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From counting to construction of BPS states in $\mathcal{N} = 4$ SYM

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Abstract

We describe a universal element $\mathbb{P}$ in the group algebra of symmetric groups, whose characters provides the counting of quarter and eighth BPS states at weak coupling in $\mathcal{N} = 4$ SYM, refined according to representations of the global symmetry group. A related projector $\mathcal{P}$ acting on the Hilbert space of the free theory is used to construct the matrix of two-point functions of the states annihilated by the one-loop dilatation operator, at finite N or in the large N limit. The matrix is given simply in terms of Clebsch-Gordan coefficients of symmetric groups and dimensions of $U(N)$ representations. It is expected, by non-renormalization theorems, to contain observables at strong coupling. Using the stringy exclusion principle, we interpret a class of its eigenvalues and eigenvectors in terms of giant gravitons. We also give a formula for the action of the one-loop dilatation operator on the orthogonal basis of the free theory, which is manifestly covariant under the global symmetry.

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1 Introduction

The half-BPS sector of $\mathcal{N} = 4$ $U(N)$ SYM has been a very fruitful area of study in the context of AdS/CFT [1, 2, 3] from a number of different perspectives. It has shown how different classes of half-BPS local operators in the large N theory can be mapped to different types of physical objects in space-time : Kaluza-Klein gravitons, strings, giant gravitons and classical space-time geometries [3, 4, 5, 6, 7, 8, 9, 10]. The study of the two-point function has been central. It defines an inner product on the space of states corresponding to the local operators. The diagonalization of the inner product in the space of half-BPS operators in terms of Young diagrams [7] has provided the tools for identifying the physical objects in different semi-classical limits. For example we expect a state describing two distinct giant gravitons to be orthogonal to a state of a single giant [6, 7].

Recent years have seen progress on the quarter and one-eighth BPS sectors. From the space-time point of view, there is a better understanding of the geometry of giant gravitons and their connections to simple harmonic oscillator systems [11, 12, 13, 14, 15]. From the point of view of gauge-invariant operators, we have a diagonalization at finite N for free Yang Mills i.e with the $g^2_{YM} = 0$ [16, 17, 18, 19]. It is known that there is a jump in the spectrum of BPS operators from zero to weak coupling. It is conjectured that there is no further jump as the coupling is tuned to the strong coupling region [13]. This makes it extremely important to understand the weak coupling sector in complete detail. The lessons we learn from these sectors will be useful for an understanding of the sixteenth BPS sector where we can access black holes with finite horizon areas.

Henceforth when we refer to BPS operators we have in mind quarter BPS operators or bosonic eighth BPS operators (more precisely eighth-BPS operators with bosonic lowest weight states) at weak coupling. The quarter BPS problem involves two matrices X_1, X_2 transforming in the adjoint of $U(N)$ and a $U(2)$ global R-symmetry which rotates these two matrices. The eighth BPS problem involves X_1, X_2, X_3 and a $U(3)$ global R-symmetry which rotates these. In our considerations below, we can keep the global symmetry general in the form $U(M)$. The general $U(M)$ case does not have a direct application in $\mathcal{N} = 4$ SYM, but may have applications elsewhere in theories with M copies of adjoints.

The explicit construction of BPS operators was studied in [20, 21]. A class of quarter BPS operators involving a small number of X_1, X_2 were constructed in the $SU(N)$ theory at weak coupling. In [16, 17], the diagonalization of the two-point function of BPS operators in the free (zero-coupling) $U(N)$ theory at general finite N was accomplished. The solution for the case of a general number n of X_1, X_2 was given in terms of the group theory of S_n. A Fourier basis constructed using Clebsch-Gordan coefficients and matrix elements of S_n was found to accomplish the diagonalization. The one-loop BPS operators can be characterized as being orthogonal, in the free field theory metric, to the descendants [21]. Based on [22] it was conjectured that these BPS operators have non-renormalized two- and three- point functions. The idea of characterizing BPS operators using orthogonality was developed in a finite N set-up, using the zero coupling inner product to give expressions for the BPS operators [16]. The expressions were based on the notion of dual bases of operators [23, 24, 25] and arrived at expressions involving inverse dimensions of $U(N)$ representations, which will play a role later in this paper. A missing ingredient was an explicit description of the descendant operators. Exploiting some developments in BMN operators [26] a characterization of the BPS operators in the $1/N$ expansion was given in [27] using the inner product in the planar limit. For related recent developments on BMN operators, see [28].

In this paper we describe the symmetrization operation on traces as a linear operator, which we call $\mathcal{P}$, acting on the orthogonal Fourier basis for the free theory. This allows us to characterize the descendants as the kernel of $\mathcal{P}$. We use the symmetrization matrix to derive a formula for the two-point function of the BPS states to all orders in the $1/N$ expansion. A manifestly finite N construction is also given. It relies on an infinite dimensional Hilbert space which contains the Hilbert spaces for any N. The geometry of intersections of subspaces in this Hilbert space plays a crucial role.

The paper is organised as follows. We will review the relevant background in Section 2. It will become clear that an explicit description of the linear operator $\mathcal{P}$ implementing the symmetrization will be important. The operation of symmetrizing traces appears in the simpler problem of counting BPS operators. In Section 3 we review the standard counting results and consider a refined counting according to representations Λ of $U(M)$. This is done with a new approach based on a universal element in $\mathbb{C}(S_n)$ related to trace symmetrization, which we denote as $\mathbb{P}$, and whose characters provide this counting for any $U(M)$. We derive a generating function which gives the structure of $\mathbb{P}$. This turns out to have a rich combinatoric structure, which we describe in Appendix E. We show in Section 4 that analysing the *fine structure* of $\mathbb{P}$ leads naturally to the symmetrization operator $\mathcal{P}$. This allows the construction of the BPS operators in Section 4.2 and leads to the matrix of two-point functions in Section 4.4. Section 5 gives a description of the finite N construction. This discussion relies on a careful description of an infinite dimensional Hilbert space $\mathcal{H}$ spanned by states $|\Lambda, M_\Lambda, R, \tau\rangle$ containing the labels of the free orthogonal basis. An inner product which we call the S_∞ inner product plays a key role. The finite N Hilbert space can be realized as a quotient of $\mathcal{H}$ by the image of a finite N projector. The S_∞ inner product induces a simple inner product on the finite N Hilbert space, which is consistent with the finie N cutoffs. Clarifying the interplay between the finite N projector and $\mathcal{P}$, viewed as an operator in $\mathcal{H}$ is the key to the finite N construction. In Section 6, we analyze the properties of the matrix of BPS two-point functions. Using the stringy exclusion principle [29, 31], we interpret some of its eigenvectors in terms of giant gravitons and their spacetime excitations.

We collect miscellaneous technical material in the Appendices. Appendix B also gives a new $U(3)$ covariant formula for the mixing of descendant operators under the action of the one-loop dilatation operator, which has a similar structure in terms of symmetric group Clebsch-Gordans as the symmetrization operator $\mathcal{P}$.

2 Review and Remarks on BPS operators

We review here recent work on the construction of quarter and eighth (bosonic) BPS operators. It is known that the spectrum of BPS operators at zero coupling $\mathcal{N} = 4$ SYM (the free theory) is different from the spectrum at weak coupling. At zero coupling, any gauge invariant operator constructed from holomorphic combinations of the three adjoint complex scalar fields X_1, X_2, X_3 is BPS. At weak coupling only those operators in the kernel of one-loop dilatation operator $\mathcal{H}_2$ are BPS [32]. In [16, 17], the diagonalization of the BPS operators in the free theory at finite N was accomplished. We will begin with a short summary of this work, where a Fourier basis was found to be central to the free field diagonalization. We will follow it with a summary of observations from [21, 16, 27] on the construction of BPS operators at one loop. This will motivate the construction of an operator $\mathcal{P}$ on the Fourier basis, which implements symmetrization of traces. In Section 3 we will obtain new results on the counting of BPS operators, which provide an independent avenue and concrete steps towards the operator $\mathcal{P}$.

2.1 The orthogonal basis in the free theory

In diagonalizing the free two-point function [16, 17], it was extremely useful to exploit the notion of Fourier transformation as applied in the context of the symmetric group. To set the stage, let us recall that the Fourier transform relation between a circle S^1 and the set of momenta $\mathbb{Z}$ can be understood in group theoretic terms as a transformation relating the group $U(1)$ to the space of inequivalent representations $\mathbb{Z}$. The transformation from permutations of n elements to representations is a generalization of this Fourier transformation to the symmetric group S_n.

Gauge invariant multi-matrix operators can be written in the form

$$\mathcal{O}_{\vec{a},\alpha} = \frac{\mathrm{tr}_n(\mathbb{X}_{\vec{a}}\ \alpha)}{N^{n/2}} = \frac{1}{N^{n/2}}(X_{a_1})^{i_1}_{i_{\alpha(1)}} \cdots (X_{a_n})^{i_n}_{i_{\alpha(n)}} \tag{2.1}$$

In the first expression, the operator $\mathbb{X}_{\vec{a}} \equiv X_{a_1} \otimes X_{a_2} \cdots \otimes X_{a_n}$ where the matrix X_{a_k} can be viewed as a linear map from the N-dimensional vector space V to itself, with matrix elements $(X_{a_k})^{i_k}_{j_k}$. The index a transforms in the fundamental M-dimensional representation of $U(M)$. By tr_n we mean a trace of operators in $V^{\otimes n}$, i.e $\mathrm{tr}_n(\mathbb{A}) = \mathbb{A}^{i_1 \cdots i_n}_{i_1 \cdots i_n}$. The permutation α takes integers $1 \cdots n$ to $\alpha(1) \cdots \alpha(n)$. In the context of representation theory, the α label for operators naturally Fourier-transforms using matrix elements $D^R_{ij}(\alpha)$ to a set of labels R, i, j, where R runs over irreducible representations (irreps.) of S_n and i, j run over states in the irrep. The state indices $\vec{a}$ of the $U(M)$ symmetry can be transformed using Schur-Weyl duality into labels Λ, M_Λ, m where Λ labels simultaneously irreps of $U(M)$ and S_n, M_Λ labels a state in the irrep Λ of $U(M)$, and m labels a state in the irrep Λ of S_n. The transformation is achieved by the Clebsch-Gordan coefficients $C^{\vec{a}}_{\Lambda, M_\Lambda, k}$ for the decomposition

$$V^{\otimes n} = \bigoplus_{\Lambda} V^{U(M)}_{\Lambda} \otimes V^{S_n}_{\Lambda} \tag{2.2}$$

This leads us to consider

$$\mathcal{O}_{\Lambda, M_\Lambda, m, R, i, j} = \sum_{\alpha, \vec{a}} D^R_{ij}(\alpha) C^{\vec{a}}_{\Lambda, M_\Lambda, k} \mathcal{O}_{\vec{a},\alpha} \tag{2.3}$$

However the labels $\vec{a}, \alpha$ are a redundant description of the gauge invariant operators.

Imposing the correct invariances leads to operators

$$\mathcal{O}_{\Lambda,M_\Lambda,R,\tau} = \frac{\sqrt{d_R}}{n!}\sum_{\alpha,\vec{a}} S^{\;R\;R\;\Lambda,\;\tau}_{\;i\;j\;m} D^R_{ij}(\alpha) C^{\vec{a}}_{\Lambda,M_\Lambda,m} \mathcal{O}_{\vec{a},\alpha} \tag{2.4}$$

where $S^{\;R\;R\;\Lambda,\;\tau}_{\;i\;j\;m}$ are Clebsch-Gordan coefficients for coupling $R\otimes R\otimes\Lambda$ to the one-dimensional irrep. of S_n. These operators also have the virtue of diagonalizing the free two-point function. We will call this the Fourier basis of the free theory. The inverse map giving the trace basis in terms of the Fourier basis is

$$\mathcal{O}_{\vec{a},\alpha} = \sum_{\Lambda,M_\Lambda,R,\tau} \sqrt{d_R} C^{\Lambda,M_\Lambda,m}_{\vec{a}} D^R_{ij}(\alpha) S^{\;R\;R\;\Lambda;\tau}_{\;i\;j\;m} \mathcal{O}_{\Lambda,M_\Lambda,R,\tau} \tag{2.5}$$

We defined $C^{\Lambda,M_\Lambda,m}_{\vec{a}} = (C^{\vec{a}}_{\Lambda,M_\Lambda,m})^*$ which satisfy standard orthogonality relations for Clebsch-Gordan coefficients [17].

Defining

$$\begin{aligned} \mathcal{S}^{\Lambda,M_\Lambda,R,\tau}_{\vec{a},\alpha} &= \sqrt{d_R} C^{\Lambda,M_\Lambda,m}_{\vec{a}} D^R_{ij}(\alpha) S^{\;R\;R\;\Lambda;\tau}_{\;i\;j\;m} \\ \mathcal{T}^{\vec{a},\alpha}_{\Lambda,M_\Lambda,R,\tau} &= \frac{\sqrt{d_R}}{n!} C^{\vec{a}}_{\Lambda,M_\Lambda,m} D^R_{ij}(\alpha) S^{\;R\;R\;\Lambda;\tau}_{\;i\;j\;m} \end{aligned} \tag{2.6}$$

we may write a more compact version of the transformations (2.4) (2.5) between trace and Fourier basis

$$\begin{aligned} \mathcal{O}_{\vec{a},\alpha} &= \mathcal{S}^{\Lambda,M_\Lambda,R,\tau}_{\vec{a},\alpha} \mathcal{O}_{\Lambda,R,M_\Lambda,\tau} \\ \mathcal{O}_{\Lambda,R,M_\Lambda,\tau} &= \mathcal{T}^{\vec{a},\alpha}_{\Lambda,M_\Lambda,R,\tau} \mathcal{O}_{\vec{a},\alpha} \end{aligned} \tag{2.7}$$

We have

$$\sum_{\vec{a},\alpha} \mathcal{T}^{\vec{a},\alpha}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} \mathcal{S}^{\Lambda_2,R_2,\tau_2,M'_{\Lambda_2}}_{\vec{a},\alpha} = \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M'_{\Lambda_2}}\delta_{R_1,R_2}\delta_{\tau_1,\tau_2} \tag{2.8}$$

and

$$\sum_{\Lambda,M_\Lambda,R,\tau} \mathcal{S}^{\Lambda,M_\Lambda,R,\tau}_{\vec{a},\alpha} \mathcal{T}^{\vec{b},\beta}_{\Lambda,M_\Lambda,R,\tau} = \sum_\gamma \frac{1}{n!}\delta_{\gamma(\vec{a}),\vec{b}}\delta_N(\beta^{-1}\gamma^{-1}\alpha\gamma) \tag{2.9}$$

The δ_N is defined as

$$\delta_N(\alpha) = \sum_{R:c_1(R)\le N} \frac{d_R\chi_R(\alpha)}{n!} \tag{2.10}$$

When $n<N$, this is the sum over all irreps of S_n, which is equal to

$$\begin{aligned} \delta(\alpha) &= 1 \quad \text{for} \quad \alpha = \text{identity element of } S_n \\ &= 0 \quad \text{otherwise} \end{aligned} \tag{2.11}$$

The right hand side of (2.9) is the identity operator in the finite N Hilbert space, which we will denote as $\mathcal{I}^{(N)}$ and return to in Section 5.

The two-point function defines an inner product on the gauge-invariant operators

$$\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = \lim_{x_1 \to \infty} x_1^{2n} \langle (\mathcal{O}_1)^\dagger (x_1) (\mathcal{O}_2)(0) \rangle \tag{2.12}$$

Here and throughout the paper we use the *zero-coupling* but *finite* N two-point function and the associated inner product, unless specified otherwise. The inner product on the trace basis evaluated by Wick contractions is:

$$\begin{aligned} \langle \mathcal{O}_{\vec{b},\beta} | \mathcal{O}_{\vec{a},\alpha} \rangle &= \sum_{\gamma,\sigma \in S_n} \delta_{\gamma(\vec{a}),\vec{b}} N^{C(\sigma)-n} \delta(\beta^{-1}\gamma^{-1}\alpha\gamma\sigma) \\ &= \sum_{\gamma \in S_n} \delta_{\gamma(\vec{a}),\vec{b}} \delta(\beta^{-1}\gamma^{-1}\alpha\gamma\Omega). \end{aligned} \tag{2.13}$$

On the Fourier basis it is diagonal [17]

$$\begin{aligned} \langle \mathcal{O}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} | \mathcal{O}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2} \rangle &= \frac{n!\mathrm{Dim}R_1}{N^n d_{R_1}} \delta_{R_1,R_2}\delta_{\tau_1,\tau_2}\delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M'_{\Lambda_2}} \\ &= \frac{\chi_{R_1}(\Omega)}{d_{R_1}} \delta_{R_1,R_2}\delta_{\tau_1,\tau_2}\delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M'_{\Lambda_2}} \end{aligned} \tag{2.14}$$

There is no d_Λ as in eq. (110) [17] because we defined our Clebsch-Gordans to be the coupling from $R \otimes R \otimes \Lambda$ to identity rather than $R \otimes R \to \Lambda$. In addition because of the $\sqrt{d_R}$ in the definition of the operator of d_R in the denominator rather than d_R^2. We explain the derivation of the result (2.14) in Appendix F.

The Ω factor

$$\Omega = \sum_\sigma N^{C_\sigma - n}\sigma \tag{2.15}$$

is a *central element* in the group algebra $\mathbb{C}(S_n)$, i.e it commutes with any element of S_n. Defining the operator $\mathcal{F}$ on the Fourier basis as

$$\begin{aligned} (\mathcal{F})^{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} &= \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M'_{\Lambda_2}}\delta_{\tau_1,\tau_2}\delta_{R_1,R_2} \frac{n!\mathrm{Dim}R_1}{N^n d_{R_1}} \\ &= \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M'_{\Lambda_2}}\delta_{\tau_1,\tau_2}\delta_{R_1,R_2} \frac{\chi_{R_1}(\Omega)}{d_{R_1}} \end{aligned} \tag{2.16}$$

We see that

$$\langle \mathcal{O}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} | \mathcal{O}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2} \rangle = (\mathcal{F})^{\Lambda_1,M_{\Lambda_1},R_1,\tau_1}_{\Lambda_2,M_{\Lambda_2},R_2,\tau_2} \tag{2.17}$$

The operator $\mathcal{G}$ is defined with inverse matrix elements

$$\begin{aligned}(\mathcal{G})^{\Lambda_1,M_{\Lambda_1},R_1,\tau_1}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2} &= \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M'_{\Lambda_2}}\delta_{\tau_1,\tau_2}\delta_{R_1,R_2}\frac{N^n d_{R_1}}{n!\mathrm{Dim}R_1} \\ &= \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M_{\Lambda_2}}\delta_{\tau_1,\tau_2}\delta_{R_1,R_2}\frac{\chi_{R_1}(\Omega^{-1})}{d_{R_1}}\end{aligned} \tag{2.18}$$

The second line is only meaningful in the region $n \leq N$, where Ω can be inverted. The equation for the inverse dimension in terms of characters $\frac{\chi_{R_1}(\Omega^{-1})}{d_{R_1}}$ is useful in two-dimensional Yang Mills theory (see e.g. equation (6.6) in [33]). In the leading large N (planar) limit, where $n \ll N$, the two-point function approaches the identity

$$\langle \mathcal{O}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}|\mathcal{O}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1}\rangle \to \langle \mathcal{O}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}|\mathcal{O}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1}\rangle_{\mathrm{planar}} = \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M_{\Lambda_2}}\delta_{\tau_1,\tau_2}\delta_{R_1,R_2} \tag{2.19}$$

and we may write

$$\langle \mathcal{O}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}|\mathcal{O}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1}\rangle = \langle \mathcal{O}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}|\mathcal{F}\mathcal{O}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1}\rangle_{\mathrm{planar}} \tag{2.20}$$

Since the large N limit of Ω is 1 the 2-point function (2.13) in the trace basis approaches

$$\langle \mathcal{O}_{\vec{b},\beta}|\mathcal{O}_{\vec{a},\alpha}\rangle_{\mathrm{planar}} = \sum_{\gamma\in S_n}\delta_{\gamma(\vec{a}),\vec{b}}\delta(\beta^{-1}\gamma^{-1}\alpha\gamma). \tag{2.21}$$

This shows that the 2-point function in this limit is zero unless the two operators have the same fields in the same trace structures. This is large N factorization.

2.2 Towards the construction of BPS operators

In this section we review earlier work on quarter and eighth-BPS operators at one-loop [22, 21, 16, 27].

These operators, which have vanishing one-loop anomalous dimensions, are annihilated by the one-loop dilatation operator $\mathcal{H}_2$ [34, 32]:

$$\mathcal{H}_2\mathcal{O}^{\mathrm{BPS}} = 0. \tag{2.22}$$

where $\mathcal{H}_2$ in the $U(3)$ sector is given by

$$\mathcal{H}_2 = -\frac{1}{2}\mathrm{tr}[X_i, X_j][\check{X}_i, \check{X}_j] \tag{2.23}$$

Equivalently, the protected operators are precisely those which are orthogonal, in the inner product defined by the zero-coupling two-point function, to the SUSY-descendant operators $\mathcal{O}^{\mathrm{D}}$:

$$\langle \mathcal{O}^{\mathrm{D}}|\mathcal{O}^{\mathrm{BPS}}\rangle = 0. \tag{2.24}$$

Indeed, from hermiticity of $\mathcal{H}_2$, we have

$$0 = \langle \mathcal{O}^{\text{any}} | \mathcal{H}_2 \mathcal{O}^{\text{BPS}} \rangle = \langle \mathcal{H}_2 \mathcal{O}^{\text{any}} | \mathcal{O}^{\text{BPS}} \rangle = \langle \mathcal{O}^{\text{D}} | \mathcal{O}^{\text{BPS}} \rangle \tag{2.25}$$

It is expected that the 2- and 3-point functions of these operators receive no further corrections at higher loops.

The descendant operators are easily found: they are those which, when written as a gauge invariant product of single traces, contain a commutator of fields [21]:

$$\mathcal{O}^{\text{D}} = \text{tr}\left([X_{a_1}, X_{a_2}] X_{a_3} X_{a_4} \ldots\right) \text{tr}(\ldots) \ldots \tag{2.26}$$

This is true even when the total number of X's denoted by n is in the region $n > N$, in which case, since there is no unique way to write an operator as a product of traces, an operator would be a descendant if there *exists a way* to write it with a commutator. Since we have an explicit set of $\mathcal{O}^{\text{D}}$, the task of finding $\mathcal{O}^{\text{BPS}}$ is just finding the orthogonal subspace according to (2.24). This has already been pointed out and used in [21]. The procedure, however, turns out to be complicated because of the non-planar inner product and does not provide a convenient way to express $\mathcal{O}^{\text{BPS}}$ when n grows large.

In order to make progress [27, 26] it is useful to use (2.20) in order to express the non-planar two-point function as

$$\langle \mathcal{O}^{\text{D}} | \mathcal{O}^{\text{BPS}} \rangle = \langle \mathcal{O}^{\text{D}} | \mathcal{F} \mathcal{O}^{\text{BPS}} \rangle_{\text{planar}}, \tag{2.27}$$

Now pick the space of operators $\mathcal{O}^{\text{S}}$ which are orthogonal to $\mathcal{O}^{\text{D}}$ in the planar inner product

$$\langle \mathcal{O}^{\text{D}} | \mathcal{O}^{\text{S}} \rangle_{\text{planar}} = 0. \tag{2.28}$$

The space of protected operators can be written as

$$\mathcal{O}^{\text{BPS}} = \mathcal{G} \mathcal{O}^{\text{S}}. \tag{2.29}$$

where $\mathcal{G}$ is defined in (2.18) as the inverse of $\mathcal{F}$. It follows simply from (2.27) that such operators will be orthogonal to $\mathcal{O}^{\text{D}}$ in the non-planar two-point function

$$\langle \mathcal{O}^{\text{D}} | \mathcal{G} \mathcal{O}^{\text{S}} \rangle = \langle \mathcal{O}^{\text{D}} | \mathcal{F} \mathcal{G} \mathcal{O}^{\text{S}} \rangle_{\text{planar}} = \langle \mathcal{O}^{\text{D}} | \mathcal{O}^{\text{S}} \rangle_{\text{planar}} = 0 \tag{2.30}$$

and thus they are protected. The action of $\mathcal{F}$, which acts diagonally by multiplication with $\frac{n!\text{Dim}R}{N^n d_R}$ in the Fourier basis is the operation

$$\mathcal{F}\ \text{tr}(\mathbb{X}_{\vec{a}}\ \alpha) = \text{tr}(\mathbb{X}_{\vec{a}}\ \Omega\ \alpha) \tag{2.31}$$

where Ω is the central element in the algebra of $\mathbb{C}(S_n)$ defined in (2.15). Indeed, using (2.5),

$$\begin{aligned}\mathcal{F}\ \mathcal{O}_{\vec{a},\alpha} &= \mathcal{F}\sqrt{d_R}C_{\vec{a}}^{\Lambda,M_\Lambda,m}D_{ij}^R(\alpha)S\ {}^{R\ R\ \Lambda;\tau}_{i\ j\ m}\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}\\ &= \frac{\chi_R(\Omega)}{d_R}\sqrt{d_R}C_{\vec{a}}^{\Lambda,M_\Lambda,m}D_{ij}^R(\alpha)S\ {}^{R\ R\ \Lambda;\tau}_{i\ j\ m}\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}\\ &= \sqrt{d_R}C_{\vec{a}}^{\Lambda,M_\Lambda,m}D_{ij}^R(\Omega\alpha)S\ {}^{R\ R\ \Lambda;\tau}_{i\ j\ m}\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}\\ &= \mathcal{O}_{\vec{a},\Omega\alpha}\end{aligned} \tag{2.32}$$

To go from the second line to the third we use the fact that Ω is a central element, hence proportional to the identity in any irrep. R by Schur's Lemma. Similar steps can be done with the $\mathcal{G}$ matrix. In the large N limit, where $n \leq N$, we can define Ω^{-1} as an element in $\mathbb{C}(S_n)$ and

$$\mathcal{G}\ \mathrm{tr}(\mathbb{X}_{\vec{a}}\ \alpha) = \mathrm{tr}(\mathbb{X}_{\vec{a}}\ \Omega^{-1}\ \alpha) \tag{2.33}$$

We now want to make a final point, which was not fully expressed or exploited in the earlier literature. The space of operators $\mathcal{O}^{\mathrm{S}}$ which are orthogonal to $\mathcal{O}^{\mathrm{D}}$ in the planar inner product is the space spanned by products of symmetrized traces. The *symmetrization* operation replaces each single trace with the sum over all permutation of elements inside the trace

$$\mathrm{symm}\,[\,\mathrm{tr}(X_{a_1}X_{a_2}\dots X_{a_n})\,] \equiv \frac{1}{n!}\sum_{\sigma\in S_n}\mathrm{tr}(X_{a_{\sigma(1)}}X_{a_{\sigma(2)}}\dots X_{a_{\sigma(n)}}) \equiv \mathrm{Str}(X_{a_1}X_{a_2}\dots X_{a_n}) \tag{2.34}$$

The operators $\mathcal{O}^{\mathrm{S}}$ are linear combinations of

$$\mathrm{Str}\,(X_{a_1}X_{a_2}\dots X_{a_k})\,\mathrm{Str}\,(X_{b_1}X_{b_2}\dots X_{b_l})\dots. \tag{2.35}$$

The orthogonality to descendants can be seen, using the property of large N factorization, as follows. An operator $\mathcal{O}^{\mathrm{S}}$, orthogonal to descendants, is a sum of individual multitrace operators. Take one multitrace term of it

$$\mathrm{tr}\,(X_aX_bA)\,B, \tag{2.36}$$

where A is a product of matrices and B is some remaining multitrace operator. Now the whole $\mathcal{O}^{\mathrm{S}}$ has to be orthogonal to a descendant

$$\mathrm{tr}\,([X_a,X_b]A)\,B. \tag{2.37}$$

But in order for that to happen the $\mathcal{O}^{\mathrm{S}}$ must also contain a term

$$\mathrm{tr}\,(X_bX_aA)\,B, \tag{2.38}$$

so that

$$\langle \mathrm{tr}\,((X_aX_b - X_bX_a)A)\,B\,|\,\mathrm{tr}\,((X_aX_b + X_bX_a)A)\,B\rangle_{\mathrm{planar}} = 0. \tag{2.39}$$

Of course, it can be that $\mathrm{tr}\,(X_bX_aA)$ is already the same as $\mathrm{tr}\,(X_bX_aA)$ either because of $a = b$ or because of cyclicity of the trace, but that does not change the fact if one is contained in $\mathcal{O}^{\mathrm{S}}$ then the other must be as well with the same coefficient. Therefore, since every two adjacent matrices in $\mathcal{O}^{\mathrm{S}}$ have to be symmetrized, we conclude that the whole $\mathcal{O}^{\mathrm{S}}$ must be built from fully symmetrized traces as in (2.35) This allows us to conclude that the space of one-loop BPS operators is spanned by

$$\mathcal{O}^{\mathrm{BPS}} = \mathcal{G}\left[\mathrm{Str}\,(X_{a_1}X_{a_2}\ldots X_{a_k})\,\mathrm{Str}\,(X_{b_1}X_{b_2}\ldots X_{b_l})\ldots\right]. \tag{2.40}$$

In this paper we will develop (2.40) and arrive at a description of the BPS operators in the $1/N$ expansion in terms of an explicitly defined matrix $\mathcal{GP}$ in the free Hilbert space of SYM. The matrix $\mathcal{P}$, related to symmetrization, will be described in Section 4.2. Incidentally, the properties of $\mathcal{P}$ will lead to a one-line proof of (2.28) in Section 4.3. We will derive the two-point function on the BPS operators as a matrix $(\mathcal{PGP})$. In Section 5 we will extend these results to finite N, where we will identify a projector $\mathcal{P}_I$, which approaches $\mathcal{P}$ in the large N limit. In the next section, we will derive some new results on the counting of BPS operators, which will lead naturally to the operator $\mathcal{P}$.

