

Catalan structures and Catalan pairs

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Abstract

A *Catalan pair* is a pair of binary relations (S, R) satisfying certain axioms. These objects are enumerated by the well-known Catalan numbers, and have been introduced in [DFPR] with the aim of giving a common language to most of the structures counted by Catalan numbers. Here, we give a simple method to pass from the recursive definition of a generic Catalan structure to the recursive definition of the Catalan pair on the same structure, thus giving an automatic way to interpret Catalan structures in terms of Catalan pairs. We apply our method to many well-known Catalan structures, focusing on the meaning of the relations S and R in each considered case.

1 Catalan pairs

Catalan numbers are a very popular sequence of integer numbers, arising in many combinatorial problems coming out from different scientific areas, including computer science, computational biology, and mathematical physics [St1, St2]. Throughout all the paper, we will refer to any combinatorial structure enumerated by Catalan numbers as to a *Catalan structure*.

Catalan pairs have been introduced in [DFPR] with the aim of giving a common language to (almost) all the Catalan structures. To reach this goal the authors of [DFPR] use an elementary mathematical tool, the Catalan pair, which is substantially a pair of binary relations satisfying certain axioms. They first prove that Catalan pairs are enumerated by Catalan numbers, according to their size. Their main goal is to prove that almost all Catalan structures can be interpreted in terms of Catalan pairs, thus providing a powerful tool to determine, in automatic way, many bijections between Catalan structures. Still in [DFPR], the authors prove several combinatorial properties of such Catalan pairs, also showing that they are related to some classes of pattern avoiding posets.

In this paper we carry on the original purpose of [DFPR], and attempt at developing a general method to determine a representation of a given Catalan structure in terms of Catalan pairs. Our method relies on the observation that most of the Catalan structures admit a recursive decomposition, which can be naturally translated onto the two binary relations defining Catalan pairs. Once we have presented our methodology, in Section 2 we apply it to furnish the interpretation of some of the most classical Catalan structures in terms of Catalan pairs. In the final section, we extend our method in order to include some other Catalan structures.

1.1 Basic definitions

In what follows we recall from [DFPR] some basic definitions of Catalan pairs. Given any set X , we denote $\mathcal{D} = \mathcal{D}(X)$ the *diagonal* of X , that is the relation $\mathcal{D} = \{(x, x) \mid x \in X\}$.

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Moreover, if θ is any binary relation on X , we denote by $\bar{\theta}$ the *symmetrization* of θ , i.e. the relation $\bar{\theta} = \theta \cup \theta^{-1}$. Given a set X of cardinality n , let $\mathcal{O}(X)$ be the set of strict order relations on X . By definition, this means that $\theta \in \mathcal{O}(X)$ when θ is an irreflexive and transitive binary relation on X .

Now let (S, R) be an ordered pair of binary relations on X . We say that (S, R) is a *Catalan pair* on X when the following axioms are satisfied:

- (i) $S, R \in \mathcal{O}(X)$;
- (ii) $\bar{R} \cup \bar{S} = X^2 \setminus \mathcal{D}$;
- (iii) $\bar{R} \cap \bar{S} = \emptyset$;
- (iv) $S \circ R \subseteq R$.

One can observe that, since S and R are both strict order relations, the two axioms (ii) and (iii) can be explicitly described by saying that, given $x, y \in X$, with $x \neq y$, exactly one of the following holds: xSy , xRy , ySx , yRx . The axiom (iv) says that if xSy and yRz , then xRz .

Two Catalan pairs (S_1, R_1) and (S_2, R_2) on the (not necessarily distinct) sets X_1 and X_2 , respectively, are said to be *isomorphic* when there exists a bijection ξ from X_1 to X_2 such that xS_1y if and only if $\xi(x)S_2\xi(y)$ and xR_1y if and only if $\xi(x)R_2\xi(y)$. We will consider Catalan pairs up to an isomorphism and, as a consequence of this definition, we say that a Catalan pair has *size* n when it is defined on a set X of cardinality n . The set of isomorphism classes of Catalan pairs of size n will be denoted $\mathcal{C}(n)$. In [DFPR] it is proved, both in an analytical and in a bijective way, that the number of Catalan pairs of size n is indeed the n th Catalan number.

2 Combinatorial interpretations of Catalan pairs

In this section we provide a general method to represent a given Catalan structure in terms of a Catalan pair. More precisely, we start from a given Catalan structure \mathcal{C} , for each object of \mathcal{C} we determine a *base set* $X_{\mathcal{C}}$, and we recursively define a pair of binary relations (S, R) on $X_{\mathcal{C}}$. We can prove that (S, R) is indeed a Catalan pair.

