(n-k+1, k)$.

2 Neighbor alignments

In this section, we give a proof of the generating function formula for the number of partial matchings avoiding neighbor alignments. Recall that a singleton of a partial matching or a set partition is the only element in a block, which corresponds to an isolated vertex in its diagram representation. For a block with at least two elements, the minimum element is called an origin, and the maximum element is called a destination, and an element in between, if any, is called a transient. An origin and a destination are also called an opener and a closer respectively by some authors. We first give a recurrence relation of $P(n,k)$.

Theorem 2.1. *For $n \geq 3$, and $1 \leq k \leq n/2$, we have*

$$P(n,k) = P(n-1,k) + (n-k)P(n-2,k-1), \tag{5}$$

with initial values $P(1,0) = 1, P(2,0) = 1, P(2,1) = 1$.

Proof. It is clear that the number of partial matchings in $\mathcal{P}(n,k)$ such that the element 1 is a singleton equals $P(n-1,k)$. So it suffices to show that the number of partial matchings in which 1 is not a singleton equals $(n-k)P(n-2,k-1)$. For a partial matching $M \in \mathcal{P}(n,k)$ in which 1 is not a singleton, after deleting the arc with left-hand endpoint 1, we are led to a partial matching in $\mathcal{P}(n-2,k-1)$.

Conversely, given a partial matching $M \in \mathcal{P}(n-2,k-1)$ with $n-2$ vertices, in order to get a partial matching with k arcs, we can add an arc into M by placing the left-hand endpoint before the first vertex of M and inserting right-hand endpoint at some position of M. Clearly, there are $n-1$ possible positions to insert the right-hand endpoint of the new arc. By abuse of language, if no confusion arises we do not distinguish a partial matching M from its diagram representation. To ensure that the insertion will not cause any neighbor alignments, we should not allow the right-hand endpoint of the inserted arc to be placed before any origin of M. Since there are $k-1$ arcs in M, thus there are $k-1$ positions that are forbidden. That is to say, we have exactly $(n-1)-(k-1) = n-k$ choices for the position of the right-hand endpoint of the inserted arc. After relabeling, we get a partial matching in $P(n,k)$. This completes the proof. ∎

As an example, let us consider a partial matching $M = \{\{1,4\},\{2\},\{3,5\},\{6\}\} \in \mathcal{P}(6,2)$. The possible positions for inserting an arc are marked by the symbol $*$ in Figure 2.Note that the positions before the vertices 1 and 3 are forbidden. The right-hand endpoint of the inserted arc is between the vertices 5 and 6.

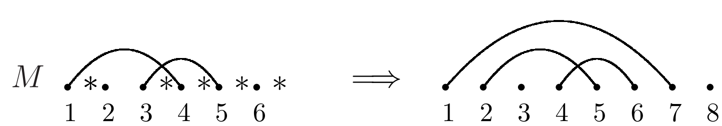

Figure 2: Possible positions for inserting an arc.

Proof of Theorem 1.1. Let $f(n,k)$ denote the coefficient of $x^k y^k$ in the expansion of

$$\prod_{i=1}^{n-k}(1+ixy).$$

It is easily verified that

$$f(n,k) = f(n-1,k) + (n-k)f(n-2,k-1),$$

with initial values

$$f(1,0) = 1, \quad f(2,0) = 1, \quad f(2,1) = 1.$$

Consequently, $P(n,k)$ and $f(n,k)$ have the same recurrence relation and the same initial values, so they are equal. This completes the proof. ∎

To conclude this section, we give a recurrence relation of the generating function of $P(n,k)$. Let

$$f_n(y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} P(n,k) y^k.$$

Corollary 2.2. *For $n \geq 3$, we have*

$$f_n(y) = f_{n-1}(y) + (n-1)y f_{n-2}(y) - y^2 f'_{n-2}(y). \tag{6}$$

3 Neighbor alignments and left nestings

This section is concerned with the generating function for partial matchings avoiding neighbor alignments and left nestings. More precisely, we establish a bijection between set partitions and partial matchings avoiding neighbor alignments and left nestings. As a consequence, we obtain the generating function in Theorem 1.2.

