NON-TRIVIAL COMPOSITIONS OF DIFFERENTIAL OPERATIONS AND DIRECTIONAL DERIVATIVE

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Abstract. In this paper we present some new results for harmonic functions and we give recurrences for an enumeration of non-trivial compositions of higher order of differential operations and Gateaux directional derivative in $\mathbb{R}^n$.

Key words: compositions of differential operations, Gateaux directional derivative, differential forms, exterior derivative, Hodge star operator, enumeration of graphs and maps

1. Non-trivial compositions of differential operations and directional derivative of the space $\mathbb{R}^3$

In the three-dimensional Euclidean space $\mathbb{R}^3$ we consider following sets

$$\mathrm{A}_0=\big\{f:\mathbb{R}^3\to\mathbb{R}\,|\,f\in C^\infty(\mathbb{R}^3)\big\}\ \text{ and }\ \mathrm{A}_1=\big\{\vec{f}:\mathbb{R}^3\to\mathbb{R}^3\,|\,f_1,f_2,f_3\in C^\infty(\mathbb{R}^3)\big\}.$$

Gradient, curl, divergence and Gateaux directional derivative in direction $\vec{e}$, for a unit vector $\vec{e}=(e_1,e_2,e_3)\in\mathbb{R}^3$, are defined in terms of partial derivative operators as follows

$$\begin{aligned}
\operatorname{grad} f &= \nabla_1 f = \tfrac{\partial f}{\partial x_1}\vec{i}+\tfrac{\partial f}{\partial x_2}\vec{j}+\tfrac{\partial f}{\partial x_3}\vec{k},\ \nabla_1:\mathrm{A}_0\longrightarrow\mathrm{A}_1\,,\\
\operatorname{curl}\vec{f} &= \nabla_2\vec{f} = \Big(\tfrac{\partial f_3}{\partial x_2}-\tfrac{\partial f_2}{\partial x_3}\Big)\vec{i}+\Big(\tfrac{\partial f_1}{\partial x_3}-\tfrac{\partial f_3}{\partial x_1}\Big)\vec{j}+\Big(\tfrac{\partial f_2}{\partial x_1}-\tfrac{\partial f_1}{\partial x_2}\Big)\vec{k},\ \nabla_2:\mathrm{A}_1\longrightarrow\mathrm{A}_1\,,\\
\operatorname{div}\vec{f} &= \nabla_3\vec{f} = \tfrac{\partial f_1}{\partial x_1}+\tfrac{\partial f_2}{\partial x_2}+\tfrac{\partial f_3}{\partial x_3},\ \nabla_3:\mathrm{A}_1\longrightarrow\mathrm{A}_0\,,\\
\operatorname{dir}_{\vec{e}} f &= \nabla_0 f = \nabla_1 f\cdot\vec{e}=\tfrac{\partial f}{\partial x_1}e_1+\tfrac{\partial f}{\partial x_2}e_2+\tfrac{\partial f}{\partial x_3}e_3,\ \nabla_0:\mathrm{A}_0\longrightarrow\mathrm{A}_0\,.
\end{aligned}$$

Let $\mathcal{A}_3=\{\nabla_1,\nabla_2,\nabla_3\}$ and $\mathcal{B}_3=\{\nabla_0,\nabla_1,\nabla_2,\nabla_3\}$. The number of compositions of the k^{th} order over the set $\mathcal{A}_3$ is $\mathtt{f}(k)=F_{k+3}$, where F_k is the k^{th} Fibonacci number (see [6] for more details). A composition of differential operations that is not 0 or $\vec{0}$ is called non-trivial. The number of non-trivial compositions of the k^{th} order over the set $\mathcal{A}_3$ is $\mathtt{g}(k)=3$, see [5]. In paper [8], it is shown that the number of compositions of the k^{th} order over the set $\mathcal{B}_3$ is $\mathtt{f}^{\mathrm{G}}(k)=2^{k+1}$.