We rely on the fact that, using a pretty classical notation, most of the Catalan structures \mathcal{C} admit a recursive decomposition as

$$\mathcal{C} = \varepsilon + x\mathcal{C} \times \mathcal{C} \tag{2.1}$$

meaning that, each element $C \in \mathcal{C}$ is the empty object of size zero, or it can be uniquely decomposed as $C = xAB$, where x is an element of unitary size belonging to the *base set* $X_{\mathcal{C}}$, and $A, B \in \mathcal{C}$. Figure 2.1 shows an example of decomposition (2.1) for the class of *Dyck paths*.

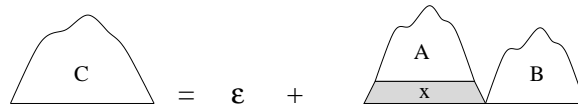


Figure 2.1: The recursive decomposition of Catalan structures illustrated for the class of Dyck paths.

From the decomposition (2.1), we can recursively define a base set $X_{\mathcal{C}}$ and a Catalan pair (S, R) on $X_{\mathcal{C}}$ for the Catalan structure \mathcal{C} in the following way:

- If $C = \varepsilon$ we have $S = \emptyset$ and $R = \emptyset$.

- Otherwise, if $C = xAB$, let (S_A, R_A) and (S_B, R_B) be the Catalan pairs on the objects A and B , with base sets X_A , and X_B , respectively. Then X_C is composed by the elements in X_A , X_B plus the new element x , and (S, R) is defined as

$$S = S_A \cup S_B \cup \{(a, x) : a \in X_A\}, \quad (2.2)$$

$$R = R_A \cup R_B \cup \{(a, b) : a \in X_A, b \in X_B\} \cup \{(x, b) : b \in X_B\} \quad (2.3)$$

The size of (S, R) is then given by the number of elements in X_C . We can trivially prove the following statement.

Proposition 2.1 *The relations S and R defined in (2.2) and (2.3) form a Catalan pair on the base set X_C . ■*

In the sequel, using Proposition 2.1, we can easily determine the relations S and R for some known Catalan structures. In particular, we will take into consideration here some examples from [St1] and [St2], involving rather different combinatorial objects.

According to our method, the relations S and R are recursively defined, but we will be able to present them in an explicit and meaningful way for each considered case; in particular we will show that each relation describes a certain combinatorial/geometrical relationship among the elements of the base set.

2.1 Perfect noncrossing matchings and Dyck paths

Our first example will be frequently recalled throughout all the paper. Given a linearly ordered set A of even cardinality, a *perfect noncrossing matching* of A is a noncrossing partition of A having all the blocks of cardinality 2. A block can be represented by means of an arch joining each couple of points. There is an obvious bijection between perfect noncrossing matchings and well formed strings of parentheses. It is known that, the class of perfect noncrossing matchings is counted by the Catalan numbers according to the number of arches.

Using this representation, we can define the following relations on the set X of arches of a given perfect noncrossing matching:

- for any $x, y \in X$, we say that xSy when x is included in y ;
- for any $x, y \in X$, we say that xRy when x is on the left of y .

The reader is invited to check that the above definition yields a Catalan pair (S, R) on the set X .

Example 2.1 Let $X = \{a, b, c, d, e, f, g\}$, and let S and R be defined as follows:

$$S = \{(b, a), (f, e), (f, d), (e, d), (g, d)\}$$

$$R = \{(a, c), (a, d), (a, e), (a, f), (a, g), (b, c), (b, d), (b, e), (b, f), (b, g), (c, d), (c, e), (c, f), (c, g), (e, g), (f, g)\}.$$

It is easy to check that (S, R) is a Catalan pair of size 7 on X , which can be represented as in Figure 2.2 (a).

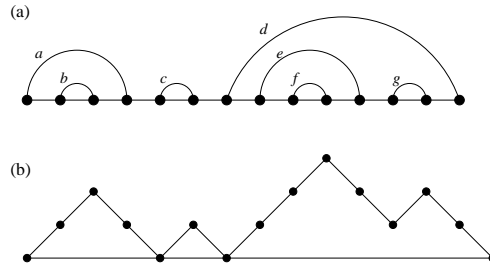


Figure 2.2: The graphical representation of a Catalan pair in terms of a noncrossing matching, and the associated Dyck path.