Theorem 3.1. *There exists a bijection between the set $\mathcal{S}(n-k, n-2k)$ and the set $\mathcal{Q}(n-1,k)$. Moreover, this bijection transforms the number of transients of a partition to the number of left crossings of a partial matching.*

Proof. Let $\pi \in \mathcal{S}(n-k, n-2k)$ be a partition of $[n-k]$ with k arcs, we wish to add $k-1$ vertices to π in order to form a partial matching $\alpha(\pi)$ in $\mathcal{Q}(n-1,k)$, that is, a partial matching on $[n-1]$ avoiding neighbor alignments and left nestings. First, we add a vertex before each origin, except for the first origin, and relabel the vertices in the natural order. Let the resulting partition be denoted by σ.

To construct a partial matching in $\mathcal{Q}(n-1,k)$ from the partition σ, we shall use the operation of changing a 2-path to a left crossing, see Figure 3 for an illustration. To be more specific, a 2-path means two arcs (i,j) and (j,k) with $i < j < k$.

It should be emphasized that at each step we have a unique choice of a 2-path to implement the switching operation. More precisely, we always try to find a 2-path consisting of

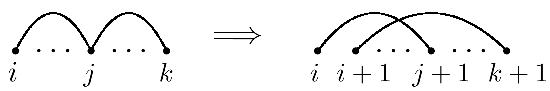

Figure 3: Change a 2-path into a left crossing.

(i,j) and (j,k) such that j is minimal. We add a vertex $i+1$ immediately after the vertex i, transform this 2-path into a left crossing $(i,j+1),(i+1,k+1)$ and relabel those vertices that are greater than i (except for j,k) so that the relabeled vertex set becomes a set of the first consecutive natural numbers. In other words, a vertex is relabeled if it is increased by one. Using this operation to a path corresponding to a block B with $r+1$ elements in a partition π, we get a left r-crossing, which is an r-crossing with consecutive left-hand endpoints. Let $\alpha(\pi)$ denote the resulting partial matching.

We claim that there are no left nestings and neighbor alignments in $\alpha(\pi)$. Recall that after the first step, a possible left nesting in σ consisting of arcs (i,j_1) and $(i+1,j_2)$ with $j_1>j_2$ can occur only when $i+1$ is a transient. Clearly, i is either a transient or an origin. After changing the 2-path for which i is a transient into a left crossing and the 2-path for which $i+1$ is a transient into a left crossing, we see that the left nesting $(i,j_1),(i+1,j_2)$ disappears. This operation is illustrated in Figure 4. A possible neighbor alignment consisting of arcs (i,j) and $(j+1,k)$ in σ can occur only when $j+1$ is a transient. After changing the 2-path for which $j+1$ is a transient into a left crossing, the arc $(j+1,k)$ becomes $(j',k+1)$ with $j'<j$, thus the neighbor alignment disappears. Hence the claim is proved.

It remains to show that there are $n-1$ vertices in $\alpha(\pi)$. Adding a vertex immediately before an origin or transforming a 2-path into a left crossing will result in an increase of the number of vertices by one. Since the left-hand endpoint of an arc is either an origin or a transient, for a partition with k arcs, after these two operations there are a total number of $k-1$ vertices added. This implies that $\alpha(\pi)\in\mathcal{Q}(n-1,k)$.

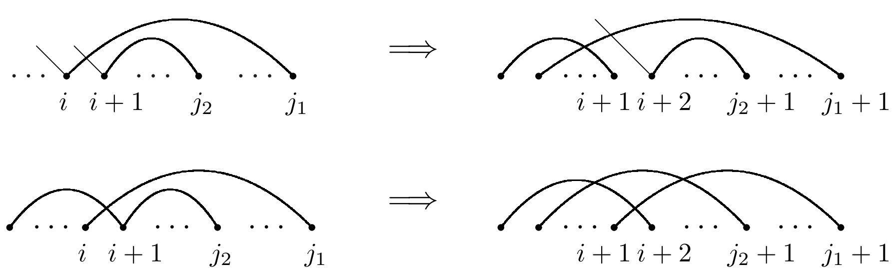

Figure 4: The two cases for the vertex i.