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According to the above results, it is natural to try to calculate the number of non-trivial compositions of differential operations from the set $\mathcal{B}_3$. Straightforward verification shows that all compositions of the second order over $\mathcal{B}_3$ are

$$\begin{aligned}
&\operatorname{dir}_{\vec{e}} \operatorname{dir}_{\vec{e}} f = \nabla_0 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{dir}_{\vec{e}} f = \nabla_1 \circ \nabla_0 f = \nabla_1\big(\nabla_1 f \cdot \vec{e}\big),\\
&\Delta f = \operatorname{div} \operatorname{grad} f = \nabla_3 \circ \nabla_1 f,\\
&\operatorname{curl} \operatorname{curl} \vec{f} = \nabla_2 \circ \nabla_2 \vec{f},\\
&\operatorname{dir}_{\vec{e}} \operatorname{div} \vec{f} = \nabla_0 \circ \nabla_3 \vec{f} = \big(\nabla_1 \circ \nabla_3 \vec{f}\big) \cdot \vec{e},\\
&\operatorname{grad} \operatorname{div} \vec{f} = \nabla_1 \circ \nabla_3 \vec{f},\\
&\operatorname{curl} \operatorname{grad} f = \nabla_2 \circ \nabla_1 f = \vec{0},\\
&\operatorname{div} \operatorname{curl} \vec{f} = \nabla_3 \circ \nabla_2 \vec{f} = 0,
\end{aligned}$$

and that only the last two are trivial. This fact leads us to use the following procedure for determining the number of non-trivial composition over the set $\mathcal{B}_3$. Let us define a binary relation σ on the set $\mathcal{B}_3$ as follows: $\nabla_i \,\sigma\, \nabla_j$ iff the composition $\nabla_j \circ \nabla_i$ is non-trivial. Relation σ induces Cayley table

σ	∇_0	∇_1	∇_2	∇_3
∇_0	1	1	0	0
∇_1	0	0	0	1
∇_2	0	0	1	0
∇_3	1	1	0	0

For convenience, we extend set $\mathcal{B}_3$ with nowhere-defined function ∇_{-1}, whose domain and range are empty set, and establish $\nabla_{-1}\,\sigma\,\nabla_i$ for $i=0,1,2,3$. Thus, graph Γ the relation σ is rooted tree with additional root ∇_{-1}

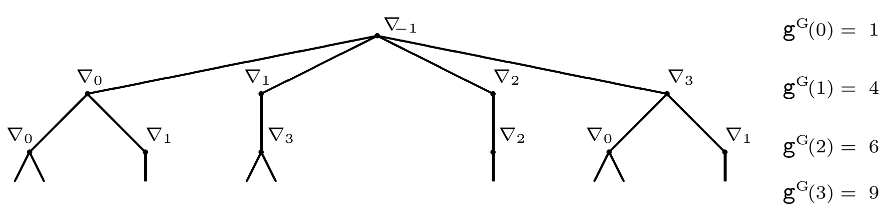


Fig. 1

Here we would like to point out that the child of ∇_i is ∇_j if composition $\nabla_j \circ \nabla_i$ is non-trivial. For any non-trivial composition $\nabla_{i_k} \circ \ldots \circ \nabla_{i_1}$ there is a unique path in the tree (Fig. 1), such that the level of vertex ∇_{i_j} is j, $1 \leq j \leq k$. Let $\mathsf{g}^{\mathrm{G}}(k)$ be the number of non-trivial compositions of the k^{th} order of functions from $\mathcal{B}_3$ and let $\mathsf{g}_i^{\mathrm{G}}(k)$ be the number of non-trivial compositions of the k^{th} order starting with ∇_i. Then we have $\mathsf{g}^{\mathrm{G}}(k) = \mathsf{g}_0^{\mathrm{G}}(k) + \mathsf{g}_1^{\mathrm{G}}(k) + \mathsf{g}_2^{\mathrm{G}}(k) + \mathsf{g}_3^{\mathrm{G}}(k)$. According to the graph Γ obtain the following equalities $\mathsf{g}_0^{\mathrm{G}}(k)=\mathsf{g}_0^{\mathrm{G}}(k-1)+\mathsf{g}_1^{\mathrm{G}}(k-1)$, $\mathsf{g}_1^{\mathrm{G}}(k)=\mathsf{g}_3^{\mathrm{G}}(k-1)$, $\mathsf{g}_2^{\mathrm{G}}(k)=\mathsf{g}_2^{\mathrm{G}}(k-1)$, $\mathsf{g}_3^{\mathrm{G}}(k)=\mathsf{g}_0^{\mathrm{G}}(k-1)+\mathsf{g}_1^{\mathrm{G}}(k-1)$. Since the only child of ∇_2 is ∇_2, we can deduce $\mathtt{g}_2^{\mathrm{G}}(k)=\mathtt{g}_2^{\mathrm{G}}(k-1)=\cdots=\mathtt{g}_2^{\mathrm{G}}(1)=1$. Putting things together we obtain the recurrence for $\mathtt{g}^{\mathrm{G}}(k)$:

$$\begin{aligned}
\mathtt{g}^{\mathrm{G}}(k) &= \mathtt{g}_0^{\mathrm{G}}(k)+\mathtt{g}_1^{\mathrm{G}}(k)+\mathtt{g}_2^{\mathrm{G}}(k)+\mathtt{g}_3^{\mathrm{G}}(k)\\
&= \big(\mathtt{g}_0^{\mathrm{G}}(k-1)+\mathtt{g}_1^{\mathrm{G}}(k-1)\big)+\mathtt{g}_3^{\mathrm{G}}(k-1)+\mathtt{g}_2^{\mathrm{G}}(k-1)+\big(\mathtt{g}_0^{\mathrm{G}}(k-1)+\mathtt{g}_1^{\mathrm{G}}(k-1)\big)\\
&= \mathtt{g}^{\mathrm{G}}(k-1)+\mathtt{g}_0^{\mathrm{G}}(k-1)+\mathtt{g}_1^{\mathrm{G}}(k-1)\\
&= \mathtt{g}^{\mathrm{G}}(k-1)+\big(\mathtt{g}_0^{\mathrm{G}}(k-2)+\mathtt{g}_1^{\mathrm{G}}(k-2)\big)+\mathtt{g}_3^{\mathrm{G}}(k-2)+\mathtt{g}_2^{\mathrm{G}}(k-2)-\mathtt{g}_2^{\mathrm{G}}(k-2)\\
&= \mathtt{g}^{\mathrm{G}}(k-1)+\mathtt{g}^{\mathrm{G}}(k-2)-1.
\end{aligned}$$

Substituting $\mathtt{t}(k)=\mathtt{g}^{\mathrm{G}}(k)-1$ into previous formula we obtain recurrence $\mathtt{t}(k)=\mathtt{t}(k-1)+\mathtt{t}(k-2)$. On the base of the initial conditions $\mathtt{g}^{\mathrm{G}}(1)=4$ and $\mathtt{g}^{\mathrm{G}}(2)=6$, i.e. $\mathtt{t}(1)=3$ and $\mathtt{t}(2)=5$, we conclude that $\mathtt{g}^{\mathrm{G}}(k)=F_{k+3}+1$.

2. Non-trivial compositions of differential operations and directional derivative of the space $\mathbb{R}^n$

We start this section by recalling some definitions of the theory of differential forms. Let $\mathbb{R}^n$ denote the n-dimensional Euclidean space and consider set of smooth functions $\mathrm{A}_0=\{f:\mathbb{R}^n\to\mathbb{R}\,|\,f\in C^\infty(\mathbb{R}^n)\}$. The set of all differential k-forms on $\mathbb{R}^n$ is a free A_0-module of rank $\binom{n}{k}$ with the standard basis $\{dx_I=dx_{i_1}\dots dx_{i_k}\,|\,1\le i_1<\ldots<i_k\le n\}$, denoted $\Omega^k(\mathbb{R}^n)$. Differential k-form ω can be written uniquely as $\omega=\sum_{I\in\mathcal{I}(k,n)}\omega_I dx_I$, where $\omega_I\in\mathrm{A}_0$ and $\mathcal{I}(k,n)$ is the set of multi-indices $I=(i_1,\ldots,i_k)$, $1\le i_1<\ldots<i_k\le n$. The complement of multi-index I is multi-index $J=(j_1,\ldots,j_{n-k})\in\mathcal{I}(n-k,n)$, $1\le j_1<\ldots<j_{n-k}\le n$, where components j_p are elements of the set $\{1,\ldots,n\}\backslash\{i_1,\ldots,i_k\}$. We have $dx_I dx_J=\sigma(I)dx_1\dots dx_n$, where $\sigma(I)$ is a signature of the permutation $(i_1,\ldots,i_k, j_1,\ldots,j_{n-k})$. Note that $\sigma(J)=(-1)^{k(n-k)}\sigma(I)$. With the notions mentioned above we define $\star_k(dx_I)=\sigma(I)dx_J$. Map $\star_k:\Omega^k(\mathbb{R}^n)\to\Omega^{n-k}(\mathbb{R}^n)$ defined by $\star_k(\omega)=\sum_{I\in\mathcal{I}(k,n)}\omega_I\star_k(dx_I)$ is Hodge star operator and it provides natural isomorphism between $\Omega^k(\mathbb{R}^n)$ and $\Omega^{n-k}(\mathbb{R}^n)$. The Hodge star operator twice applied to a differential k-form yields $\star_{n-k}(\star_k\omega)=(-1)^{k(n-k)}\omega$. So for the inverse of the operator $\star_k$ holds ${\star_k}^{-1}(\psi)=(-1)^{k(n-k)}\star_{n-k}(\psi)$, where $\psi\in\Omega^{n-k}(\mathbb{R}^n)$.

