Robin inequality for 7−free integers

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Abstract

Recall that an integer is $t-$free iff it is not divisible by p^t for some prime p. We give a method to check Robin inequality $\sigma(n) < e^\gamma n \log\log n$, for $t-$free integers n and apply it for $t = 6, 7$. We introduce Ψ_t, a generalization of Dedekind Ψ function defined for any integer $t \geq 2$ by

$$\Psi_t(n) := n \prod_{p|n} (1 + 1/p + \cdots + 1/p^{t-1}).$$

If n is $t-$free then the sum of divisor function $\sigma(n)$ is $\leq \Psi_t(n)$. We characterize the champions for $x \mapsto \Psi_t(x)/x$, as primorial numbers. Define the ratio $R_t(n) := \frac{\Psi_t(n)}{n \log\log n}$. We prove that, for all t, there exists an integer $n_1(t)$, such that we have $R_t(N_n) < e^\gamma$ for $n \geq n_1$, where $N_n = \prod_{k=1}^n p_k$. Further, by combinatorial arguments, this can be extended to $R_t(N) \leq e^\gamma$ for all $N \geq N_n$, such that $n \geq n_1(t)$. This yields Robin inequality for $t = 6$, 7. For t varying slowly with N, we also derive $R_t(N) < e^\gamma$.

Keywords: Dedekind Ψ function, Robin inequality, Riemann Hypothesis, Primorial numbers

I. Introduction

The Riemann Hypothesis (RH), which describes the non trivial zeroes of Riemann ζ function has been deemed the Holy Grail of Mathematics by several authors [1], [7]. There exist many equivalent formulations in the literature [5]. The one of concern here is that of Robin [12], which is given in terms of $\sigma(n)$ the sum of divisor function

$$\sigma(n) < e^\gamma n \log\log n,$$

for $n \geq 5041$. Recall that an integer is $t-$free iff it is not divisible by p^t for some prime p. The above inequality was checked for many infinite families of integers in [3], for instance $5-$free integers. In the present work we introduce a method to check the inequality for $t-$free integers for larger values of t and apply it to $t = 6, 7$. The idea of our method is to introduce the generalized Dedekind Ψ function defined for any integer $t \geq 2$ by

$$\Psi_t(n) := n \prod_{p|n} (1 + 1/p + \cdots + 1/p^{t-1}).$$

If $t = 2$ this is just the classical Dedekind function which occurs in the theory of modular forms [4], in physics [10], and in analytic number theory [9]. By construction, if n is $t-$free then the sum of divisors function $\sigma(n)$ is $\leq \Psi_t(n)$. To see this note that the multiplicative function σ satisfies for any integer a in the range $t > a \geq 2$

$$\sigma(p^a) = 1 + p + \cdots + p^a,$$

when the multiplicative function Ψ_t satisfies

$$\Psi_t(p^a) = p^a + \cdots + 1 + \cdots + 1/p^{t-1-a}.$$

It turns out that the structure of champion numbers for the arithmetic function $x \mapsto \Psi_t(x)/x$ is much easier to understand than that of $x \mapsto \sigma(x)/x$, the super abundant numbers. They are exactly the so-called primorial numbers (product of first consecutive primes). We prove that, in order to maximize the ratio R_t it is enough to consider its value at primorial integers. Once this reduction is made, bounding above unconditionally R_t is easy by using classical lemmas on partial eulerian products. We conclude the article by some results on $t-$free integers $N \geq N_n$, valid for t varying slowly with N.

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II. Reduction to primorial numbers

Define the primorial number N_n of index n as the product of the first n primes

$$N_n = \prod_{k=1}^{n} p_k,$$

so that $N_0 = 1, N_1 = 2, N_2 = 6, \cdots$ and so on. The primorial numbers (OEIS sequence $A002110$ [11]) play the role here of superabundant numbers in [12] or primorials in [8]. They are champion numbers (ie left to right maxima) of the function $x \mapsto \Psi_t(x)/x$:

$$\frac{\Psi_t(m)}{m} < \frac{\Psi_t(n)}{n} \text{ for any } m < n. \tag{1}$$

We give a rigorous proof of this fact.

Proposition 1: The primorial numbers and their multiples are exactly the champion numbers of the function $x \mapsto \Psi_t(x)/x$.

