Dynamics in parallel of double Boolean automata circuits

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Introduction

In this paper, we give some results concerning the dynamics of *double Boolean automata circuits* (DBAC's for short), namely, networks associated to interaction graphs composed of two *side-circuits* that share a node. More precisely, a double circuit of *left-size* $\ell \in \mathbb{N}$ and of *right-size* $r \in \mathbb{N}$ is a graph that we denote by $\mathbb{D}_{\ell,r}$. It has $n = \ell + r - 1$ nodes. Nodes that are numbered from 0 to $\ell - 1$ belong to the left-circuit and the others plus the node 0 (that belongs to both side-circuits) belong to the right-circuit. Node 0 is the only node with in- and out-degree 2. All other nodes have in- and out-degree 1.

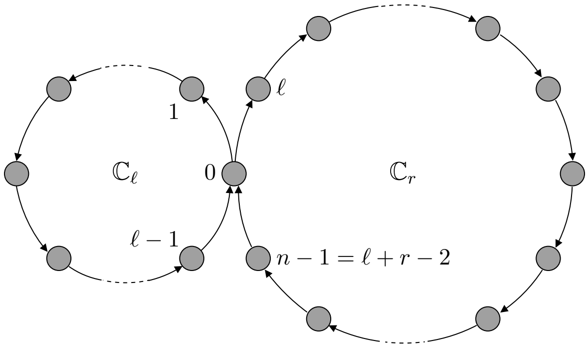

Figure 1: Double circuit $\mathbb{D}_{\ell,r}$.

A DBAC $D_{\ell,r} = (\mathbb{D}_{\ell,r}, F)$ is a network defined by *(i)* its *interaction graph*, a double-circuit $\mathbb{D}_{\ell,r}$, and *(ii)* a *global transition function* F that updates the states of all nodes in parallel and that is defined as follows by the *local transition functions* f_i of nodes $i < n$:

$$\forall x \in \{0,1\}^n,$$
$$\begin{cases} F(x)_i = f_i(x_{i-1}), \ \forall i \notin \{0, \ell\}, \\ F(x)_\ell = f_\ell(x_0), \\ F(x)_0 = f_0(x_{\ell-1}, x_{\ell+r-2}) = f_0^L(x_{\ell-1}) \star f_0^R(x_{\ell+r-2}) \text{ where } \star \in \{\wedge, \vee\}. \end{cases} \tag{1}$$

All local transition functions are supposed to be non-constant. Thus, $\forall i < n, f_i$, f_0^L, $f_0^R \in \{id, neg\}$ where $id : a \mapsto a$ and $neg : a \mapsto \neg a$, $\forall a \in \{0,1\}$. As a consequence all local transition functions are locally monotone. All arcs (i,j) entering a node j whose local transition function is id (resp. neg), with respect to node i, are labeled with a $+$ sign (resp. a $-$ sign) and called *positive arcs* (resp. *negative arcs*). A side-circuit with an *even* number of negative arcs (resp. *odd* number of negative arcs) is called a *positive (side-) circuit* (resp. a *negative (side-) circuit*).

Given a configuration $x = (x_0, \ldots, x_{n-1}) \in \{0,1\}^n$ of a DBAC $D_{\ell,r}$, we use the following notation:

$$x^L = (x_0, \ldots, x_{\ell-1}) \qquad \text{and} \qquad x^R = (x_0, x_\ell, \ldots, x_{n-1}).$$

A configuration $x(t) = (x_0(t), \ldots, x_{n-1}(t)) \in \{0,1\}^n$ such that $\forall k \in \mathbb{N}$, $F^{k \cdot p}(x(t)) = x(t+k \cdot p) = x(t)$ is said to have period p. If $x(t)$ has period p and does not also have period $d < p$, then $x(t)$ is said to have *exact* period p. An attractor of period $p \in \mathbb{N}$, or p-attractor, is the set of configurations belonging to the orbit of a configuration that has p as exact period. Attractors of period 1 are called fixed points. The graph whose nodes are the configurations $x \in \{0,1\}^n$ of a network and whose arcs represent the transitions $(x(t), x(t+1) = F(x(t)))$ is called the transition graph of the network.

In [1], the authors showed the following results:

Proposition 1

1. *The transition graphs of two* DBAC*s with same side-signs and side-sizes are isomorphic, whatever the definition of f_0 (i.e., whether $\star = \vee$ or $\star = \wedge$ in the definition (1) of F above).*

2. *Attractor periods of a* DBAC *divide the sizes of the positive side-circuits if there are some and do not divide the sizes of the negative side-circuits if there are some.*

3. *If both side-circuits of a* DBAC $D_{\ell,r}$ *have the same sign, then, attractor periods divide the sum* $N = \ell + r$.