Conversely, given a partial matching in $\mathcal{Q}(n-1,k)$, in order to obtain a partition in $\mathcal{S}(n-k,n-2k)$, we must delete $k-1$ vertices. First, we change left crossings to 2-paths from right to left until there are no more left crossings. More precisely, suppose that the two arcs (i,j_1) and $(i+1,j_2)$ form a left crossing which is the rightmost one in the sense that i is the largest among all the first origins of left crossings. For such a left crossing, delete the vertex $i+1$, and change the two arcs (i,j_1) and $(i+1,j_2)$ to (i,j_1-1) and (j_1-1,j_2-1).

Then, relabel the vertices, that is, reduce the labels of vertices larger than i (except j, k) by 1. After we have eliminated all the left crossings, we further delete the singleton immediately before each origin, if there is any, except for the singleton immediately before the first origin. Finally, we relabel the vertices by the natural order.

Since there are neither neighbor alignments nor left nestings in a partial matching in $\mathcal{Q}(n-1,k)$ with k arcs, for any arc (i,j), we have only two possibilities: (1) The vertex $i-1$ is a singleton. (2) There exists a vertex k such that $(i-1,k)$ and (i,j) form a left crossing. Transforming a left crossing into a 2-path or deleting a vertex immediately before an origin will result in a decrease of the number of vertices by one. Thus after these two operations there are a total number of $k-1$ vertices deleted, and so we are led to a partition of $[n-k]$ with k arcs. It is easily seen that the number of transients of M equals the number of left crossings of $\alpha(M)$. This completes the proof. ∎

For example, let $\pi = \{\{1,5\},\{2,3,4,7\},\{6,8\}\} \in \mathcal{S}(8,3)$. There are 5 arcs in π, so we must add 4 vertices in order to get a partial matching in $\mathcal{Q}(12,5)$. We first add a singleton before the arc $(2,3)$ and a singleton before the arc $(6,8)$. Then change the two 2-paths into left crossings from left to right. An illustration of the above procedure is shown in Figure 5.

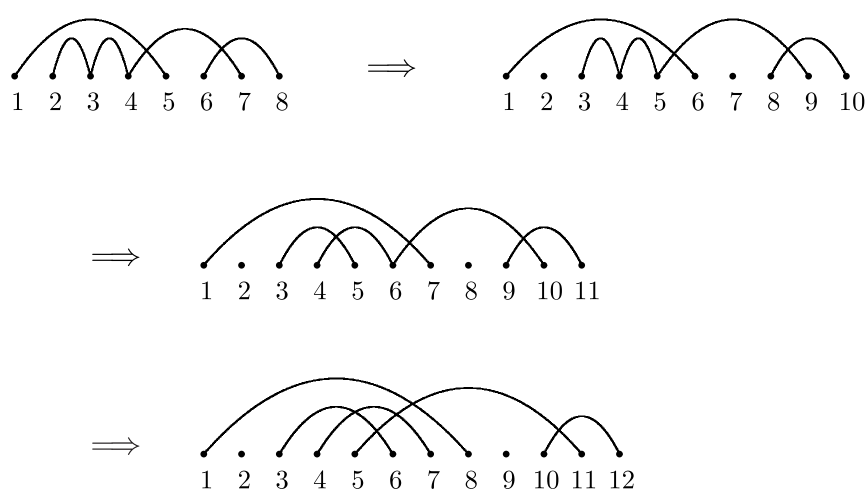

Figure 5: The bijection α.

Let $g_n(y)$ be the generating function for the numbers $Q(n-1,k)$. From Theorem 3.1 we see that

$$g_n(y) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} S(n-k, n-2k) y^k,$$

where $S(n,k)$ are the Stirling numbers of the second kind. It is worth mentioning that the generating function for the numbers $g_n(1)$ has been given in OEIS [13], that is,

$$\sum_{n\geq 1} g_n(1) x^n = \sum_{n\geq 1} \frac{x^n}{\prod_{k=1}^{n}(1-kx^2)}.$$

From the generating function of Stirling numbers of the second kind, it is straightforward to deduce that

$$\sum_{n\geq 1} g_n(y)x^n = \sum_{n\geq 1} \frac{x^n}{\prod_{k=1}^{n}(1-kx^2y)}.$$

Below is the recurrence relation of $g_n(y)$ which follows from the recurrence of $S(n,k)$.