Proof: The proof is by induction on n. The induction hypothesis H_n is that the statement is true up to N_n. Sloane sequence $A002110$ begins $1, 2, 4, 6 \ldots$ so that H_2 is true. Assume H_n true. Let $N_n \leq m < N_{n+1}$ denote a generic integer. The prime divisors of m are $\leq p_n$. Therefore $\Psi_t(m)/m \leq \Psi_t(N_n)/N_n$ with equality iff m is a multiple of N_n. Further $\Psi_t(N_n)/N_n < \Psi_t(N_{n+1})/N_{n+1}$. The proof of H_{n+1} follows. ■

In this section we reduce the maximization of $R_t(n)$ over all integers n to the maximization over primorials.

Proposition 2: Let n be an integer ≥ 2. For any m in the range $N_n \leq m < N_{n+1}$ one has $R_t(m) < R_t(N_n)$.

Proof: Like in the preceding proof we have

$$\Psi_t(m)/m \leq \Psi_t(N_n)/N_n.$$

Since $0 < \log\log N_n \leq \log\log m$, the result follows. ■

III. Ψ_t at primorial numbers

We begin by an easy application of Mertens formula.

Proposition 3: For n going to ∞ we have

$$\lim R_t(N_n) = \frac{e^\gamma}{\zeta(t)}.$$

Proof: Writing $1+1/p = (1-1/p^2)/(1-1/p)$ in the definition of $\Psi(n)$ we can combine the Eulerian product for $\zeta(t)$ with Mertens formula

$$\prod_{p \leq x} (1 - 1/p)^{-1} \sim e^\gamma \log(x)$$

to obtain

$$\Psi(N_n) \sim \frac{e^\gamma}{\zeta(t)} \log(p_n).$$

Now the Prime Number Theorem [6, Th. 6, Th. 420] shows that $x \sim \theta(x)$ for x large, where $\theta(x)$ stands for Chebyshev's first summatory function:

$$\theta(x) = \sum_{p \leq x} \log p.$$

This shows that, taking $x = p_n$ we have

$$p_n \sim \theta(p_n) = \log(N_n).$$

The result follows. ∎

This motivates the search for explicit upper bounds on $R_t(N_n)$ of the form $\frac{e^\gamma}{\zeta(t)}(1 + o(1))$. In that direction we have the following bound.

Proposition 4: For n large enough to have $p_n \geq 20000$, we have

$$\frac{\Psi_t(N_n)}{N_n} \leq \frac{\exp(\gamma + 2/p_n)}{\zeta(t)}(\log\log N_n + \frac{1.1253}{\log p_n}).$$

We prepare for the proof of the preceding Proposition by some Lemmas. First an upper bound on a partial Eulerian product from [13, (3.30) p.70].

Lemma 1: For $x \geq 2$, we have

$$\prod_{p \leq x}(1 - 1/p)^{-1} \leq e^\gamma(\log x + \frac{1}{\log x}).$$

Next an upper bound on the tail of the Eulerian product for $\zeta(t)$.

Lemma 2: For $n \geq 2$ we have

$$\prod_{p > p_n}(1 - 1/p^t)^{-1} \leq \exp(2/p_n).$$

Proof: Use Lemma 6.4 in [3] with $x = p_n$. Bound $\frac{t}{t-1}x^{1-t}$ above by $2/x$. ∎

Lemma 3: For $n \geq 2263$, we have

$$\log p_n < \log\log N_n + \frac{0.1253}{\log p_n}.$$

Proof: If $n \geq 2263$, then $p_n \geq 20000$. By [13], we know then that

$$\log N_n > p_n(1 - \frac{1}{8p_n}).$$

On taking log's we obtain

$$\log\log N_n > \log p_n - \frac{0.1253}{p_n},$$

upon using

$$\log(1 - \frac{x}{8}) > -0.1253x$$

for x small enough. In particular $x < 1/20000$ is enough. ∎

We are now ready for the proof of Proposition 4.

Proof:

Write

$$\frac{\Psi_t(N_n)}{N_n} = \prod_{k=1}^{n}\frac{1 - 1/{p_k}^t}{1 - 1/p_k} = \frac{\prod_{p > p_n}(1 - 1/p^t)^{-1}}{\zeta(t)}\prod_{p \leq p_n}(1 - 1/p)^{-1}$$

and use both Lemmas to derive

$$\frac{\Psi_t(N_n)}{N_n} \leq \frac{\exp(\gamma + 2/p_n)}{\zeta(t)}(\log p_n + \frac{1}{\log p_n}).$$

Now we get rid of the first $\log$ in the RHS by Lemma 3.

The result follows.

■

So, armed with this powerful tool, we derive the following significant Corollaries.
For convenience let

$$f(n) = (1 + \frac{1.1253}{\log p_n \log\log N_n}).$$

Corollary 1: Let $n_0 = 2263$. Let $n_1(t)$ denote the least $n \geq n_0$ such that $e^{2/p_n} f(n) < \zeta(t)$. For $n \geq n_1(t)$ we have $R_t(N_n) < e^\gamma$.