4. *If both side-circuits of a* DBAC $D_{\ell,\ell}$ *have the same sign and size, then* $D_{\ell,\ell}$ *behaves as an isolated circuit of that size and sign (i.e., the sub-transition graph generated by the periodic configurations of* $D_{\ell,\ell}$ *is isomorphic to the transition graph of an isolated circuit with the same sign and size).*

5. *A* DBAC *has as many fixed points as it has positive side-circuits.*

6. *If both side-circuits of a* DBAC $D_{\ell,r}$ *are positive and* p *divides* ℓ *and* r*, then number of attractors of period* p *is given by* $\mathtt{A}_p^+$ *(sequence* `A1037` *of the OEIS [3]), namely, the number of attractors of period* p *of an isolated positive circuit of size a multiple of* p.

As a result of the first two points of Proposition 1, we may focus on *canonical* instances of DBACs. Thus, from now on, we will suppose that $\star = \vee$ and $\forall i \neq 0,\ f_i = id$. If the left-circuit is positive (resp. negative), we will suppose that the arc $(\ell - 1, 0)$ is positive (resp. negative) and $f_0^L = id$ (resp. $f_0^L = neg$) and similarly for the right-circuit. Thus, the only possible negative arcs on the DBACs we will study will be the arcs $(\ell - 1, 0)$ and $(n - 1, 0)$.

The last point of Proposition 1 yields a description of the dynamics of a doubly positive DBAC (in terms of combinatorics only but [1] also gives a characterisation the configurations of period p, $\forall p \in \mathbb{N}$ for all types of DBACs). Thus, here, we will focus on the cases where the DBACs have at least one negative side-circuit.

For a negative-positive or a negative-negative DBAC $D_{\ell,r}$, we will use the following notations and results. The function μ is the Mobiüs function. It appears in the expressions below because of the Mobiüs inversion formula [2].

- The number of configurations of period p is written:

$$\mathtt{C}_p(\ell, r).$$

- The number of configurations of period exactly p is written and given by:

$$\mathtt{C}_p^*(\ell, r) \ = \ \sum_{q|p} \mu(\frac{p}{q}) \cdot \mathtt{C}_p(\ell, r).$$

- The number of attractors of period p (where p is an attractor period) is written and given by:

$$\mathtt{A}_p(\ell, r) \;=\; \frac{\mathtt{C}^*_p(\ell, r)}{p} \;=\; \frac{1}{p} \cdot \sum_{q|p} \mu(\frac{p}{q}) \cdot \mathtt{C}_p(\ell, r).$$

  In particular, $C_1(\ell, r) = C^*_1(\ell, r) = 1$.

- The total number of attractors is written and given by:

$$\mathtt{T}(\ell, r) \;=\; \sum_{p \text{ attractor period}} \mathtt{A}_p(\ell, r).$$

For the sake of clarity, we will write $\mathtt{A}_p(\ell, r) = \mathtt{A}^{\pm}_p(\ell, r)$ (resp. $\mathtt{A}_p(\ell, r) = \mathtt{A}^{=}_p(\ell, r)$) and $\mathtt{T}(\ell, r) = \mathtt{T}^{\pm}(\ell, r)$ (resp. $\mathtt{T}(\ell, r) = \mathtt{T}^{=}(\ell, r)$) when $D_{\ell,r}$ will be a negative-positive DBAC (resp. a negative-negative DBAC).

1 Positive-Negative

We first concentrate on DBACs whose left-circuit is negative and whose right-circuit is positive. From Proposition 1, we know that all possible attractor periods

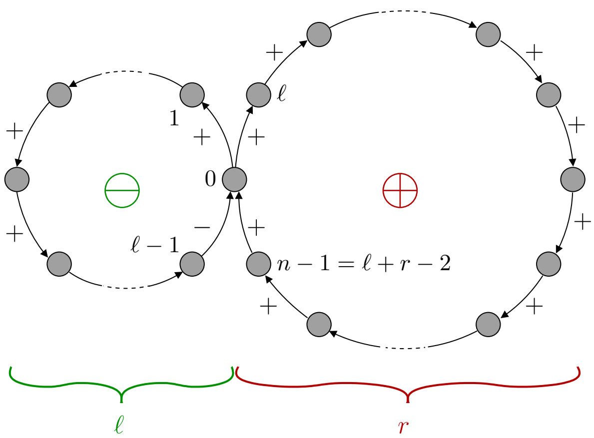

Figure 2: Interaction graph of a negative-positive DBAC. All arcs are positive (resp. all local transition functions are equal to id) except for the arc $(\ell - 1, 0)$ (resp. except for the local transition function $f^L_0 = neg$).

of such DBACs divide r and do not divide ℓ. We also know that these networks have exactly one fixed point. In the sequel, we focus on attractor periods $p > 1$.