Corollary 3.2. *For $n \geq 3$, $g_n(y)$ has the following recurrence relation*

$$g_n(y) = g_{n-1}(y) + (n-2)y \cdot g_{n-2}(y) - 2y^2 \cdot g'_{n-2}(y).$$

4 Neighbor alignments and left, right nestings

In this section, we obtain the bivariate generating function for the number of partial matchings of $[n+k-1]$ with k arcs that avoid neighbor alignments, left nestings and right nestings. This generating function turns out to be identical to the generating function for the number of self-modified ascent sequences of length n with largest element $k-1$ or $3\bar{1}52\bar{4}$-avoiding permutations of $[n]$ that have k right-to-left minima due to Bousquet-Mélou, Claesson, Dukes and Kitaev [1].

Recall that $\mathcal{R}(n,k)$ denotes the set of partial matchings of $[n]$ with k arcs that avoid neighbor alignments and both left and right nestings. We shall give a bijection between $\mathcal{R}(2n-k-1, n-k)$ and the set of compositions of $n-k$ into $\binom{k+1}{2}$ components, from which we can deduce the generating function for the numbers $R(n+k-1,k)$. Denote the set of compositions of n into k components (possibly empty) by $\mathcal{C}(n,k)$.

Theorem 4.1. *There is a bijection between the set $\mathcal{R}(2n-k-1, n-k)$ and the set $\mathcal{C}(n-k, \binom{k+1}{2})$.*

Proof. Let $M \in \mathcal{R}(2n-k-1, n-k)$ be a partial matching with $2n-k-1$ vertices and $n-k$ arcs containing no left nestings, no right nestings and no neighbor alignments. We aim to construct a composition $\beta(M)$ in $\mathcal{C}(n-k, \binom{k+1}{2})$. Clearly, there are $k-1$ singletons in M. These $k-1$ singletons separate the vertices into k intervals, the first interval is the interval before the first singleton and the $(i+1)$-st interval is the interval between the i-th and $(i+1)$-st singletons, the k-th interval is the interval after the last singleton.

By the following procedure, we can associate the i-th $(1 \leq i \leq k)$ interval with an integer composition, that is, a sequence $s^{(i)}$ of nonnegative integers of length $k-i+1$. For the origins in the i-th interval, their corresponding destinations have $k-i+1$ choices to be in the i-th, $(i+1)$-st,…, and the k-th interval. If there are r $(r \geq 0)$ destinations in the j-th $(i \leq j \leq k)$ interval, then the $(j-i+1)$-st entry of the sequence $s^{(i)}$ is set to be r. Since there are $n-k$ destinations in M, thus putting all these k sequences together, we get a composition $s = (s^{(1)}, s^{(2)}, \ldots, s^{(k)})$ of $n-k$ into $k + (k-1) + \cdots + 1 = \binom{k+1}{2}$ components.

Conversely, given a composition of $n-k$ with $\binom{k+1}{2}$ components, we may break it into sequences of length $k, k-1, \ldots, 1$ respectively, and we denote the i-th sequence by

$$s^{(i)} = (s_i^{(i)}, s_{i+1}^{(i)}, \ldots, s_k^{(i)}),$$

where $1 \leq i \leq k$. Denote the sum of the elements of $s^{(i)}$ by $|s^{(i)}|$. We now proceed to construct the diagram, or the linear representation of a partial matching, based on the sequences $s^{(1)}, s^{(2)}, \ldots, s^{(k)}$. First, we draw $k-1$ singletons on a line to form k intervals such that the first interval is the one before the first singleton, the $(i+1)$-st interval is that between the i-th and $(i+1)$-st singleton, and the k-th interval is the one after the last singleton. Then we need to determine the origins and the destinations in each interval. We put $|s^{(i)}|$ origins and $s_i^{(1)} + s_i^{(2)} + \cdots + s_i^{(i)}$ destinations in the i-th interval, where all the destinations are located after all the origins. So there are $(k-1) + 2(n-k) = 2n-k-1$ vertices. Next, we label the vertices from left to right by using the numbers $1, 2, \ldots, 2n-k-1$.