An equivalent way to represent perfect noncrossing matchings is to use Dyck paths: just interpret the leftmost element of an arch as an up step and the rightmost one as a down step. For instance, the matching represented in Figure 2.2 (a) corresponds to the Dyck path depicted in Figure 2.2 (b). Coming back to Catalan pairs, the relations S and R are suitably interpreted using the notion of a tunnel. A *tunnel* in a Dyck path [E] is a horizontal segment joining the midpoints of an up step and a down step, remaining below the path and not intersecting the path anywhere else. Now define S and R on the set X of the tunnels of a Dyck paths by declaring, for any $x, y \in X$:

- xSy when x lies above y ;
- xRy when x is completely on the left of y .

See again Figure 2.2 for an example illustrating the above definition.

2.2 Plane trees

Let \mathcal{T}_n be the set of plane trees having n edges. It is known that, the number of the elements of the set \mathcal{T}_n is the n th Catalan number. We say that a node b is a *descendant* of a node a when b belongs to the subtree of root a . In this situation, we also say that a is an *ancestor* of b . For any two nodes b and c , we define their *minimum common ancestor* to be the root of the minimum subtree containing both b and c . Finally, we will say that b lies *on the left of* c when, called a the minimum common ancestor of b and c , $a \notin \{b, c\}$ and b is on the left of c .

Given $t \in \mathcal{T}_n$, let X denote the set of nodes of t other than the root. Define two relations S and R on X as follows:

- xSy when x is a descendant of y ;
- xRy when x lies on the left of y .

Then the pair (S, R) is indeed a Catalan pair on X , and it induces the well known (see [St1]) bijection between plane trees and Dyck paths. Figure 2.3 depicts the plane tree corresponding to the Catalan pair (S, R) considered in Example 2.1.

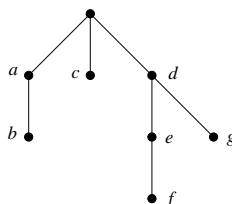


Figure 2.3: The plane tree corresponding to the Catalan pair represented in Figure 2.2.

2.3 Pattern avoiding permutations

Let n, m be two positive integers with $m \leq n$, and let $\pi = \pi(1) \cdots \pi(n) \in S_n$ and $\nu = \nu(1) \cdots \nu(m) \in S_m$. We say that π *contains* the pattern ν if there exist indices $i_1 < i_2 < \dots < i_m$ such that $(\pi(i_1), \pi(i_2), \dots, \pi(i_m))$ is in the same relative order as $(\nu(1), \dots, \nu(m))$. If π does not contain ν , we say that π is ν -*avoiding*. See [B] for plenty of information on pattern avoiding permutations. For instance, if $\nu = 123$, then $\pi = 524316$ contains ν , while $\pi = 632541$ is ν -avoiding.

We denote by $S_n(\nu)$ the set of ν -avoiding permutations of S_n . It is known that, for each pattern $\nu \in S_3$, $|S_n(\nu)| = C_n$ (see, for instance, [B]).

It is possible to give a description of the classes of 312-avoiding permutations and 321-avoiding permutations in terms of Catalan pairs. In this way, we are able to determine a description of the class $S_n(\nu)$ in terms of Catalan pair, for any $\nu \in S_3$. From the interpretation of $S_n(312)$ follows the interpretation of $S_n(231)$ by inverse and the interpretation of $S_n(213)$ by symmetry. From the interpretation of $S_n(321)$ follows the interpretation of $S_n(123)$ by symmetry and from the interpretation of $S_n(231)$ follows the interpretation of $S_n(132)$ by symmetry.

2.3.1 312-avoiding permutations

Let $X = \{1, 2, \dots, n\}$, for every permutation $\pi \in S_n$ we define the following relations S and R on X :

- iSj when $i < j$ and (i, j) is an inversion in π (i.e. $\pi(i) > \pi(j)$);
- iRj when $i < j$ and (i, j) is a noninversion in π (i.e. $\pi(i) < \pi(j)$).

Proposition 2.2 *The permutation $\pi \in S_n$ is 312-avoiding if and only if (S, R) is a Catalan pair of size n .*

Proof. The axioms (i) to (iii) in the definition of a Catalan pair are satisfied by (S, R) for any permutation π , as the reader can easily check. Moreover, π is 312-avoiding if and only if, given any three positive integers $i < j < k$, it can never happen that both (i, j) and (i, k) are inversions and (j, k) is a noninversion. This happens if and only if $S \circ R$ and S are disjoint. But, from the above definitions of S and R , it must be $S \circ R \subseteq R \cup S$, hence $S \circ R \subseteq R$. The axiom (iv) in the definition of a Catalan pair is satisfied by (S, R) . ■

Example. We consider the following 312-avoiding permutation π :

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 6 & 4 \end{pmatrix}$$

This configuration defines the following Catalan pair (S, R) :

$$S = \{(1, 2), (4, 6), (5, 6)\} \ ;$$

$$R = \{(1, 3), (1, 4), (1, 5), (1, 6), (2, 3), (2, 4), (2, 5), (2, 6), \\ (3, 4), (3, 5), (3, 6), (4, 5)\}.$$

□

The present interpretation of 312-avoiding permutations can be connected with the previous ones using Dyck paths and perfect noncrossing matchings, giving rise to a very well-known bijection, whose origin is very hard to be traced back (see, for instance, [P]). We leave all the details to the interested reader.