A differential 0-form is a function $f(x_1,x_2,\ldots,x_n)\in\mathrm{A}_0$. We define df to be the differential 1-form $df=\sum_{i=1}^n\frac{\partial f}{\partial x_i}dx_i$. Given a differential k-form $\sum_{I\in\mathcal{I}(k,n)}\omega_I dx_I$, the exterior derivative $d_k\omega$ is differential $(k+1)$-form $d_k\omega=\sum_{I\in\mathcal{I}(k,n)}d\omega_I dx_I$. The exterior derivative d_k is a linear map from k-forms to $(k+1)$-forms which obeys Leibnitz rule: If ω is a k-form and ψ is a l-form, then $d_{k+l}(\varphi\psi)=d_k\omega\,\psi+(-1)^k\varphi\,d_l\psi$. The exterior derivative has a property that $d_{k+1}(d_k\omega)=0$ for any differential k-form ω.

For $m=[n/2]$ and $k=0,1,\ldots,m$ let us consider the following sets of functions

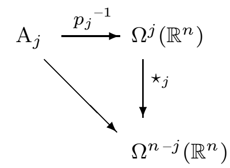
$$\mathrm{A}_k=\big\{\vec{f}:\mathbb{R}^n\to\mathbb{R}^{\binom{n}{k}}\,|\,f_1,\ldots,f_{\binom{n}{k}}\in C^\infty(\mathbb{R}^n)\big\}.$$

Let $p_k : \Omega^k(\mathbb{R}^n) \to \mathrm{A}_k$ be presentation of differential forms in coordinate notation. Let us define functions φ_i $(0 \le i \le m)$ and φ_{n-j} $(0 \le j < n-m)$ as follows

$$\varphi_i = p_i : \Omega^i(\mathbb{R}^n) \longrightarrow \mathrm{A}_i$$

and

$$\varphi_{n-j} = p_j \star_j^{-1} : \Omega^{n-j}(\mathbb{R}^n) \longrightarrow \mathrm{A}_j.$$



Then, according to [7], the combination of Hodge star operator and the exterior derivative generates one choice of differential operations $\nabla_k = \varphi_k\, d_{k-1}\, \varphi_{k-1}^{-1}$, $1 \le k \le n$, in n-dimensional space $\mathbb{R}^n$.

$\mathcal{A}_n$ $(n=2m)$:

$$\nabla_1 = p_1\, d_0\, p_0^{-1} : A_0 \to A_1$$
$$\nabla_2 = p_2\, d_1\, p_1^{-1} : A_1 \to A_2$$
$$\vdots$$
$$\nabla_i = p_i\, d_{i-1}\, p_{i-1}^{-1} : A_{i-1} \to A_i$$
$$\vdots$$
$$\nabla_m = p_m\, d_{m-1}\, p_{m-1}^{-1} : A_{m-1} \to A_m$$
$$\nabla_{m+1} = p_{m-1} \star_{m-1}^{-1} d_m\, p_m^{-1} : A_m \to A_{m-1}$$
$$\nabla_{m+2} = p_{m-2} \star_{m-2}^{-1} d_{m+1} \star_{m-1} p_{m-1}^{-1} : A_{m-1} \to A_{m-2}$$
$$\vdots$$
$$\nabla_{n-j} = p_j \star_j^{-1} d_{n-(j+1)} \star_{j+1} p_{j+1}^{-1} : A_{j+1} \to A_j$$
$$\vdots$$
$$\nabla_{n-1} = p_1 \star_1^{-1} d_{n-2} \star_2 p_2^{-1} : A_2 \to A_1$$
$$\nabla_n = p_0 \star_0^{-1} d_{n-1} \star_1 p_1^{-1} : A_1 \to A_0,$$