Proof:

Let $n \geq n_0$. We need to check that

$$\exp(2/p_n)(1 + \frac{1.1253}{\log p_n \log\log N_n}) \leq \zeta(t).$$

which, for fixed t holds for n large enough. Indeed $\zeta(t) > 1$ and the LHS goes monotonically to 1^+ for n large.

■

We give a numerical illustration of Corollary 1 in Table 1.

t	$n_1(t)$	$N_{n_1(t)}$
3	10	6.5×10^9
4	24	2.4×10^{34}
5	79	4.1×10^{163}
6	509	5.8×10^{1551}
7	10 596	2.5×10^{48337}

TABLE I: The numbers in Corollary 1.

We can extend this Corollary to all integers $\geq n_0$ by using the reduction of preceding section.

Corollary 2: For all $N \geq N_n$ such that $n \geq n_1(t)$ we have $R_t(N) < e^\gamma$.

Proof: Combine Corollary 1 with Proposition 2. ■

We are now in a position to derive the main result of this note.

Theorem 1: If N is a $7-$free integer, then $\sigma(N) < Ne^\gamma \log\log N$.

Proof: If N is $\geq N_n$ with $n \geq n_1(7)$, then the above upper bound holds for $\Psi_7(N)$ by Corollary 2, hence for $\sigma(N)$ by the remark in the Introduction. If not, we invoke the results of [2], who checked Robin inequality for $5040 < N \leq 10^{10^{10}}$, and observe that all $7-$free integers are > 5040. ■

IV. Varying t

We begin with an easy Lemma.

Lemma 4: Let t be a real variable. For t large, we have $\zeta(t) = 1 + \frac{1}{2^t} + o(\frac{1}{2^t})$.

Proof: By definition, for $t > 1$ we may write

$$\zeta(t) = \sum_{n=1}^{\infty} \frac{1}{n^t}$$

so that

$$\zeta(t) \geq 1 + \frac{1}{2^t}.$$

In the other direction, we write

$$\zeta(t) = 1 + \frac{1}{2^t} + \frac{1}{3^t} + \sum_{n=4}^{\infty} \frac{1}{n^t},$$

and compare the remainder of the series expansion of the ζ function with an integral:

$$\sum_{n=4}^{\infty}\frac{1}{n^t}<\int_3^{\infty}\frac{du}{u^t}=\frac{3}{(t-1)3^t}=O(\frac{1}{3^t}).$$

The result follows. ∎

We can derive a result when t grows slowly with n.

Theorem 2: Let S_n be a sequence of integers such that $S_n \geq N_n$ for n large, and such that S_n is $t-$free with $t=o(\log\log n)$. For n large enough, Robin inequality holds for S_n.

Proof: For Corollary 2 to hold we need

$$e^{2/p_n}f(n)<\zeta(t)$$

to hold, or , taking logs, the exact bound

$$2/p_n+\log f(n)<\log\zeta(t),$$

or up to $o(1)$ terms

$$2/p_n+\frac{1.1253}{\log p_n\log\log N_n}\leq\log\zeta(t).$$

In the LHS, the dominant term is of order $1/(\log p_n)^2$, since, like in the proof of Proposition 3, we may write $p_n\sim\log N_n$. Now $p_n\sim n\log n$ by [6, Th. 8], entailing $\log p_n\sim\log n$ and $(\log p_n)^2\sim(\log n)^2$. In the RHS, with the hypothesis made on t we have, by Lemma 4, the estimate $\log\zeta(t)\sim\frac{1}{2^t}$. The result follows after comparing logarithms of both sides. ∎

V. Conclusion

In this article we have proposed a technique to check Robin inequality for $t-$free integers for some values of t. The main idea has been to investigate the complex structure of the divisor function σ though the sequence of Dedekind psi functions ψ_t. The latter are simpler for the following reasons

- $\Psi_t(n)$ solely depends on the prime divisors of n and not on their multiplicity
- the champions of Ψ_t are the primorials instead of the colossally abundant numbers
- Ψ_t is easier to bound for n large because of connections with Eulerian products

Further, $\sigma(n)\leq\Psi_t(n)$ for $t-$free integers n. We checked Robin inequality for $t-$free integers for $t=6,7$ and $t=o(\log\log n)$. It is an interesting and difficult open problem to apply Theorem 2 to superabundant numbers or colossally abundant numbers for instance. We do not believe it is possible. New ideas are required to prove Robin inequality in full generality.

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