Finally, we should match the $n-k$ origins and the $n-k$ destinations to form $n-k$ arcs. For i from 1 to k, the right-hand endpoints of the arcs with origins in the i-th interval are determined as follows. As the initial step, for each j ($i \leq j \leq k$), we choose the first $s_j^{(i)}$ available destinations (i.e., the destinations that have not been processed) in the j-th interval. It is easy to check that there are $s_i^{(i)} + s_{i+1}^{(i)} + \cdots + s_k^{(i)} = |s^{(i)}|$ destinations that have been chosen so far. Then we match these $|s^{(i)}|$ destinations with the $|s^{(i)}|$ origins in the i-th interval to form an $|s^{(i)}|$-crossing. This construction ensures that there are neither left nestings nor right nestings. Furthermore, the positions of singletons guarantee that there are no neighbor alignments. Therefore we get a partial matching in $\mathcal{R}(2n-k-1, n-k)$. This implies that the above mapping β is a bijection, and hence the proof is complete. ∎

For example, let

$$M = \{\{1,6\},\{2,7\},\{3\},\{4,8\},\{5,14\},\{9\},\{10,11\},\{12\},\{13,15\}\}$$

which belongs to $\mathcal{R}(15,6)$. The three singletons $3, 9, 12$ separate the vertices into 4 intervals, namely, $\{1,2\}$, $\{4,5,6,7,8\}$, $\{10,11\}$, $\{13,14,15\}$. For the origins $\{1,2\}$ in the first interval, their destinations are both in the second interval, so $s^{(1)} = (0,2,0,0)$. Similarly, we have $s^{(2)} = (1,0,1)$, $s^{(3)} = (1,0)$, $s^{(4)} = (1)$. So the corresponding composition of 6 into $4+3+2+1 = 10$ components is $(0,2,0,0,1,0,1,1,0,1)$.

Conversely, given the composition $(0,2,0,0,1,0,1,1,0,1)$, we split it into 4 sequences, $s^{(1)} = (s_1^{(1)}, s_2^{(1)}, s_3^{(1)}, s_4^{(1)}) = (0,2,0,0)$, $s^{(2)} = (s_2^{(2)}, s_3^{(2)}, s_4^{(2)}) = (1,0,1)$, $s^{(3)} = (s_3^{(3)}, s_4^{(3)}) = (1,0)$, $s^{(4)} = (s_4^{(4)}) = (1)$. The construction of the corresponding partial matching is illustrated in Figure 6.

Proof of Theorem 1.3. Note that the coefficient of x^n in

$$\sum_{n\geq 1} \frac{x^n}{(1-xy)^{\binom{n+1}{2}}}$$

equals

$$\sum_{k=0}^{n} \binom{\binom{k}{2}+n-1}{n-k} y^{n-k},$$

which equals the number of compositions of $n-k$ into $\binom{k+1}{2}$ components. Thus the result follows from Theorem 4.1. ∎

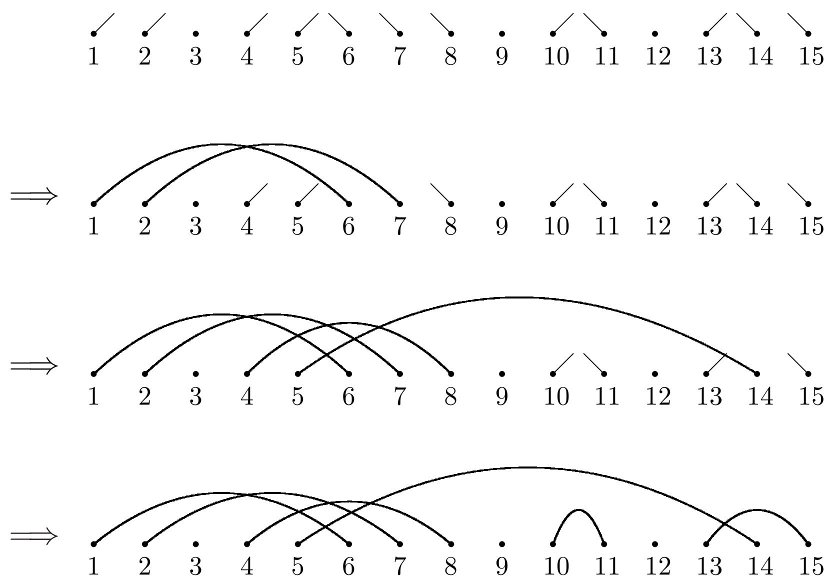

Figure 6: The bijection β.