2.3.2 321-avoiding permutations

Each permutation $\pi = (\pi(1)\dots\pi(n)) \in S_n$ can be naturally represented on the Cartesian plane. In particular, each element $\pi(i)$ of the permutation is the point $(i, \pi(i))$ on the Cartesian plane, for any $1 \leq i \leq n$.

For example, we represent the permutations of $S_3(321)$ on the Cartesian plane as in Figure 2.4:

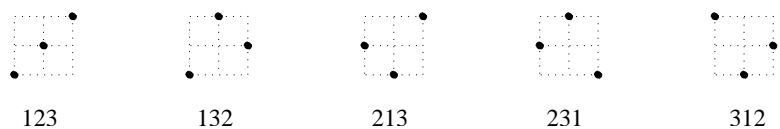


Figure 2.4: A Graphical representation of $S_3(321)$ on the Cartesian plane.

Let $X = \{a_1, a_2, \dots, a_n\}$ be the set of the points on the Cartesian plane representing a permutation $\pi \in S_n(321)$. If $x, y \in X$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$, we set $x \prec y$ when $x_1 < y_1$ and $x \triangleleft y$ when $x_2 < y_2$.

Using this representation, we can define the following relations on the set X . Given $x, y \in X$, a *cover* of $\{x, y\}$ is any point c of X having the following properties (see Figure 2.5):

- $x \triangleleft c$ and $y \triangleleft c$;
- $c \prec x$ and $c \prec y$.

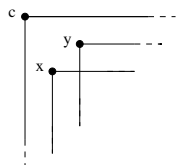


Figure 2.5: A graphical representation of the cover of (x, y) .

For any $x, y \in X$, we say that:

- xRy when there is no cover of $\{x, y\}$, $x \triangleleft y$ and $x \prec y$;
- xSy when $(x, y) \notin \overline{R}$ and $x \prec y$.

We can observe that the definition of the relation S consists in two distinct cases which are depicted in Figure 2.6.

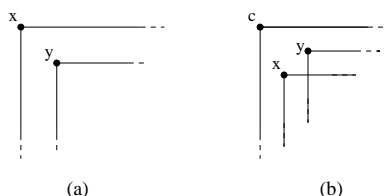


Figure 2.6: A graphical representation of distinct cases of the relation S .

Proposition 2.3 (S, R) is Catalan pair on the set X .

Proof. The axioms (i) to (iii) in the definition of a Catalan pair are satisfied by (S, R) , as the reader can easily check.

Let $x, y, z \in X$ such that xSy and yRz . From the above definition of the relation S it follows that the configuration $xSyRz$ can be represented by two distinct cases, as in Figure 2.7.

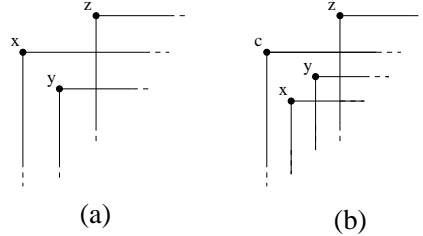


Figure 2.7: A graphical representations of the configuration $xSyRz$.

Both configurations (a) and (b) satisfy the relation xRz , hence $S \circ R \subseteq R$. The axiom (iv) in the definition of a Catalan pair is satisfied by (S, R) . ■

Example. We consider the following 321-avoiding permutation π :

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 4 & 5 \end{pmatrix}$$

The Figure 2.8 shows the graphical representation of π .

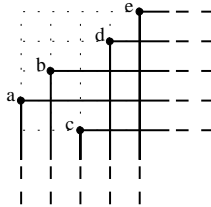


Figure 2.8: A graphical representation of π .