POSITIVE

NEGATIVE

$\ell \backslash r$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	$\mathtt{T}^-_\ell$
1	1	2	2	3	3	5	5	8	10	15	19	31	41	64	94	143	211	329	1
2		1	2	3	3	4	5	8	10	14	19	31	41	63	94	143	211	328	1
3			1	3	3	6	5	8	8	15	19	33	41	64	91	143	211	332	2
4				1	3	4	5	11	10	14	19	24	41	63	94	156	211	328	2
5					1	5	5	8	10	26	19	31	41	64	70	143	211	329	4
6						1	5	8	8	14	19	63	41	63	91	143	211	232	6
7							1	8	10	15	19	31	41	158	94	143	211	329	10
8								1	10	14	19	24	41	63	94	411	211	328	16
9									1	15	19	33	41	64	91	143	211	1098	30
10										1	19	31	41	63	70	143	211	328	52
11											1	31	41	64	94	143	211	329	94
12												1	41	63	91	156	211	232	172
13													1	64	94	143	211	329	316
14														1	94	143	211	328	586
15															1	143	211	332	1096
16	$gcd(\ell,r) =$															1	211	328	2048
17	1	2	3	4	5	6	7	8	9								1	329	3856
18																		1	7286
$\mathtt{T}^{\pm}_{\ell,r}$	1	2	2	3	3	6	5	11	10	26	19	63	41	158	94	411	211	1098	
$\mathtt{T}^+_r$	2	3	4	6	8	14	20	36	60	108	188	352	632	1182	2192	4116	7712	14602	
$\frac{\sqrt{3}^r}{2^{r-1}} \times \mathtt{T}^+_r$	3.464	4.5	5.196	6.75	7.794	11.812	14.614	22.781	32.881	51.258	77.272	125.297	194.825	315.56	506.8	1337.27	2192.77	3588.9	

Table 1: Total number of attractors of a negative-positive DBAC $D_{\ell,r}$ (obtained by computer simulations). Each colour corresponds to a value of $gcd(\ell,r)$. The line $\mathtt{T}^+_r$ (resp. the column $\mathtt{T}^-_\ell$) gives the total number of attractors of an isolated positive (resp. negative) circuit of size r (resp. ℓ).

Characterisation of configurations of period p

Let p be a divisor of $r = p \cdot q$ that does not divide $\ell = k \cdot p + d$, $d = \ell \ mod\ p$. We write $\Delta_p = gcd(p, \ell) = gcd(p, d)$. Let $x = x(t)$ be a configuration of period p: $\forall m \in \mathbb{N},\ x(t+m\cdot p) = x(t)$. Note that to describe the dynamics of $D_{\ell,r}$, it suffices to describe the behaviour of node 0 (see [1]). The configuration x satisfies the following:

$$\begin{aligned} x_0(t) &= x_0(t + k \cdot p) \\ &= \neg x_{\ell-1}(t + k \cdot p - 1) \vee x_{n-1}(t + k \cdot p - 1) \\ &= \neg x_{\ell-k\cdot p}(t) \vee x_0(t + (k - q) \cdot p) \\ &= \neg x_d(t) \vee x_0(t) \end{aligned}$$

Thus, if $x_0(t) = 0$, then $x_d(t) = 1$ (and also, because $x_0(t) = \neg x_0(t - d) \vee x_0(t)$, if $x_0(t) = 0$, then $x_0(t - d) = 1$). From this, we may derive the following characterisation:

Proposition 2 *Let $p \in \mathbb{N}$ be divisor of $r = p \cdot q$ that does not divide ℓ. A configuration $x = x(t)$ has period p if and only if there exists a circular word $w \in \{0,1\}^p$ of size p that does not contain the sub-sequence $0u0$, $u \in \{0,1\}^{d-1}$ and that satisfies:*

$$x^L = w^k w[0 \dots d-1] \qquad \text{and} \qquad x^R = w^q$$

where $w[0 \dots m] = w_0 \dots w_m$. More precisely, the word w satisfies:

$$\forall i < p,\ w_i = x_0(t + i).$$

Consequently, the property for a configuration to be of period p depends only on $d = \ell \ mod\ p$ and on p (and not on ℓ nor on r).