3 Counting and the universal element in $\mathbb{C}(S_n)$

3.1 Chiral ring counting

Here we briefly review some known results about counting of BPS operators.

The expected number of bosonic gauge invariant eighth-BPS operators with given R-charges (n_1, n_2, n_3) can be calculated using either the Chiral ring counting arguments [13] or alternatively using the description of the BPS sector as a Hilbert space of N 3D harmonic oscillators [35]. In the chiral ring gauge invariant operators made from X_1, X_2, X_3 are counted modulo equivalences following from

$$[X_1, X_2] = [X_2, X_3] = [X_3, X_1] = 0, \tag{3.1}$$

It is argued that this leads to a counting of traces of diagonal matrices [13] :

$$\mathrm{tr}\left((X_1)^{a_1}(X_2)^{b_1}(X_3)^{c_1}\right)\mathrm{tr}\left((X_1)^{a_2}(X_2)^{b_2}(X_3)^{c_2}\right)\ldots\mathrm{tr}\left((X_1)^{a_k}(X_2)^{b_k}(X_3)^{c_k}\right) \tag{3.2}$$

or as a symmetric polynomial of the N eigenvalues of the three matrices. This counting of states also arises from the collective coordinate quantization of the coordinates of the

BPS moduli space determined by (3.1) upon reduction to $0+1$ dimensions on S^3 [35]. In the limit of large N the counting is given by the partition function $Z(x_1, x_2, x_3)$:

$$Z(x_1,x_2,x_3) = \prod_{i+j+k>0}^{\infty} \frac{1}{1-x_1^i x_2^j x_3^k} = \sum_{n_1,n_2,n_3=0}^{\infty} Z_{n_1 n_2 n_3}\, x_1^{n_1} x_2^{n_2} x_3^{n_3}, \tag{3.3}$$

with numbers $Z_{n_1 n_2 n_3}$ providing the number of operators with charges (n_1, n_2, n_3). This will be valid as long as $n_1 + n_2 + n_3 \leq N$.

For a $U(N)$ gauge group at finite N, the counting is given by the generating function $Z_N(x_1, x_2, x_3)$ which can be expressed as:

$$Z(\nu; x_1,x_2,x_3) = \prod_{i,j,k=0}^{\infty} \frac{1}{1-\nu x_1^i x_2^j x_3^k} = 1 + \sum_{N=1}^{\infty} Z_N(x_1,x_2,x_3)\nu^N \tag{3.4}$$

$$Z_N(x_1,x_2,x_3) = \sum_{n_1,n_2,n_3=0}^{\infty} Z_{N;n_1 n_2 n_3}\, x_1^{n_1} x_2^{n_2} x_3^{n_3} \tag{3.5}$$

In terms of counting this is equivalent to limiting the number of traces k appearing in the operators such as (3.2) to at most N (this is certainly not true in the actual construction of BPS states at finite N as can be easily checked using the examples of Appendix C). This limit arises from the fact that there are only N distinct eigenvalues to build the symmetric polynomials, or, in the 3D harmonic oscillator picture, from the fact that the collection of labels (a_i, b_i, c_i) indicate the excitation numbers of k particles, and we are limited to at most N excited particles.

For the purposes of this paper we would like one extra refinement, that is, to distinguish operators in different representations Λ of the global symmetry group $U(3)$, which rotates the three matrices X_a. We will represent $\Lambda = [\lambda_1, \lambda_2, \lambda_3]$ by a Young diagram with at most three rows of lengths $\lambda_1 \geq \lambda_2 \geq \lambda_3$ of $U(3)$. Operators transforming in the representation Λ contain a total number of matrices equal to the number of boxes in the Young diagram.

$$\lambda_1 + \lambda_2 + \lambda_3 = n_1 + n_2 + n_3 \equiv n \tag{3.6}$$

We denote by $\mathcal{M}_\Lambda$ the multiplicity of Λ. Each representation Λ contributes to the full partition function the *Schur polynomial* in variables x_i, therefore the partition function can be expanded as [36, 27]:

$$Z(x_1,x_2,x_3) = \sum_{\Lambda} \mathcal{M}_\Lambda\, \chi_\Lambda(x_1,x_2,x_3). \tag{3.7}$$

This can be inverted to calculate $\mathcal{M}_\Lambda$ (see for example (A.23) in [37]):

$$\mathcal{M}_{[\lambda_1,\lambda_2,\lambda_3]} = \left[Z(x_1,x_2,x_3) \prod_{i<j} (x_i - x_j) \right]_{x_1^{\lambda_1+2} x_2^{\lambda_2+1} x_3^{\lambda_3}} \tag{3.8}$$

where the square brackets with the subscript is an instruction to take the coefficient of the indicated monomial from the function inside. Equivalently we can define

$$\mathcal{M}(x_1, x_2, x_3) \equiv Z(x_1, x_2, x_3) \left(1 - \frac{x_2}{x_1}\right) \left(1 - \frac{x_3}{x_1}\right) \left(1 - \frac{x_3}{x_2}\right) \tag{3.9}$$

and write

$$\mathcal{M}_{[\lambda_1,\lambda_2,\lambda_3]} = \Big[\mathcal{M}(x_1, x_2, x_3)\Big]_{x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3}}. \tag{3.10}$$

Organizing the operators into representations of $U(3)$ works equally well for finite N. We can define $\mathcal{M}_{N;\Lambda}$ as the multiplicity of the Λ representation in the space of operators at finite N. We can again define

$$\mathcal{M}_N(x_1, x_2, x_3) = Z_N(x_1, x_2, x_3) \left(1 - \frac{x_2}{x_1}\right) \left(1 - \frac{x_3}{x_1}\right) \left(1 - \frac{x_3}{x_2}\right) \tag{3.11}$$

and write for finite N multiplicities $\mathcal{M}_{N;[\lambda_1,\lambda_2,\lambda_3]}$

$$\mathcal{M}_{N;[\lambda_1,\lambda_2,\lambda_3]} = \Big[\mathcal{M}_N(x_1, x_2, x_3)\Big]_{x_1^{\lambda_1} x_2^{\lambda_2} x_3^{\lambda_3}} \tag{3.12}$$

We refer the reader to (D.2) in Appendix D for some examples of $\mathcal{M}_\Lambda$.

We can also refine the counting for specified $U(1)$ charges n_1, n_2, n_3 into those belonging to specified $U(3)$ representations Λ. Let $Z^\Lambda_{n_1 n_2 n_3}$, $Z^\Lambda_{N;n_1 n_2 n_3}$ be these refined multiplicities at large and finite N respectively. They are given by

$$\begin{aligned} Z^\Lambda_{n_1 n_2 n_3} &= \mathcal{M}_\Lambda \; g(\Lambda, [n_1], [n_2], [n_3]) \\ Z^\Lambda_{N;n_1 n_2 n_3} &= \mathcal{M}_{N;\Lambda} \; g(\Lambda, [n_1], [n_2], [n_3]) \end{aligned} \tag{3.13}$$

where $g([n_1], [n_2], [n_3]; \Lambda)$ standard group theoretic numbers, namely the Littlewood-Richardson coefficient for combining three single-row Young diagrams of lengths n_1, n_2, n_3 to give the Young diagram Λ (see for example [37]).

The representation multiplicities $\mathcal{M}_\Lambda$, $\mathcal{M}_{N;\Lambda}$ thus provide a refinement of the counting $Z_{n_1 n_2 n_3}$, $Z_{N;n_1 n_2 n_3}$:

$$\begin{aligned} Z_{n_1,n_2,n_3} &= \sum_\Lambda Z^\Lambda_{n_1 n_2 n_3} \\ Z_{N;n_1 n_2 n_3} &= \sum_\Lambda Z^\Lambda_{N;n_1 n_2 n_3} \end{aligned} \tag{3.14}$$

In the rest of the paper we will focus on these representation multiplicities $\mathcal{M}_\Lambda, \mathcal{M}_{N;\Lambda}$.

3.1.1 Universality from counting formulae

The above arguments can be run for the case of quarter BPS operators with $U(2)$ symmetry. We have the partition functions

$$\begin{aligned} Z(x_1, x_2) &= \prod_{i+j>0}^{\infty} \frac{1}{1 - x_1^i x_2^j} = \sum_{n_1, n_2=0}^{\infty} Z_{n_1 n_2}\, x_1^{n_1} x_2^{n_2} \\ Z_N(x_1, x_2) &= \left[\prod_{i,j=0}^{\infty} \frac{1}{1 - \nu x_1^i x_2^j} \right]_{\nu^N} = \sum_{n_1, n_2=0}^{\infty} Z_{N; n_1 n_2}\, x_1^{n_1} x_2^{n_2} \end{aligned} \tag{3.15}$$

and representation multiplicities

$$\begin{aligned} \mathcal{M}_{[\lambda_1, \lambda_2]} &= \Big[Z(x_1, x_2)(x_1 - x_2) \Big]_{x_1^{\lambda_1+1} x_2^{\lambda_2}} \\ \mathcal{M}_{N;[\lambda_1, \lambda_2]} &= \Big[Z_N(x_1, x_2)(x_1 - x_2) \Big]_{x_1^{\lambda_1+1} x_2^{\lambda_2}} \end{aligned} \tag{3.16}$$

These can be usefully expressed in the form

$$\begin{aligned} \mathcal{M}_{[\lambda_1, \lambda_2]} &= Z_{\lambda_1, \lambda_2} - Z_{\lambda_1+1, \lambda_2-1} \\ \mathcal{M}_{N;[\lambda_1, \lambda_2]} &= Z_{N;\lambda_1, \lambda_2} - Z_{N;\lambda_1+1, \lambda_2-1} \end{aligned} \tag{3.17}$$

An important point is that these expressions in fact follow from the ones for $U(3)$ simply by setting $\lambda_3 = 0$. If we consider generating functions with M variables, as appropriate for $U(M)$ symmetry, for sufficiently large M, they contain all the information about the multiplicities $\mathcal{M}_\Lambda$ for any Young diagrams Λ having n boxes, with $n < M$. Of course when $M < n$, we are only interested in Young diagrams with less than or equal to n rows. All Young diagrams Λ with n boxes correspond to the complete set of representations of S_n, so we may expect structures entirely in S_n which contain information about the multiplicities $\mathcal{M}_\Lambda, \mathcal{M}_{N;\Lambda}$ for all Λ with n boxes which are irreps. of S_n. In the next section, we will show that $\mathcal{M}_\Lambda$ is related to a universal element $\mathbb{P}$ and $\mathcal{M}_{N;\Lambda}$ to a finite N version thereof $\mathbb{P}^{(N)}$.

3.2 Universal element

We will now present an alternative method of counting the BPS operators and multiplicities $\mathcal{M}_\Lambda$ using the symmetric group techniques. We will develop a notion of *universal element* $\mathbb{P}$ in the symmetric group algebra whose characters give the multiplicities. The element is easy to define but the precise coefficients of different permutations in $\mathbb{P}$ (which only depend on the conjugacy class of these permutations) are not easy to infer directly from the definition. We will find that the generating function of BPS states $\mathcal{M}(x_1, x_2, x_3)$ in (3.10) can, after a change of variables, be used to derive a generating function for these coefficients. Analysing the structure of $\mathbb{P}$ will lead to the construction of BPS operators in Section 4.

First consider the space of abstract strings of n operators X_1, X_2, X_3 without putting any traces:

$$X_{a_1} \otimes X_{a_2} \otimes \ldots \otimes X_{a_n}, \tag{3.18}$$

where a_i take values 1 to 3. Most of what we will discuss is true for any global symmetry group $U(M)$ with indices running over 1 to M, in the case at hand we will specialize to $M = 2$ for the quarter BPS sector and $M = 3$ for the eighth-BPS sector. Each "letter" X_a can be thought of as a state in the vector space V_M, which is the fundamental of $U(M)$. The whole string is then a state in the tensor product representation $V_M^{\otimes n}$.

There is the natural action of the symmetric group S_n on these states where the permutation σ shuffles the indices:

$$X_{a_{\sigma(1)}} \otimes X_{a_{\sigma(2)}} \otimes \ldots \otimes X_{a_{\sigma(n)}}. \tag{3.19}$$

By Schur-Weyl duality the tensor product representation can be decomposed into irreducible representations of both $U(3)$ and S_n:

$$V_M^{\otimes n} = \oplus_\Lambda V_\Lambda^{U(M)} \otimes V_\Lambda^{S_n}. \tag{3.20}$$

This immediately tells us that if our operators were defined by ordered strings of X_a as in (3.18) then the multiplicity of the $U(M)$ representation Λ would be just d_Λ – the dimension of $V_\Lambda^{S_n}$.

In our case of interest the operators are instead products of traces like (3.2), where the order of operators inside the trace does not matter. Let us still view them as states in $V_M^{\otimes n}$, but with extra structure, for example:

$$\mathrm{tr}(X_{a_1} X_{a_2}) \mathrm{tr}(X_{a_3} X_{a_4}) \mathrm{tr}(X_{a_5}). \tag{3.21}$$

The groups $U(3)$ and S_n act on the indices just like before. The only difference now between counting states like (3.21) and those in (3.18) is that if we change the ordering of the operators within a trace or if we swap all operators between two traces of the same length it is still considered to be the same state. This restriction can be formulated in terms of the S_n action: in our example we want to consider the states which are invariant under permutations $\sigma = (12)$, $\sigma = (34)$ and $\sigma = (13)(24)$.

Consider a general trace structure which we represent by a partition $p = [1^{p_1}, 2^{p_2}, \ldots]$. This means there are p_i traces of length i, and we have $\sum_i i p_i = n$. We will first count how many states there are with the given p and later sum over it. The set of permutations which leave the state invariant given the trace structure p will form a subgroup of S_n which we denote $G(p)$. It consists of all permutations within traces:

$$G_1(p) = \times_i (S_i)^{p_i}, \tag{3.22}$$

with each $S_i \subset S_n$ understood to be embedded in S_n in such a way to permute its corresponding trace. Regarding S_n as the permutations of integers $1 \cdots n$, the 1-cycles correspond to $S_1^{p_1}$ acting on $1 \cdots p_1$, the 2-cycles correspond to $S_2^{p_2}$ acting on $p_1 + 1 \cdots p_1 + 2p_2$. $G(p)$ also contains permutations which exchange traces:

$$G_2(p) = \times_i S_{p_i}, \tag{3.23}$$

where each S_{p_i} permutes all the elements in the p_i traces of the same length i. The full group which leaves the state invariant is the one generated by $G_1(p)$ and $G_2(p)$, which can be seen as a semi-direct product[3]:

$$G(p) = G_1(p) \ltimes G_2(p). \tag{3.24}$$

For completeness note that:

$$|G_1(p)| = \prod_i (i!)^{p_i}\,, \quad |G_2(p)| = \prod_i (p_i)!\,, \quad |G(p)| = |G_1(p)|\,|G_2(p)|\,. \tag{3.25}$$

Note that we have chosen a specific embedding of $G(p)$. Different permutations α in the conjugacy class p give different embeddings $G(\alpha)$ of the same group in S_n. We can now define a projector which will project onto the subspace of $V_M^{\otimes n}$ invariant under $G(p)$:

$$\begin{aligned} \mathbb{P}_p &= \frac{1}{|G(p)|} \sum_{\sigma \in G(p)} \sigma \\ &= \frac{1}{|G_1(p)||G_2(p)|} \sum_{\sigma \in G_1(p)} \sum_{\tau \in G_2(p)} \sigma\tau. \end{aligned} \tag{3.26}$$

Choosing different α also leads to different $\mathbb{P}_\alpha$, a special case of which is $\mathbb{P}_p:$ we will return to this freedom in the next section. The element $\mathbb{P}_p$ is exactly what we need to account for the multiplicity of distinct operators with the trace structure p: the dimension of the space projected to is just the trace of the projection operator

$$\mathcal{M}_p = \mathrm{tr}_{V_M^{\otimes n}}(\mathbb{P}_p). \tag{3.27}$$

[3] It is a semi-direct product in the sense that G_2 provides an automorphism of G_1 by conjugation in the S_n: $g_2 G_1 g_2^{-1} = G_1$ for $g_2 \in G_2$. Then group multiplication of two elements in $G_1 \ltimes G_2$ can be defined as $(g_1, g_2) \cdot (h_1, h_2) = (g_1(g_2 h_1 g_2^{-1}), g_2 h_2)$ where $g_1, h_1 \in G_1$ and $g_2, h_2 \in G_2$.

What we in fact want to know is how many times a particular $U(M)$ representation Λ appears in this space. For that we also project onto the $V_\Lambda^{U(M)}$ and divide by its size:

$$\mathcal{M}_{\Lambda,p} = \frac{1}{\mathrm{Dim}\Lambda}\mathrm{tr}_{V_M^{\otimes n}}(\mathbb{P}_p\mathbb{P}_\Lambda), \tag{3.28}$$

where $\mathrm{Dim}\Lambda$ is the dimension of $U(M)$ representation Λ. Because of the Schur-Weyl duality (3.20) we know that $\mathbb{P}_\Lambda$ just projects onto the $V_\Lambda^{U(M)} \otimes V_\Lambda^{S_n}$ factor, and so:

$$\begin{aligned}\mathcal{M}_{\Lambda,p} &= \frac{1}{\mathrm{Dim}\Lambda}\mathrm{tr}_{V_\Lambda^{U(M)}\otimes V_\Lambda^{S_n}}(\mathbb{P}_p)\\ &= \frac{1}{\mathrm{Dim}\Lambda}\mathrm{tr}_{V_\Lambda^{U(M)}}(\mathbb{I})\mathrm{tr}_{V_\Lambda^{S_n}}(\mathbb{P}_p)\\ &= \chi_\Lambda(\mathbb{P}_p).\end{aligned} \tag{3.29}$$

Here χ_Λ is the symmetric group character and this is an expression entirely in terms of S_n quantities! The second line follows from the fact that $\mathbb{P}_p$ consists of permutations, therefore, it acts only in the $V_\Lambda^{S_n}$ factor. Then the trace factors into $U(M)$ and S_n parts, with the trace over identity in the $U(M)$ canceling $\mathrm{Dim}\Lambda$. This result relates nicely to the fact we mentioned earlier, that for states living in $V_3^{\otimes n}$, the multiplicity of representation Λ would be just $d_\Lambda = \chi_\Lambda(\mathbb{I})$. Instead here because of the invariance of products of traces under some permutations captured by $\mathbb{P}_p$ we lose some states, which is reflected in counting by taking trace of the projector $\mathbb{P}_p$ instead of identity.

We are just a short step away now from the formula for the total multiplicity $\mathcal{M}_\Lambda$. All we need to do is to sum over all partitions p describing different trace structures. Also note that the precise form of $\mathbb{P}_p$ depends on how the $G(p)$ group is embedded in S_n, i.e. how the traces are ordered. However, the counting does not depend on that, and so we want to make things more uniform by summing over all embeddings. That gives us the final form for what we will call the *universal element* $\mathbb{P}$ in the algebra of the symmetric group $\mathbb{C}(S_n)$:

$$\begin{aligned}\mathbb{P} &= \sum_{p\vdash n}\left(\frac{1}{n!}\sum_{\gamma\in S_n}\gamma\mathbb{P}_p\gamma^{-1}\right)\\ &= \sum_{p\vdash n}\left(\frac{1}{n!\,|G(p)|}\sum_{\gamma\in S_n}\sum_{\sigma\in G(p)}\gamma\sigma\gamma^{-1}\right).\end{aligned} \tag{3.30}$$

Note that $\mathbb{P}$ depends on n but we will leave that dependence implicit.

Now the formula for counting multiplicities is just

$$\boxed{\mathcal{M}_\Lambda = \chi_\Lambda(\mathbb{P}).} \tag{3.31}$$

Just to emphasize, this is exactly the same $\mathcal{M}_\Lambda$ as in (3.8), but now we have expressed it as a character of an element $\mathbb{P}$ of $\mathbb{C}(S_n)$. The first nice thing about this result is that the same $\mathbb{P}$ will give multiplicity for any $U(M)$ representation Λ. When the global symmetry is $U(2)$, we are only interested in $\Lambda = [\lambda_1, \lambda_2]$, i.e in Young diagrams with at most two rows; for $U(3)$ we apply the formula for $\Lambda = [\lambda_1, \lambda_2, \lambda_3]$. Second, the $\mathbb{P}$ itself is an interesting object to study. It is universal in the sense that it is defined uniquely for every S_n and there is no additional data that goes into it. Since $\mathbb{P}$ contains averaging over conjugation by γ, its terms will be sums over conjugacy classes, and we will find that the coefficients weighting these conjugacy classes are integers (after multiplication by $n!$) with interesting properties.

Before we go on let us provide a short example of $\mathbb{P}$ and $\mathcal{M}_\Lambda$'s to get a better feeling for them. Take $n = 4$, states made from four matrices. Using (3.30) we can explicitly calculate:

$$\mathbb{P}(S_4) = \frac{1}{4!}\left(3\Sigma_{[4]} + 3\Sigma_{[3,1]} + 7\Sigma_{[2,2]} + 7\Sigma_{[2,1,1]} + 15\Sigma_{[1,1,1,1]}\right), \tag{3.32}$$

where Σ_p is the sum over all elements in the conjugacy class p:

$$\Sigma_p \equiv \sum_{\sigma\in p} \sigma. \tag{3.33}$$

Now if we go and compute the characters of $\mathbb{P}$ we get:

$$\begin{aligned} \mathcal{M}_{[4]} &= \chi_{[4]}(\mathbb{P}) = 5 \\ \mathcal{M}_{[3,1]} &= \chi_{[3,1]}(\mathbb{P}) = 2 \\ \mathcal{M}_{[2,2]} &= \chi_{[2,2]}(\mathbb{P}) = 2. \end{aligned} \tag{3.34}$$

These are indeed the right multiplicities. Converting to the more familiar numbers of operators for different (n_1, n_2, n_3) according to (3.7) gives $Z_{400} = 5$, $Z_{310} = 7$, $Z_{220} = 9$.

Note that $\chi_\Lambda(\mathbb{P}_p)$ has some vanishing properties which are easy to understand by thinking about symmetries and trace structures. When p is a single cycle, then the character vanishes for any Λ with more than one row. When it has two cycles, the character is zero for any Λ with more than two rows. This reflects the well-known fact that the simplest quarter (eighth) BPS operator has two (three) traces at large N.

We refer the reader to Appendix D for explicit examples of $\mathbb{P}$ at various n.

3.3 Generating function for $\mathbb{P}$

As already mentioned, $\mathbb{P}$ will be an important ingredient in the construction of operators in this paper, so we would like to know more about it than the formula (3.30), from which it is hard to see what the sums evaluate to. First, we find that it can always be written as

$$\mathbb{P}(S_n) = \frac{1}{n!} \sum_{p \vdash n} t_p \Sigma_p, \tag{3.35}$$

where the coefficients t_p are *integers*. These are numbers assigned to each possible partition p, and, therefore, have various combinatorial properties, which might be of interest in themselves. For certain simple sequences of partitions they belong to integer sequences tabulated in [38]. We discuss them further in Appendices D, E and provide more explicit examples there.