This configuration defines the following Catalan pair (S, R) on the $X = \{a, b, c, d, e\}$:

$$S = \{(a, c), (b, c)\} \quad ;$$

$$R = \{(a, b), (a, d), (a, e), (b, d), (b, e), (c, d), (c, e), (d, e)\}.$$

□

2.4 Sequences of integers counted by Catalan numbers

There are many classes of integers sequences satisfying special constraints which are enumerated by Catalan numbers. In [St1] we can find some examples, among which we focus on the following ones:

- (1) the set of sequences $a_1 a_2 \dots a_n$ of integers with $i \leq a_i \leq n$ and such that if $i \leq j \leq a_i$, then $a_j \leq a_i$.
- (2) the set of sequences $1 \leq a_1 \leq a_2 \leq \dots \leq a_n \leq n$ of integers with exactly one fixed point, i.e., exactly one index $1 \leq f \leq n$ for which $a_f = f$.

Let us study these two classes separately:

- (1) Let $X = \{a_1, a_2, \dots, a_n\}$ be the set of integers which form the sequence. For instance, for $n = 3$ we have the following five sequences on $X = \{1, 2, 3\}$:

$$123 \quad 133 \quad 223 \quad 323 \quad 333$$

We can define the relations S and R on the set X , as follows.

For any $a_i, a_j \in X$, with $1 \leq i, j \leq n$:

- $a_i R a_j$ when $i < j$ and $a_i < a_j$;
- $a_i S a_j$ when $j < i$ and $a_i \leq a_j$.

Proposition 2.4 (S, R) is Catalan pair on the set X .

Proof. The axioms (i) to (iii) in the definition of a Catalan pair are satisfied by (S, R) , as the reader can easily check.

Let $a_i, a_j, a_k \in X$ with $i < j < k$ such that $a_j S a_i$ and $a_i R a_k$. From the above definitions of the relations S and R it follows that the relation $a_j S a_i$ is satisfied when $i < j$ and $a_j \leq a_i$, while the relation $a_i R a_k$ is satisfied when $i < k$ and $a_i < a_k$. Then $a_j < a_k$ and since $j < k$ it follows that the relation $a_j R a_k$ is satisfied, hence $S \circ R \subseteq R$. The axiom (iv) in the definition of a Catalan pair is satisfied by (S, R) . ■

Example. We consider the following sequence:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ 5 & 2 & 4 & 4 & 5 & 6 \end{pmatrix}$$

This configuration defines the following Catalan pair (S, R) on the set $X = \{a_1, a_2, \dots, a_6\}$:

$$S = \{(a_2, a_1), (a_3, a_1), (a_4, a_1), (a_5, a_1), (a_4, a_3)\} ;$$

$$R = \{(a_1, a_6), (a_2, a_3), (a_2, a_4), (a_2, a_5), (a_2, a_6), (a_3, a_5), (a_3, a_6), (a_4, a_5), (a_4, a_6), (a_5, a_6)\}.$$

□

- (2) As above, let $X = \{a_1, a_2, \dots, a_n\}$ be the set of integers which form the sequence. For instance, for $n = 3$ we have the following five sequences on $X = \{1, 2, 3\}$:

$$111 \quad 112 \quad 222 \quad 223 \quad 333$$

Let f be the fixed point of X , i.e. the index such that $a_f = f$; for any index $y \leq f$ we consider the integer $a'_y = a_y - y$ and for any index z with $f < z \leq n$ we consider the integer $a'_z = z - a_z$. In this way, for any sequence $a_1 a_2 \dots a_n$ we have the corresponding sequence $a'_1 a'_2 \dots a'_n$.

Using this representation, we can define the following relations on the set X . Let i, j be the indexes of the integers $a_i, a_j \in X$, we can describe the following cases:

- If $i, j \leq f$ then:
 - $a_i S a_j$ when $i < j$, $a'_i > a'_j$ and $j = \min\{k : i < k \leq f, a'_k = a'_j\}$;

$$a_i R a_j \text{ when } \begin{cases} i < j, a'_i > a'_j \text{ and} \\ \exists w \text{ with } i < w < j \text{ such that } a'_w = a'_j; & (1) \\ \\ i < j \text{ and } a'_i \leq a'_j. & (2) \end{cases}$$

Roughly speaking, we say that $a_i S a_j$ if $i < j$ and $a'_i > a'_j$ where a'_j is the first integer in the sequence $a'_{i+1} a'_{i+2} \dots a'_f$ having that value and we say that $a_i R a_j$ for (1) if $i < j$ and $a'_i > a'_j$ where a'_j is not the first integer in the sequence $a'_{i+1} a'_{i+2} \dots a'_f$ having that value, but exists an integer a'_w with $i < w < j$ such that $a'_w = a'_j$.