5 Partitions with no right nestings

The objective of this section is to construct a bijection between the set $\mathcal{T}(n-k+1,k)$ of partitions of $[n-k+1]$ with k arcs but with no right nestings and the set $\mathcal{R}(n,k)$ of partial matchings of $[n]$ with k arcs but with no left nestings, right nestings and neighbor alignments. In fact, we only need to establish a correspondence between a sequence of $n-2k+1$ compositions and the set $\mathcal{T}(n-k+1,k)$. Combining the bijection β in Section 4 from compositions to partial matchings without left, right nestings and neighbor alignments, we obtain the desired bijection between the set $\mathcal{R}(n,k)$ and the set $\mathcal{T}(n-k+1,k)$. Since in the previous section we have computed the generating function for the numbers $R(n,k)$, we are led to the generating function for $T(n,k)$ as stated in Theorem 1.4.

Theorem 5.1. *There exists a bijection between the set $\mathcal{R}(n,k)$ and the set $\mathcal{T}(n-k+1,k)$. Moreover, this bijection transforms the number of left crossings of a partial matching into the number of transients of a partition.*

Proof. Let $M \in \mathcal{R}(n,k)$, that is, M is a partial matching of $[n]$ with k arcs but with no left nestings, right nestings and neighbor alignments. We wish to construct a partition $\gamma(M) \in \mathcal{T}(n-k+1,k)$. The construction consists of two steps. The first step is to associate the partial matching $M \in \mathcal{R}(n,k)$ with a sequence of $n-2k+1$ compositions. The second step is to obtain the desired partition $\gamma(M) \in \mathcal{T}(n-k+1,k)$ from those compositions.

Given a partial matching $M \in \mathcal{R}(n,k)$, intuitively the $n-2k$ singletons of M break the vertices of M into $n-2k+1$ intervals since the vertices are assumed to be arranged in increasing order on the horizontal line. As in the construction of the bijection β between compositions and partial matchings, we associate the i-th interval with a composition $s^{(i)} = (s_i^{(i)}, \ldots, s_{n-2k+1}^{(i)})$, where $s_j^{(i)}(i \leq j \leq n-2k+1)$ is the number of arcs with origins in the i-interval and destinations in the j-th interval.

The procedure to generate the partition $\gamma(M)$ can be described as follows. We start with $n-2k+1$ empty intervals by putting down $n-2k$ singletons on a line. Then we determine the left-hand and right-hand endpoints of every interval so that all the arcs are consequently determined by the endpoints.

To reach this goal, we define a k-path to be a sequence of k arcs of the form $(i_1, i_2), (i_2, i_3), \ldots, (i_k, i_{k+1})$, where $i_1 < i_2 < \cdots < i_{k+1}$. For i from 1 to $n-2k+1$, we construct an $|s^{(i)}|$-path $(i_1, i_2), (i_2, i_3), \ldots, (i_{|s^{(i)}|}, i_{|s^{(i)}|+1})$ via the following steps.

Step 1. We put the origin i_1 of this path immediately before the leftmost right-hand endpoint that has been constructed in the i-th interval. If there are no right-hand endpoints in the i-th interval, just put i_1 before the i-th singleton.

Step 2. According to the composition $s^{(i)}$, we determine the positions of the right-hand endpoints $i_2, \ldots, i_{|s^{(i)}|+1}$ of this path. More precisely, we assign $s_j^{(i)}(i \leq j \leq n-2k+1)$ right-hand endpoints to the j-th $(i \leq j \leq n-2k+1)$ interval.

The $|s^{(i)}|$-path is constructed by inserting the arcs $(i_1, i_2), (i_2, i_3), \ldots, (i_{|s^{(i)}|}, i_{|s^{(i)}|+1})$ one by one. Precisely, by inserting an arc to an interval we mean inserting the right-hand endpoint of this arc to this interval. We claim that the positions of the right-hand endpoints $i_2, \ldots, i_{|s^{(i)}|+1}$ of the $|s^{(i)}|$-path in each interval are uniquely determined subject to the constraint that no right nestings are allowed. To prove this claim, we show that for each arc (i_s, i_{s+1}) $(1 \leq s \leq |s^{(i)}|)$, there is one and only one position to insert the right-hand endpoint i_{s+1}.