As a result of the characterisation above, we may focus on negative-positive DBACs $D_{\ell,r}$ such that $\ell < r$ (*i.e.*, $\ell = \ell \ mod\ r$) and to count the number of attractors of period p we may focus on DBACs $D_{\ell,r}$ such that $\ell < p$ (*i.e.*, $\ell = d = \ell \ mod\ p$). In other words, from Proposition 2:

$$A_p^{\pm}(\ell, r) = A_p^{\pm}(\ell \ mod\ r, r) = A_p^{\pm}(\ell \ mod\ p, r).$$

Combinatorics

From Proposition 2 each configuration x of period p is associated with a circular word $w \in \{0,1\}^p$ that does not contain the sub-sequence $0u0$, $u \in \{0,1\}^{d-1}$, $d = \ell \ mod\ p$. It is easy to check that this word w can be written as an interlock of a certain number N of circular words $w^{(1)}, w^{(2)}, \dots, w^{(N)}$ of size $T = p/N$ that do not contain the sub-sequence 00 (see figure 3). More precisely, the size of a word $w^{(j)}$ satisfies $T \cdot d = K \cdot p$ for a certain K such that T and K are minimal. In other words, $T \cdot d = lcm(d, p) = \frac{d \cdot p}{gcd(d,p)}$ and thus $T = \frac{p}{\Delta_p}$. Consequently, $N = \Delta_p$ and we obtain the following lemma which explains why, in each column of Table 1, every cell of a same colour contains the same number.

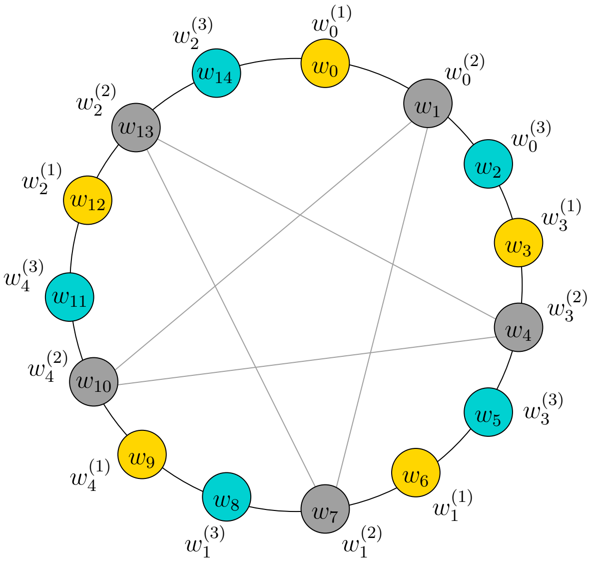

Figure 3: The circular word $w = w_0 \ldots w_{p-1} = x_0(t) \ldots x_0(t+p-1)$ mentioned in Proposition 2 that characterises a configuration $x(t)$ of period $|w| = p$. In this example, $p = 15$, $d = \ell \ mod \ p = 6$ so that w is made of an interlock of $\Delta_p = gcd(d, p) = 3$ words $w^{(j)} = w_0^{(j)} \ldots w_4^{(j)}$ of size $p/\Delta_p = 5$.

Lemma 1 *Let p be a divisor of $r = p \cdot q$ that does not divide ℓ. The number of configurations of period p, $C_p(\ell, r)$, depends only on $\Delta_p = gcd(\ell, p)$ and on p. Thus, we write:*

$$C_p(\ell, r) \;\; = \;\; C_{p,\Delta_p}.$$

The number of circular words of size n that do not contain the sub-sequence 00 is counted by the Lucas sequence (sequence `A204` of the OEIS [3]):

$$\begin{cases} L(1) = 1 \\ L(2) = 2 \\ L(n) = L(n-1) + L(n-2) = \phi^n + \overline{\phi}^n = \phi^n + (-\frac{1}{\phi})^n, \end{cases}$$

where $\phi = \frac{1+\sqrt{5}}{2} \sim 1.61803399$ is the golden ratio and $\overline{\phi} = 1 - \phi = \frac{1-\sqrt{5}}{2} \sim -0.61803399$. Among the properties of ϕ that will be useful to us in the sequel are the following:

$$\phi^2 = 1 + \phi \qquad \text{and} \qquad \overline{\phi} = -\frac{1}{\phi}.$$

Thus, to build a circular word $w \in \{0,1\}^p$ without the sub-sequence $0u0$, $u \in \{0,1\}^{\ell-1}$, one needs to chose Δ_p among $L(\frac{p}{\Delta_p})$ words $w^{(j)}$ of size $\frac{p}{\Delta_p}$ without the sub-sequence 00. As a result holds Proposition 3 below:

Proposition 3 *The number of configurations of period p is given by:*

$$C_{p,\Delta_p} = L(\frac{p}{\Delta_p})^{\Delta_p}.$$

Consequently, the number of attractors of period p is given by:

$$\mathtt{A}_p^{\pm}(\ell, r) = A_{p,\Delta_p} = \frac{1}{p} \cdot \sum_{q|p} \mu(\frac{p}{q}) \cdot L(\frac{p}{\Delta_p})^{\Delta_p},$$

where $\Delta_p = gcd(\ell, p)$.