- If $i \leq f < j$ then $a_i R a_j$.
- If $i, j > f$ then:

$$a_i S^{-1} a_j (a_j S a_i) \text{ when } i < j, a'_i < a'_j \text{ and } i = \max\{k : f < k < j, a'_k = a'_i\};$$

$$a_i R a_j \text{ when } \begin{cases} i < j, a'_i < a'_j \text{ and} \\ \exists w \text{ with } i < w < j \text{ such that } a'_w = a'_i; & (1) \\ \\ i < j \text{ and } a'_i \geq a'_j. & (2) \end{cases}$$

Roughly speaking, we say that $a_i S^{-1} a_j$ (or equivalently $a_j S a_i$) if $i < j$ and $a'_i < a'_j$ where a'_i is the last integer in the sequence $a'_{f+1} a'_{f+2} \dots a'_{j-1}$ having that value and we say that $a_i R a_j$ for (1) if $i < j$ and $a'_i < a'_j$ where a'_i is not the last integer in the sequence $a'_{f+1} a'_{f+2} \dots a'_{j-1}$ having that value, but exists an integer a'_w with $i < w < j$ such that $a'_w = a'_i$.

Proposition 2.5 Let $a'_1 a'_2 \dots a'_n$ be the corresponding sequence of $a_1 a_2 \dots a_n$ of integers with exactly one fixed point $f = a_f$. If there are indexes $x < y < z < f$ with $a'_x, a'_z > a'_y$ and $a'_x > a'_z$ then there is an index w with $x < w < y$ such that $a'_w = a'_z$. ■

Proposition 2.6 (S, R) is Catalan pair on the set X .

Proof. The axioms (i) to (iii) in the definition of a Catalan pair are satisfied by (S, R) , as the reader can easily check.

Let $a_1 a_2 \dots a_n$ be the sequence of integers with exactly one fixed point $f = a_f$ and let $a'_1 a'_2 \dots a'_n$ be the corresponding sequence of $a_1 a_2 \dots a_n$.

Let $a_i, a_j, a_z \in X$ such that $a_i S a_j$ and $a_j R a_z$.

Consider the case $i < j < z < f$. From the above definitions of the relations S and R it follows that the relation $a_i S a_j$ is satisfied when $i < j, a'_i > a'_j$ and $j = \min\{k : i < k \leq f, a'_k = a'_j\}$, while the relation $a_j R a_z$ can be satisfied by two distinct cases: the case (1) or the case (2). Suppose that $a_j R a_z$ is satisfied by the case (1), when $j < z, a'_j > a'_z$ and there is an index w with $j < w < z$ such that $a'_w = a'_z$. Then $a'_i > a'_z$ and since $i < j$ it follows that there is an index w with $i < w < z$ such that $a'_w = a'_z$, hence the relation $a_i R a_z$ is satisfied by the case (1) of R .

Now, suppose that $a_j R a_z$ is satisfied by the case (2) of the relation R , when $j < z$ and $a'_j \leq a'_z$. If $a'_j = a'_z$ then $a'_i > a'_z$ and since $i < j < z$ it follows that the relation $a_i R a_z$ is satisfied by the case (1) of R . If $a'_j < a'_z$, it must be either $a'_i \leq a'_z$ or $a'_i > a'_z$. If

$a'_i \leq a'_z$, since $i < z$ it follows that the relation $a_i R a_z$ is satisfied by the case (2) of R . If $a'_i > a'_z$, from the Proposition 2.5 it follows that there is an index w with $i < w < j$ such that $a'_w = a'_z$, since $i < j < z$, the relation $a_i R a_z$ is satisfied by the case (1) of R . Thus, we can conclude that, in every case, $a_i R a_z$, hence $S \circ R \subseteq R$ for $i < j < z < f$.

The case $i, j, z > f$ can be treated analogously and the case $i, j < f < z$ is obvious, hence the axiom (iv) in the definition of a Catalan pair is satisfied by (S, R) . ■

Example. We consider the following sequence:

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 & a_8 \\ 2 & 4 & 4 & 5 & 5 & 5 & 6 & 6 \end{pmatrix}$$

The corresponding sequence of $a_1 a_2 \dots a_n$ is the sequence $a'_1 a'_2 \dots a'_n$ defined as follows:

$$\begin{pmatrix} a'_1 & a'_2 & a'_3 & a'_4 & a'_5 & a'_6 & a'_7 & a'_8 \\ 1 & 2 & 1 & 1 & 0 & 1 & 1 & 2 \end{pmatrix}$$