Consider an *exponential generating function* $t(\vec{y})$ for t_p in the sense that:

$$t(y_1, y_2, y_3, \ldots) = \sum_p t_{p=[1^{p_1} 2^{p_2} 3^{p_3} \ldots]} \frac{y_1^{p_1}}{p_1!} \frac{y_2^{p_2}}{p_2!} \frac{y_3^{p_3}}{p_3!} \ldots \tag{3.36}$$

so that power of y_i indicates the number of cycles of length i in the partition. We claim that this generating function is in closed form:

$$t(\vec{y}) = \exp\left(\sum_{d=1}^{\infty} \frac{1}{d}\left(e^{d\sum_i y_{di}} - 1\right)\right) = \prod_{d=1}^{\infty} \exp\left(\frac{1}{d}\left(e^{d\sum_i y_{di}} - 1\right)\right) \tag{3.37}$$

or more explicitly

$$t(\vec{y}) = \exp\left(\left(e^{y_1+y_2+y_3+\ldots} - 1\right) + \frac{1}{2}\left(e^{2y_2+2y_4+2y_6+\ldots} - 1\right) + \frac{1}{3}\left(e^{3y_3+3y_6+3y_9+\ldots} - 1\right) + \ldots\right). \tag{3.38}$$

This is as close as we can get to an explicit formula for $\mathbb{P}$.

We will prove that (3.37) generates the coefficients of $\mathbb{P}$ correctly by verifying the counting formula (3.31) with the assumed $t(\vec{y})$ against the known result (3.10). The expression $\chi_\Lambda(\mathbb{P})$ can be evaluated directly from the generating function $t(\vec{y})$ with the help of Frobenius character formula. It states that a symmetric group character can be written as [37]:

$$\chi_\Lambda(p) = \left[\prod_{i<j}(x_i - x_j) \prod_k \left(x_1^k + x_2^k + \ldots + x_n^k\right)^{p_k}\right]_{x_1^{n-1+\lambda_1} x_2^{n-2+\lambda_2} \ldots}, \tag{3.39}$$

where p is a partition. Using this we can express $\chi_\Lambda(\mathbb{P})$ as:

$$\begin{aligned}
\chi_\Lambda(\mathbb{P}) &= \frac{1}{n!}\sum_{p\vdash n} t_p \chi_\Lambda(\Sigma_p) \\
&= \sum_{p\vdash n} \frac{t_p}{|\mathrm{Sym}(p)|}\chi_\Lambda(p) \\
&= \left[\prod_{i<j}(x_i - x_j)\sum_{p\vdash n}\frac{t_p}{|\mathrm{Sym}(p)|}\prod_k \left(x_1^k + x_2^k + \ldots + x_n^k\right)^{p_k}\right]_{x_1^{n-1+\lambda_1}x_2^{n-2+\lambda_2}\ldots}.
\end{aligned} \tag{3.40}$$

In the second line we used the fact that Σ_p contains $n!/|\mathrm{Sym}(p)|$ elements all in conjugacy class p, where $\mathrm{Sym}(p)$ is the automorphism group of permutations in p and

$$|\mathrm{Sym}(p)| = \prod_i (p_i!) i^{p_i}. \tag{3.41}$$

Let us now perform a change of variables

$$y_k = \frac{1}{k}\left(x_1^k + x_2^k + \ldots + x_n^k\right). \tag{3.42}$$

We get

$$\begin{aligned}
\chi_\Lambda(\mathbb{P}) &= \left[\prod_{i<j}(x_i - x_j)\sum_{p\vdash n}\frac{t_p}{|\mathrm{Sym}(p)|}\prod_k k^{p_k}(y_k)^{p_k}\right]_{x_1^{n-1+\lambda_1}x_2^{n-2+\lambda_2}\ldots} \\
&= \left[\prod_{i<j}(x_i - x_j)\sum_{p\vdash n} t_p \prod_k \frac{(y_k)^{p_k}}{p_k!}\right]_{x_1^{n-1+\lambda_1}x_2^{n-2+\lambda_2}\ldots} \\
&= \left[\prod_{i<j}(x_i - x_j)\times t(\vec{y})\right]_{x_1^{n-1+\lambda_1}x_2^{n-2+\lambda_2}\ldots}
\end{aligned} \tag{3.43}$$

with exactly the generating function $t(\vec{y})$, with the understanding that the y_k variables have to be substituted according to (3.42).

We can evaluate the function $t(\vec{y})$ in terms of x_k variables:

$$t(\vec{x}) \equiv t\left(y_k = \frac{1}{k}\sum_{i=1}^{n}(x_i)^k\right) = \exp\left\{\sum_{d=1}^{\infty}\frac{1}{d}\left[\exp\left(d\sum_k \frac{1}{dk}\sum_i (x_i)^{dk}\right) - 1\right]\right\}. \tag{3.44}$$

Let us simplify this using $\sum_{k=1}(x^k)/k = \log(1/(1-x))$ and some other infinite series relationships:

$$
\begin{aligned}
t(\vec{x}) &= \exp\left\{\sum_{d=1}^{\infty}\frac{1}{d}\left[\exp\left(\sum_i \log\frac{1}{1-(x_i)^d}\right)-1\right]\right\} \\
&= \exp\left\{\sum_{d=1}^{\infty}\frac{1}{d}\left[\prod_{i=1}\frac{1}{1-(x_i)^d}-1\right]\right\} \\
&= \exp\left\{\sum_{d=1}^{\infty}\frac{1}{d}\left[\sum_{i+j+k+\ldots>0}(x_1^i x_2^j x_3^k\ldots)^d\right]\right\} \\
&= \exp\left\{\sum_{i+j+k+\ldots>0}\log\left(\frac{1}{1-(x_1^i x_2^j x_3^k\ldots)}\right)\right\} \\
&= \prod_{i+j+k+\ldots>0}\frac{1}{1-(x_1^i x_2^j x_3^k\ldots)}.
\end{aligned}
\tag{3.45}
$$

We find, in fact, that $t(\vec{x})$ in terms of x_i variables is precisely the partition function $Z(x_1, x_2, x_3)$ from (3.3) only with arbitrary number of variables

$$
t(x_1, x_2, x_3; x_{i>3} = 0) = Z(x_1, x_2, x_3). \tag{3.46}
$$

Evaluating (3.43) for the specific case of counting $U(3)$ representations $\Lambda = [\lambda_1, \lambda_2, \lambda_3]$ we then immediately find

$$
\chi_{[\lambda_1,\lambda_2,\lambda_3]}(\mathbb{P}) = \left[Z(x_1, x_2, x_3)\prod_{i<j}(x_i - x_j)\right]_{x_1^{\lambda_1+2}x_2^{\lambda_2+1}x_3^{\lambda_3}}. \tag{3.47}
$$

This is precisely $\mathcal{M}_{[\lambda_1,\lambda_2,\lambda_3]}$ according to (3.8)! That means we have explicitly shown $\chi_\Lambda(\mathbb{P}) = \mathcal{M}_\Lambda$ and, therefore, proved (3.37).

3.4 Finite N counting

We can also introduce a finite-N version of the universal element, denoted as $\mathbb{P}^{(N)}$, and defined as

$$
\mathbb{P}^{(N)} = \sum_{p\vdash n}^{(N)}\left(\frac{1}{n!}\sum_{\gamma\in S_n}\gamma\mathbb{P}_p\gamma^{-1}\right) \tag{3.48}
$$

The superscript on the summation symbol indicates that the summation over partitions $p = [1^{p_1}2^{p_2}\cdots]$ is restricted by $\sum p_i \leq N$. This constraint follows from the description of the chiral ring counting in terms of N particles in a simple harmonic oscillator, which can be described as the counting of products of symmetrized traces with no more than N factors. By repeating the arguments explained in deriving (3.31), we obtain

$$\mathcal{M}_{N;\Lambda} = \chi_\Lambda(\mathbb{P}^{(N)}). \tag{3.49}$$

As in (3.35) there is an expansion in conjugacy classes

$$\mathbb{P}^{(N)} = \frac{1}{n!}\sum_{p\vdash n} t_p \Sigma_p \tag{3.50}$$

Let us introduce conjugate variable ν to N again and define a deformed generating function $t(\nu;\vec{y})$:

$$t(\nu;\vec{y}) = \frac{1}{1-\nu}\exp\left(\sum_{d=1}^{\infty}\frac{\nu^d}{d}\left(e^{d\sum_i y_{di}}-1\right)\right). \tag{3.51}$$

Then the picking off the coefficient of ν^N in $t(\nu;\vec{y})$ will provide a generating function $t_N(\vec{y})$ for the coefficients t_p in $\mathbb{P}^{(N)}$:

$$t(\nu;\vec{y}) = \sum_N \nu^N t_N(\vec{y}). \tag{3.52}$$

This can be proved by noting first that the derivation (3.45) with $t(\nu;\vec{y})$ instead of $t(\vec{y})$ gives $\mathcal{M}(\nu;x_1,x_2,x_3)$. Then we see that

$$\begin{aligned}\chi_\Lambda(\mathbb{P}^{(N)}) &= \left[\chi_\Lambda(\mathbb{P}(\nu))\right]_{\nu^N}\\ &= \left[t(\nu;\vec{y})(x_1-x_2)(x_1-x_3)(x_2-x_3)\right]_{\nu^N x_1^{\lambda_1+2}x_2^{\lambda_2+1}x_3^{\lambda_3}}\\ &= \mathcal{M}_{N;\Lambda},\end{aligned} \tag{3.53}$$

where by $\mathbb{P}(\nu)$ we just mean the object generated by $t(\nu;\vec{y})$. This is again in agreement with Chiral ring counting results.

4 From counting to operators on the Hilbert space

The counting of BPS operators in Section 3 used a universal element $\mathbb{P}\in\mathbb{C}(S_n)$ defined in (3.30). It is a sum of projectors $\mathbb{P}_p$ over conjugacy classes p. Equivalently it can be viewed as a sum over the symmetric group of projectors $\mathbb{P}_\alpha$, where $\alpha\in S_n$ is a permutation specifying a concrete embedding of the cycle structure p. We examine this in Section 4.1 and also derive a related operator $\mathbf{p}$. The operator $\mathbf{p}$ is then used in Section 4.2 to construct an operator $\mathcal{P}$ which acts on the free basis $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ and produces linear combinations of symmetrized traces. With the help of $\mathcal{P}$ we then construct the BPS basis valid at $N \geq n$. We then pause to discuss the properties of $\mathcal{P}$ in Section 4.3 and conclude with the calculation of the two-point function on the BPS states in Section 4.4.

In Section 5 we will also see how the operator $\mathcal{P}$ allows the construction of BPS operators at finite N.

4.1 A p- map from $\mathbb{C}(S_n) \to \mathbb{C}(S_n)$

As we remarked in Section 3 the element $\mathbb{P}_p$ in $\mathbb{C}(S_n)$, which counts the number of $U(M)$ representations Λ among trace structures specified by the cycle structure p, depends on a choice of embedding of the group $G(p)$ in S_n. Different permutations α in the conjugacy class p are associated with different groups $G(\alpha)$. We can define an element

$$\mathbb{P}_\alpha = \frac{1}{|G(\alpha)|} \sum_{\sigma \in G(\alpha)} \sigma \tag{4.1}$$

There is a related linear map $\mathbf{p} : \mathbb{C}(S_n) \to \mathbb{C}(S_n)$

$$\mathbf{p}(\alpha) = \mathbb{P}_\alpha = \sum_\beta \mathbf{p}_{\beta,\alpha} \beta \tag{4.2}$$

A trace $\mathrm{tr}_n(\mathbb{X}_{\vec{a}}\alpha)$ is symmetrized by the action of $\mathbb{P}_\alpha$

$$\mathrm{symm}[\mathrm{tr}_n(\mathbb{X}_{\vec{a}}\alpha)] = \sum_\beta \mathbf{p}_{\beta,\alpha} \mathrm{tr}_n(\mathbb{X}_{\beta(\vec{a})}\alpha) \tag{4.3}$$

We extend the definition to sums of traces by linearity, i.e the symmetrization of a sum of multi-traces is the sum of terms obtained by symmetrization each multi-trace.

The universal element in the previous section (3.30) is a linear combination

$$\begin{aligned}
\mathbb{P} &= \frac{1}{n!} \sum_{p \vdash n} \sum_{\gamma \in S_n} \gamma \mathbb{P}_p \gamma^{-1} \\
&= \sum_{\alpha \in S_n} \frac{|\mathrm{Sym}(\alpha)|}{n!} \mathbb{P}_\alpha \\
&= \sum_{\alpha,\beta} \frac{|\mathrm{Sym}(\alpha)|}{n!} \mathbf{p}_{\beta,\alpha} \beta
\end{aligned} \tag{4.4}$$

We can define the coefficients in the Fourier transformed basis for $\mathbb{C}(S_n)$.

$$\mathbf{p}^{Rij}_{Skl} \equiv \sum_{\alpha,\beta} D^S_{kl}(\beta) \mathbf{p}_{\beta,\alpha} D^R_{ij}(\alpha) \tag{4.5}$$

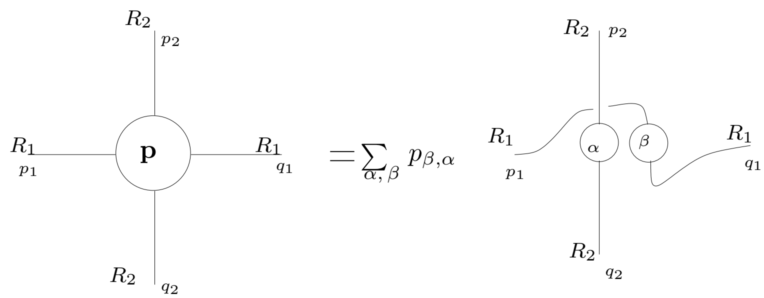

Figure 1: Diagram for $\mathbf{p}$

Then

$$\mathbf{p}_{\beta,\alpha} = \sum_{R,S,i,j,k,l} \frac{d_R}{n!}\frac{d_S}{n!}\ D^S_{kl}(\beta)\ \mathbf{p}_{S\ k\ l}^{R\ i\ j}\ D^R_{ij}(\alpha) \tag{4.6}$$

$$\mathbf{p}(\alpha) = \sum_{R,S,i,j,k,l} \frac{d_R}{n!}\frac{d_S}{n!}\mathbf{p}^{Rij}_{Skl}\ D^R_{ij}(\alpha)\ D^S_{kl}(\beta)\ \beta \tag{4.7}$$

Figure 3 gives a diagrammatic expression of the equation (4.5). On the right hand side we use the standard diagrammatic form for operators. On the left we we have short-hand which we will use subsequently.

4.2 The $\mathcal{P}$ operator and the BPS basis

In this section we proceed with the explicit construction of the BPS operators, annihilated by the one loop dilatation operator $\mathcal{H}_2$. As explained in Section 2.2, in order to do that, we have to consider operators of the form (2.40). The construction involves two tasks: symmetrization and the application of $\mathcal{G}$. A priori it is not clear how to accomplish both in a way to arrive at a closed form expression. The reason is that if we start off with a naive basis of multitrace operators, the symmetrization is easy to implement, but the action of $\mathcal{G}$ (2.33) is complicated, involving the inverse Ω^{-1} which is not well defined for $n > N$. On the other hand if we start with the diagonal basis of the free theory $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$, the $\mathcal{G}$ action is straightforward (2.18), but it is not clear how to impose the symmetrization. Here we will show how the $\mathbf{p}_{\beta,\alpha}$ defined in the previous section allows us to do just that. The procedure described in this section is illustrated in the Appendix C with concrete examples of $\Lambda = [2,2]$ and $\Lambda = [3,2]$.

We define the following basis for the BPS operators

$$\mathcal{O}^{\rm BPS}_{\Lambda,M_\Lambda,R,\tau} = \mathcal{G}\ \mathrm{symm}\left[\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}\right]. \tag{4.8}$$

In order to evaluate this expression, the key step is to calculate how the symmetrization acts on $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$. We use (4.3) and the fact that it acts linearly:

$$\begin{aligned}
\text{symm}\left[\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}\right] &= \frac{\sqrt{d_R}}{n!}\sum_{\alpha,\vec{a}} S^{R\ R\ \Lambda,\ \tau}_{i\ j\ m} D^R_{ij}(\alpha) C^{\vec{a}}_{\Lambda,M_\Lambda,m}\text{symm}\left[\text{tr}(\mathbb{X}_{\vec{a}}\ \alpha)\right] \\
&= \frac{\sqrt{d_R}}{n!}\sum_{\alpha,\beta,\vec{a}} S^{R\ R\ \Lambda,\ \tau}_{i\ j\ m} D^R_{ij}(\alpha) C^{\vec{a}}_{\Lambda,M_\Lambda,m}\mathbf{p}_{\beta,\alpha}\text{tr}(\mathbb{X}_{\beta(\vec{a})}\ \alpha) \\
&= \frac{\sqrt{d_R}}{n!}\sum_{\alpha,\beta,\vec{a}} S^{R\ R\ \Lambda,\ \tau}_{i\ j\ m} D^R_{ij}(\alpha) D^\Lambda_{mm'}(\beta) C^{\vec{a}}_{\Lambda,M_\Lambda,m'}\mathbf{p}_{\beta,\alpha}\text{tr}(\mathbb{X}_{\vec{a}}\ \alpha)
\end{aligned} \tag{4.9}$$

Now re-express the trace in terms of the Fourier basis, use orthogonality of the $U(M)\times S_n$ Clebsch, the fusion property of the S_n Clebsch :

$$\begin{aligned}
\text{symm}\left[\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}\right] &= \frac{\sqrt{d_R}}{n!}\sum_{\alpha,\beta,\vec{a}} S^{R\ R\ \Lambda,\ \tau}_{i\ j\ m} D^R_{ij}(\alpha) D^\Lambda_{mm'}(\beta) C^{\vec{a}}_{\Lambda,M_\Lambda,m'}\mathbf{p}_{\beta,\alpha} \\
&\quad\times \sum_{\Lambda_1,\tilde{M}_\Lambda,R_1,\tau_1} \sqrt{d_{R_1}} C^{\Lambda_1,\tilde{M}_{\Lambda_1},m_1}_{\vec{a}} D^{R_1}_{ab}(\alpha) S^{\ R_1\ \ R_1\ \ \Lambda_1;\tau_1}_{\ a\ \ \ b\ \ \ m_1} \mathcal{O}_{\Lambda_1,\tilde{M}_\Lambda,R_1,\tau_1} \\
&= \sum_{R_1,\tau_1}\frac{\sqrt{d_R d_{R_1}}}{n!}\sum_{\alpha,\beta,S,\tau_S} S^{R\ R\ \Lambda,\ \tau}_{i\ j\ m} D^R_{ij}(\alpha) D^\Lambda_{mm_1}(\beta) D^{R_1}_{ab}(\alpha)\mathbf{p}_{\beta,\alpha} S^{\ R_1\ \ R_1\ \ \Lambda;\tau_1}_{\ a\ \ \ b\ \ \ m_1}\mathcal{O}_{\Lambda,M_\Lambda,R_1,\tau_1} \\
&= \sum_{R_1,\tau_1}\frac{\sqrt{d_R d_{R_1}}}{n!}\sum_{\alpha,\beta,S,\tau_S} d_S S^{R\ R\ \Lambda,\ \tau}_{i\ j\ m} S^{R\ S\ R_1,\tau_S}_{i\ k\ a} S^{R\ S\ R_1,\tau_S}_{j\ l\ b} D^S_{kl}(\alpha) D^\Lambda_{mm_1}(\beta)\mathbf{p}_{\beta,\alpha} \\
&\quad\times S^{\ R_1\ \ R_1\ \ \Lambda;\tau_1}_{\ a\ \ \ b\ \ \ m_1}\mathcal{O}_{\Lambda,M_\Lambda,R_1,\tau_1} \\
&= \sum_{S,\tau_S,R_1,\tau_1}\frac{\sqrt{d_R d_{R_1}}d_S}{n!}\mathbf{p}^{S\ k\ l}_{\Lambda\, mm_1} S^{R\ R\ \Lambda,\ \tau}_{i\ j\ m} S^{R\ S\ R_1,\tau_S}_{i\ k\ a} S^{R\ S\ R_1,\tau_S}_{j\ l\ b} S^{R_1\ R_1\ \Lambda,\tau_1}_{a\ b\ m_1}\mathcal{O}_{\Lambda,M_\Lambda,R_1,\tau_1}.
\end{aligned} \tag{4.10}$$

In the final line we recognized the sum over α,β as a Fourier transform of the $\mathbf{p}_{\beta,\alpha}$

This allows us to define a matrix $\mathcal{P}$:

$$\begin{aligned}
&(\mathcal{P})^{\Lambda_2,M_{\Lambda_2},R_2,\tau_2}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} \\
&= \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M_{\Lambda_2}}\sum_{S,\tau}\sum_{m_\Lambda,m'_\Lambda}\frac{d_S\sqrt{d_{R_1}d_{R_2}}}{n!} S^{R_1\ \ S\ \ R_2;\ \tau}_{i_1\ k_1\ j_1}\mathbf{p}^{S\ \ k_1\ \ k_2}_{\Lambda\ m_\Lambda\ m'_\Lambda} S^{R_1\ \ S\ \ R_2;\ \tau}_{i_2\ k_2\ j_2} S^{R_1\ \ R_1\ \ \Lambda;\ \tau_1}_{i_1\ \ i_2\ \ m_\Lambda} S^{R_2\ \ R_2\ \ \Lambda;\ \tau_2}_{j_1\ \ j_2\ \ m'_\Lambda}
\end{aligned} \tag{4.11}$$

and write the symmetrization as

$$\mathcal{O}^{\rm S}_{\Lambda,M_\Lambda,R,\tau} \equiv \text{symm}\left[\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}\right] = \sum_{R_1,\tau_1}(\mathcal{P})^{\Lambda,M_\Lambda,R_1,\tau_1}_{\Lambda,M_\Lambda,R,\tau}\mathcal{O}_{\Lambda,M_\Lambda,R_1,\tau_1}. \tag{4.12}$$

We have thus been able to express the symmetrization of individual traces as the transformation of the free orthogonal basis by a matrix $\mathcal{P}$.

Let us make a remark about the notation. Since $\mathcal{P}$ is diagonal in the M_Λ label and also independent of it, we will often use a shorthand matrix $(\mathcal{P})^{\Lambda_2,R_2,\tau_2}_{\Lambda_1,R_1,\tau_1}$ understood as $\mathcal{P}$ without the $\delta_{M_{\Lambda_1},M_{\Lambda_2}}$ factor:

$$(\mathcal{P})^{\Lambda_2,M_{\Lambda_2},R_2,\tau_2}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} \equiv \delta_{M_{\Lambda_1},M_{\Lambda_2}}(\mathcal{P})^{\Lambda_2,R_2,\tau_2}_{\Lambda_1,R_1,\tau_1}. \tag{4.13}$$

On the other hand, even though $\mathcal{P}$ is diagonal Λ, it does depend on it, so we will keep the Λ label. The same applies to the matrices $\mathcal{F}$ and $\mathcal{G}$. Note also that whenever a repeated Λ or M_Λ index appears, the summation convention *does not* apply.

Evaluation of (4.8) is now straightforward. Using (4.12) the BPS operators can be written as

$$\mathcal{O}^{\mathrm{BPS}}_{\Lambda,M_\Lambda,R,\tau} = \sum_{R_1,\tau_1}(\mathcal{G}\mathcal{P})^{\Lambda,R_1,\tau_1}_{\Lambda,R,\tau}\mathcal{O}_{\Lambda,M_\Lambda,R_1,\tau_1} \tag{4.14}$$

or more explicitly

$$\mathcal{O}^{\mathrm{BPS}}_{\Lambda,M_\Lambda,R,\tau} = \sum_{R_1,\tau_1}(\mathcal{P})^{\Lambda,R_1,\tau_1}_{\Lambda,R,\tau}\frac{N^n d_{R_1}}{n!\mathrm{Dim}R_1}\mathcal{O}_{\Lambda,M_\Lambda,R_1,\tau_1} \tag{4.15}$$

The equation (4.14) is a key equation for BPS operators and we will devote the rest of the paper to analyzing these BPS operators in detail. We have written an expression for the space of operators annihilated by the one-loop dilatation operator

$$\mathcal{H}_2\mathcal{O}^{\mathrm{BPS}}_{\Lambda,M_\Lambda,R,\tau} = 0 \tag{4.16}$$

in terms of known group theory quantities. They are nicely packaged in $\mathcal{P}$ (4.11) involving Clebsch-Gordan coefficients and $\mathbf{p}_{\beta,\alpha}$, and in the Fourier basis of operators (2.4) for the free theory. To be more precise, the above space of operators provides an overcomplete basis. The labels of $\{\Lambda, M_\Lambda, R, \tau\}$ uniquely label the free basis, but we know that the kernel of $\mathcal{H}_2$ has lower dimensionality. A linearly independent basis can be found by analysing the inner product on the the BPS operators. We will give an expression for the inner product in Section 4.4 and discuss its eigenvalues and eigenvectors in the context of the dual space-time physics in Section 6.

The expressions (4.14), (4.15) should be understood as valid for $n \leq N$. They already contains hints on the problems we might encounter at $n > N$. The $\mathrm{Dim}R$ factor in the denominator will be 0 whenever $c_1(R) > N$ making the basis ill-defined. This is related to the fact that Ω^{-1} appearing in (2.33) is not well defined as $n > N$. We will solve the finite N problem in Section 5.

$$\mathcal{P}^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2} = \sum_{R,\tau} \frac{d_R\sqrt{d_{R_1}d_{R_2}}}{n!}$$

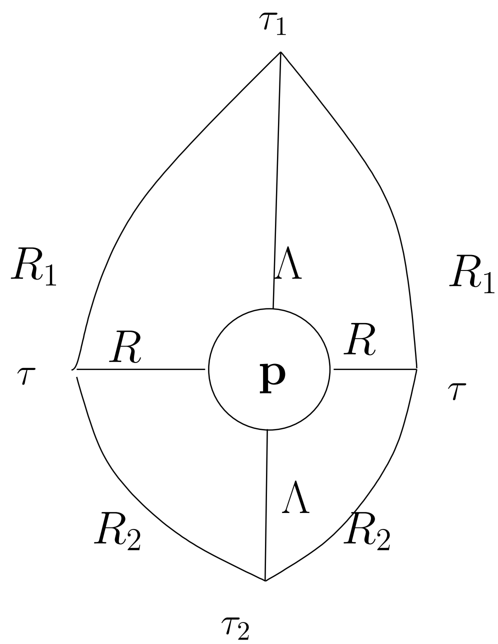

Figure 2: Definition of $\mathcal{P}$

4.3 Properties of $\mathcal{P}$

The matrix $\mathcal{P}$ will be a crucial object throughout the rest of the paper and so we pause here to note a few of its properties. In addition to this section we provide further analysis of $\mathcal{P}$ in Appendix A.