Number of configurations of period p

Let us develop the expression for $C_p(\ell, r) = C_{p,\Delta_p}$:

$$\begin{aligned}
C_{p,\Delta_p} &= L(\tfrac{p}{\Delta_p})^{\Delta_p} \\
&= L(\phi^{\frac{p}{\Delta_p}} + (-\phi)^{-\frac{p}{\Delta_p}})^{\Delta_p} \\
&= \textstyle\sum_{k \leq \Delta_p} \binom{\Delta_p}{k} \cdot \phi^{\frac{p \cdot k}{\Delta_p}} \cdot (-\phi)^{-p + \frac{p \cdot k}{\Delta_p}} \\
&= (-\phi)^{-p} \cdot \textstyle\sum_{k \leq \Delta_p} \binom{\Delta_p}{k} \cdot \phi^{2 \cdot \frac{p \cdot k}{\Delta_p}} \cdot (-1)^{\frac{p \cdot k}{\Delta_p}} \\
&= \overline{\phi}^p \cdot \textstyle\sum_{k \leq \Delta_p} \binom{\Delta_p}{k} \cdot (-\phi^2)^{\frac{p \cdot k}{\Delta_p}} \\
&= \overline{\phi}^p \cdot ((-\phi^2)^{\frac{p}{\Delta_p}} + 1)^{\Delta_p} \\
&= (-1)^p \cdot |\overline{\phi}|^p \cdot ((-1)^{\frac{p}{\Delta_p}} \cdot (\phi^2)^{\frac{p}{\Delta_p}} + 1)^{\Delta_p}.
\end{aligned}$$

If p is odd, $\frac{p}{\Delta_p}$ cannot be even. Thus, there are three cases only:

1. p and $\frac{p}{\Delta_p}$ are odd. Thus, because Δ_p is necessarily also odd:

$$C_{p,\Delta_p} \;=\; -|\overline{\phi}|^p \cdot (-(\phi^2)^{\frac{p}{\Delta_p}} + 1)^{\Delta_p} \;=\; |\overline{\phi}|^p \cdot ((\phi^2)^{\frac{p}{\Delta_p}} - 1)^{\Delta_p}. \tag{2}$$

2. p is even and $\frac{p}{\Delta_p}$ is odd. Thus, because Δ_p is necessarily even:

$$C_{p,\Delta_p} \;=\; |\overline{\phi}|^p \cdot (-(\phi^2)^{\frac{p}{\Delta_p}} + 1)^{\Delta_p} \;=\; |\overline{\phi}|^p \cdot ((\phi^2)^{\frac{p}{\Delta_p}} - 1)^{\Delta_p}. \tag{3}$$

3. p and $\frac{p}{\Delta_p}$ are both even. Thus:

$$C_{p,\Delta_p} \;=\; |\overline{\phi}|^p \cdot ((\phi^2)^{\frac{p}{\Delta_p}} + 1)^{\Delta_p}. \tag{4}$$

To sum up, we give below Proposition 4 whose last part can be derived from the relation between the Euler totient $\psi(\cdot)$ and the Mobiüs function $\mu(\cdot)$, $\psi(n) = \sum_{m|n}(n/m)\cdot\mu(m)$, and from the following equations where $\Delta_q = gcd(q,\ell)$:

$$T^{\pm}(\ell,r) \;=\; \sum_{p|r} A_p(\ell,r) = \sum_{p|r}\sum_{q|p}\frac{1}{p}\cdot\mu(\frac{p}{q})\cdot C_{q,\Delta_q} \;=\; \frac{1}{r}\cdot\sum_{p|r}\sum_{q|p} C_{q,\Delta_q}\cdot\frac{r}{(p/q)\cdot q}\cdot\mu(\frac{p}{q})$$

$$=\; \frac{1}{r}\cdot\sum_{q|r} C_{q,\Delta_q}\sum_{k|\frac{r}{q}}\frac{r}{k\cdot q}\cdot\mu(k) \;=\; \frac{1}{r}\cdot\sum_{q|r}\psi(\frac{r}{q})\cdot C_{q,\Delta_q}.$$

Proposition 4 *Let $\Delta_p = gcd(\ell,p)$ where p is a divisor of r that does not divide ℓ. Then, the number of configurations of period p is given by:*

$$C_{p,\Delta_p} \;=\; \begin{cases} |\overline{\phi}|^p\cdot((\phi^2)^{\frac{p}{\Delta_p}}-1)^{\Delta_p} & \textit{if } \frac{p}{\Delta_p} \textit{ is odd,} \\ |\overline{\phi}|^p\cdot((\phi^2)^{\frac{p}{\Delta_p}}+1)^{\Delta_p} & \textit{if } \frac{p}{\Delta_p} \textit{ is even.} \end{cases}$$

In particular, $C_{r,gcd(\ell,r)}$ counts the total number of periodic configurations of the network. The number of attractors of period p and the total number of attractors are respectively given by:

$$\mathtt{A}^{\pm}_p(\ell,r) \;=\; \mathtt{A}^{\pm}_{p,\Delta_p} \;=\; \frac{1}{p}\cdot\sum_{q|p}\mu(\frac{p}{q})\cdot C_{q,\Delta_q}, \quad \textit{and} \quad \mathtt{T}^{\pm}(\ell,r) \;=\; \frac{1}{r}\cdot\sum_{p|r,\ \neg(p|\ell)}\psi(\frac{r}{p})\cdot C_{p,\Delta_p}.$$