This configuration defines the following Catalan pair (S, R) on the set $X = \{a_1, a_2, \dots, a_8\}$:

$$S = \{(a_1, a_5), (a_2, a_3), (a_2, a_5), (a_3, a_5), (a_4, a_5), (a_8, a_7)\} ;$$

$$R = \{(a_1, a_2), (a_1, a_3), (a_1, a_4), (a_1, a_6), (a_1, a_7), (a_1, a_8), (a_2, a_4), (a_2, a_6), (a_2, a_7), (a_2, a_8), (a_3, a_4), (a_3, a_6), (a_3, a_7), (a_3, a_8), (a_4, a_6), (a_4, a_7), (a_4, a_8), (a_5, a_6), (a_5, a_7), (a_5, a_8), (a_6, a_7), (a_6, a_8)\}.$$

□

2.5 Staircase shape

A *staircase shape* (see [St2]) A is depicted in Figure 2.9. In particular $|b| = |l| = n$ which is the *size* of the staircase shape having n steps of the form \lrcorner . Two staircase shapes of size n are said to be different one the other when they are divided into exactly n rectangles in two different ways. The following Figure 2.10 shows the case $n = 3$.

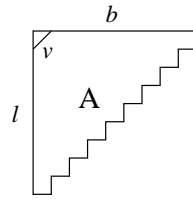


Figure 2.9: A graphical representation of staircase shape A.

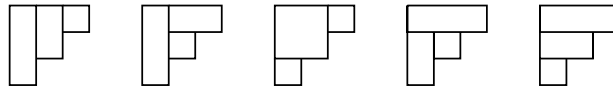


Figure 2.10: A graphical representation of all staircase shapes of size 3.

Proposition 2.7 *Each staircase shape of size n , with n rectangles, has n steps and each step belong to one and only one rectangle.* ■

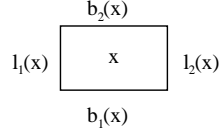


Figure 2.11: A graphical representation of the labelled sides of x .

Let $X(A)$ be the set of the rectangles which tiles the staircase shape A . For any $x \in X(A)$, the sides of the rectangle x are labelled as in Figure 2.11.

Proposition 2.8 *Each staircase shape A of size n admits the unique decomposition in Figure 2.12, where:*

- ε is the empty staircase shape;
- $\varphi \in X(A)$ is the junction rectangle containing the angle v ;
- L and U are staircase shapes of size m and m' respectively , with $m, m' < n$.

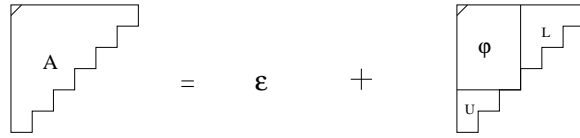


Figure 2.12: A graphical representation of the unique decomposition of staircase shape A .

Proof. Let A be a staircase shape of size n , with $n \geq 1$. Just locate the junction rectangle $\varphi \in X(A)$ to determine the decomposition of A . The junction rectangle of the configuration A is the rectangle of the set $X(A)$ which contains the angle v .

From Proposition 2.7, it follows that $|l_1(\varphi)| = k + 1$ and $|b_1(\varphi)| = n - k$ with $0 \leq k \leq n - 1$. By the way, the staircase shape L of size k , lies completely on the right of $l_1(\varphi)$, while the staircase shape U of size $n - (k + 1)$, lies under $b_1(\varphi)$.

The uniqueness of the decomposition of A is related to the existence of only one junction rectangle $\varphi \in X(A)$. ■

At this point, we can define the following relations on the set X of rectangles which tile a staircase shape. For any $x, y \in X$, we set:

- xSy when $l_2(x)$ (or the extension of $l_2(x)$) intersects $b_1(y)$;
- xRy when x is completely on the left of $l_1(y)$ (or the extension of $l_1(y)$).

Proposition 2.9 *(S, R) is Catalan pair on the set X .*

Proof. The axiom (i) in the definition of a Catalan pair is satisfied by (S, R) , as the reader can easily check.

Let $x, y \in X$ be two distinct rectangles of a staircase shape A . From Proposition 2.8, it follows that the staircase shape A admits a unique decomposition, then we can consider the following cases:

- a) $x = \varphi$,
it must be either $y \in L$ or $y \in U$. If $y \in L$ then xRy , otherwise if $y \in U$ then ySx ;

- b) $y = \varphi$,
it must either $x \in L$ or $x \in U$. If $x \in L$ then yRx , otherwise if $x \in U$ then xSy .

Only one relation between x and y can hold in A , hence the axioms (ii) and (iii) in the definition of a Catalan pair are satisfied by (S, R) .