$\mathcal{A}_n$ $(n=2m+1)$:

$$\nabla_1 = p_1\, d_0\, p_0^{-1} : A_0 \to A_1$$
$$\nabla_2 = p_2\, d_1\, p_1^{-1} : A_1 \to A_2$$
$$\vdots$$
$$\nabla_i = p_i\, d_{i-1}\, p_{i-1}^{-1} : A_{i-1} \to A_i$$
$$\vdots$$
$$\nabla_m = p_m\, d_{m-1}\, p_{m-1}^{-1} : A_{m-1} \to A_m$$
$$\nabla_{m+1} = p_m \star_m^{-1} d_m\, p_m^{-1} : A_m \to A_m$$
$$\nabla_{m+2} = p_{m-1} \star_{m-1}^{-1} d_{m+1} \star_m p_m^{-1} : A_m \to A_{m-1}$$
$$\nabla_{m+3} = p_{m-2} \star_{m-2}^{-1} d_{m+2} \star_{m-1} p_{m-1}^{-1} : A_{m-1} \to A_{m-2}$$
$$\vdots$$
$$\nabla_{n-j} = p_j \star_j^{-1} d_{n-(j+1)} \star_{j+1} p_{j+1}^{-1} : A_{j+1} \to A_j$$
$$\vdots$$
$$\nabla_{n-1} = p_1 \star_1^{-1} d_{n-2} \star_2 p_2^{-1} : A_2 \to A_1$$
$$\nabla_n = p_0 \star_0^{-1} d_{n-1} \star_1 p_1^{-1} : A_1 \to A_0.$$

List of differential operations in $\mathbb{R}^n$

Formulae for the number of compositions of differential operations from the set $\mathcal{A}_n$ and corresponding recurrences are given by Malešević in [6, 7], see also appropriate integer sequences in [14] and [15]. The following theorem provides a natural characterization of the number of non-trivial compositions of differential operations from the set $\mathcal{A}_n$. For the proof we refer reader to [6].

Theorem 2.1. *All non-trivial compositions of differential operations from the set $\mathcal{A}_n$ are given in the following form*

$$(*) \qquad (\nabla_i\circ)\nabla_{n+1-i} \circ \nabla_i \circ \cdots \circ \nabla_{n+1-i} \circ \nabla_i$$

where $2i$, $2(i-1) \neq n$, $1 \le i \le n$. Term in bracket is included in if the number of differential operations is odd and left out otherwise.

Theorem 2.2. *Let $\mathsf{g}(k)$ be the number of non-trivial compositions of the k^{th} order of differential operations from the set $\mathcal{A}_n$. Then we have*

$$\mathsf{g}(k) = \begin{cases} n & : \ 2 \nmid n\,; \\ n & : \ 2 \mid n\ ,\ \ k=1\,; \\ n-1 & : \ 2 \mid n\ ,\ \ k=2\,; \\ n-2 & : \ 2 \mid n\ ,\ \ k>2\,. \end{cases}$$

Hodge dual to the exterior derivative $d_k : \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$ is codifferential δ_{k-1}, a linear map $\delta_{k-1} : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$, which is a generalization of the divergence, defined by

$$\delta_{k-1} = (-1)^{n(k-1)+1} \star_{n-(k-1)} d_{n-k} \star_k = (-1)^k \star_{k-1}^{-1} d_{n-k} \star_k \,.$$

Note that $\nabla_{n-j} = (-1)^{j+1} p_j \, \delta_j \, p_{j+1}^{-1}$, for $0 \leq j < n-m-1$. The codifferential can be coupled with the exterior derivative to construct the Hodge Laplacian, also known as the Laplace-de Rham operator, $\Delta_k : \Omega^k(\mathbb{R}^n) \to \Omega^k(\mathbb{R}^n)$, a harmonic generalization of Laplace differential operator, given by $\Delta_0 = \delta_0 d_0$ and $\Delta_k = \delta_k d_k + d_{k-1}\delta_{k-1}$, for $1 \leq k \leq m$, see [9]. The operator Δ_0 is actually the negative of the Laplace-Beltrami (scalar) operator. A k-form ω is called harmonic if $\Delta_k(\omega) = 0$. We say that $\vec{f} \in A_k$ is a harmonic function if $\omega = {p_k}^{-1}(\vec{f})$ is harmonic k-form. If $k \geq 1$ harmonic function $\vec{f}$ is also called harmonic field.