- The operator $\mathcal{P}$ can be seen essentially as the operator $\mathbf{p}_{\beta,\alpha}$ transformed appropriately into the basis of free operators labeled by $|\Lambda, M_\Lambda, R, \tau\rangle$. As is apparent from the definition (4.11), the transformation is done by applying four Clebsch-Gordan coefficients. We can also visualize this transformation by expressing it diagrammatically (see Figure 2). We can also write the matrix $\mathcal{P}$ as a sum over conjugacy classes $[\alpha]$ in $\mathbb{C}(S_n)$, which replaces the sum over R in (4.12). See Appendix A.

- The matrix $(\mathcal{P})^{\Lambda_1,M_{\Lambda_1},R_1,\tau_1}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}$ is proportional to $\delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M'_{\Lambda_2}}$ and is symmetric under exchange of R_1, τ_1 with R_2, τ_2.

$$(\mathcal{P})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2} = (\mathcal{P})^{\Lambda,R_2,\tau_2}_{\Lambda,R_1,\tau_1} \tag{4.17}$$

This means that it has the hermiticity property

$$\mathcal{P}^\dagger = \mathcal{P}$$

under any inner product where the $(\Lambda, M_\Lambda, R, \tau)$ label an orthonormal basis. The free field inner product does not have this property, except in the planar limit ($n \ll N$) where it simplifies to (2.19). In the finite N construction of Section 5 we will define an inner product which has this property for any n.

- Since $\mathcal{P}$ is constructed as the operator on the free field Hilbert space which projects to symmetrized traces, we expect it satisfies the projector equation

$$\mathcal{P}^2 = \mathcal{P}$$

We prove this equation directly in the Appendix A.

- The hermiticity of $\mathcal{P}$ in the planar inner product gives an elegant way to see that the symmetrised traces are orthogonal to the descendants in the planar limit - a property we first proved in Section 2.2. States in $\mathrm{Ker}(\mathcal{P})$ are descendants. States in $\mathrm{Im}(\mathcal{P})$ are symmetrised traces. The hermiticity of the projector $\mathcal{P}$ ensures that $\mathrm{Im}(\mathcal{P})$ is orthogonal to $\mathrm{Ker}(\mathcal{P})$. Explicitly

$$0 = \langle \mathcal{P}\mathcal{O}^{\mathrm{D}} | \mathcal{O}^{\mathrm{S}} \rangle_{\mathrm{planar}} = \langle \mathcal{O}^{\mathrm{D}} | \mathcal{P}\mathcal{O}^{\mathrm{S}} \rangle_{\mathrm{planar}} = \langle \mathcal{O}^{\mathrm{D}} | \mathcal{O}^{\mathrm{S}} \rangle_{\mathrm{planar}} \tag{4.18}$$

- For each Λ, M_Λ, the states $\mathcal{O}_{\Lambda, M_\Lambda, R, \tau}$ span a space of dimension

$$\sum_R C(R, R, \Lambda) \tag{4.19}$$

where $C(R, R, \Lambda)$ is the multiplicity of the one-dimensional irrep. in the S_n tensor product $R \times R \times \Lambda$. Note the symmetry under transposition of the Young diagram

$$\begin{aligned} C(R, R, \Lambda) &= \frac{1}{n!} \sum_\sigma \chi_R(\sigma) \chi_R(\sigma) \chi_\Lambda(\sigma) \\ &= \frac{1}{n!} \sum_\sigma (-1)^\sigma \chi_{R^T}(\sigma) (-1)^\sigma \chi_{R^T}(\sigma) \chi_\Lambda(\sigma) \\ &= C(R^T, R^T, \Lambda) \end{aligned} \tag{4.20}$$

We expect that, given a choice of orthogonal basis labelled by τ in $\mathrm{Inv}(R \otimes R \otimes \Lambda)$, there is a corresponding choice τ^T in $\mathrm{Inv}(R^T \otimes R^T \otimes \Lambda)$, such that

$$(\mathcal{P})^{\Lambda, R_1, \tau_1}_{\Lambda, R_2, \tau_2} = \pm (\mathcal{P})^{\Lambda, R_1^T, \tau_1^T}_{\Lambda, R_2^T, \tau_2^T} \tag{4.21}$$

This will be demonstrated in examples by the explicit construction in Appendix C, using bases in τ space constructed as eigenstates of generalized Casimirs defined in [39].

4.3.1 Counting Check

The counting of states in a given irrep. Λ of $U(M)$ is given, using (3.30) by

$$\mathcal{M}_\Lambda = \sum_{p\vdash n} \chi_\Lambda(\mathbb{P}_p) = \sum_{\alpha\in S_n} \chi_\Lambda(\mathbb{P}_\alpha)\frac{|\mathrm{Sym}(\alpha)|}{n!} \tag{4.22}$$

It is useful to write this as

$$\mathcal{M}_\Lambda = \sum_{\alpha,\beta\in S_n} \chi_\Lambda(\mathbb{P}_\alpha)\frac{1}{n!}\delta(\beta\alpha\beta^{-1}\alpha^{-1}) \tag{4.23}$$

On the other hand $(\mathcal{P})^{\Lambda,M_\Lambda,R_2,\tau_2}_{\Lambda,M_\Lambda,R_1,\tau_1}$ at fixed Λ, M_Λ is a projector matrix to the space of symmetrized traces. The dimension of this space should be the large-N number of BPS operators $\mathcal{M}_\Lambda$. In general, the dimension of the projection must be given by the trace of the projector matrix. We can thus perform a consistency check by calculating the trace of $\mathcal{P}$:

$$\begin{aligned}
\sum_{R,\tau}(\mathcal{P})^{\Lambda,M_\Lambda,R,\tau}_{\Lambda,M_\Lambda,R,\tau} &= \sum_{\tau,\tau_1,S} \mathbf{p}_{\beta,\alpha} D^\Lambda_{i_3j_3}(\beta) D^S_{k_1k_2}(\alpha)\frac{d_Rd_S}{n!} S^{R\ R\ S,\tau}_{i_2\ j_2\ k_2} S^{R\ R\ S,\tau}_{i_1\ j_1\ k_1} S^{R\ R\ \Lambda,\tau_1}_{i_1\ i_2\ i_3} S^{R\ R\ \Lambda,\tau_1}_{j_1\ j_2\ j_3} \\
&= \sum_{\tau,\tau_1,S}\sum_{\sigma_1,\sigma_2} \mathbf{p}_{\beta,\alpha} D^\Lambda_{i_3j_3}(\beta) D^S_{k_1k_2}(\alpha)\frac{d_Rd_S}{n!}\frac{1}{n!} D^R_{i_2i_1}(\sigma_1) D^R_{j_2j_1}(\sigma_1) D^S_{k_2k_1}(\sigma_1) \\
&\qquad \frac{1}{n!} D^R_{i_1j_1}(\sigma_2) D^R_{i_2j_2}(\sigma_2) D^\Lambda_{i_3j_3}(\sigma_2) \\
&= \sum_{\sigma_1,\sigma_2\in S_n} \frac{1}{n!}\chi_\Lambda(\mathbb{P}_{\sigma_1}\sigma_2)\delta(\sigma_1\sigma_2\sigma_1^{-1}\sigma_2^{-1})
\end{aligned} \tag{4.24}$$

Now given the form of $\mathbb{P}_{\sigma_1}$ we know that if σ_2 commutes with σ_1 then $\mathrm{tr}(\mathbb{P}_{\sigma_1}\sigma_2) = \mathrm{tr}(\mathbb{P}_{\sigma_1})$. This is then equal to $\mathcal{M}_\Lambda$ according to (4.23) as required.

4.4 The two-point function for BPS operators

Given the expression of BPS operators as (4.14) we may easily evaluate the two-point function matrix.

$$\begin{aligned}
\langle \mathcal{O}^{\mathrm{BPS}}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} | \mathcal{O}^{\mathrm{BPS}}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}\rangle &= \langle \mathcal{GPO}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} | \mathcal{GPO}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}\rangle \\
&= \langle \mathcal{O}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} | \mathcal{PGFGPO}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}\rangle_{\mathrm{planar}} \\
&= \langle \mathcal{O}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} | \mathcal{PGPO}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}\rangle_{\mathrm{planar}} \\
&= \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M'_{\Lambda_2}}(\mathcal{PGP})^{\Lambda_1,R_1,\tau_1}_{\Lambda_1,R_2,\tau_2}
\end{aligned} \tag{4.25}$$

We have used the fact that the two-point function of $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ is diagonal in the labels and given by the matrix $\mathcal{F}$ in equation (2.17), which is an inverse of $\mathcal{G}$. This leads to the formula

$$\boxed{\langle \mathcal{O}^{\rm BPS}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} | \mathcal{O}^{\rm BPS}_{\Lambda_2,M_{\Lambda_2},R_2,\tau_2} \rangle = \delta_{\Lambda_1,\Lambda_2} \delta_{M_{\Lambda_1},M'_{\Lambda_2}} (\mathcal{PGP})^{\Lambda_1,R_1,\tau_1}_{\Lambda_1,R_2,\tau_2}} \tag{4.26}$$

for the two-point function on of BPS operators, valid at all orders in the $1/N$ expansion. This is the main result of this paper.

Since the Clebsch-Gordan coefficients of the symmetric group can be chosen to be real [40], the matrix $\mathcal{PGP}$ is a real symmetric matrix. We will develop a manifestly finite N construction in the next section to arrive at $\mathcal{P}_I \mathcal{G}_N \mathcal{P}_I$. In Section 6 we will use analyticity arguments to show that the matrix $\mathcal{PGP}$ can be used to extract finite N properties.

5 Finite N and geometry in Hilbert spaces

The derivation of the two-point function (4.26) is valid to all orders in the $1/N$ expansion but requires $N \geq n$. Here we will describe a framework which allows the derivation of a formula for the 2-point function at finite N, with no restriction on n. This treatment contains some general insights, notably the geometry of the interplay between different projectors in appropriate Hilbert spaces, which should be useful in thinking more generally about finite N issues. e.g in the sixteenth BPS case in $\mathcal{N} = 4$ or in more general theories.

5.1 Finite N Hilbert space as a quotient and BPS operators from intersections

Start with an infinite dimensional Hilbert space $\mathcal{H}$, graded by a natural number n. For each n, there are finitely many states. The states are labeled by $|\Lambda, M_\Lambda, R, \tau\rangle$ for each natural number n. Λ, R are Young diagrams with n boxes, M_Λ a state in the $U(M)$ representation corresponding to Λ, τ runs over the multiplicity $C(R, R, \Lambda)$ which is the number of times the one-dimensional representation of S_n appears in the tensor product $R \otimes R \otimes \Lambda$. We can regard the states as made of gauge invariant polynomials in variables $(\mathcal{X}_1)^i_j, (\mathcal{X}_2)^i_j, (\mathcal{X}_3)^i_j$ where i, j extend from 1 to ∞. These are traces e.g $\mathrm{tr}(\mathcal{X}_1\mathcal{X}_2\mathcal{X}_3)$. The relation between the traces and the group theoretic labels is given by the equation (2.7) involving the matrices $\mathcal{S}, \mathcal{T}$, involved in the free-field diagonalization problems solved in [16, 17], constructed from Clebsch-Gordan coefficients of $U(M) \times S_n$, matrix elements and Clebsch-Gordan coefficients of S_n. They make no reference to the $U(N)$ gauge group hence they make sense in the $N = \infty$ set-up we are describing here. We can define a simple inner product on $\mathcal{H}$

$$\langle \Lambda', M'_{\Lambda'}, R', \tau' | \Lambda, M_\Lambda, R, \tau \rangle_{S_\infty} = \delta_{\Lambda',\Lambda} \delta_{M_\Lambda, M'_{\Lambda'}} \delta_{R,R'} \delta_{\tau,\tau'}, \tag{5.1}$$

which we will call the S_∞ inner product. Note that its form matches the planar limit inner product (2.19), but there is a subtle difference in that $\langle . | . \rangle_{S_\infty}$ is defined on a manifestly $N = \infty$ Hilbert space $\mathcal{H}$, while the planar product is defined as the limit of the finite large N inner product for traces involving a total of n matrices chosen from X_1, X_2, X_3 when $n \ll N$.

Finite N physics can be obtained by replacing the formal variables $(\mathcal{X}_a)^i_j$ with matrix elements $(X_a)^i_j$ of $N \times N$ matrices X_a. It was shown in [36, 16] that the finite N counting of gauge-invariant operators can be achieved simply by cutting off R with the condition that its first column does not exceed N in length, i.e $c_1(R) \leq N$. It is useful therefore to introduce a projection operator

$$\mathcal{I}^{(N)} | \Lambda, M_\Lambda, R, \tau \rangle = \begin{cases} |\Lambda, M_\Lambda, R, \tau\rangle & \text{if } \ \mathrm{c_1(R) \leq N} \\ 0 & \text{if } \ \mathrm{c_1(R) > N} \end{cases} \tag{5.2}$$

The finite N Hilbert space is isomorphic to $\mathcal{H}/\mathrm{Ker}(\mathcal{I}^{(N)})$. We find that a very useful way to think about finite N physics is to consider projectors in $\mathcal{H}$. Alongside $\mathcal{I}^{(N)}$ we will also be interested in $\mathcal{P}$, as defined by the equation (4.11). In the finite N Hilbert space, we do not think of $\mathcal{P}$ as a linear operator defined by its action on traces, because the traces do not form a good basis, hence the emphasis on (4.11). Understanding the interplay between $\mathcal{P}$ and $\mathcal{I}^{(N)}$ will be the key to understanding finite N physics of BPS operators. Since $\mathcal{P}$ is real and symmetric in the Λ, R, τ basis, it is a Hermitian operator with respect to the S_∞ metric. The operator $\mathcal{I}^{(N)}$ is also Hermitian with respect to the same metric. In the following we will describe the construction of a new orthogonal projector $\mathcal{P}_I$, where orthogonality is meant with respect to the S_∞ metric.

The S_∞ metric induces a metric on the quotient $\mathcal{H}/\mathrm{Ker}(\mathcal{I}^{(N)})$, which is isomorphic to the finite N Hilbert space $\mathcal{H}^{(N)}$ of finite N matrices. On this finite N quotient it is

$$\langle \Lambda_1, M_{\Lambda_1}, R_1, \tau_1 | \mathcal{I}^{(N)} | \Lambda_2, M_{\Lambda_2}, R_2, \tau_2 \rangle_{S_\infty} = \delta_{\Lambda_1 \Lambda_2} \delta_{M_{\Lambda_1}, M_{\Lambda_2}} \delta_{R_1 R_2} \delta_{\tau_1 \tau_2} (\mathcal{I}^{(N)})_{R_1 R_2} \tag{5.3}$$

This metric can be evaluated on the trace basis using (2.9) to be

$$\begin{aligned} \langle \mathcal{O}_{\vec{b},\beta} | \mathcal{I}^{(N)} | \mathcal{O}_{\vec{a},\alpha} \rangle_{S_\infty} &= \sum_\gamma \frac{1}{n!} \delta_{\gamma(\vec{a}),\vec{b}} \delta_N(\beta^{-1}\gamma^{-1}\alpha\gamma) \\ &= \sum_\gamma \delta_{\gamma(\vec{a}),\vec{b}} \sum_{R: c_1(R) \leq N} d_R \chi_R(\beta^{-1}\gamma^{-1}\alpha\gamma) \end{aligned} \tag{5.4}$$

For operators with $n \ll N$, this is just the delta function expressing large N factorization. Operators only overlap if they are a product of like traces with like field content.

Whether at large N or finite N, the descendants are in $\mathrm{Ker}(\mathcal{P})$. Whenever a single trace contains commutators, it is a descendant and it is annihilated by the symmetrization operation $\mathcal{P}$. For example in the case of $\Lambda = [2,2]$ in the Appendix C, we have the expression $\frac{1}{2}\mathcal{O}_{[3,1]} - \frac{1}{\sqrt{2}}\mathcal{O}_{[2,2]} + \frac{1}{2}\mathcal{O}_{[2,1^2]}$ for the descendant in the Fourier basis. For finite N matrices, the last term is identically zero, so we may equally write $\frac{1}{2}\mathcal{O}_{[3,1]} - \frac{1}{\sqrt{2}}\mathcal{O}_{[2,2]}$. Equivalently, without ever talking about finite N matrices, but only about projectors in $\mathcal{H}$, we obtain the same expression by considering the image of $\mathcal{I}^{(N)}(1-\mathcal{P})$ in $\mathcal{H}$ which contains the linear combination $\frac{1}{2}\mathcal{O}_{[3,1]} - \frac{1}{\sqrt{2}}\mathcal{O}_{[2,2]}$. We look for a hermitian (under the S_∞ metric) projection operator $\mathcal{P}_I$ which will annihilate the descendants, so we have $\mathcal{P}_I\mathcal{I}^{(N)}(1-\mathcal{P}) = 0$. We also would like this projector to map states in $\mathcal{H}^{(N)}$ to $\mathcal{H}^{(N)}$, so that $\mathcal{I}^{(N)}\mathcal{P}_I = \mathcal{P}_I\mathcal{I}^{(N)} = \mathcal{P}_I$. Combining these we have $\mathcal{P}\mathcal{P}_I = \mathcal{P}_I\mathcal{P} = \mathcal{P}_I$. Together, we see that these equations characterise a projector for the intersection

$$\mathrm{Im}(\mathcal{P}) \cap \mathrm{Im}(\mathcal{I}^{(N)}) = \mathrm{Im}(P_I) \tag{5.5}$$

An equivalent way to arrive at this is to consider the orthogonal complement in $\mathcal{H}$ to $\mathrm{Ker}(I^{(N)}) \cup \mathrm{Ker}(\mathcal{P})$ in the S_∞ metric. The dimension of this complement is equal to the number of independent BPS operators, since imposing the finite N condition in $\mathcal{H}$ sets to zero states in $\mathrm{Ker}(I^{(N)})$ and setting descendants to zero imposes the vanishing of states in $\mathrm{Ker}(\mathcal{P})$. This orthogonal complement is $\mathrm{Im}(I^{(N)}) \cap \mathrm{Im}(\mathcal{P})$ so we have the orthogonal decomposition

$$\mathcal{H} = \big(\mathrm{Ker}(I^{(N)}) \cup \mathrm{Ker}(\mathcal{P})\big) \oplus \big(\mathrm{Im}(\mathcal{I}^{(N)}) \cap \mathrm{Im}(\mathcal{P})\big)\,. \tag{5.6}$$

A natural question arises how, given two projectors $\mathcal{I}^{(N)}$ and $\mathcal{P}$, one explicitly constructs a hermitian projector for their intersection space

$$\mathcal{P}_I \equiv \mathcal{P}_{\mathrm{Im}(\mathcal{I}^{(N)})\cap\mathrm{Im}(\mathcal{P})}. \tag{5.7}$$

Let us consider the null space of $\mathcal{I}^{(N)}(1-\mathcal{P})\mathcal{I}^{(N)}$. It will be composed of two orthogonal subspaces: the null space of $\mathcal{I}^{(N)}$ itself and the null space of $(1-\mathcal{P})$ inside of $\mathrm{Im}(\mathcal{I}^{(N)})$:

$$\begin{aligned} \mathrm{Ker}(\mathcal{I}^{(N)}(1-\mathcal{P})\mathcal{I}^{(N)}) &= \mathrm{Ker}(\mathcal{I}^{(N)}) \oplus \big(\mathrm{Im}(\mathcal{I}^{(N)}) \cap \mathrm{Ker}(1-\mathcal{P})\big) \\ &= \mathrm{Ker}(\mathcal{I}^{(N)}) \oplus \big(\mathrm{Im}(\mathcal{I}^{(N)}) \cap \mathrm{Im}(\mathcal{P})\big)\,. \end{aligned} \tag{5.8}$$

This means if we had a projector for $\mathrm{Ker}(\mathcal{I}^{(N)}(1-\mathcal{P})\mathcal{I}^{(N)})$ then we would just multiply by $\mathcal{I}^{(N)}$ on each side to extract the second component

$$\mathcal{P}_I = \mathcal{I}^{(N)}\mathcal{P}_{\mathrm{Ker}(\mathcal{I}^{(N)}(1-\mathcal{P})\mathcal{I}^{(N)})}\mathcal{I}^{(N)}. \tag{5.9}$$

The question of constructing $\mathcal{P}_I$ is then reduced to the question of, given an arbitrary matrix X, constructing the projector for its null space $\mathcal{P}_{\mathrm{Ker}(X)}$. It can be written as follows

$$\mathcal{P}_{\mathrm{Ker}(X)} = \frac{1}{\prod_{i=1}^{k}(-\lambda_i)} \prod_{i=1}^{k} (X - \lambda_i), \tag{5.10}$$

where i runs over the *non-zero eigenvalues* of X and $k = \mathrm{Rank}(X)$. This, in turn, can be rewritten in terms of Schur polynomials

$$\mathcal{P}_{\mathrm{Ker}(X)} = \frac{1}{\chi_{[1^k]}(X)} \sum_{j=0}^{k} (-1)^{k-j} X^{k-j} \chi_{[1^j]}(X). \tag{5.11}$$

The characters $\chi_{[1^j]}(X)$ are the Schur Polynomials for Young diagrams with j rows of length 1 or one column of length j. This manipulation then allows us to write the intersection projector $\mathcal{P}_I$ fairly explicitly in terms of matrix multiplications as

$$\begin{aligned} \mathcal{P}_I &= \frac{1}{\chi_{[1^k]}(X)} \sum_{j=0}^{k} (-1)^{k-j} \left(\mathcal{I}^{(N)} X^{k-j} \mathcal{I}^{(N)} \right) \chi_{[1^j]}(X), \\ X &\equiv \mathcal{I}^{(N)} (1 - \mathcal{P}) \mathcal{I}^{(N)}, \\ k &= \mathrm{Rank}(X). \end{aligned} \tag{5.12}$$

Also note that using the decomposition (5.8) we can express the number of operators $\mathcal{M}_{N;\Lambda}$ as

$$\begin{aligned} \mathcal{M}_{N;\Lambda} &= |\mathrm{Im}(\mathcal{I}^{(N)}) \cap \mathrm{Im}(\mathcal{P})| \\ &= |\mathrm{Ker}(\mathcal{I}^{(N)}(1-\mathcal{P})\mathcal{I}^{(N)})| - |\mathrm{Ker}(\mathcal{I}^{(N)})| \\ &= \mathrm{Rank}(\mathcal{I}^{(N)}) - \mathrm{Rank}(\mathcal{I}^{(N)}(1-\mathcal{P})\mathcal{I}^{(N)}). \end{aligned} \tag{5.13}$$

5.2 Finite N construction of BPS operators

We will prove that the BPS operators at finite N are given by the image of $\mathcal{G}P_I$ as an operator in $\mathcal{H}$.

We first restate the construction of the BPS operators at large N in language that will generalize at finite N. The large N construction starts with two key observations. The relation between the planar inner product $\langle .|.\rangle_{S_\infty}$ and the $1/N$ corrected inner product $\langle .|.\rangle$ is given by (see (2.20)):

$$\langle \mathcal{O}_1 | \mathcal{O}_2 \rangle = \langle \mathcal{O}_1 | \mathcal{F} \mathcal{O}_2 \rangle_{S_\infty} \tag{5.14}$$

The second observation is that descendant operators contain commutators $[X_a, X_b]$ inside a trace. These operators are annihilated by the operator $\mathcal{P}$ we constructed. From these facts and the Hermiticity of $\mathcal{P}$ with respect to S_∞ we easily see that the operators in the image of $\mathcal{GP}$ are orthogonal to descendants, and thus BPS:

$$\begin{aligned}&\langle \mathcal{O}^{\mathrm{D}}|\mathcal{GPO}\rangle = \langle \mathcal{O}^{\mathrm{D}}|\mathcal{FGPO}\rangle_{S_\infty}\\ &= \langle \mathcal{O}^{\mathrm{D}}|\mathcal{PO}\rangle_{S_\infty} = 0 = \langle \mathcal{PO}^{\mathrm{D}}|\mathcal{O}\rangle_{S_\infty} = 0\end{aligned} \tag{5.15}$$

To develop a finite version of this argument, we need to carefully consider the definition of $\mathcal{G}$ at finite N. Define $\mathcal{G}_N \equiv \mathcal{I}^{(N)}\mathcal{G}\mathcal{I}^{(N)}$ which is manifestly zero for $c_1(R) > N$ and well defined at finite N. Acting on a Hilbert space $\mathcal{H}$ with states $|\Lambda, M_\Lambda, R, \tau\rangle$ it is non-zero only on the operators obeying $c_1(R) \le N$, i.e where the Young diagram R has first column of length smaller than N. The second ingredient is the projector $\mathcal{P}_I$ defined in (2.20).

Note the following useful properties of $\mathcal{G}, \mathcal{F}, \mathcal{I}^{(N)}, \mathcal{P}, \mathcal{P}_I$

$$\begin{aligned}&\mathcal{G}_N\mathcal{F} = \mathcal{F}\mathcal{G}_N = \mathcal{I}^{(N)}\\ &\mathcal{P}_I\mathcal{I}^{(N)} = \mathcal{I}^{(N)}\mathcal{P}_I = \mathcal{P}_I\\ &\mathcal{P}_I\mathcal{P} = \mathcal{P}\mathcal{P}_I = \mathcal{P}_I\\ &\mathcal{FGP}_I = \mathcal{FGI}_N\mathcal{P}_I = \mathcal{P}_I\end{aligned} \tag{5.16}$$

Consider operators $\mathcal{G}_N\mathcal{P}_I\mathcal{O}$. These operators are manifestly well-defined at finite N and have the following 2-point function with descendants

$$\begin{aligned}\langle \mathcal{O}^{\mathrm{D}}|\mathcal{G}_N\mathcal{P}_I\mathcal{O}\rangle &= \langle \mathcal{O}^{\mathrm{D}}|\mathcal{F}\mathcal{G}_N\mathcal{P}_I\mathcal{O}\rangle_{S_\infty}\\ &= \langle \mathcal{O}^{\mathrm{D}}|\mathcal{I}^{(N)}\mathcal{P}_I\mathcal{O}\rangle_{S_\infty}\\ &= \langle \mathcal{O}^{\mathrm{D}}|\mathcal{P}_I\mathcal{O}\rangle_{S_\infty}\\ &= \langle \mathcal{O}^{\mathrm{D}}|\mathcal{P}\mathcal{P}_I\mathcal{O}\rangle_{S_\infty}\\ &= \langle \mathcal{PO}^{\mathrm{D}}|\mathcal{P}_I\mathcal{O}\rangle_{S_\infty}\\ &= 0\end{aligned} \tag{5.17}$$

In the first line we have used the fact that the metric of the free theory is related to the S_∞ metric by the matrix $\mathcal{F}$. In the fourth line, we have used the fact that $\mathcal{P}$ is Hermitian with respect to the S_∞ metric, and annihilates the descendants. This shows that the $\mathcal{G}_N\mathcal{P}_I\mathcal{O}$ construct BPS operators.

$$\mathcal{O}^{\mathrm{BPS}} = \mathcal{P}_I\mathcal{G}_N\mathcal{O} \tag{5.18}$$

The number of these operators is $|\mathrm{Span}(\mathcal{P}_I)|$, which as explained in section 5.1, correctly matches the finite N counting.