Upper bounds

From equations (2), (3) and (4), one can derive that C_{p,Δ_p} and thus $\mathtt{A}^{\pm}_p(\ell,r) = \mathtt{A}^{\pm}_{p,\Delta_p}$ are maximal when Δ_p is minimal (*i.e.*, $\Delta_p = 1$), if p is odd and if p is even, on the contrary, C_{p,Δ_p} and $\mathtt{A}^{\pm}_p(\ell,r)$ are maximal when Δ_p is maximal (*i.e.*, $\Delta_p = \frac{p}{2}$). Thus, we have:

$$\begin{aligned} C_{p,\Delta_p} \;&\leq\; |\overline{\phi}|^p\cdot((\phi^2)^{\frac{p}{\Delta_p}}+1)^{\Delta_p} \\ &\leq\; |\overline{\phi}|^p\cdot((\phi^4)+1)^{\frac{p}{2}} \;=\; C_{p,\frac{p}{2}} \\ &=\; \tfrac{(3+3\cdot\phi)^{\frac{p}{2}}}{\phi^p} \\ &=\; (\tfrac{3+3\cdot\phi}{1+\phi})^{\frac{p}{2}} \\ &=\; 3^{\frac{p}{2}}, \end{aligned}$$

In addition, $\mathtt{A}^{\pm}_{p,\Delta_p} \leq \frac{1}{p} \cdot F_p(\sqrt{3})$ where $F_p(a) = \sum_{d|p} \mu(\frac{p}{d}) \cdot a^d$. Now, it can be shown that:

$$\forall a > \phi,\ \forall p \neq 2, \quad a^{p-1} \ < \ F_p(a) = \sum_{d|p} \mu(\frac{p}{d}) \cdot a^d \ < \ a^p.$$

Consequently,

$$\forall p \neq 2, \quad \mathtt{A}_{p,\Delta_p} \ \leq \ \frac{1}{p} \cdot F_p(\sqrt{3}) \ < \ 2 \cdot (\frac{\sqrt{3}}{2})^p \cdot \frac{1}{p} \cdot F_p(2) = 2 \cdot (\frac{\sqrt{3}}{2})^p \cdot \mathtt{A}^+_p. \qquad (5)$$

where $\mathtt{A}^+_p$ refers to the number of attractors of period p of an isolated positive circuit whose size is a multiple of p [1]. Then, from (5), we may derive the following result:

Proposition 5 *For all $p \neq 2$, the number of attractors of period p satisfies:*

$$\mathtt{A}^{\pm}_p(\ell, r) = \mathtt{A}_{p,\Delta_p} \ < \ 2 \cdot (\frac{\sqrt{3}}{2})^p \cdot \mathtt{A}^+_p$$

and for $p = 2$: $\mathtt{A}_{2,\Delta_2} = 1 = \mathtt{A}^+_2$.

Let us denote by T^+_n the number of attractors of a positive circuit of size n. From Proposition 5, for all r, the *total* number of attractors of a negative-positive DBAC satisfies:

$$\mathtt{T}^{\pm}_{\ell,r} < 2 \cdot (\frac{\sqrt{3}}{2})^r \cdot \mathtt{T}^+_r.$$

However, computer simulations (*cf* Table 1, last line) show that this bound on $\mathtt{T}^{\pm}_{\ell,r}$ is too large. We leave open the problem of finding a better bound.

2 Negative-Negative

We now concentrate on doubly negative DBACs. The canonical DBAC we will use in the discussion below is defined in Figure 4.

Let $p \in \mathbb{N}$ be a possible attractor period of $D_{\ell,r}$ (p divides $N = \ell + r$ but divides neither ℓ nor r). Without loss of generality, suppose $\ell\ mod\ p > r\ mod\ p = d$. Because p divides $\ell + r$, it holds that $\ell\ mod\ p = p - d$. Then, for any configuration $x = x(t) \in \{0, 1\}^n$ of period p, we have the following:

$$\begin{aligned} x_0(t) &= \neg x_{\ell-1}(t-1) \vee \neg x_{n-1}(t-1) \\ &= \neg x_0(t-\ell) \vee \neg x_0(t-r) \\ &= \neg x_0(t+r) \vee \neg x_0(t+\ell) \\ &= \neg x_0(t+d) \vee \neg x_0(t-d) \end{aligned}$$

As a consequence, if $x_0(t) = 0$, then $x_0(t+d) = x_0(t-d) = 1$ and if $x_0(t) = 1$, then either $x_0(t+d) = 0$, or $x_0(t-d) = 0$. Thus, the circular word $w =$