For function $\vec{f} \in A_k$, $1 \leq k \leq m$, according to Proposition 4.15. from [3], holds $\Delta_k(p_k^{-1}\vec{f}) = 0$ iff $\delta_{k-1}(p_k^{-1}\vec{f}) = 0$ and $d_k(p_k^{-1}\vec{f}) = 0$. In fact, we obtain the following

Lemma 2.1. *Let $\vec{f} \in A_k$, $1 \leq k \leq m$, then*

$$\Delta_k(p_k^{-1}\vec{f}) = 0 \Longleftrightarrow \nabla_{n-(k-1)}(\vec{f}) = 0 \wedge \nabla_{k+1}(\vec{f}) = 0.$$

For harmonic function $f \in A_0$ we have $\Delta_0 f = \delta_0 d_0 f = 0$, hence $\nabla_n \circ \nabla_1 f = 0$ and finally $(\nabla_1 \circ)\nabla_n \circ \nabla_1 \circ \cdots \circ \nabla_n \circ \nabla_1 f = 0$. We can now rephrase Theorem 2.1 for harmonic functions.

Theorem 2.3. *All compositions of the second and higher order in $(*)$ acting on harmonic function $f \in A_0$ are trivial. All compositions of the first and higher order in $(*)$ acting on harmonic field $\vec{f} \in A_k$, $1 \leq k \leq m$, are trivial.*

We say that $\boldsymbol{f} \in A_k$, $0 \leq k \leq m$, is coordinate-harmonic function or that $\boldsymbol{f}$ satisfies harmonic coordinate condition, if all its coordinates are harmonic functions. Malešević [5] showed that all compositions of the third and higher order in $(*)$ acting on coordinate-harmonic function $\boldsymbol{f}$ are trivial in $\mathbb{R}^3$. Based on the previous statement for coordinate-harmonic functions in $\mathbb{R}^3$ we formulate.

Conjecture 2.1. *All compositions of the third and higher order in $(*)$ acting on coordinate-harmonic function $\boldsymbol{f} \in A_k$, $0 \leq k \leq m$, are trivial.*

One approach to a coordinate investigation of Conjecture 2.1 in $\mathbb{R}^4$ can be found in Gilbert N. Lewis and Edwin B. Wilson papers [1], [2] (see also [4]). Similar problem for coordinate-harmonic functions can be considered in Discrete Exterior Calculus [10, 11] and Combinatorial Hodge Theory [12], [13].

Let $f \in \mathrm{A}_0$ be a scalar function and $\vec{e} = (e_1, \ldots, e_n) \in \mathbb{R}^n$ be a unit vector. Gateaux directional derivative in direction $\vec{e}$ is defined by

$$\mathrm{dir}_{\vec{e}}\, f = \nabla_0 f = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\, e_k : A_0 \longrightarrow A_0 \,.$$

Let us extend set of differential operations $\mathcal{A}_n = \{\nabla_1,\ldots,\nabla_n\}$ with directional derivative to the set $\mathcal{B}_n = \mathcal{A}_n \cup \{\nabla_0\} = \{\nabla_0, \nabla_1, \ldots, \nabla_n\}$. Recurrences for counting compositions of differential operations from the set $\mathcal{B}_n$ can be found in [8]. Corresponding integer sequences are given in [14].

The number of non-trivial compositions of differential operations from the set $\mathcal{B}_n$ is determined by the binary relation ν, defined by:

$$\nabla_i \nu \nabla_j \text{ iff } (i=0 \wedge j=0) \vee (i=0 \wedge j=1) \vee (i=n \wedge j=0) \vee (i+j=n+1 \wedge 2i \neq n).$$

Applying Theorem 2.2 to cases $i = 2, \ldots, n-1$ we conclude that the number of non-trivial compositions of the k^{th} order starting with $\nabla_2,\ldots,\ \nabla_{n-1}$ can be express by formula

$$\mathtt{j}(k) = \mathtt{g}(k) - 2 = \begin{cases} n-2 & : \ 2 \nmid n\,; \\ n-2 & : \ 2 \mid n\ ,\ \ k=1\,; \\ n-3 & : \ 2 \mid n\ ,\ \ k=2\,; \\ n-4 & : \ 2 \mid n\ ,\ \ k>2\,. \end{cases}$$