The two point function on BPS operators is the matrix $\mathcal{P}_I \mathcal{G}_N \mathcal{P}_I$:

$$\begin{aligned}
&\langle \mathcal{G}_N \mathcal{P}_I \mathcal{O}_{\Lambda_2, M'_{\Lambda_2}, R_2, \tau_2} | \mathcal{G}_N \mathcal{P}_I \mathcal{O}_{\Lambda_1, M_{\Lambda_1}, R_1, \tau_1} \rangle \\
&= \langle \mathcal{O}_{\Lambda_2, M'_{\Lambda_2}, R_2, \tau_2} | \mathcal{P}_I \mathcal{G}_N \mathcal{F} \mathcal{G}_N \mathcal{P}_I \ \mathcal{O}_{\Lambda_1, M_{\Lambda_1}, R_1, \tau_1} \rangle_{S_\infty} \\
&= (\mathcal{P}_I \mathcal{G}_N I^{(N)} \mathcal{P}_I)^{\Lambda_2, M_{\Lambda_2}, R_2, \tau_2}_{\Lambda_1, M_{\Lambda_1}, R_1, \tau_1} \\
&= (\mathcal{P}_I \mathcal{G}_N \mathcal{P}_I)^{\Lambda_2, M_{\Lambda_2}, R_2, \tau_2}_{\Lambda_1, M_{\Lambda_1}, R_1, \tau_1} \\
&= \delta_{\Lambda_1, \Lambda_2} \delta_{M_{\Lambda_1}, M'_{\Lambda_2}} (\mathcal{P}_I \mathcal{G}_N \mathcal{P}_I)^{\Lambda_1, R_2, \tau_2}_{\Lambda_1, R_1, \tau_1}
\end{aligned} \tag{5.19}$$

It is interesting to note that the geometry of intersections $\mathrm{Im}(\mathcal{I}^{(N)}) \cap \mathrm{Im}(\mathcal{P})$ has played a key role. The projector $\mathcal{P}$ was constructed from the universal element $\mathbb{P}$ which gave the large N counting. Although it is easy to give indirect arguments, based on points already discussed, that the above finite N construction with $\mathcal{P}_I$ agrees with the finite N counting from $\mathbb{P}^{(N)}$, there was no direct finite N construction following directly from $\mathbb{P}^{(N)}$, the way the large N construction based on $\mathcal{P}$ followed from $\mathbb{P}$. This was not apriori obvious and is intriguing.

6 The matrix $\mathcal{P}_I \mathcal{G} \mathcal{P}_I$ and space-time physics

According to non-renormalization theorems [22, 21, 41], we expect that the two-point functions of BPS states, constructed in this paper as the matrix $\mathcal{P}_I \mathcal{G} \mathcal{P}_I$ acting in the free Hilbert space, will be unchanged as the coupling g^2_{YM} is increased from weak to strong coupling. We will attempt to extract aspects of the 2-point functions at one-loop which are most likely to have a strong coupling interpretation.

The set of eigenvalues, and their multiplicities, being independent of a choice of basis are of particular interest. In the regime $n < N$, the matrix $\mathcal{P}_I \mathcal{G} \mathcal{P}_I$ simplifies to $\mathcal{P} \mathcal{G} \mathcal{P}$. We will also argue that, by analytic continuation, that information about $\mathcal{P}_I \mathcal{G} \mathcal{P}_I$ can be extracted from $\mathcal{P} \mathcal{G} \mathcal{P}$. In particular, there is a simple way to get the characteristic equation of the former from the latter. The behaviour of the spectrum for n near N is specially interesting in view of the stringy exclusion principle (SEP) [29, 30] and the related property of the growth in size of giant gravitons with angular momentum [31]. The matching of these spacetime phenomena with gauge theory operators is well-understood in the half-BPS case using a Young diagram classification of the operators [7]. We will use the physics of the SEP to make a start towards the identification of gauge theory operators corresponding to BPS operators beyond the half-BPS case. We will then turn to a discussion of how to find a labelling analogous to the Young diagrams of the half-BPS case in the case of quarter and eighth-BPS operators at hand.

6.1 Eigenvalues of $\mathcal{PGP}$ and $\mathcal{P}_I\mathcal{G}\mathcal{P}_I$

The matrix $\mathcal{PGP}$ is derived as the matrix of two-point functions of BPS operators calculated in the regime where $N \geq n$. As described in Section 5, when $N < n$ the two-point function of the well-defined BPS operators is instead given by a modified matrix $\mathcal{P}_I\mathcal{G}_N\mathcal{P}_I$. However, the eigenvalues of $\mathcal{PGP}$, even though calculated in the large N regime, are explicit functions of N. Therefore, one may ask, what happens to the eigenvalues as we continuously decrease N below n. For an example of the functions see (C.17) in Appendix C.

First, let us review the half-BPS case $\Lambda = [n]$. Since $\mathcal{P} = \mathbb{I}$, the BPS operators are just rescaled free operators

$$\mathcal{O}^{\mathrm{BPS}}_{\Lambda=[n];R} = \frac{N^n d_R}{n!\mathrm{Dim}R}\mathcal{O}_{\Lambda=[n];R} \tag{6.1}$$

and the two-point function

$$(\mathcal{PGP})^{\Lambda=[n];R_1}_{R_2} = \frac{N^n d_{R_1}\delta_{R_1,R_2}}{n!\mathrm{Dim}R_1}. \tag{6.2}$$

The eigenvectors coincide then with the orthogonal half-BPS basis labelled by Young diagram R [7]. The eigenvalues $\lambda_R \equiv \frac{N^n d_R}{n!\mathrm{Dim}R}$ are equal to N^n divided by a polynomial $\prod_{i,j}(N - i + j)$, where the product runs over boxes in the Young diagram R, with i labelling the row and j labelling the column. The eigenvalue λ_R diverges when the first column has length $c_1(R)$ in the region $c_1(R) > N$. The eigenvalue of a BPS state at large N, when analytically continued to a regime where that BPS state is not part of the finite N Hilbert space, *diverges*. In the original half-BPS construction or in the free orthogonal basis the states were normalized so that the eigenvalues continued to *zero*. In our construction of the general quarter and bosonic eighth-BPS states, the divergent normalization is more natural.

Interestingly, we find the same behavior to be general for any Λ. That is, if we calculate eigenvalues at large N and then take the limit for any finite N, the divergent eigenvalues will correspond to the states that drop out from the Hilbert space. On the other hand, each finite (but non-zero) eigenvalue corresponds to a state that remains BPS. Equivalently, we can read off the finite N characteristic equation by factoring out a common denominator and dropping terms in the numerator which vanish at finite N (see the discussion around (C.29)). In fact, the surviving finite eigenvalues of $\mathcal{PGP}$ with the corresponding eigenvectors precisely match those of $\mathcal{P}_I\mathcal{G}_N\mathcal{P}_I$, calculated in the manifestly finite N framework. In the discussion here we are not interested in zero eigenvalues of $\mathcal{PGP}$, because they just span $\mathrm{Ker}(\mathcal{P})$ and do not depend on N. Also, since $\mathrm{Ker}(\mathcal{P})$

is contained in $\mathrm{Ker}(\mathcal{P}_I)$, the zero eigenvalues of $\mathcal{PGP}$ will always correspond to zero eigenvalues of $\mathcal{P}_I\mathcal{G}_N\mathcal{P}_I$.

This correspondence between eigenstates of $\mathcal{PGP}$ and $\mathcal{P}_I\mathcal{G}_N\mathcal{P}_I$ can be explained as follows. Consider the matrix $\mathcal{PGP}$ at a value $N_\epsilon = N + \epsilon$ approaching some finite N in the regime $N < n$. The matrix $\mathcal{G}$ will contain finite and divergent parts:

$$\mathcal{G} = \mathcal{G}_N + O(\epsilon^{-1}). \tag{6.3}$$

The finite directions are those in $\mathrm{Im}(\mathcal{I}^{(N)})$ and the divergent ones are in $\mathrm{Im}(1-\mathcal{I}^{(N)})$,where the projector $\mathcal{I}^{(N)}$ is defined in (5.2). Now, $\mathcal{PGP}$ can be viewed as a metric induced by $\mathcal{G}$ on the subspace $\mathrm{Im}(\mathcal{P})$. The divergent direction of $\mathcal{G}$ will induce a divergent direction on $\mathrm{Im}(\mathcal{P})$ which is a projection of $\mathrm{Im}(1 - \mathcal{I}^{(N)})$ to $\mathrm{Im}(\mathcal{P})$ and can be expressed as $\mathrm{Im}(\mathcal{P}(1 - \mathcal{I}^{(N)})\mathcal{P})$. Therefore, as $\epsilon \to 0$ there will be a set of eigenvectors of $\mathcal{PGP}$ with divergent eigenvalues which span $\mathrm{Im}(\mathcal{P}(1-\mathcal{I}^{(N)})\mathcal{P})$. The remaining non-zero eigenvalues, however, should be finite and correspond to non-divergent directions. Since eigenvectors with different eigenvalues have to be orthogonal (in the S_∞ metric which is used in the subsequent discussion), they will span a space in $\mathrm{Im}(\mathcal{P})$ which is orthogonal to $\mathrm{Im}(\mathcal{P}(1 - \mathcal{I}^{(N)})\mathcal{P})$ and that is precisely

$$\mathrm{Im}(\mathcal{P}) \cap \mathrm{Im}(\mathcal{I}^{(N)}) = \mathrm{Im}(\mathcal{P}_I). \tag{6.4}$$

That means that the finite-eigenvalue eigenvectors of $\mathcal{PGP}$ live in $\mathrm{Im}(\mathcal{P}_I)$ and on that subspace $\mathcal{PGP} = \mathcal{P}_I\mathcal{G}_N\mathcal{P}_I$. Therefore, the finite eigenvalues of the two operators match. It is also plausible, and justified by examples we have studied, that the precise forms of the eigenvectors of $\mathcal{P}_I\mathcal{G}_N\mathcal{P}_I$ can be recovered as limits of those of $\mathcal{PGP}$.

As an additional evidence that interesting strong-coupling spacetime physics can be expected from the eigenvalues of $\mathcal{PGP}$, note that the 2-point function in the half-BPS case is $\frac{\chi_R(\Omega^{-1})}{d_R}$. The $\frac{1}{N}$ expansion of Ω^{-1} is related to conjugacy classes of S_n and in turn to Casimirs [33, 42, 39]. In the spacetime picture they are related to conserved charges related to multipole moments of LLM geometries [43].

6.2 Identifying giant graviton states

We now turn to a discussion of the physics of BPS states which disappear as n is reduced to values below N. We expect on physical grounds [31, 11] that there will be states in the region of $n = N + k$ with k order 1 as $N \to \infty$, which can be interpreted as *single* quarter or eighth-BPS giant gravitons along with excitations of the space-time background : single and multi-particle gravitons.

In the half-BPS sector, with Young diagrams of type $\Lambda = [n, 0]$, there are Young diagrams R which have a long row of order N, and a few rows with order 1 boxes.

These correspond to a giant graviton large in the AdS_5 directions along with background spacetime excitations. Then there are states close to a sphere giant (large in the S^5 directions) with one column of length order N, and a few columns with order 1 boxes. There are also states consisting of multiple giants, each having a finite fraction of the total angular momentum, along with background excitations. These correspond to Young diagrams with a few long columns along with shorter columns. There are also generic states corresponding to Young diagrams of shape far from the above types.

More generally, we can consider a sequence of representations of the form $\Lambda = [n-\lambda, \lambda]$, where λ is order 1. We expect that a single giant graviton will have well-defined $U(2)$ quantum numbers. As n increases for fixed λ, the total energy of the giant will increase, and the energy available for excitations of the background spacetime will increase.

The distinguishing characteristic of the states which consist of a giant expanding in the S^5, along with order 1 spacetime excitations is that as $k = n - N$ increases from 0, their size will grow as the angular momentum increases and they will start disappearing from the spectrum. In fact all the states that disappear as k increases from 0, will be precisely the states consisting of a sphere giant along with space-time excitations. Consider then the difference

$$\Delta\mathcal{M}_{N=n-k;[n-\lambda,\lambda]} \equiv \mathcal{M}_{N=\infty;[n-\lambda,\lambda]} - \mathcal{M}_{N=n-k;[n-\lambda,\lambda]} \tag{6.5}$$

This will measure the number of states in the irrep. $[n-\lambda, \lambda]$ which disappear due to finite N effects. A first consequence of our discussion is that this number should become independent of n as n increases. We have accumulated extensive numerical evidence for this behaviour. We will return to these properties of $\Delta\mathcal{M}_{N=n-k;[n-\lambda,\lambda]}$ and related counting formulae in future work.

Along the lines of the discussion based on continuity in section 6.1, we expect that states that are close to the cutoff, will have finite but large eigenvalues of $\mathcal{P}_I\mathcal{G}\mathcal{P}_I$ (or $\mathcal{P}\mathcal{G}\mathcal{P}$). Similarly we would expect that the states with the smallest eigenvalues of $\mathcal{P}_I\mathcal{G}\mathcal{P}_I$ will be associated with single AdS giants and their excitations.

Clearly an important open problem is to find a precise description of the eigenstates of $\mathcal{P}_I\mathcal{G}\mathcal{P}_I$ which will allow a more precise identification of the giant graviton states, notably to distinguish which states just correspond to single or multiple quarter-BPS giants and which correspond to giants along with excitations of the background spacetime. We turn to a discussion of avenues towards this goal.

6.3 Orthogonal BPS basis

Eigenvectors of the 2-point function matrix are precisely the ones that allow an identification to semi-classical states [6, 7, 10] in the half-BPS case. The same line of argument has been used above for states close to giant gravitons. We expect that the same can be applied to corresponding to deformations of the bulk geometry, i.e quarter and eighth-BPS generalizations of LLM.

In Section 4.4 we have derived the two-point function on the BPS states as the matrix $\mathcal{PGP}$:

$$\langle \mathcal{O}^{\rm BPS}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} | \, \mathcal{O}^{\rm BPS}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2} \rangle = \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M'_{\Lambda_2}} (\mathcal{PGP})^{\Lambda_1,R_1,\tau_1}_{\Lambda_1 R_2,\tau_2}. \tag{6.6}$$

Consider an eigenvector of $\mathcal{PGP}$ with eigenvalue $\lambda_\mu(N)$ and components $B^{\Lambda;R\tau}_{\mu,\nu}$. The label ν is a multiplicity label for eigenstates with a fixed eigenvalue λ_μ. The eigenvector equation is :

$$\sum_{R_2,\tau_2} (\mathcal{PGP})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2} B^{\Lambda;R_2\tau_2}_{\mu,\nu} = \lambda_\mu B^{\Lambda;R_1\tau_1}_{\mu,\nu}. \tag{6.7}$$

We know there will be $\mathcal{M}_{\Lambda;N}$ of eigenvectors for which λ_μ is non-zero and finite. The remaining eigenvectors having zero eigenvalues are just the ones spanning the kernel of $\mathcal{P}$ and so they correspond to the descendant non-BPS operators, while the eigenvectors with divergent eigenvalues drop out of the finite N Hilbert space. An orthogonal basis for the BPS operators is then

$$\mathcal{O}^{\rm BPS}_{\Lambda,M_\Lambda;\mu,\nu} = \sum_{R,\tau} B^{\Lambda;R\tau}_{\mu,\nu} \mathcal{O}^{\rm BPS}_{\Lambda,M_\Lambda;R,\tau} \tag{6.8}$$

with the two-point function given by

$$\langle \mathcal{O}^{\rm BPS}_{\Lambda,M_\Lambda;\mu_1,\nu_1} | \, \mathcal{O}^{\rm BPS}_{\Lambda,M_\Lambda;\mu_2,\nu_2} \rangle = B^{\Lambda;R_1\tau_1}_{\mu_1,\nu_1} (\mathcal{PGP})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2} B^{\Lambda;R_2\tau_2}_{\mu_2,\nu_2} = \lambda_{\mu_1} B^{\Lambda;R_1\tau_1}_{\mu_1,\nu_1} B^{\Lambda;R_1\tau_1}_{\mu_2,\nu_2}. \tag{6.9}$$

These states will be orthogonal as long as the eigenvalues differ $\lambda_{\mu_1} \neq \lambda_{\mu_2}$. In the half-BPS case, $\mathcal{P} = \mathbb{I}$ and $\mathcal{PGP} \sim ({\rm Dim}R)^{-1}$ while the states are labeled by R and so all eigenvalues are distinct. Also for all cases $\Lambda = [n,m]$ with $m \geq 2$ that we checked explicitly we did not find any degeneracy among the eigenvalues. The only exception seems to be the sector $\Lambda = [n,1]$. With only one X operator in the traces symmetrization doesn't act and $\mathcal{PGP} \sim ({\rm Dim}R)^{-1}$ again. But now there is a possible multiplicity in τ and two distinct states with the same R will be eigenstates with the *same* eigenvalue $({\rm Dim}R)^{-1}$. This happens first at $\Lambda = [5,1]$ where we find operators $\mathcal{O}_{R=[3,2,1],\tau=1}, \mathcal{O}_{R=[3,2,1],\tau=2}$. If it is true that this pattern is general, namely that the cases of degenerate eigenvalues for fixed Λ only arise with Λ of the form $[n,1]$, then for most Λ the eigenvalues of $\mathcal{PGP}$ can provide a way to label the orthogonal eigenvectors of the 2-point function.

Even though we can write the orthogonal basis formally as (6.8), finding the eigenvectors $B^{\Lambda;R\tau}_\mu$ in practice is of course a very difficult task. We provide an example result in the Appendix C.1.3 for the simplest nontrivial case $\Lambda = [2,2]$. In general they come out to be complicated expressions involving N. We will not have much more to say about the precise form of the eigenvectors.

6.4 Towards a complete set of labels for the BPS operators

We will now describe a matrix related to $\mathcal{PGP}$ whose diagonalisation should be simpler and which may provide a more practical way of arriving at a labelling of the eigenstates of $\mathcal{PGP}$. Consider the large N expansion of $\mathcal{G}$:

$$\mathcal{G} \equiv \mathbb{I} + \frac{1}{N}\mathcal{G}_1 + O\left(\frac{1}{N^2}\right). \tag{6.10}$$

$$(\mathcal{G}_1)^{R_1}_{R_2} = -\delta_{R_1,R_2}\frac{\chi_{R_1}(\Sigma_{[2]})}{d_R} \tag{6.11}$$

with $\Sigma_{[2]}$ being the sum over all permutations with cycle structure [2]. Note that $\frac{\chi_{R_1}(\Sigma_{[2]})}{d_R}$ is just a number and not an N-dependent function. The leading terms of $\mathcal{PGP}$ are then

$$\mathcal{PGP} = \mathcal{P} + \frac{1}{N}\mathcal{PG}_1\mathcal{P} + O\left(\frac{1}{N^2}\right). \tag{6.12}$$

$\mathcal{P}$ and $\mathcal{PG}_1\mathcal{P}$ commute due to $\mathcal{P}^2 = \mathcal{P}$ and they can be simultaneously diagonalized. This implies that the eigenstates of $\mathcal{PGP}$ in the limit $N \to \infty$ will approach those of simultaneous eigenstates of $\mathcal{P}$ and $\mathcal{PG}_1\mathcal{P}$. The BPS states will correspond to the subspace of $\mathcal{P}$ eigenvalue 1 and their eigenvalues will be

$$\lambda_\mu = 1 + \frac{1}{N}\widetilde{\lambda}_\mu + O\left(\frac{1}{N^2}\right) \tag{6.13}$$

where the $\widetilde{\lambda}_\mu$ are the eigenvalues of $\mathcal{PG}_1\mathcal{P}$.

It is conceivable that, for fixed Λ, the eigenvalues of $\mathcal{PGP}$ do not cross when plotted as a function of N. Indeed, this is the case in the examples we have studied concretely. In that case, the eigenvalues of $\mathcal{PG}_1\mathcal{P}$ would be adequate as a way to label the states and identify them systematically with giant gravitons and their excitations in the regime $n \sim N$, and with LLM-like geometries in the regime $n \sim N^2$. This deserves further investigation.

Progress on the dictionary between space-time giant graviton and LLM-type states will also require the computation of appropriately normalized three-point functions. For example we may expect the three point function of an operator corresponding to a giant graviton along with some spacetime excitations to be of order one when the other operators in the 3-point function are the giant in question and the small operator for the space-time excitation. The techniques of [48] using correlators in connection with the asymptotics of SUGRA solutions should be useful in relating correlators of eigenstates of $\mathcal{PGP}$ to bulk spacetime.

7 Summary and Outlook

We have described an elegant method for a refined counting of quarter and bosonic eighth-BPS states, where states are organized according to representations of $U(2)$ or $U(3)$. The method extends to general $U(M)$. It relies on the construction of an element $\mathbb{P}$ living in the group algebra of S_n. Exploiting the Schur-Weyl duality relation between representations of $U(M)$ and symmetric groups, the characters of $\mathbb{P}$ in different representations of S_n provide the multiplicities of different $U(M)$ representations Λ among BPS states constructed from n matrices, chosen from $X_1, X_2, \cdots X_M$. This result is given in (3.31). The coefficients of different conjugacy classes in the expansion of $\mathbb{P}$ match in simple cases known integer sequences, and generalize them (see Appendix E). Recent results on the generating functions for BPS states [13, 35] in terms of simple harmonic oscillator states in M dimensions, after a judicious change of variables (3.42), provide the generating functions for these coefficients of conjugacy classes in $\mathbb{P}$.

The fine-structure of $\mathbb{P}$ encoded in a map $\mathbf{p} : \mathbb{C}(S_n) \to \mathbb{C}(S_n)$ with matrix elements $\mathbf{p}_{\beta,\alpha}$ lead to the construction of a linear operator $\mathcal{P}$ acting on the Hilbert space of the free $\mathcal{N} = 4$ SYM theory (i.e the theory at $g^2_{YM} = 0$) which projects to the symmetrized traces which are BPS states in the planar limit, annihilated by the one-loop dilatation operator [32]. The matrix elements of $\mathcal{P}$ are given in terms of a representation theoretic basis for the free theory, which diagonalizes the free two-point functions at finite N [16, 17]. The free basis is a Fourier transform of the trace basis. The expression for $\mathcal{P}$ in this Fourier basis is given in terms of Clebsch-Gordan coefficients and matrix elements of the symmetric group (4.11).

BPS states to all orders in the large N expansion are given in (4.14) as the image of $\mathcal{G}\mathcal{P}$, where $\mathcal{G}$ has simple matrix elements in the Fourier basis, given in terms of inverse dimensions of representations of $U(N)$. The two-point functions of the BPS states are given by the matrix $\mathcal{P}\mathcal{G}\mathcal{P}$ (4.26). The matrix $\mathcal{P}$ also has the virtue that its kernel is spanned by the descendants, which are not annihilated by $\mathcal{H}_2$. As a result it allows a manifestly finite N construction of BPS states (5.18) and their two-point functions (5.19) $\mathcal{P}_I\mathcal{G}\mathcal{P}_I$. The eigenstates of $\mathcal{P}_I\mathcal{G}\mathcal{P}_I$ can be used to construct higher point correlators in a manifestly finite N setting. Analyticity makes it possible to reconstruct eigenvalues and, conjecturally, eigenvectors of the $\mathcal{P}_I\mathcal{G}\mathcal{P}_I$ from the matrix $\mathcal{P}\mathcal{G}\mathcal{P}$. Non-renormalization theorems allow us to argue that the eigenvalues of the matrix $\mathcal{P}\mathcal{G}\mathcal{P}$ contain information about gauge theory operators for giant gravitons [11] and their spacetime excitations and LLM-like geometries [44, 45, 46, 47]. The use of the stringy exclusion principle allows some progress in this direction. An important open problem is to find a labelling of the eigenvectors of $\mathcal{P}\mathcal{G}\mathcal{P}$, which will allow the identification of different types of semiclassical objects : giant gravitons and their spacetime excitations, LLM geometries and their fluctuations. We discussed some avenues in this direction in section 6. The importance of the matrix $\mathcal{PGP}$ cannot be overstated[4].

The route leading from $\mathbb{P}$, which furnishes results in counting, to the matrix $\mathcal{P}$ and the two-point functions $\mathcal{PGP}$ is very interesting. It seems to capture a sort of categorification [49], where numbers (multiplicities) are promoted to explicit constructions of quantum states in string theory, in a manner that can make semiclassical limits and space-time physics transparent. It will be interesting to find out if a similar route can be followed in more general systems in string theory, to lead from results in counting to explicit operators, states and semi-classical limits.

We end by mentioning some additional avenues for further research.

- Generalization of $\mathbb{P}$, $\mathcal{P}$ and $\mathcal{P}_I\mathcal{G}\mathcal{P}_I$ to the case of the most general eighth-BPS states with global symmetry group $U(3|2)$. Other generalizations to consider are for the $SU(N), O(N), Sp(N)$ gauge groups.

- Developing the identification of eighth-BPS giant graviton states further and finding operators corresponding to strings attached to giants, generalizing the use of the Young diagram description to characterize strings attached to half-BPS giants [50, 51, 52, 53, 54]

- The matrix $\mathcal{PGP}$ can be described in other bases which diagonalize the free 2-point functions. The transformation between the restricted Schur basis [50, 55, 56] and the Fourier basis is described in [57]. A similar transformation can be used to connect to the Brauer algebra basis [19, 58]. This should shed light on the construction of [59] which leads to an elegant description of a subset of quarter-BPS states in terms of projectors in Brauer algebras. This could provide another angle on the problem of finding a neat labelling of the eigenstates of $\mathcal{PGP}$ which connects to spacetime physics.

- By exploiting a result of [60] on the one-loop dilatation operator for the $U(2)$ sector, the structure of Clebsch-Gordan coefficients and symmetric group elements which we used to describe $\mathcal{P}$ can also be used to describe the action of the one-loop dilatation operator on the finite N Fourier basis in a manifestly $U(2)$ covariant way (see Appendix B). The study of this mixing in the Fourier basis was initiated in [61]. The equation (B.10) should shed light on the one-loop mixing problem of the descendants, which are in Ker($\mathcal{P}$). The mixing problem at planar level and beyond

[4] We will in fact confess that an earlier title of this paper which did not make the final cut was :"The Matrix : $\mathcal{PGP}$."

has been a very active subject of research, notably in connection with integrability. A few references, by no means complete, giving the flavour of the subject are [26, 62, 63, 34, 64, 65, 66, 67]

- A better understanding of the quarter and eighth-BPS sector is expected to have implications for black hole physics [68, 69]. The identification of gauge theory operators, involving impurities added to the eigenstates of $\mathcal{PGP}$, corresponding to black hole solutions will be interesting to explore. In [70], there was evidence for commuting variables in the counting of sixteenth BPS states, with analogies to the eighth BPS sector. This suggests something akin to the $\mathcal{P}$ operator could be useful for the sixteenth BPS case.