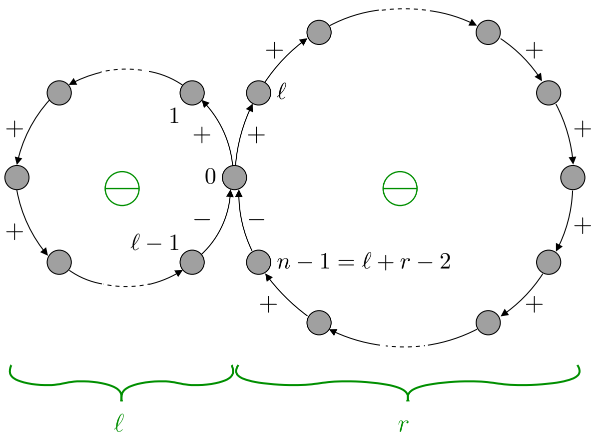

Figure 4: Interaction graph of a negative-negative DBAC. All arcs are positive (resp. all local transition functions are equal to id) except for the arcs $(\ell-1,0)$ and $(n-1,0)$ (resp. except for the local transition functions $f_0^L = neg$ and $f_0^R = neg$).

$x_0(t)\dots x_0(t+p-1)$ contains neither the sub-sequence $0u0$ nor the sub-sequence $1u1u'1$ $(u,u' \in \{0,1\}^{d-1})$.

Let $\Delta = gcd(\ell, r)$. As in the previous section, w can be written as an interlock of $\Delta_p = gcd(d,p) = gcd(\Delta,p)$ words $w^{(j)}$ of size p/Δ_p that do not contain the sub-sequences 00 and 111. As one may show by induction, the number of such words is counted by the Perrin sequence [4], sequence `A1608` of the OEIS [3]:

$$\begin{cases} P(0) = 3, \\ P(1) = 0, \\ P(2) = 2, \\ P(n) = P(n-2) + P(n-3) = \alpha^n + \beta^n + \overline{\beta}^n, \end{cases}$$

where α, $\beta = \frac{1}{2} \cdot (-\alpha + i \cdot \sqrt{\frac{3}{\alpha} - 1})$ and $\overline{\beta}$ are the three roots of $x^3 - x - 1 = 0$, and α, the only real root of this equation, is called the *plastic number* [5].

Using similar arguments to those used in the previous section, we derive Proposition 6 below. This proposition explains why, in Table 3, all cells of a same diagonal (*i.e.,* when $N = \ell + r$ is kept constant) that have the same colour also contain the same number: the number of attractors depends only on $N = \ell + r$ and on $\Delta = gcd(\ell, r)$ and not on ℓ nor r. Equations in Proposition 6 exploit, in particular, the fact that if p divides ℓ or r then $C_{p,\Delta_p} = 0$ (because then $P(\frac{p}{\Delta_p}) = P(1) = 0$).

Proposition 6 *Let $N = \ell + r$ and let $p \in \mathbb{N}$ be a possible attractor period of $D_{\ell,r}$ (p divides N but divides neither ℓ nor r). Let also $\Delta = gcd(\ell, r)$ and $\Delta_p = gcd(\Delta, p)$.*

Then, the number of configurations of period p of the doubly negative DBAC $D_{\ell,r}$ *depends only on p and Δ_p. It is given by:*

$$C_p(\ell,r) = C_{p,\Delta_p} = P(\frac{p}{\Delta_p})^{\Delta_p}.$$

The number of p-attractors and the total number of attractors of a doubly negative DBAC $D_{\ell,r}$ *are respectively given by:*

$$\mathtt{A}^{=}_p(\ell,r) = \mathtt{A}^{=}_{p,\Delta_p} = \frac{1}{p} \cdot \sum_{q|p} \mu(\frac{p}{q}) \cdot P(\frac{q}{\Delta_q})^{\Delta_q},$$

$$\mathtt{T}^{=}(\ell,r) = \mathtt{T}^{=}_{N,\Delta} = \frac{1}{N} \cdot \sum_{p|N} \psi(\frac{N}{p}) \cdot P(\frac{p}{\Delta_p})^{\Delta_p}.$$

The expression for $\mathtt{T}^{=}_{N,\Delta}$ in Proposition 6 above simplifies into the following if $K = \frac{N}{\Delta}$ is a prime:

$$\mathtt{T}^{=}_{N,\Delta} = \frac{1}{N} \cdot \sum_{q|\Delta,\ gcd(q,K)=1} \psi(q) \cdot P(K)^{\frac{\Delta}{q}}.$$