Suppose that we wish to insert the arc $e = (i_s, i_{s+1})$ to the j-th interval, where the left-hand endpoint i_s of e is determined already. We proceed to determine the position of i_{s+1}. If there are no right-hand endpoints to the right of i_s in this interval, then we insert i_{s+1} immediately before the j-th singleton. Otherwise, we assume that there are t right-hand endpoints $r_1, r_2, \ldots, r_{t-1}, r_t$ to the right of i_s in the j-th interval. As will be seen, there is a unique position to insert e to the j-th interval such that no right nestings will be formed.

The strategy of inserting e can be easily described as follows. We begin with the position immediately to the left of r_1. If i_{s+1} can be placed in this position without causing any right nestings, then this is the position we are looking for. Otherwise, we consider the position immediately before r_2 as the second candidate.

Like the case for r_1, if putting i_{s+1} immediately before r_2 does not cause any right nestings, then it is the desired choice. Otherwise, we consider the position immediately before r_3 as the third candidate. Then we continue this process until we find a position such that after inserting e creates no right nestings.

To see that the above process will terminate at some point, we assume that i_{s+1} cannot be inserted immediately before r_i, and we assume that inserting i_{s+1} immediately after r_i also yields a right nesting. Then this right nesting caused by the insertion of i_{s+1} immediately after r_i must be formed by the arc e and the arc whose right-hand endpoint is immediately after r_i. This means that there is a right-hand endpoint after r_i. Since the number of right-hand endpoints in every interval is finite, we conclude that there always exits a position such that inserting i_{s+1} does not cause any right nestings.

It is still necessary to show that there is a unique choice for the position of i_{s+1}. Assume that we have found a position immediately before the vertex r_{i_1} such that the insertion of e does not cause any right nestings. It can be shown that all the positions to the right of r_{i_1} cannot be chosen for the insertion of e. Otherwise, suppose that the position immediately after the vertex r_{i_2} is a feasible choice, where $r_{i_1} < r_{i_2}$.

We now proceed to find a right nesting that implies a contradiction. If i_{s+1} can be inserted immediately before r_{i_1}, then the arc e and the arc $e_1 = (l_1, r_{i_1})$ form a crossing, that is, $i_s < l_1$; On the other hand, if i_{s+1} can be inserted immediately after r_{i_2}, then e and $e_2 = (l_2, r_{i_2})$ form a crossing as well, that is, $l_2 < i_s$. This implies that $l_1 > l_2$. So the arcs e_1 and e_2 form a nesting.

In order to find a right nesting, we consider the distance between r_{i_1} and r_{i_2}. If $r_{i_1} + 1 = r_{i_2}$, then the two arcs e_1 and e_2 form a right nesting. If $r_{i_1} + 2 = r_{i_2}$, namely, there is a vertex r_{i_1+1} between r_{i_1} and r_{i_2}, then the arc with right-hand endpoint r_{i_1+1} forms a right nesting with the arc e_1 or e_2. We now consider the case that there are more than one vertices between r_{i_1} and r_{i_2}. Since in every step of the inserting process no right nestings are formed, the left-hand endpoint l_3 of the arc $e_3 = (l_3, r_{i_1+1})$ must be to the right of l_1, and the position of the left-hand endpoint l_4 of the arc $e_4 = (l_4, r_{i_2-1})$ must be to the left of l_2. Thus we deduce that e_3 and e_4 form a nesting as well, and the distance between the right-hand endpoints of e_3 and e_4 has decreased by two compared with the distance between the right-hand endpoints of e_1 and e_2. See Figure 7 for an illustration.

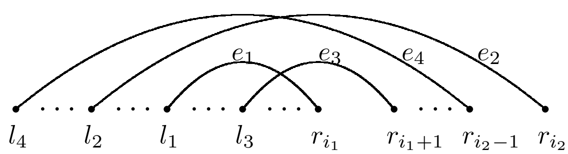

Figure 7: The uniqueness of inserting an arc.

Iterating the above process by checking the distance between the point r_{i_1+1} and the point r_{i_2-1}, we can always find a right nesting. This is a contradiction, which means that there is a unique choice for the insertion of e without causing right nestings.

By the above uniqueness property, we can insert the arcs $(i_1, i_2), (i_2, i_3), \ldots, (i_{|s^{(i)}|}, i_{|s^{(i)}|+1})$ one by one to construct a unique $|s^{(i)}|$-path. After $n - 2k + 1$ steps, we get a partition with no right nestings. Finally, delete every singleton that is immediately to the left of an origin, except for the first origin. Examining the number of points as in the construction of the bijection α in Section 3, we are led to a partition $\gamma(M)$ on $[n-k+1]$ without right nestings.