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A Further properties of $\mathcal{P}$ and related operators

The operators $\mathcal{P}$ represents the action of the symmetrization operation in the basis $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ which diagonalizes the free two-point function. We describe the properties of $\mathcal{P}$ and of some closely related projectors which are used in counting of BPS operators.

A.1 Glossary of operators related to $\mathcal{P}$

$\mathbb{P}$: Introduced in (3.30). A universal element in $\mathbb{C}(S_n)$. Key counting result (3.31).

$\mathbb{P}_p$: Introduced in (3.26). It is a projector in $\mathbb{C}(S_n)$ which depends on the partition p. Relation to $\mathbb{P}$ given in (3.30).

$\mathbb{P}_\alpha$: Introduced in (4.1), also a projecctor in $\mathbb{C}(S_n)$. It is a slightly refined version of $\mathbb{P}_p$ with a specified embedding of p.

$\mathbf{p}$: Introduced in (4.2). It is a linear map from $\mathbb{C}(S_n)$ to $\mathbb{C}(S_n)$ with $\mathbf{p}_{\beta,\alpha}$ as the matrix elements of the map. $\mathbf{p}^{R,i,j}_{S,k,l}$ is the Fourier transform of $\mathbf{p}_{\beta,\alpha}$.

$\mathbb{P}^{(N)}$: Introduced in (3.48). The finite N version of the universal element $\mathbb{P}$ captures in symmetric group language the fact that the chiral ring counting is equivalent to counting states of N particles in a simple harmonic oscillator.

$\mathcal{P}$: Introduced in (4.11). It is a matrix built from $\mathbf{p}$, using Clebsch-Gordan coefficients of S_n. Key property: it is a projector implementing trace-symmetrization in the Fourier-basis.

$\mathcal{P}_{[\alpha]}$: Introduced in (A.10). A summand that appears in the expression for $\mathcal{P}$. Also a projector in the Fourier basis which symmetrizes only the given trace structure $[\alpha]$.

A.2 Poperties of $\mathbf{p} : \mathbb{C}(S_n) \to \mathbb{C}(S_n)$

The equation $\mathbb{P}_\alpha^2 = \mathbb{P}_\alpha$ implies that

$$\sum_\beta \mathbf{p}_{\beta,\alpha}\mathbf{p}_{\beta^{-1}\tau,\alpha} = \mathbf{p}_{\tau,\alpha} \tag{A.1}$$

We can convert the equation (A.1) into an equation in terms of the Fourier transformed quantity.

$$\mathbf{p}^{R_1 p_1 q_1}_{R_2 p_1 q_2} = \sum_{S_1,S_2,i_1,i_2,j_1,j_2} \frac{d_{S_1} d_{S_2}}{n! d_{R_1}} \mathbf{p}^{S_1\ i_1\ j_1}_{R_2\ p_2\ m}\ \mathbf{p}^{S_2\ i_2\ j_2}_{R_2\ m\ q_2}\ S^{S_1,S_2;R_1,\tau}_{\ j_1\ j_2\ q_1} S^{S_1,S_2;R_1,\tau}_{\ i_1\ i_2\ p_1} \tag{A.2}$$

Figure 3 gives a diagrammatic expression of the equation.

The coefficients $\mathbf{p}_{\beta,\alpha}$ have the symmetry

$$\mathbf{p}_{\beta,\alpha} = \mathbf{p}_{\gamma\beta\gamma^{-1},\gamma\alpha\gamma^{-1}} \tag{A.3}$$

Hence $\sum_{\alpha,\beta} \mathbf{p}_{\beta,\alpha}\ \beta \otimes \alpha$ defines an element of $\mathbb{C}(S_n) \otimes \mathbb{C}(S_n)$ which commutes with the diagonal $\mathbb{C}(S_n)$, and contains all the information about the counting $\chi_\Lambda(\mathbb{P}_\alpha)$ of irreps Λ among the symmetrized traces of trace structure detrmined by the cycles of α. Such elements were considered in [39] in order to resolve the τ multiplicity of symmetric group Clebsch-Gordans.

$$\mathbf{p}_{\beta^{-1},\alpha} = \mathbf{p}_{\beta,\alpha} \tag{A.4}$$

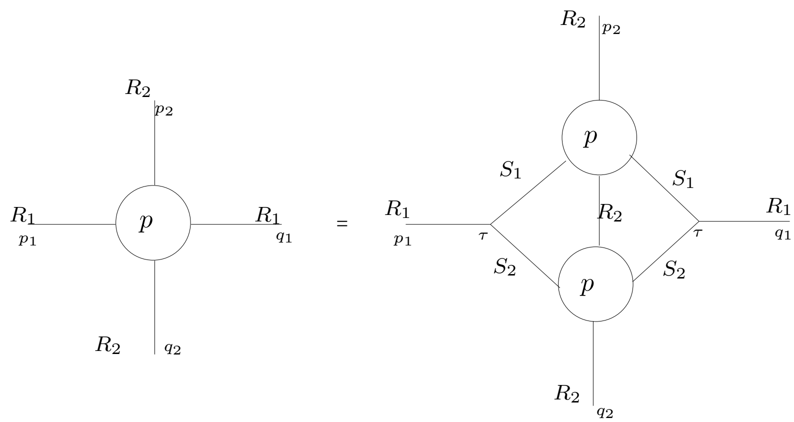

Figure 3: projector relation

Using $D^S_{ij}(\beta^{-1}) = D^S_{ji}(\beta)$. This means that

$$\mathbf{p}_{S\ k\ l}^{R\ i\ j} = \mathbf{p}_{S\ l\ k}^{R\ i\ j} \tag{A.5}$$

We also have

$$\mathbf{p}_{\beta,\alpha} = \mathbf{p}_{\gamma\beta,\alpha} = \mathbf{p}_{\beta\gamma,\alpha} \quad \text{for} \quad \gamma \in G(\alpha) \tag{A.6}$$

where $G(\alpha)$ is defined in section 4.1.

A.3 Projectors $\mathcal{P}_{[\alpha]}$

The matrix $\mathcal{P}$ in (4.11) can alternatively be written as:

$$(\mathcal{P})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2} = \frac{\sqrt{d_{R_1}d_{R_2}}}{n!}\sum_{\alpha\in S_n} D^\Lambda_{ij}(\mathbb{P}_\alpha)\left(D^{R_1}_{k_1l_1}(\alpha)S^{R_1\ R_1\ \Lambda,\ \tau_1}_{\ k_1\ l_1\ i}\right)\left(D^{R_2}_{k_2l_2}(\alpha)S^{R_2\ R_2\ \Lambda,\ \tau_2}_{\ k_2\ l_2\ j}\right). \tag{A.7}$$

This is seen by writing out $\mathbf{p}^{S\ k_1\ k_2}_{\Lambda\ m_\Lambda\ m'_\Lambda} = \sum_\alpha D^\Lambda_{m_\Lambda m'_\Lambda}(\mathbb{P}_\alpha)D^S_{k_1k_2}(\alpha)$ and then performing the sum over the representation S. Figure 4 expresses the RHS of (A.7) in a diagramatic form.

The first thing to note is that the argument of the α sum in (A.7) is invariant under conjugation $\alpha \to \gamma\alpha\gamma^{-1}$. That is because

$$D^\Lambda_{ij}(\mathbb{P}_{\gamma\alpha\gamma^{-1}}) = D^\Lambda_{ij}(\gamma\mathbb{P}_\alpha\gamma^{-1}) = D^\Lambda_{i'j'}(\mathbb{P}_\alpha)D^\Lambda_{ii'}(\gamma)D^\Lambda_{jj'}(\gamma) \tag{A.8}$$

and

$$D^{R_1}_{k_1l_1}(\gamma\alpha\gamma^{-1})S^{R_1\ R_1\ \Lambda,\ \tau_1}_{\ k_1\ l_1\ i} = D^{R_1}_{k'_1l'_1}(\alpha)D^{R_1}_{k_1k'_1}(\gamma)D^{R_1}_{l_1l'_1}(\gamma)S^{R_1\ R_1\ \Lambda,\ \tau_1}_{\ k_1\ l_1\ i} = D^{R_1}_{k_1l_1}(\alpha)S^{R_1\ R_1\ \Lambda,\ \tau_1}_{\ k_1\ l_1\ i_1}D^\Lambda_{i_1i}(\gamma^{-1}) \tag{A.9}$$

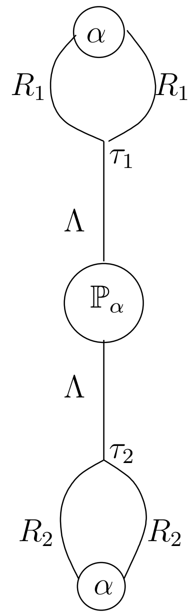

Figure 4: The operator $\mathcal{P}_\alpha$

and so the resulting $D^\Lambda(\gamma)$ matrices multiply to the identity. This invariance means that the sum over α can be really seen as a sum over the conjugacy classes $[\alpha]$ each coming with a factor of $n!/|\mathrm{Sym}(\alpha)|$. It is then convenient to define a new matrix $\mathcal{P}_{[\alpha]}$:

$$(\mathcal{P}_{[\alpha]})^{\Lambda,R_1,\tau_1}_{\Lambda,R_2,\tau_2} \equiv \frac{\sqrt{d_{R_1}d_{R_2}}}{|\mathrm{Sym}(\alpha)|} D^{\Lambda}_{ij}(\mathbb{P}_\alpha)\left(D^{R_1}_{k_1l_1}(\alpha)S^{R_1\;R_1\;\Lambda,\;\tau_1}_{k_1\;l_1\;i}\right)\left(D^{R_2}_{k_2l_2}(\alpha)S^{R_2\;R_2\;\Lambda,\;\tau_2}_{k_2\;l_2\;j}\right), \tag{A.10}$$

where the right hand side can be evaluated with any permutation α in the same conjugacy class, and thus $\mathcal{P}_{[\alpha]}$ only depends on $[\alpha]$. The full matrix $\mathcal{P}$ is then just a sum over conjugacy classes or partitions:

$$\mathcal{P} = \sum_{[\alpha]\vdash n} \mathcal{P}_{[\alpha]} \tag{A.11}$$

The newly introduced matrix $\mathcal{P}_{[\alpha]}$ can be seen as $\mathbb{P}_\alpha$ or $\mathbb{P}_p$ projector in $\mathbb{C}(S_n)$ appropriately transformed to the $|\Lambda, R, \tau\rangle$ basis in the same way that $\mathcal{P}$ is a transformed $\mathbb{P}$. The most remarkable fact is that $\mathcal{P}_{[\alpha]}$ itself is a projector with each $[\alpha]$ projecting to a *distinct* subspace! That is

$$\mathcal{P}_{[\alpha]}\mathcal{P}_{[\beta]} = \mathcal{P}_{[\alpha]}\delta_{[\alpha],\,[\beta]}. \tag{A.12}$$

This can be shown by evaluating

$$\sum_{R,\tau} d_R \left(D^{R}_{k_1l_1}(\alpha)S^{R\;R\;\Lambda,\;\tau}_{k_1\;l_1\;i}\right)\left(D^{R}_{k_2l_2}(\beta)S^{R\;R\;\Lambda,\;\tau}_{k_2\;l_2\;j}\right) = \sum_{\sigma\in S_n} D^{\Lambda}_{ij}(\sigma)\delta(\alpha\sigma\beta^{-1}\sigma^{-1}) \tag{A.13}$$

and then using this result to take the product:

$$
\begin{aligned}
&\sum_{R,\tau}(\mathcal{P}_{[\alpha]})^{\Lambda,R_1,\tau_1}_{\Lambda,R,\tau}(\mathcal{P}_{[\beta]})^{\Lambda,R,\tau}_{\Lambda,R_2,\tau_2}\\
&\quad=\frac{\sqrt{d_{R_1}d_{R_2}}}{|\mathrm{Sym}(\alpha)|^2}\sum_{\sigma\in S_n}\delta(\alpha\sigma\beta^{-1}\sigma^{-1})\left(D^{R_1}_{k_1l_1}(\alpha)S^{R_1\ R_1\ \Lambda,\ \tau_1}_{\ k_1\ l_1\ i}\right)D^{\Lambda}_{ij}(\mathbb{P}_\alpha\sigma\mathbb{P}_\beta)\left(D^{R_2}_{k_2l_2}(\beta)S^{R_2\ R_2\ \Lambda,\ \tau_2}_{\ k_2\ l_2\ j}\right)\\
&\quad=\frac{\sqrt{d_{R_1}d_{R_2}}}{|\mathrm{Sym}(\alpha)|^2}\sum_{\sigma\in S_n}\delta(\alpha\sigma\beta^{-1}\sigma^{-1})\left(D^{R_1}_{k_1l_1}(\alpha)S^{R_1\ R_1\ \Lambda,\ \tau_1}_{\ k_1\ l_1\ i}\right)D^{\Lambda}_{ij}(\mathbb{P}_\alpha)\left(D^{R_2}_{k_2l_2}(\alpha)S^{R_2\ R_2\ \Lambda,\ \tau_2}_{\ k_2\ l_2\ j}\right)\\
&\quad=(\mathcal{P}_{[\alpha]})^{\Lambda,R_2,\tau_2}_{\Lambda,R_1,\tau_1}\sum_{\sigma\in S_n}\frac{\delta(\alpha\sigma\beta^{-1}\sigma^{-1})}{|\mathrm{Sym}(\alpha)|}\\
&\quad=(\mathcal{P}_{[\alpha]})^{\Lambda,R_2,\tau_2}_{\Lambda,R_1,\tau_1}\,\delta_{[\alpha],\,[\beta]}.
\end{aligned}
\tag{A.14}
$$

In the third line we used the fact that with the delta function present we could replace $\beta=\sigma^{-1}\alpha\sigma$ and that $\mathbb{P}_\alpha^2=\mathbb{P}_\alpha$.

It follows from (A.12) and (A.11) that $\mathcal{P}$ itself is a projector

$$\mathcal{P}^2=\mathcal{P} \tag{A.15}$$

which is a nice consistency check.

It is interesting to ask what is the significance of the individual projectors $\mathcal{P}_{[\alpha]}$, that is, what operation on the traces do they correspond to. More precisely, let us define a linear operation on the Hilbert space

$$\mathrm{symm}_{[\sigma]}\left[\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}\right]=\sum_{R_1,\tau_1}(\mathcal{P}_{[\sigma]})^{\Lambda,M_\Lambda,R_1,\tau_1}_{\Lambda,M_\Lambda,R,\tau}\mathcal{O}_{\Lambda,M_\Lambda,R_1,\tau_1} \tag{A.16}$$

and ask what

$$\mathrm{symm}_{[\sigma]}[\mathrm{tr}_n(X_{\vec{a}}\alpha)] \tag{A.17}$$

evaluates to. From the derivation (4.9) and (4.10) it can be seen that in order to get $\mathcal{P}_{[\sigma]}$ on the RHS rather than $\mathcal{P}$ the sum over α has to run only over the conjugacy class $[\sigma]$. That implies that the following action is needed on the trace basis instead of (4.3):

$$\mathrm{symm}_{[\sigma]}[\mathrm{tr}_n(X_{\vec{a}}\alpha)]=\delta_{[\sigma],[\alpha]}\sum_\beta\mathbf{p}_{\beta,\alpha}\mathrm{tr}_n(X_{\beta(\vec{a})}\alpha). \tag{A.18}$$

The interpretation is clear: $\mathrm{symm}_{[\sigma]}$ and its representation on the Fourier basis $\mathcal{P}_{[\sigma]}$ acts by symmetrizing only the multitraces with the given trace structure $[\sigma]$. All traces with different structures are annihilated. This explains (A.12): if $[\alpha]$ and $[\beta]$ are different, there are no traces which survive. Also (A.11) has a straightforward interpretation that full symmetrization is performed by symmetrizing all trace structures.

B One-loop dilatation operator in the Fourier basis

In this section we derive an expression for the one-loop dilatation operator $\mathcal{H}_2$ in the quarter-BPS sector acting on the Fourier basis. We use the same methods that led us to the $\mathcal{P}$ matrix for symmetrization. The goal is to find the matrix $(\mathcal{H}_2)^{\Lambda_1,M_{\Lambda_1},R_1,\tau_1}_{\Lambda_2,M'_{\Lambda_2},R_2,\tau_2}$ such that

$$\mathcal{H}_2\mathcal{O}_{\Lambda,M_\Lambda,R,\tau} = \sum_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} (\mathcal{H}_2)^{\Lambda_1,\tilde{M}_{\Lambda_1},R_1,\tau_1}_{\Lambda,M_\Lambda,R,\tau}\mathcal{O}_{\Lambda_1,\tilde{M}_{\Lambda_1},R_1,\tau_1} \tag{B.1}$$

and to express it in terms of symmetric group objects. The matrix $(\mathcal{H}_2)$ could allow us to go beyond the analysis of BPS states (which correspond to its zero-eigenvalue eigenvectors) and to calculate the anomalous scaling dimensions of the non-BPS operators.

Inspecting the derivation of $\mathcal{P}$ in (4.9), (4.10) we find that the main ingredient there is the action of the linear operator of interest on the $\mathcal{O}_{\vec{a},\alpha}$ basis. In the case of $\mathcal{P}$ that was the symmetrization operator

$$\mathrm{symm}[\mathrm{tr}_n(\mathbb{X}_{\vec{a}}\alpha)] = \sum_\beta \mathbf{p}_{\beta,\alpha}\mathrm{tr}_n(\mathbb{X}_{\beta(\vec{a})}\alpha). \tag{B.2}$$

The $\mathcal{P}$ matrix is then a transformation of this action to the $|\Lambda, M_\Lambda, R, \tau\rangle$ basis.

The one-loop dilatation operator in the $U(3)$ sector acting on multitrace operators can be written as

$$\mathcal{H}_2 = -\frac{1}{2}\mathrm{tr}([X_i, X_j][\check{X}_i, \check{X}_j]) \tag{B.3}$$

with

$$(\check{X}_a)^i_j \equiv \frac{\partial}{\partial (X_a)^j_i}. \tag{B.4}$$

This can be translated to the $\mathcal{O}_{\vec{a},\alpha}$ basis as [60]

$$\mathcal{H}_2\mathrm{tr}_n(\mathbb{X}_{\vec{a}}\alpha) = \sum_{i\neq j}\mathrm{tr}_n(\mathbb{X}_{P_{ij}(\vec{a})}\,(i\alpha(j))'\alpha\,). \tag{B.5}$$

Here P_{ij} is an element of the symmetric group algebra

$$P_{ij} \equiv \mathbb{I} - (ij), \tag{B.6}$$

with the action on $\mathbb{X}_{\vec{a}}$ defined as $\mathbb{X}_{P_{ij}(\vec{a})} = \mathbb{X}_{\vec{a}} - \mathbb{X}_{(ij)(\vec{a})}$. The term $(i\alpha(j))'$ is a permutation interchanging i and $\alpha(j)$ which multiplies α, *except* for when $i = \alpha(j)$. In that case it should be understood as a factor of N:

$$(ij)' \equiv \begin{cases} (ij) & \text{if } \; i \neq j, \\ N & \text{if } \; i = j. \end{cases} \tag{B.7}$$

It is an element of $\mathbb{C}(S_n)$ when N is viewed as $N \in \mathbb{C}$. We can view (B.5) as expressing the fact that $\mathcal{H}_2$ is determined by an element

$$\sum_{\alpha}\sum_{i\neq j} \alpha \otimes (i\alpha(j))' \otimes P_{ij} \tag{B.8}$$

in $\mathbb{C}(S_n)\otimes\mathbb{C}(S_n)\otimes\mathbb{C}(S_n)$, which is an analog of the operator $\mathbf{p}$ in $\mathbb{C}(S_n)\otimes\mathbb{C}(S_n)$ introduced before.

Now that we have expressed the $\mathcal{H}_2$ action in terms of $\mathbb{C}(S_n)$ quantities, we can transform it to the Fourier basis. Repeating the derivation along the lines of (4.9), (4.10):

$$\begin{aligned}
\mathcal{H}_2\mathcal{O}_{\Lambda,M_\Lambda,R,\tau} &= \frac{\sqrt{d_R}}{n!}\sum_{k\neq l}\sum_{\alpha,\vec{a}} S^{R\;R\;\Lambda,\,\tau}_{\,i\;\,j\;\,m} D^R_{ij}(\alpha) C^{\vec{a}}_{\Lambda,M_\Lambda,m} \mathrm{tr}_n(\mathbb{X}_{P_{kl}(\vec{a})}\,(k\alpha(l))'\alpha\,) \\
&= \frac{\sqrt{d_R}}{n!}\sum_{k\neq l}\sum_{\alpha,\vec{a}} S^{R\;R\;\Lambda,\,\tau}_{\,i\;\,j\;\,m} D^R_{ij}(\alpha) C^{\vec{a}}_{\Lambda,M_\Lambda,m} \\
&\qquad\times \sum_{\Lambda_1,\tilde{M}_{\Lambda_1},R_1,\tau_1} \sqrt{d_{R_1}}\, C^{\Lambda_1,\tilde{M}_{\Lambda_1},m_1}_{P_{kl}(\vec{a})} D^{R_1}_{i_1j_1}((k\alpha(l))'\alpha) S^{\;R_1\;\;R_1\;\;\Lambda_1,\tau_1}_{\;i_1\;\;j_1\;\;m_1} \mathcal{O}_{\Lambda_1,\tilde{M}_{\Lambda_1},R_1,\tau_1} \\
&= \sum_{R_1,\tau_1} \mathcal{O}_{\Lambda,M_\Lambda,R_1,\tau_1} \frac{\sqrt{d_R d_{R_1}}}{n!} \\
&\qquad\times \sum_{k\neq l}\sum_{\alpha} S^{R\;R\;\Lambda,\,\tau}_{\,i\;\,j\;\,m} D^R_{ij}(\alpha) D^{\Lambda}_{mm_1}(P_{kl}) D^{R_1}_{i_1j_1}((k\alpha(l))'\,\alpha) S^{\;R_1\;\;R_1\;\;\Lambda,\tau_1}_{\;i_1\;\;j_1\;\;m_1} \\
&\equiv \sum_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} (\mathcal{H}_2)^{\Lambda_1,\tilde{M}_{\Lambda_1},R_1,\tau_1}_{\Lambda,M_\Lambda,R,\tau} \mathcal{O}_{\Lambda_1,\tilde{M}_{\Lambda_1},R_1,\tau_1}.
\end{aligned} \tag{B.9}$$

We have thus calculated the transformation matrix to be:

$$\begin{aligned}
(\mathcal{H}_2)^{\Lambda_1,M_{\Lambda_1},R_1,\tau_1}_{\Lambda_2,\tilde{M}_{\Lambda_2},R_2,\tau_2} &= \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},\tilde{M}_{\Lambda_2}} \frac{\sqrt{d_{R_1}d_{R_2}}}{n!} \\
&\times \sum_{k\neq l}\sum_{\alpha\in S_n} D^{\Lambda_1}_{m_1m_2}(P_{kl}) \left(D^{R_1}_{i_1j_1}((k\alpha(l))'\,\alpha)\, S^{\;R_1\;\;R_1\;\;\Lambda_1,\tau_1}_{\;i_1\;\;j_1\;\;m_1}\right) \left(D^{R_2}_{i_2j_2}(\alpha)\, S^{\;R_2\;\;R_2\;\;\Lambda_1,\tau_2}_{\;i_2\;\;j_2\;\;m_2}\right).
\end{aligned} \tag{B.10}$$

This expression should be compared to (A.7). One can see the $D^\Lambda(P_{kl})$ appearing in the same place where we had $D^\Lambda(\mathbb{P}_\alpha)$ before, because that is the permutation acting on $\vec{a}$ in the trace basis. A new ingredient here is the $(k\alpha(l))'$ operator acting on α, which was absent in the symmetrization action.

C Examples of BPS operators

C.1 $\Lambda = [2, 2]$

In this section we will provide an example of explicit calculation of BPS operators in the representation $\Lambda = [2, 2]$. Since the operators throughout the section will all have the same Λ, also M_Λ will be the highest weight state and $\tau = 1$ because there is no multiplicity, we will abbreviate

$$\mathcal{O}_R \equiv \mathcal{O}_{\Lambda=[2,2],M_\Lambda=\mathrm{HWS},R,\tau=1}. \tag{C.1}$$

C.1.1 Large N operators

First we will assume $N \geq 3$, but will keep all $1/N$ corrections. Then in the next section we will also explain what happens for lower N.

We start with the basis in the free theory. Using (2.4) the operators are:

$$\begin{aligned}
\mathcal{O}_{R=[3,1]} &= \frac{1}{4\sqrt{3}N^2}\left(-\mathrm{tr}(X_i X^i X_j X^j) + \mathrm{tr}(X_i X_j)\mathrm{tr}(X^i X^j) + \mathrm{tr}(X_i X_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\right) \\
\mathcal{O}_{R=[2,2]} &= \frac{1}{2\sqrt{6}N^2}\left(\mathrm{tr}(X_i X^i X_j X^j) + \mathrm{tr}(X_i X_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\right) \\
\mathcal{O}_{R=[2,1,1]} &= \frac{1}{4\sqrt{3}N^2}\left(-\mathrm{tr}(X_i X^i X_j X^j) - \mathrm{tr}(X_i X_j)\mathrm{tr}(X^i X^j) + \mathrm{tr}(X_i X_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\right).
\end{aligned} \tag{C.2}$$

Here for compactness we are using upper-lower index notation meaning $X_i X^i = X_1 X_2 - X_2 X_1$, which is a singlet in the quarter-BPS sector.