In particular, if $K = \frac{N}{\Delta} = 2$ or $K = \frac{N}{\Delta} = 3$, then because $P(2) = 2$ and $P(3) = 3$, the following holds:

$$\mathtt{T}^{=}(\tfrac{N}{2},\tfrac{N}{2}) = \mathtt{T}^{=}_{N,\frac{N}{2}} = \frac{1}{N} \cdot \sum_{q|\frac{N}{2},\ gcd(q,2)=1} \psi(q) \cdot 2^{\frac{N}{2\cdot q}},$$

$$\mathtt{T}^{=}(\tfrac{N}{3},\tfrac{2N}{3}) = \mathtt{T}^{=}{}_{N,\frac{N}{3}} = \frac{1}{N} \cdot \sum_{q|\frac{N}{3},\ gcd(q,3)=1} \psi(q) \cdot 3^{\frac{N}{3\cdot q}}.$$

From the computer simulations we performed (see Tables 2 and 3), we observe the following:

1. Given ℓ, $\mathtt{T}^{=}(\ell,r)$ is maximal when $r = \ell$.
2. Given an integer N which is not a multiple of 3, $\mathtt{T}^{=}_{N,\Delta}$ is maximal when Δ is maximal (in particular, if N is even without being a multiple of 3, then $\mathtt{T}^{=}_{N,\Delta} \leq \mathtt{T}^{=}_{N,\frac{N}{2}}$).
3. Given an integer N which is a multiple of 3, $\mathtt{T}^{=}_{N,\Delta}$ is maximal when $\Delta = \frac{N}{3}$.

We leave the proofs of these three points as an open problem.

NEGATIVE

ℓ \ r (NEGATIVE)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1															
2	1	1														
3	1	1	2													
4	1	2	1	2												
5	2	1	2	2	4											
6	1	1	3	3	2	6										
7	2	2	3	2	4	3	10									
8	2	3	2	8	3	4	6	16								
9	3	2	2	3	5	9	7	7	30							
10	2	4	3	4	17	7	7	10	11	52						
11	4	3	5	6	7	7	11	11	16	19	94					
12	3	4	9	2	7	42	11	33	17	23	28	172				
13	5	6	7	7	11	11	16	19	24	28	39	46	316			
14	6	7	7	10	11	17	105	23	28	38	46	60	75	586		
15	7	7	10	11	4	17	24	28	44	125	60	66	97			
16	7	10	11	33	19	23	28	278	46	60	75	88				
17	11	11	16	19	24	28	39	46	60	75	97					
18	11	17	17	23	28	6	46	60	729	96						
19	16	19	24	28	39	46	60	75	97							
20	19	23	28	32	125	60	75	88								
21	24	28	44	46	60	66	10									
22	28	38	46	60	75	96										
23	39	46	60	75	97											
24	46	60	66	88												
25	60	75	97													
26	75	96														
27	97															

$\ell + r$ prime
$\ell + r$ even
$\ell + r$ odd

$\ell = r$

$\ell = 2 \times r$

Table 2: Total number of attractors of a negative-negative DBAC $D_{\ell,r}$ (obtained by computer simulations).

NEGATIVE

ℓ \ r (NEGATIVE)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	$\mathtt{T}_\ell^-$
1	1																1
2	1	1															1
3	1	1	2														2
4	1	2	1	2													2
5	2	1	2	2	4												4
6	1	1	3	3	2	6											6
7	2	2	3	2	4	3	10										10
8	2	3	2	8	3	4	6	16									16
9	3	2	2	3	5	9	7	7	30								30
10	2	4	3	4	17	7	7	10	11	52							52
11	4	3	5	6	7	7	11	11	16	19	94						94
12	3	4	9	2	7	42	11	33	17	23	28	172					172
13	5	6	7	7	11	11	16	19	24	28	39	46	316				316
14	6	7	7	10	11	17	105	23	28	38	46	60	75	586			586
15	7	7	10	11	4	17	24	28	44	125	60	66	97				1096
16	7	10	11	33	19	23	28	278	46	60	75	88					2048
17	11	11	16	19	24	28	39	46	60	75	97						3856
18	11	17	17	23	28	6	46	60	729	96							7286
19	16	19	24	28	39	46	60	75	97								13798
20	19	23	28	32	125	60	75	88									26216
21	24	28	44	46	60	66	10										49940
22	28	38	46	60	75	96											95326
23	39	46	60	75	97												182362
24	46	60	66	88													349536
25	60	75	97														671092
26	75	96															1290556
27	97																2485534

$gcd(\ell, r) =$ 1, 2, 3, 4, 5, 6, 7, 8, 9

$\ell = r$

$\ell = 2 \times r$

$\ell = 4 \times r$

Table 3: Total number of attractors of a negative-negative DBAC $D_{\ell,r}$ (obtained by computer simulations). Each colour corresponds to a value of $gcd(\ell, r)$. The last column gives the total number $\mathtt{T}_\ell^-$ of attractors of an isolated negative circuit.

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