Conversely, given a partition $\pi \in \mathcal{T}(n+1-k, k)$ with k arcs that has no right nestings, we wish to construct a partial matching in $\mathcal{R}(n, k)$. Clearly, we should add $k - 1$ vertices into π. First of all, we add a vertex before each origin except the first origin. At this point, the number of added vertices is the number of non-singleton blocks of π minus one. Assume that the new partition π' has m singletons which split the vertex set into $m+1$ intervals.

In each interval, there is at most one origin. Assume that the origin in the i-th interval is the origin of an r-path, then we associate the i-th interval with a composition $t^{(i)} = (t^{(i)}_i, \ldots, t^{(i)}_{m+1})$ of the integer r, where $t^{(i)}_j (i \leq j \leq m+1)$ is the number of right-hand endpoints of this r-path in the j-th interval. From these $m+1$ compositions, using the map β in Section 4 from compositions to partial matchings without left, right nestings and neighbor alignments, we obtain a partial matching of $\mathcal{R}(n,k)$ with k arcs. It is easily seen that the number of left crossings of M equals the number of transients of $\gamma(M)$. This completes the proof. ∎

Figure 8 gives an example of a partial matching M without left, right nestings and neighbor alignments. It also illustrates the process to construct a partition $\gamma(M)$ without right nestings. There are two singletons in M which create three intervals. The first interval is associated with the composition $s^{(1)} = (0, 2, 1)$, which is transformed into a 3-path of $\gamma(M)$. The second interval is associated with the composition $s^{(2)} = (2, 2)$, which is transformed into a 4-path of $\gamma(M)$. The third interval is associated with the composition $s^{(3)} = (1)$, which is transformed into a 1-path of $\gamma(M)$.

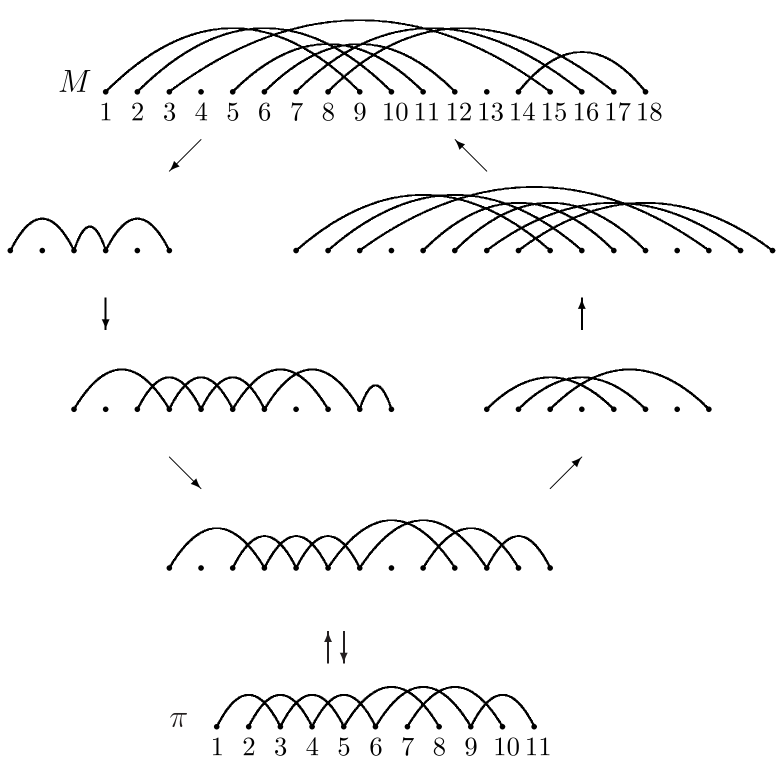

Figure 8: The bijection γ.

To conclude, we remark that in general the number of partitions of $[n]$ avoiding right crossings is not equal to the number of partitions of $[n]$ avoiding right nestings. It would be interesting to find the generating function for the number of partitions of $[n]$ without right crossings.

Acknowledgments. This work was supported by the 973 Project, the PCSIRT Project of the Ministry of Education, and the National Science Foundation of China.

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