The symmetrization matrix $\mathcal{P}$ can be evaluated using (4.11) or (A.7). In the $\Lambda = [2, 2]$ sector we get:

$$\mathcal{P} = \begin{pmatrix} \frac{3}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{3}{4} \end{pmatrix}. \tag{C.3}$$

It can be verified that transforming the free operator basis with this matrix indeed symmetrizes terms in each trace. In this example that just means that the $\mathrm{tr}(X_i X_j X^i X^j)$ is dropped.

$$\begin{pmatrix} \mathcal{O}^{\mathrm{S}}_{[3,1]} & \mathcal{O}^{\mathrm{S}}_{[2,2]} & \mathcal{O}^{\mathrm{S}}_{[2,1,1]} \end{pmatrix} = \begin{pmatrix} \mathcal{O}_{[3,1]} & \mathcal{O}_{[2,2]} & \mathcal{O}_{[2,1,1]} \end{pmatrix} \begin{pmatrix} \frac{3}{4} & \frac{1}{2\sqrt{2}} & -\frac{1}{4} \\ \frac{1}{2\sqrt{2}} & \frac{1}{2} & \frac{1}{2\sqrt{2}} \\ -\frac{1}{4} & \frac{1}{2\sqrt{2}} & \frac{3}{4} \end{pmatrix} \tag{C.4}$$

$$
\begin{aligned}
\mathcal{O}^{\mathrm{S}}_{[3,1]} &= \frac{1}{4\sqrt{3}N^2}\left(\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j) + \mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\right) \\
\mathcal{O}^{\mathrm{S}}_{[2,2]} &= \frac{1}{2\sqrt{6}N^2}\left(\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\right) \\
\mathcal{O}^{\mathrm{S}}_{[2,1,1]} &= \frac{1}{4\sqrt{3}N^2}\left(-\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j) + \mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\right).
\end{aligned}
\tag{C.5}
$$

Clearly, there are only 2 independent symmetric operators. This corresponds to the fact that at large N there are only 2 BPS operators when the coupling is turned on, compared to 3 operators in the free theory. It is also worth noting that the null eigenvector

$$
\begin{pmatrix} \frac{1}{2} \\ \frac{-1}{\sqrt{2}} \\ \frac{1}{2} \end{pmatrix}
\tag{C.6}
$$

corresponding to $\frac{1}{2}\mathcal{O}_{[3,1]} - \frac{1}{\sqrt{2}}\mathcal{O}_{[2,2]} + \frac{1}{2}\mathcal{O}_{[2,1^2]}$ is the descendant operator. This is a general fact that the descendants can be characterized as $\mathrm{Ker}(\mathcal{P})$.

We can write down the precise expressions for the BPS operators using (4.14):

$$
\mathcal{O}^{\mathrm{BPS}}_{\Lambda,M_\Lambda,R,\tau} = \sum_{R_1,\tau_1} (\mathcal{G}\mathcal{P})^{\Lambda,R_1,\tau_1}_{\Lambda,R,\tau}\,\mathcal{O}_{\Lambda,M_\Lambda,R_1,\tau_1}
\tag{C.7}
$$

with the diagonal matrix

$$
\mathcal{G} = \mathrm{diag}\left\{\frac{N^n d_R}{n!\,\mathrm{Dim}R}\right\} = \frac{N^3}{N^2-1}\begin{pmatrix} \frac{1}{N+2} & 0 & 0 \\ 0 & \frac{1}{N} & 0 \\ 0 & 0 & \frac{1}{N-2} \end{pmatrix}
\tag{C.8}
$$

Let us arbitrarily pick $\mathcal{O}^{\mathrm{BPS}}_{[3,1]}$ and $\mathcal{O}^{\mathrm{BPS}}_{[2,1,1]}$ as the two linearly independent operators. In terms of the free operators we get

$$
\begin{aligned}
\mathcal{O}^{\mathrm{BPS}}_{[3,1]} &= \frac{N^3}{N^2-1}\left(\frac{3}{4}\frac{\mathcal{O}_{[3,1]}}{N+2} + \frac{1}{2\sqrt{2}}\frac{\mathcal{O}_{[2,2]}}{N} - \frac{1}{4}\frac{\mathcal{O}_{[2,1,1]}}{N-2}\right), \\
\mathcal{O}^{\mathrm{BPS}}_{[2,1,1]} &= \frac{N^3}{N^2-1}\left(-\frac{1}{4}\frac{\mathcal{O}_{[3,1]}}{N+2} + \frac{1}{2\sqrt{2}}\frac{\mathcal{O}_{[2,2]}}{N} + \frac{3}{4}\frac{\mathcal{O}_{[2,1,1]}}{N-2}\right).
\end{aligned}
\tag{C.9}
$$

Explicitly in terms of products of traces they are

$$\begin{aligned}
\mathcal{O}^{\mathrm{BPS}}_{[3,1]} &= \frac{1}{4\sqrt{3}(N^2-1)(N^2-4)} \big(2(N-1)\mathrm{tr}(X_i X^i X_j X^j) \\
&\quad + N(N-1)\mathrm{tr}(X_i X_j)\mathrm{tr}(X^i X^j) + (N^2-2N-2)\mathrm{tr}(X_i X_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\big)\,, \\
\mathcal{O}^{\mathrm{BPS}}_{[2,1,1]} &= \frac{1}{4\sqrt{3}(N^2-1)(N^2-4)} \big(-2(N+1)\mathrm{tr}(X_i X^i X_j X^j) \\
&\quad - N(N+1)\mathrm{tr}(X_i X_j)\mathrm{tr}(X^i X^j) + (N^2+2N-2)\mathrm{tr}(X_i X_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\big)\,.
\end{aligned} \tag{C.10}$$

To the leading order in N the operators $\mathcal{O}^{\mathrm{BPS}}_R$ are equal to the symmetric combinations $\mathcal{O}^{\mathrm{S}}_R$, but there are further $1/N$ corrections which make the state exactly annihilated by the one-loop dilatation operator. Note how the antisymmetric combinations $\mathrm{tr}(X_i X_j X^i X^j)$ appear in the exact operator, although only in the subleading-N order.

C.1.2 Finite N cutoff

The expressions for BPS operators (C.9), (C.10) are valid whenever $N \geq 3$. Clearly something goes wrong when $N = 2$ as indicated by $N-2$ in the denominator – this is where the finite-N cutoff takes effect. We will demonstrate here how it is dealt with according to the general procedure described in Section 5. It turns out that at $N = 2$ there is only one BPS operator remaining.

The cutoff is best seen in the free operator basis: $\mathcal{O}_{\Lambda,M_\Lambda,R,\tau}$ becomes 0 whenever Young diagram R has more than N rows. In our case one of the three free operators drops out:

$$\mathcal{O}_{[2,1,1]} = 0. \tag{C.11}$$

Note that implies a relationship between products of traces, using (C.2):

$$-\,\mathrm{tr}(X_i X^i X_j X^j) - \mathrm{tr}(X_i X_j)\mathrm{tr}(X^i X^j) + \mathrm{tr}(X_i X_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) = 0. \tag{C.12}$$

In the expressions for BPS operators exactly these free operators which vanish are accompanied by 0 in denominator because of $\mathrm{Dim}R = 0$.

It is not immediately obvious how then to find correct BPS combinations. One could try to just naively cut off the terms in (C.9) and write

$$\begin{aligned}
\mathcal{O}^{\mathrm{BPS}}_{[3,1]} &\sim \frac{3}{4}\frac{\mathcal{O}_{[3,1]}}{N+2} + \frac{1}{2\sqrt{2}}\frac{\mathcal{O}_{[2,2]}}{N}, \\
\mathcal{O}^{\mathrm{BPS}}_{[2,1,1]} &\sim -\frac{1}{4}\frac{\mathcal{O}_{[3,1]}}{N+2} + \frac{1}{2\sqrt{2}}\frac{\mathcal{O}_{[2,2]}}{N},
\end{aligned}$$

however, this is *not* correct. If it were true, it would seem that we are still left with two BPS operators, however, it turns out that even at $N = 2$ neither of them is annihilated by one-loop dilatation operator!

What we have to do instead is to pick linear combinations of $\mathcal{O}_R^{\mathrm{BPS}}$ which do not depend on these $\mathcal{O}_R$ that are set to zero. These linear combinations will be "protected" from what happens to the terms with $\mathrm{Dim}R = 0$ in the denominator. This procedure corresponds exactly to picking vectors in the intersection space $\mathrm{Im}(\mathcal{P}_I)$ described in Section 5, where it is shown to give all the BPS operators. In our example we have to take the combination:

$$\begin{aligned}\mathcal{O}^{\mathrm{BPS};(N=2)} = \mathcal{O}^{\mathrm{BPS}}_{[3,1]} + \frac{1}{3}\mathcal{O}^{\mathrm{BPS}}_{[2,1,1]} &= \frac{N^3}{N^2-1}\left(\frac{2}{3}\frac{\mathcal{O}_{[3,1]}}{N+2} + \frac{\sqrt{2}}{3}\frac{\mathcal{O}_{[2,2]}}{N}\right)\\ &= \frac{4}{9}(\mathcal{O}_{[3,1]} + \sqrt{2}\mathcal{O}_{[2,2]})\end{aligned} \tag{C.13}$$

This is the single remaining BPS operator at $N = 2$. To be more precise, in the language of Section 5, we find the projector $\mathcal{P}_I$ for the intersection of $\mathrm{Im}(\mathcal{I}_{(N)})$ and $\mathrm{Im}(\mathcal{P})$ to be

$$\mathcal{P}_I^{(N=2)} = \begin{pmatrix} \frac{2}{3} & \frac{\sqrt{2}}{3} & 0 \\ \frac{\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{C.14}$$

Then we get the BPS operators by acting with $(\mathcal{G}\mathcal{P}_I)$ on the full Hilbert space, which gives precisely the one-dimensional space spanned by the operator (C.13).

There is a related subtlety worth pointing out. It is true that counting BPS operators is the same as counting states in the chiral ring, where the matrices X_i can be taken to be diagonal. One may wonder whether this is the same as counting independent products of traces, where the terms in each trace are symmetrized, that is, the $\mathcal{O}^{\mathrm{S}}_{\Lambda;R,\tau}$ operators. It turns out that this is *not* the case when the finite-N cutoff applies. In our present example at $N = 2$ there are still two independent symmetric operators: $\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)$ and $\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)$. However, one has to eliminate one linear combination - the one that is equal to $\mathrm{tr}(X_iX^iX_jX^j)$ by the relationship (C.12). This leaves us with one operator.

C.1.3 Orthogonal BPS basis and $\mathcal{P}\mathcal{G}\mathcal{P}$ eigenvalues

In order to find an orthogonal basis for the BPS operators we have to look for the eigenvectors of the two-point function matrix on the BPS operator space $\mathcal{P}\mathcal{G}\mathcal{P}$. In the case of our present example $\Lambda = [2,2]$ we can calculate the matrix to be:

$$\mathcal{PGP} = \frac{N^2}{(N^2-1)(N^2-4)} \begin{pmatrix} \frac{3N^2-4N-2}{4} & \frac{N^2-2N-2}{2\sqrt{2}} & -\frac{N^2+2}{4} \\ \frac{N^2-2N-2}{2\sqrt{2}} & \frac{N^2-2}{2} & \frac{N^2-2N-2}{2\sqrt{2}} \\ -\frac{N^2+2}{4} & \frac{N^2-2N-2}{2\sqrt{2}} & \frac{3N^2-4N-2}{4} \end{pmatrix}. \tag{C.15}$$

Its characteristic polynomial is

$$\det(\mathcal{PGP} - \lambda\mathbb{I}) = \frac{\lambda}{(N^2-4)(N^2-1)^2}\left(-(N^2-4)(N^2-1)^2\lambda^2 + 2N^2(N^2-1)^2\lambda - N^6\right) \tag{C.16}$$

and we find eigenvalues

$$\begin{aligned} \lambda_{1,2} &= \frac{N^2}{(N^2-4)(N^2-1)}\left(N^2 - 1 \mp \sqrt{2N^2+1}\right), \\ \lambda_3 &= 0. \end{aligned} \tag{C.17}$$

The eigenvectors with zero eigenvalues are the same as those of $\mathcal{P}$ and span the non-BPS combinations. We are interested in the other two eigenvectors

$$\begin{aligned} \mathcal{O}^{\text{BPS}}_{(1,2)} &= \frac{-3N \mp 2\sqrt{2N^2+1}}{N}\mathcal{O}^{\text{BPS}}_{[3,1]} + \frac{-\sqrt{2}(N+1 \pm \sqrt{2N^2+1})}{N}\mathcal{O}^{\text{BPS}}_{[2,2]} + \frac{N-2}{N}\mathcal{O}^{\text{BPS}}_{[2,1,1]} \\ &= \frac{N^2}{N^2-1}\left((-3N \mp 2\sqrt{2N^2+1})\frac{\mathcal{O}_{[3,1]}}{N+2} - \sqrt{2}(N+1 \pm \sqrt{2N^2+1})\frac{\mathcal{O}_{[2,2]}}{N} + \mathcal{O}_{[2,1,1]}\right). \end{aligned} \tag{C.18}$$

These two operators provide us with an *orthogonal basis* for the BPS sector. The normalization that we picked so far is arbitrary, except that we adjusted it to allow for a nice limit to $N = 2$ as we will see shortly.

C.1.4 $N = 2$ limit revisited

As discussed already, at $N = 2$ there is only one BPS operator in the $\Lambda = [2,2]$ representation given by the combination (C.13). On the other hand, the operators $\mathcal{O}^{\text{BPS}}_{(1)}$, $\mathcal{O}^{\text{BPS}}_{(2)}$ provide an N-dependent basis of orthogonal states from a calculation done assuming $N \geq n$. According to the discussion in Section 6.1 we can in fact take the limit of the orthogonal basis to $N \to 2$ and find the one remaining BPS state as the surviving eigenvector with finite eigenvalue. We will see an example of that here.

Let us simply take the expressions for eigenvalues and eigenvectors (C.17), (C.18) and calculate the limit $N \to 2$. We find

$$\lambda_1^{(N=2)} = \frac{8}{9}, \quad \lambda_2^{(N=2)} \sim \frac{2}{N-2} \to \infty \tag{C.19}$$

and

$$
\begin{aligned}
\mathcal{O}^{\mathrm{BPS};(N=2)}_{(1)} &= -4\mathcal{O}_{[3,1]} - 4\sqrt{2}\mathcal{O}_{[2,2]} + \frac{4}{3}\mathcal{O}_{[2,1,1]}, \\
\mathcal{O}^{(N=2)}_{(2)} &= \frac{4}{3}\mathcal{O}_{[2,1,1]}.
\end{aligned}
\tag{C.20}
$$

The operator $\mathcal{O}_{[2,1,1]}$ is identically zero in this case and so can be dropped from the operators. The first eigenvector $\mathcal{O}^{\mathrm{BPS};(N=2)}_{(1)}$, which has a finite eigenvalue, is indeed the right BPS state (C.13) up to an overall normalization. The second eigenvector, with divergent eigenvalue, disappears from the Hilbert space. This reflects a pattern that we expect to be general.

C.1.5 $N \to \infty$ limit revisited

Another interesting limit of the orthogonal BPS operators that we can take is $N \to \infty$. We know already that at large N the leading term of the BPS operators is always a symmetrized trace. So a priori we can take any orthogonal combination of $\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)$ and $\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)$ to be the large-N basis. What is interesting about taking the limit of (C.18) is that it provides us with a unique *preferred* basis. We get

$$
\begin{aligned}
\mathcal{O}^{\mathrm{BPS};(N=\infty)}_{(1,2)} &= (-3 \mp 2\sqrt{2})\mathcal{O}_{[3,1]} + (-\sqrt{2} \mp 2)\mathcal{O}_{[2,2]} + \mathcal{O}_{[2,1,1]} \\
&= -\frac{2 \pm \sqrt{2}}{2\sqrt{3}N^2}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j) - \frac{1 \pm \sqrt{2}}{\sqrt{3}N^2}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j).
\end{aligned}
\tag{C.21}
$$

Both eigenvalues approach 1, because $\mathcal{P}\mathcal{G}\mathcal{P} \to \mathcal{P}$. These states can also be found as the eigenstates of $\mathcal{P}\mathcal{G}_1\mathcal{P}$, as discussed in Section 6.4. The $\mathcal{G}_1$ matrix is in this case:

$$
\mathcal{G}_1 = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.
\tag{C.22}
$$

C.2 $\Lambda = [3,2]$

Here we work out the calculations for the $\Lambda = [3,2]$ sector. In this section we again abbreviate the operators:

$$
\begin{aligned}
\mathcal{O}_{R,\tau} &\equiv \mathcal{O}_{\Lambda=[3,2],M_\Lambda=\mathrm{HWS},R,\tau} \\
\mathcal{O}_{R} &\equiv \mathcal{O}_{\Lambda=[3,2],M_\Lambda=\mathrm{HWS},R,\tau=1}
\end{aligned}
\tag{C.23}
$$

with the $U(3)$ label M_Λ in the highest weight state.

The free operators:

$$
\begin{aligned}
\mathcal{O}_{R=[4,1]} &= \frac{1}{2\sqrt{10}N^{5/2}}\Bigg(-\mathrm{tr}(X_1X_iX^iX_jX^j) - \frac{2}{3}\mathrm{tr}(X_iX^iX_jX^j)\mathrm{tr}(X_1) \\
&\quad +\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^iX^j) + \frac{1}{3}\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) \\
&\quad +\frac{2}{3}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)\mathrm{tr}(X_1) + \frac{1}{3}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\mathrm{tr}(X_1)\Bigg), \\
\mathcal{O}_{R=[3,2]} &= \frac{1}{4\sqrt{2}N^{5/2}}\Bigg(-\mathrm{tr}(X_1X_iX^iX_jX^j) - \frac{2}{3}\mathrm{tr}(X_iX^iX_jX^j)\mathrm{tr}(X_1) \\
&\quad -\frac{2}{3}\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) \\
&\quad -\frac{1}{3}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)\mathrm{tr}(X_1) - \frac{2}{3}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\mathrm{tr}(X_1)\Bigg), \\
\mathcal{O}_{R=[3,1^2],\tau=1} &= \frac{1}{4\sqrt{5}N^{5/2}}\big(\mathrm{tr}(X_1X_iX^iX_jX^j) - 2\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) \\
&\quad +\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)\mathrm{tr}(X_1)\big) \\
\mathcal{O}_{R=[3,1^2],\tau=2} &= \frac{1}{\sqrt{30}N^{5/2}}\big(\mathrm{tr}(X_iX^iX_jX^j)\mathrm{tr}(X_1) + \mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^iX^j) \\
&\quad -\frac{1}{2}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\mathrm{tr}(X_1)\Bigg), \\
\mathcal{O}_{R=[2^2,1]} &= \frac{1}{4\sqrt{2}N^{5/2}}\Bigg(-\mathrm{tr}(X_1X_iX^iX_jX^j) + \frac{2}{3}\mathrm{tr}(X_iX^iX_jX^j)\mathrm{tr}(X_1) \\
&\quad -\frac{2}{3}\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) \\
&\quad -\frac{1}{3}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)\mathrm{tr}(X_1) + \frac{2}{3}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\mathrm{tr}(X_1)\Bigg), \\
\mathcal{O}_{R=[2,1^3]} &= \frac{1}{2\sqrt{10}N^{5/2}}\Bigg(\mathrm{tr}(X_1X_iX^iX_jX^j) - \frac{2}{3}\mathrm{tr}(X_iX^iX_jX^j)\mathrm{tr}(X_1) \\
&\quad +\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^iX^j) - \frac{1}{3}\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) \\
&\quad -\frac{2}{3}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)\mathrm{tr}(X_1) + \frac{1}{3}\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\mathrm{tr}(X_1)\Bigg).
\end{aligned}
\tag{C.24}
$$

The symmetrization matrix:

$$
\mathcal{P} = \begin{pmatrix}
\frac{2}{3} & -\frac{\sqrt{5}}{6} & \frac{1}{5\sqrt{2}} & \frac{2}{5\sqrt{3}} & -\frac{1}{6\sqrt{5}} & \frac{1}{15} \\
-\frac{\sqrt{5}}{6} & \frac{7}{12} & \frac{1}{2\sqrt{10}} & \frac{1}{\sqrt{15}} & -\frac{1}{12} & \frac{1}{6\sqrt{5}} \\
\frac{1}{5\sqrt{2}} & \frac{1}{2\sqrt{10}} & \frac{9}{10} & 0 & \frac{1}{2\sqrt{10}} & -\frac{1}{5\sqrt{2}} \\
\frac{2}{5\sqrt{3}} & \frac{1}{\sqrt{15}} & 0 & \frac{3}{5} & -\frac{1}{\sqrt{15}} & \frac{2}{5\sqrt{3}} \\
-\frac{1}{6\sqrt{5}} & -\frac{1}{12} & \frac{1}{2\sqrt{10}} & -\frac{1}{\sqrt{15}} & \frac{7}{12} & \frac{\sqrt{5}}{6} \\
\frac{1}{15} & \frac{1}{6\sqrt{5}} & -\frac{1}{5\sqrt{2}} & \frac{2}{5\sqrt{3}} & \frac{\sqrt{5}}{6} & \frac{2}{3}
\end{pmatrix}, \tag{C.25}
$$

which has rank 4, corresponding to four BPS operators in this sector. The $\mathcal{G}$ matrix:

$$
\mathcal{G} = \frac{N^4}{(N^2-1)(N^2-4)} \begin{pmatrix}
\frac{N-2}{N+3} & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{N-2}{N} & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{N+2}{N} & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{N+2}{N-3}
\end{pmatrix} \tag{C.26}
$$

The BPS operators are again given by

$$
\mathcal{O}^{\mathrm{BPS}}_{R,\tau} = \sum_{R_1,\tau_1} (\mathcal{G}\mathcal{P})_{R,\tau}^{R_1,\tau_1} \mathcal{O}_{R_1,\tau_1} \tag{C.27}
$$

The resulting combinations are best seen by inspecting matrices $\mathcal{P}$ and $\mathcal{G}$ and not worth writing out explicitly.

The two-point function matrix on the BPS states will be

$$
\mathcal{P}\mathcal{G}\mathcal{P} = \frac{N^3}{(N^2-1)(N^2-4)(N^2-9)} \times
\begin{pmatrix}
\frac{10N^3-37N^2+11N+36}{15} & -\frac{5N^3-23N^2+4N+54}{6\sqrt{5}} & \frac{N^3-3N^2-6}{5\sqrt{2}} & \frac{2N^3-4N^2+4N-18}{5\sqrt{3}} & -\frac{N^3-3N^2-4N-18}{6\sqrt{5}} & \frac{N^3+11N}{15} \\
 & \frac{7N^3-16N^2-37N+72}{12} & \frac{N^3-15N+12}{2\sqrt{10}} & \frac{N^3+N^2-13N+9}{\sqrt{15}} & -\frac{N^3-19N}{12} & \frac{N^3+3N^2-4N+18}{6\sqrt{5}} \\
 & & \frac{9N^3-75N}{10} & -\frac{6N^2-18}{5\sqrt{6}} & \frac{N^3-15N-12}{2\sqrt{10}} & -\frac{N^3+3N^2+6}{5\sqrt{2}} \\
 & & & \frac{3N^3-19N}{5} & -\frac{N^3-N^2-13N-9}{\sqrt{15}} & \frac{2N^3+4N^2+4N+18}{5\sqrt{3}} \\
 & \ldots & & & \frac{7N^3+16N^2-37N-72}{12} & \frac{5N^3+23N^2+4N-54}{6\sqrt{5}} \\
 & & & & & \frac{10N^3+37N^2+11N-36}{15}
\end{pmatrix} \tag{C.28}
$$

The characteristic polynomial of this matrix is:

$$
\begin{aligned}
\det(\mathcal{P}\mathcal{G}\mathcal{P} - \lambda\mathbb{I}) = {} & \frac{\lambda^2}{(N^2-9)(N^2-4)^3(N^2-1)^4} \big((N^2-9)(N^2-4)^3(N^2-1)^4\lambda^4 \\
& -4N^4(N^2-4)^3(N^2-1)^3\,\lambda^3 + 3N^6(2N^4-3N^2+4)(N^2-4)(N^2-1)^2\,\lambda^2 \\
& -2N^{10}(2N^4-3N^2+4)(N^2-1)\,\lambda + N^{16}\big)\,,
\end{aligned} \tag{C.29}
$$

The λ^2 prefactor indicates two zero-eigenvalue states corresponding to the kernel of $\mathcal{P}$. The remaining fourth-degree characteristic polynomial equation can not be solved explicitly, but it is useful for inspecting what happens at finite values of N. When N approaches critical values, some eigenvalues of $\mathcal{PGP}$ run off to infinity while the number of eigenvalues that remain finite is equal to the number of surviving BPS operators. One can see that (C.29) reduces to a third degree polynomial when $N \to 3$ and to a first degree polynomial when $N \to 2$. We can conclude the following multiplicities of BPS operators depending on N:

N	# of BPS states
> 3	4
3	3
2	1
1	0

Finally, we find the orthogonal BPS states in the limit of $N \to \infty$. That is, the preferred basis of symmetrized traces. We calculate the $O(1/N)$ term of $\mathcal{G}$:

$$\mathcal{G} = \mathbb{I} + \frac{1}{N}\mathcal{G}_1 + O(1/N^2), \tag{C.30}$$

$$\mathcal{G}_1 = \begin{pmatrix} -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 5 \end{pmatrix}. \tag{C.31}$$

Then the eigenvectors of

$$\mathcal{P}\mathcal{G}_1\mathcal{P} = \mathcal{P} + \frac{1}{N}\mathcal{P}\mathcal{G}_1\mathcal{P} + O(1/N^2) \tag{C.32}$$

approach the eigenvectors of $\mathcal{P}\mathcal{G}_1\mathcal{P}$ as $N \to \infty$. Since $\mathcal{P}$ and $\mathcal{P}\mathcal{G}_1\mathcal{P}$ commute, they can be simultaneously diagonalized. That means that we can resolve the degeneracy of the symmetrized traces which all have $\mathcal{P}$ eigenvalue $+1$, by finding combinations of them which are at the same time eigenstates of $\mathcal{P}\mathcal{G}_1\mathcal{P}$, but with different eigenvalues. We find these eigenstates reexpressed in terms of multitraces to be:

$$\begin{aligned}
\mathcal{O}^{\mathrm{BPS};(N=\infty)}_{(1)} &= 10\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^iX^j) + (-\sqrt{19}+3\sqrt{11})\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) \\
&+ \frac{1}{2}(3\sqrt{19}+\sqrt{11})\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)\mathrm{tr}(X_1) + \frac{1}{2}(-3+\sqrt{209})\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\mathrm{tr}(X_1), \\
\mathcal{O}^{\mathrm{BPS};(N=\infty)}_{(2)} &= 10\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^iX^j) + (-\sqrt{19}-3\sqrt{11})\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) \\
&+ \frac{1}{2}(3\sqrt{19}-\sqrt{11})\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)\mathrm{tr}(X_1) + \frac{1}{2}(-3-\sqrt{209})\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\mathrm{tr}(X_1), \\
\mathcal{O}^{\mathrm{BPS};(N=\infty)}_{(3)} &= 10\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^iX^j) + (\sqrt{19}+3\sqrt{11})\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) \\
&+ \frac{1}{2}(-3\sqrt{19}+\sqrt{11})\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)\mathrm{tr}(X_1) + \frac{1}{2}(-3-\sqrt{209})\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\mathrm{tr}(X_1), \\
\mathcal{O}^{\mathrm{BPS};(N=\infty)}_{(4)} &= 10\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^iX^j) + (\sqrt{19}-3\sqrt{11})\mathrm{tr}(X_1X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j) \\
&+ \frac{1}{2}(-3\sqrt{19}-\sqrt{11})\mathrm{tr}(X_iX_j)\mathrm{tr}(X^iX^j)\mathrm{tr}(X_1) + \frac{1}{2}(-3+\sqrt{209})\mathrm{tr}(X_iX_j)\mathrm{tr}(X^i)\mathrm{tr}(X^j)\mathrm{tr}(X_1).
\end{aligned} \tag{C.33}$$

C.3 Computational remarks

The main challenge in the calculations was getting the $\mathcal{P}$ matrix. In our implementation we used *Mathematica* to evaluate the expression (A.7). It involves three ingredients: calculation of $\mathbb{P}_\alpha$, the representation matrices $D^R_{ij}(\alpha)$ and the Clebsch-Gordan (CG) coefficients $S^{R\ R\ \Lambda,\ \tau}_{i\ j\ k}$. We evaluated the algebra element $\mathbb{P}_\alpha$ directly using the definition (4.1). The representation matrices were calculated using the method described in Section 7-7 of [40].