Let $x, y, z \in X$ be two distinct rectangles of a staircase shape A , such that xSy and yRz . From Proposition 2.8, it follows that the staircase shape A admits a unique decomposition. The most interesting case is when $y = \varphi$. Since xSy and yRz , it follows that $x \in U$ and $z \in L$, hence xRz . The axiom (iv) in the definition of a Catalan pair is satisfied by (S, R) . ■

Example. Let $X = \{a, b, c, d, e, f, g\}$ be the set of the rectangles which tile the staircase shape of size 7 represented in Figure 2.13.

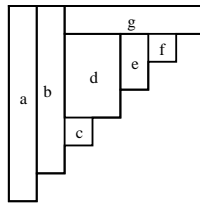


Figure 2.13: An example of staircase shape of size 7.

This configuration defines the following Catalan pair (S, R) on the set $X = \{a, b, c, d, e, f, g\}$:

$$S = \{(c, d), (c, g), (d, g), (e, g), (f, g)\} ;$$

$$R = \{(a, b), (a, c), (a, d), (a, e), (a, f), (a, g), (b, c), (b, d), (b, g), (b, e), (b, f), (c, e), (c, f), (d, e), (d, f), (e, f)\}.$$

□

3 Further work

At the end of the paper, we would like to take into considerations the Catalan structures which do not admit a recursive decomposition as (2.1). Among these ones, the most popular are perhaps the parallelogram polyominoes, the 2-colored Motzkin paths, the binary trees [St1].

By means of the following example, concerning the class of parallelogram polyominoes, we show that it is possible to adapt our method to include also these "more complex" combinatorial structures.

A parallelogram polyomino with semi-perimeter $n + 1$ is defined by two distinct non intersecting lattice paths of length $n + 1$ beginning in $(0, 0)$ and using only horizontal and vertical unit steps. These paths, called the upper and the lower path, respectively, meet in only two points, the beginning point and the ending point of both, see Figure 3.14.

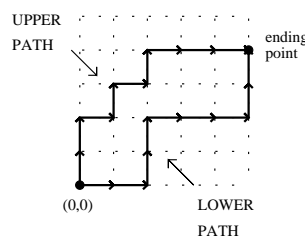


Figure 3.14: A parallelogram polyomino with semi-perimeter 9.

The class \mathcal{P} of the parallelogram polyominoes is counted by the Catalan numbers according to the semi-perimeter, but \mathcal{P} does not admit a decomposition as (2.1). For this class, a simpler decomposition is given by that depicted in Figure 3.15, where x is a single element of unitary size belonging to the base set X and A, B, C, D are parallelogram polyominoes of lower size.

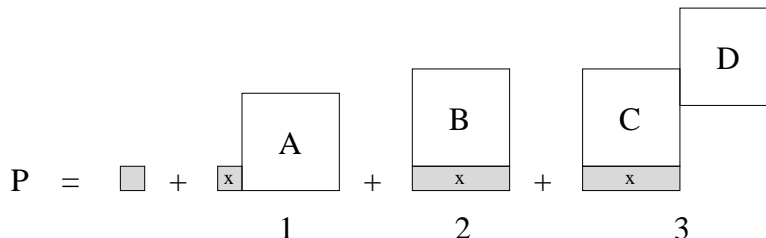


Figure 3.15: Recursive decomposition of the parallelogram polyominoes.

Figure 3.15 shows that we have three distinct operations, denoted 1, 2 and 3 respectively, which can be applied on the class of parallelogram polyominoes. As we did for decomposition (2.1), also in this case we can recursively define Catalan pairs (S, R) on the class of parallelogram polyominoes.

In particular, if P is a parallelogram polyomino, and it is not the single cell, then it can be uniquely decomposed according to 1, 2 or 3:

1. if the last operation applied on P is operation 1, let (S_A, R_A) be the Catalan pair on the base set for A , then S and R are defined as follows:

$$S = S_A, \quad R = R_A \cup \{(x, a) : a \in A\}.$$

2. if the last operation applied on P is operation 2, let (S_B, R_B) be the Catalan pair on the base set for B , then S and R are defined as follows:

$$S = S_B \cup \{(b, x) : b \in B\}, \quad R = R_B.$$

3. if the last operation applied on P is operation 3, let $(S_C, R_C), (S_D, R_D)$ be the Catalan pairs on C, D , respectively, then S and R are defined as follows:

$$S = S_C \cup S_D \cup \{(c, x) : c \in C\}, \quad R = R_C \cup R_D \cup \{(c, d) : c \in C, d \in D\} \cup \{(x, d) : d \in D\}$$

The construction of the base set X follows the method previously described and is straightforward.

Applying the previous technique it is then possible to automatically determine the Catalan pairs associated with all the structures which satisfy a decomposition like that in Figure 3.15.

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