Let $\mathtt{g}^{\mathrm{G}}(k)$ be the number of non-trivial the k^{th} order compositions of operations from the set $\mathcal{B}_n$. Let $\mathtt{g}_0^{\mathrm{G}}(k)$, $\mathtt{g}_1^{\mathrm{G}}(k)$ and $\mathtt{g}_n^{\mathrm{G}}(k)$ be the numbers of non-trivial the k^{th} order compositions starting with ∇_0, ∇_1 and ∇_n, respectively. Then we have $\mathtt{g}^{\mathrm{G}}(k) = \mathtt{g}_0^{\mathrm{G}}(k) + \mathtt{g}_1^{\mathrm{G}}(k) + \mathtt{j}(k) + \mathtt{g}_n^{\mathrm{G}}(k)$. Denote $\widetilde{\mathtt{g}}^{\mathrm{G}}(k) = \mathtt{g}_0^{\mathrm{G}}(k) + \mathtt{g}_1^{\mathrm{G}}(k) + \mathtt{g}_n^{\mathrm{G}}(k)$. Hence, the following three recurrences are true $\mathtt{g}_0^{\mathrm{G}}(k) = \mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1)$, $\mathtt{g}_1^{\mathrm{G}}(k) = \mathtt{g}_n^{\mathrm{G}}(k-1)$, $\mathtt{g}_n^{\mathrm{G}}(k) = \mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1)$. Thus, the recurrence for $\widetilde{\mathtt{g}}^{\mathrm{G}}(k)$ is of the form

$$\begin{aligned} \widetilde{\mathtt{g}}^{\mathrm{G}}(k) &= \mathtt{g}_0^{\mathrm{G}}(k) + \mathtt{g}_1^{\mathrm{G}}(k) + \mathtt{g}_n^{\mathrm{G}}(k) \\ &= \big(\mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1)\big) + \mathtt{g}_n^{\mathrm{G}}(k-1) + \big(\mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1)\big) \\ &= \widetilde{\mathtt{g}}^{\mathrm{G}}(k-1) + \mathtt{g}_0^{\mathrm{G}}(k-1) + \mathtt{g}_1^{\mathrm{G}}(k-1) \\ &= \widetilde{\mathtt{g}}^{\mathrm{G}}(k-1) + \big(\mathtt{g}_0^{\mathrm{G}}(k-2) + \mathtt{g}_1^{\mathrm{G}}(k-2)\big) + \mathtt{g}_n^{\mathrm{G}}(k-2) \\ &= \widetilde{\mathtt{g}}^{\mathrm{G}}(k-1) + \widetilde{\mathtt{g}}^{\mathrm{G}}(k-2). \end{aligned}$$

With initial conditions $\widetilde{\mathtt{g}}^{\mathrm{G}}(1) = 3$, $\widetilde{\mathtt{g}}^{\mathrm{G}}(2) = 5$ we deduce $\widetilde{\mathtt{g}}^{\mathrm{G}}(k) = F_{k+3}$. Therefore, we have proved the following theorem.

Theorem 2.4. *The number of non-trivial compositions of the k^{th} order over the set $\mathcal{B}_n$ is*

$$\mathtt{g}^{\mathrm{G}}(k) = F_{k+3} + \mathtt{j}(k) = \begin{cases} F_{k+3} + n - 2 & : \ 2 \nmid n\,; \\ n+1 & : \ 2 \mid n\ ,\ \ k=1\,; \\ n+2 & : \ 2 \mid n\ ,\ \ k=2\,; \\ F_{k+3} + n - 4 & : \ 2 \mid n\ ,\ \ k>2\,. \end{cases}$$

Corollary 2.1. *In the case $n = 3$ follows formula $\mathtt{g}^{\mathrm{G}}(k) = F_{k+3} + 1$ from the first section.*

Remark 2.1. *The values of function* $\mathrm{g}^{\mathrm{G}}(k)$ *are given in* [14] *as the following sequences* A001611 $(n=3)$*,* A000045 $(n=4)$*,* A157726 $(n=5)$*,* A157725 $(n=6)$*,* A157729 $(n=7)$*,* A157727 $(n=8)$*,* A187107 $(n=9)$*,* A187179 $(n=10)$ *for* $k>2$*.*

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