Finally, the CG coefficients $S^{R\ R\ \Lambda,\ \tau}_{i\ j\ k}$ were calculated using the recursion relationships according to the method described in Section 7-14 of [40]. This is the most computationally intensive procedure. We also found that the relevant equations (7-226), (7-233) needed a small clarification for dealing with multiplicities, which is worth pointing out here. The coefficients $K^{\gamma\ \alpha\ \beta}_{i\ a\ b}$ used in the recursion relationship have, in fact, *two* extra multiplicity indexes. The equation (7-226) should read as:

$$S^{\gamma\ \ \alpha\ \ \beta,\ \tau}_{ij\ ab\ ef} = \sum_{\tau'} K^{\gamma\ \alpha\ \beta,\ \tau'}_{i\ a\ e}(\tau)\ S^{\gamma_i\ \alpha_a\ \beta_e,\ \tau'}_{j\ \ b\ \ f}, \tag{C.34}$$

where τ' runs over the multiplicity associated with $(\gamma_i, \alpha_a, \beta_e)$. Then $K^{\gamma\ \alpha\ \beta,\ \tau'}_{i\ a\ e}(\tau)$ is the τ'th solution of the equation (7-233) which should read as:

$$\begin{aligned}
&K^{\gamma\ \alpha\ \beta,\ \tau'}_{i\ c\ g} S^{\gamma_i\ \alpha_c\ \beta_g,\ \tau'}_{jt\ dr\ hs}(f^\alpha_{cd}f^\beta_{gh} - f^\gamma_{ij}) + K^{\gamma\ \alpha\ \beta,\ \tau'}_{i\ c\ h} S^{\gamma_i\ \alpha_c\ \beta_h,\ \tau'}_{jt\ dr\ gs} f^\alpha_{cd}g^\beta_{hg} \\
&+ K^{\gamma\ \alpha\ \beta,\ \tau'}_{i\ d\ g} S^{\gamma_i\ \alpha_d\ \beta_g,\ \tau'}_{jt\ cr\ hs} g^\alpha_{dc}f^\beta_{gh} + K^{\gamma\ \alpha\ \beta,\ \tau'}_{i\ d\ h} S^{\gamma_i\ \alpha_d\ \beta_h,\ \tau'}_{jt\ cr\ gs} g^\alpha_{dc}g^\beta_{hg} = K^{\gamma\ \alpha\ \beta,\ \tau'}_{j\ c\ g} S^{\gamma_j\ \alpha_c\ \beta_g,\ \tau'}_{it\ dr\ hs} g^\gamma_{ij},
\end{aligned} \tag{C.35}$$

with sums over $i, j, c, d, g, h, t, r, s$ and the extra multiplicity τ'.

The *Mathematica* code is available from the authors upon request.

D Multiplicity tables

In this appendix we give some counting information.

First we write out the evaluated universal element as defined in (3.30). We organize the terms in increasing n.

$$
\begin{aligned}
\mathbb{P}(S_2) &= \frac{1}{2!}\left(2\Sigma_{[2]} + 2\Sigma_{[1,1]}\right),\\
\mathbb{P}(S_3) &= \frac{1}{3!}\left(2\Sigma_{[3]} + 3\Sigma_{[2,1]} + 5\Sigma_{[1,1,1]}\right),\\
\mathbb{P}(S_4) &= \frac{1}{4!}\left(3\Sigma_{[4]} + 3\Sigma_{[3,1]} + 7\Sigma_{[2,2]} + 7\Sigma_{[2,1,1]} + 15\Sigma_{[1,1,1,1]}\right),\\
\mathbb{P}(S_5) &= \frac{1}{5!}\left(2\Sigma_{[5]} + 4\Sigma_{[4,1]} + 5\Sigma_{[3,2]} + 7\Sigma_{[3,1,1]} + 12\Sigma_{[2,2,1]} + 20\Sigma_{[2,1,1,1]} + 52\Sigma_{[1,1,1,1,1]}\right),\\
\mathbb{P}(S_6) &= \frac{1}{6!}\Big(4\Sigma_{[6]} + 3\Sigma_{[5,1]} + 9\Sigma_{[4,2]} + 9\Sigma_{[4,1,1]} + 8\Sigma_{[3,3]} + 10\Sigma_{[3,2,1]} + 20\Sigma_{[3,1,1,1]}\\
&\qquad +31\Sigma_{[2,2,2]} + 31\Sigma_{[2,2,1,1]} + 67\Sigma_{[2,1,1,1,1]} + 203\Sigma_{[1,1,1,1,1,1]}\Big),\\
\mathbb{P}(S_7) &= \frac{1}{7!}\Big(2\Sigma_{[7]} + 5\Sigma_{[6,1]} + 5\Sigma_{[5,2]} + 7\Sigma_{[5,1,1]} + 7\Sigma_{[4,3]} + 15\Sigma_{[4,2,1]} + 25\Sigma_{[4,1,1,1]}\\
&\qquad +13\Sigma_{[3,3,1]} + 19\Sigma_{[3,2,2]} + 27\Sigma_{[3,2,1,1]} + 67\Sigma_{[3,1,1,1,1]} + 59\Sigma_{[2,2,2,1]} + 97\Sigma_{[2,2,1,1,1]}\\
&\qquad +255\Sigma_{[2,1,1,1,1,1]} + 877\Sigma_{[1,1,1,1,1,1,1]}\Big),\\
\mathbb{P}(S_8) &= \frac{1}{8!}\Big(4\Sigma_{[8]} + 3\Sigma_{[7,1]} + 11\Sigma_{[6,2]} + 11\Sigma_{[6,1,1]} + 5\Sigma_{[5,3]} + 10\Sigma_{[5,2,1]} + 20\Sigma_{[5,1,1,1]}\\
&\qquad +16\Sigma_{[4,4]} + 13\Sigma_{[4,3,1]} + 38\Sigma_{[4,2,2]} + 38\Sigma_{[4,2,1,1]} + 82\Sigma_{[4,1,1,1,1]} + 21\Sigma_{[3,3,2]}\\
&\qquad +33\Sigma_{[3,3,1,1]} + 43\Sigma_{[3,2,2,1]} + 87\Sigma_{[3,2,1,1,1]} + 255\Sigma_{[3,1,1,1,1,1]} + 164\Sigma_{[2,2,2,2]}\\
&\qquad +164\Sigma_{[2,2,2,1,1]} + 352\Sigma_{[2,2,1,1,1,1]} + 1080\Sigma_{[2,1,1,1,1,1,1]} + 4140\Sigma_{[1,1,1,1,1,1,1,1]}\Big).
\end{aligned}
\tag{D.1}
$$

In (D.2) we give the multiplicities $\mathcal{M}_\Lambda$ for various $U(3)$ representations of the BPS states organized by total charge n. According to (3.31) they are equal to $\chi_\Lambda(\mathbb{P})$, using the elements in (D.1).

n	$U(3)$ representations
1	$1\,[1]$
2	$2\,[2]$
3	$3\,[3] \oplus 1\,[2,1]$
4	$5\,[4] \oplus 2\,[3,1] \oplus 2\,[2,2]$
5	$7\,[5] \oplus 5\,[4,1] \oplus 4\,[3,2]$ $\oplus 1\,[2,2,1]$
6	$11\,[6] \oplus 8\,[5,1] \oplus 10\,[4,2] \oplus 2\,[3,3]$ $\oplus 1\,[4,1,1] \oplus 2\,[3,2,1] \oplus 2\,[2,2,2]$
7	$15\,[7] \oplus 15\,[6,1] \oplus 17\,[5,2] \oplus 10\,[4,3]$ $\oplus 2\,[5,1,1] \oplus 7\,[4,2,1] \oplus 1\,[3,3,1] \oplus 4\,[3,2,2]$
8	$22\,[8] \oplus 23\,[7,1] \oplus 32\,[6,2] \oplus 20\,[5,3] \oplus 12\,[4,4]$ $\oplus 5\,[6,1,1] \oplus 14\,[5,2,1] \oplus 9\,[4,3,1] \oplus 11\,[4,2,2] \oplus 2\,[3,3,2]$

(D.2)

In (D.3) we list combinations of R, τ which are allowed at various values of Λ to write the free operators $\mathcal{O}_{\Lambda,M_\Lambda;R,\tau}$. The τ multiplicity for any Λ, R is given by $c(R,R,\Lambda)$. We skip $\Lambda = [n]$ because in that case R runs over all partitions of n with multiplicity 1, and we will constrain ourselves to the $U(2)$ sector $\Lambda = [\lambda_1, \lambda_2]$. Note that the number of free operators is larger than $\mathcal{M}_\Lambda$.

Λ	Operators
$\Lambda = [2,1]$	$\mathcal{O}_{R=[2,1],\tau=1}$
$\Lambda = [3,1]$	$\mathcal{O}_{[3,1]}, \mathcal{O}_{[2,1,1]}$
$\Lambda = [2,2]$	$\mathcal{O}_{[3,1]}, \mathcal{O}_{[2,2]}, \mathcal{O}_{[2,1,1]}$
$\Lambda = [4,1]$	$\mathcal{O}_{[4,1]}, \mathcal{O}_{[3,2]}, \mathcal{O}_{[3,1,1]}, \mathcal{O}_{[2,2,1]}, \mathcal{O}_{[2,1,1,1]}$
$\Lambda = [3,2]$	$\mathcal{O}_{[4,1]}, \mathcal{O}_{[3,2]}, \mathcal{O}_{[3,1,1],\tau=1}, \mathcal{O}_{[3,1,1],\tau=2}, \mathcal{O}_{[2,2,1]}, \mathcal{O}_{[2,1,1,1]}$
$\Lambda = [5,1]$	$\mathcal{O}_{[5,1]}, \mathcal{O}_{[4,2]}, \mathcal{O}_{[4,1,1]}, \mathcal{O}_{[3,2,1],\tau=1}, \mathcal{O}_{[3,2,1],\tau=2}, \mathcal{O}_{[3,1,1,1]}, \mathcal{O}_{[2,2,1,1]}, \mathcal{O}_{[2,1,1,1,1]}$
$\Lambda = [4,2]$	$\mathcal{O}_{[5,1]}, \mathcal{O}_{[4,2],\tau=1}, \mathcal{O}_{[4,2],\tau=2}, \mathcal{O}_{[4,1,1],\tau=1}, \mathcal{O}_{[4,1,1],\tau=2}, \mathcal{O}_{[3,3]}, \mathcal{O}_{[3,2,1],\tau=1}, \mathcal{O}_{[3,2,1],\tau=2},$ $\mathcal{O}_{[3,2,1],\tau=3}, \mathcal{O}_{[3,1,1,1],\tau=1}, \mathcal{O}_{[3,1,1,1],\tau=2}, \mathcal{O}_{[2,2,2]}, \mathcal{O}_{[2,2,1,1],\tau=1}, \mathcal{O}_{[2,2,1,1],\tau=2}, \mathcal{O}_{[2,1,1,1,1]}$
$\Lambda = [3,3]$	$\mathcal{O}_{[4,1,1]}, \mathcal{O}_{[3,2,1],\tau=1}, \mathcal{O}_{[3,2,1],\tau=2}, \mathcal{O}_{[3,1,1,1]}$

(D.3)

E Integer sequences in coefficients of $\mathbb{P}$

The integer coefficients of $\mathbb{P}$ which we called t_p contain a wealth of combinatoric data. The first few examples of t_p can be seen in (D.1). Here we explore a few known integer sequences found among them. For more information about the sequences we will refer to [38].

An easily identifiable two-dimensional class of coefficients is found by considering $p = [j^n]$. For example:

$$t_{[1^n]} = \{1, 2, 5, 15, 52, 203, 877, 4140, \ldots\} \tag{E.1}$$

$$t_{[2^n]} = \{2, 7, 31, 164, 999, 6841, \ldots\} \tag{E.2}$$

$$t_{[3^n]} = \{2, 8, 42, 268, 1994, 16852, \ldots\} \tag{E.3}$$

$$t_{[4^n]} = \{3, 16, 111, 931, 9066, 99925, \ldots\}. \tag{E.4}$$

According to the formula (3.37) we can express the generating functions for this class of sequences as

$$t(y_j) = \exp\left(\sum_{d|j} \frac{1}{d}\left(e^{d\,y_j} - 1\right)\right), \tag{E.5}$$

where d are divisors of j. For example:

$$t(y_1) = \exp\left(e^{y_1} - 1\right) \tag{E.6}$$

$$t(y_2) = \exp\big((e^{y_2} - 1) + (e^{2y_2} - 1)/2\big) \tag{E.7}$$

$$t(y_3) = \exp\big((e^{y_3} - 1) + (e^{3y_3} - 1)/3\big) \tag{E.8}$$

$$t(y_4) = \exp\big((e^{y_4} - 1) + (e^{2y_4} - 1)/2 + (e^{4y_4} - 1)/4\big). \tag{E.9}$$

We find the sequences $t_{[1^n]}, t_{[2^n]}, t_{[3^n]}, t_{[4^n]}$ are identified in the [38] as *A000110*, *A002872*, *A002874*, *A141003* respectively. The whole class of such coefficients $t_{[j^n]}$ are in fact called "Sorting numbers" [71].

A couple of interesting special cases are worth mentioning. For $j = 1$ the sequence is the Bell or exponential numbers. It counts the total number of ways that n distinguishable elements can be divided into any number of indistinguishable subsets.

For $j = 2$ we have a description as "number of partitions of $2n$ objects invariant under a permutation consisting of n 2-cycles".

It is also noteworthy that the individual factors in (3.37) have a combinatoric and group theoretic meaning.

$$\exp \frac{1}{d}\left(x^d - 1\right) = \sum_{k=0}^{\infty} B(d,k)\frac{x^k}{k!} \tag{E.10}$$

The numbers $B(d, i)$ are discussed in [72] which defines them as

$$B(i,k) = \frac{i^k}{(ik)!} \sum_{\sigma \in S_{ik}} a_i(\sigma)^k \tag{E.11}$$

where $a_i(\sigma)$ is the number of i-cycles in the symmetric group element σ.

They are related to Stirling numbers of the second kind $S(k,j)$.

$$B(i,k) = i^k \sum_{j=1}^{k} \frac{S(k,j)}{i^j} \tag{E.12}$$

The Stirling numbers of the second kind $S(k,j)$ [73] count the number of ways to partition a set of k elements into j non-empty subsets. An explicit formula is

$$S(k,j) = \frac{1}{j!}\sum_{i=0}^{j}(-1)^{j-i}\binom{j}{i} i^k. \tag{E.13}$$

F Fourier basis two-point function

Here we present a derivation of (2.14) in the conventions of this paper. The Fourier basis operators are defined as (2.4).

$$\mathcal{O}_{\Lambda,M_\Lambda,R,\tau} = \frac{\sqrt{d_R}}{n!}\sum_{\alpha,\vec{a}} S\,{}^{R}_{i}\,{}^{R}_{j}\,{}^{\Lambda,}_{m}\,{}^{\tau}\, D^R_{ij}(\alpha) C^{\vec{a}}_{\Lambda,M_\Lambda,m}\mathcal{O}_{\vec{a},\alpha} \tag{F.1}$$

The two-point function in the trace basis can be evaluated from Wick contractions to be:

$$\begin{aligned}
\langle \mathcal{O}_{\vec{b},\beta}|\mathcal{O}_{\vec{a},\alpha}\rangle &= \frac{1}{N^n}\langle \mathrm{tr}_n(\mathbb{X}_{\vec{b}}\beta)|\mathrm{tr}_n(\mathbb{X}_{\vec{a}}\alpha)\rangle \\
&= \sum_{\gamma\in S_n} \delta_{\gamma(\vec{a}),\vec{b}} N^{C(\beta^{-1}\gamma^{-1}\alpha\gamma)-n} \\
&= \sum_{\gamma,\sigma\in S_n} \delta_{\gamma(\vec{a}),\vec{b}} N^{C(\sigma)-n}\delta(\beta^{-1}\gamma^{-1}\alpha\gamma\sigma) \\
&= \sum_{\gamma\in S_n} \delta_{\gamma(\vec{a}),\vec{b}}\delta(\beta^{-1}\gamma^{-1}\alpha\gamma\Omega).
\end{aligned} \tag{F.2}$$

Here $C(\sigma)$ is the number of cycles in permutation σ and we used the definition (2.15)

$$\Omega = \sum_{\sigma} N^{C(\sigma)-n}\sigma. \tag{F.3}$$

We can evaluate now the two-point function of operators (F.1) using (F.2):

$$
\begin{aligned}
&\langle \mathcal{O}_{\Lambda_2,M_{\Lambda_2},R_2,\tau_2} | \mathcal{O}_{\Lambda_1,M_{\Lambda_1},R_1,\tau_1} \rangle \\
&= \frac{\sqrt{d_{R_1} d_{R_2}}}{(n!)^2} \sum_{\vec{a},\alpha,\vec{b},\beta,\gamma} S\,{}^{R_2}_{i_2}\,{}^{R_2}_{j_2}\,{}^{\Lambda_2,}_{m_2}\,{}^{\tau_2} S\,{}^{R_1}_{i_1}\,{}^{R_1}_{j_1}\,{}^{\Lambda_1,}_{m_1}\,{}^{\tau_1} D^{R_2}_{i_2 j_2}(\beta) D^{R_1}_{i_1 j_1}(\alpha) \left(C^{\vec{b}}_{\Lambda_2,M_{\Lambda_2},m_2} \right)^* C^{\vec{a}}_{\Lambda_1,M_{\Lambda_1},m_1} \\
&\qquad \times \delta_{\gamma(\vec{a}),\vec{b}} \delta(\beta^{-1}\gamma^{-1}\alpha\gamma\Omega) \\
&= \frac{\sqrt{d_{R_1} d_{R_2}}}{(n!)^2} \sum_{\vec{a},\alpha,\gamma} S\,{}^{R_2}_{i_2}\,{}^{R_2}_{j_2}\,{}^{\Lambda_2,}_{m_2}\,{}^{\tau_2} S\,{}^{R_1}_{i_1}\,{}^{R_1}_{j_1}\,{}^{\Lambda_1,}_{m_1}\,{}^{\tau_1} D^{R_2}_{i_2 j_2}(\gamma^{-1}\alpha\gamma\Omega) D^{R_1}_{i_1 j_1}(\alpha) \left(C^{\gamma(\vec{a})}_{\Lambda_2,M_{\Lambda_2},m_2} \right)^* C^{\vec{a}}_{\Lambda_1,M_{\Lambda_1},m_1} \\
&= \frac{\sqrt{d_{R_1} d_{R_2}}}{(n!)^2} \sum_{\vec{a},\alpha,\gamma} S\,{}^{R_2}_{i_2}\,{}^{R_2}_{j_2}\,{}^{\Lambda_2,}_{m_2}\,{}^{\tau_2} S\,{}^{R_1}_{i_1}\,{}^{R_1}_{j_1}\,{}^{\Lambda_1,}_{m_1}\,{}^{\tau_1} D^{R_2}_{k i_2}(\gamma) D^{R_2}_{kl}(\alpha) D^{R_2}_{l j_2}(\gamma) \frac{\chi_{R_2}(\Omega)}{d_{R_2}} D^{R_1}_{i_1 j_1}(\alpha) \\
&\qquad \times D^{\Lambda_2}_{m_2' m_2}(\gamma) \left(C^{\vec{a}}_{\Lambda_2,M_{\Lambda_2},m_2'} \right)^* C^{\vec{a}}_{\Lambda_1,M_{\Lambda_1},m_1} \\
&= \frac{\chi_{R_2}(\Omega)}{d_{R_2}} \delta_{\Lambda_1,\Lambda_2} \delta_{M_{\Lambda_1},M_{\Lambda_2}} \delta_{R_1,R_2} S\,{}^{R_2}_{i_2}\,{}^{R_2}_{j_2}\,{}^{\Lambda_2,}_{m_2}\,{}^{\tau_2} S\,{}^{R_2}_{i_1}\,{}^{R_2}_{j_1}\,{}^{\Lambda_2,}_{m_1}\,{}^{\tau_1} \sum_{\tau'} S\,{}^{R_2}_{i_1}\,{}^{R_2}_{j_1}\,{}^{\Lambda_2,}_{m_1}\,{}^{\tau'} S\,{}^{R_2}_{i_2}\,{}^{R_2}_{j_2}\,{}^{\Lambda_2,}_{m_2}\,{}^{\tau'} \\
&= \frac{\chi_{R_2}(\Omega)}{d_{R_2}} \delta_{\Lambda_1,\Lambda_2} \delta_{M_{\Lambda_1},M_{\Lambda_2}} \delta_{R_1,R_2} \delta_{\tau_1,\tau_2}
\end{aligned}
\tag{F.4}
$$

in agreement with (2.14). The following steps were performed:

- In the second line the β and $\vec{b}$ sums were performed using delta functions.
- In the third line we expanded the product $D^{R_2}_{i_2 j_2}(\gamma^{-1}\alpha\gamma\Omega)$ and used:

$$D^R_{ij}(\gamma^{-1}) = D^R_{ji}(\gamma), \tag{F.5}$$

$$D^R_{ij}(\Omega) = \delta_{ij} \frac{\chi_R(\Omega)}{d_R}, \tag{F.6}$$

  because Ω is a central element. We also use the following transformation property of the CG coefficient:

$$C^{\gamma(\vec{a})}_{\Lambda_2,M_{\Lambda_2},m_2} = C^{\vec{a}}_{\Lambda_2,M_{\Lambda_2},m_2'} D^{\Lambda_2}_{m_2' m_2}(\gamma). \tag{F.7}$$

  This is easily understood using the bracket notation, where $C^{\vec{a}}_{\Lambda,M_\Lambda,m}$ is the overlap between $V^{\otimes n}$ basis $|\vec{a}\rangle$ and the $V^{U(M)}_\Lambda \otimes V^{S_n}_\Lambda$ basis $|\Lambda, M_\Lambda, m\rangle$:

$$
\begin{aligned}
C^{\gamma(\vec{a})}_{\Lambda,M_\Lambda,m} &\equiv \langle \gamma(\vec{a}) | \Lambda, M_\Lambda, m \rangle \\
&= \langle \vec{a} | \gamma | \Lambda, M_\Lambda, m \rangle \\
&= \langle \vec{a} | \Lambda, M_\Lambda, m' \rangle \langle \Lambda, M_\Lambda, m' | \gamma | \Lambda, M_\Lambda, m \rangle \\
&\equiv C^{\vec{a}}_{\Lambda,M_\Lambda,m'} D^\Lambda_{m'm}(\gamma).
\end{aligned}
\tag{F.8}
$$

Here it is important to note that according to our definition of $\gamma(\vec{a})$ the permutation γ acts by the *right action* on the vector space $|\vec{a}\rangle$, that is:

$$\langle\gamma(\vec{a})| = \langle\vec{a}|\gamma, \qquad |\gamma(\vec{a})\rangle = \gamma^{-1}|\vec{a}\rangle. \tag{F.9}$$

- In the fourth line we have performed the sums over $\vec{a}, \alpha, \gamma$ using the symmetric group identities:

$$\sum_{\alpha\in S_n} D^{R_1}_{i_1 j_1}(\alpha) D^{R_2}_{i_2 j_2}(\alpha) = \frac{n!}{d_{R_1}} \delta_{R_1,R_2}\delta_{i_1,i_2}\delta_{j_1,j_2} \tag{F.10}$$

$$\sum_{\alpha\in S_n} D^{R_1}_{i_1 j_1}(\alpha) D^{R_2}_{i_2 j_2}(\alpha) D^{R_3}_{i_3 j_3} = n! \sum_{\tau} S^{\,R_1\ R_2\ R_3,\ \tau}_{\,i_1\ i_2\ i_3} S^{\,R_1\ R_2\ R_3,\ \tau}_{\,j_1\ j_2\ j_3}. \tag{F.11}$$

The Clebsch-Gordan coefficients $S^{\,R_1\ R_2\ R_3,\ \tau}_{\,i_1\ i_2\ i_3}$ are again for coupling representations $R_1 \otimes R_2 \otimes R_3 \to 1$ and *not* as more usual $R_1 \otimes R_2 \to R_3$. They are related by:

$$S^{\,R_1\ R_2\ R_3,\ \tau}_{\,i_1\ i_2\ i_3} = \frac{1}{\sqrt{d_{R_3}}} S^{\,R_1\ R_2;\ R_3\tau}_{\,i_1\ i_2;\ i_3}. \tag{F.12}$$

Also we have used that our $C^{\vec{a}}_{\Lambda,M_\Lambda,m}$ are defined in an orthonormal basis and thus

$$\sum_{\vec{a}} \left(C^{\vec{a}}_{\Lambda_2,M_{\Lambda_2},m_2}\right)^* C^{\vec{a}}_{\Lambda_1,M_{\Lambda_1},m_1} = \delta_{\Lambda_1,\Lambda_2}\delta_{M_{\Lambda_1},M_{\Lambda_2}}\delta_{m_1,m_2}. \tag{F.13}$$

- Finally, in the last line we used the orthogonality of the Clebsch-Gordan coefficients:

$$\sum_{i,j,k} S^{\,R_1\ R_2\ R_3,\ \tau_1}_{\,i\ \ j\ \ k} S^{\,R_1\ R_2\ R_3,\ \tau_2}_{\,i\ \ j\ \ k} = \delta_{\tau_1,\tau_2}. \tag{F.14}$$

A useful reference for properties of CG coefficients is [40].

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