Triangulations of nearly convex polygons *†

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Abstract.— *Counting Euclidean triangulations with vertices in a finite set $\mathcal{C}$ of the convex hull* Conv$(\mathcal{C})$ *of $\mathcal{C}$ is difficult in general, both algorithmically and theoretically. The aim of this paper is to describe nearly convex polygons, a class of configurations for which this problem can be solved to some extent. Loosely speaking, a nearly convex polygon is an infinitesimal perturbation of a weakly convex polygon (a convex polygon with edges subdivided by additional points). Our main result shows that the triangulation polynomial, enumerating all triangulations of a nearly convex polygon, is defined in a straightforward way in terms of polynomials associated to the "perturbed" edges.*

1 Introduction

Given a finite subset $\mathcal{C}$ of the Euclidean plane $\mathbf{R}^2$, calculating the number of triangulations of the convex hull Conv$(\mathcal{C})$ using only Euclidean triangles with vertices in $\mathcal{C}$ seems to be difficult and has attracted some interest, both from an algorithmic and a theoretical point of view, see for instance [1], [2], [3], [4], [5], [7], [9], [10], [11].

An important and well understood special case is given by the n vertices of a strictly convex polygon. The associated number of triangulations is the Catalan number C_{n-2}.

In a first part of the paper, we consider convex polygons having perhaps collinear vertices, called weakly convex polygons. We are not aware of the existence of formulae giving the number of triangulations for such polygons. Thinking of weakly convex polygons as strictly convex polygons with edges subdivided by additional vertices, we call *edges* the edges of the underlying strictly convex polygon. Edges are thus maximal straight segments contained in the boundary of such polygons.

Denoting by $\mathcal{C}$ the set of all vertices, we define the *weight* of an edge E as the number of connected components of $E \setminus (E \cap \mathcal{C})$. Weights of successive edges form a finite sequence a_1, a_2... a_l of total sum $n = \#\mathcal{C}$.

We show (Theorem 3.2) that there exists a sequence of polynomials $p_m(t)$, $m \geq 1$, called *maximal edge polynomials*, such that the number of maximal triangulations (*i.e.* involving all vertices of $\mathcal{C}$) equals

$$\tau_{\max}(\mathcal{C}) = \sum_{k\geq 2} b_k C_{k-2}, \tag{1}$$

where the coefficients b_k are defined by $\prod_{i=1}^{l} p_{a_i}(t) = \sum_k b_k t^k$.

We deduce that the triangulation polynomial of a configuration (which takes into account non-maximal triangulations) verifies formally the same formula as the previous

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one, replacing maximal edge polynomials by *complete edge polynomials*. This has the perhaps surprising consequence that enumerative properties of triangulations do not depend on the particular cyclic order of the edges.

In a second part, we define *nearly convex polygons* as small perturbations of weakly convex ones. Our main result, Theorem 4.5, establishes the existence of near-edge polynomials such that the previous formulae continue to hold. Factorization of near-edges, a useful arithmetical property, allows classification of small near-edges.

Near-edge polynomials are difficult to compute in general except in a few special cases. We plan to describe algorithms in a future paper dealing with computation aspects. Some details are given in section 5.

This article is organized as follows. We introduce first definitions and recall the strictly convex case in section 2, prove formulae for weakly convex polygons in section 3, expose the nearly convex setting and prove our main result in section 4. Finally, section 5 contains a few remarks and open problems.

2 Triangulations of planar configurations

A *(planar) configuration of points* is a finite subset $\mathcal{C} = \{P_1,\ldots,P_n\}$ of the oriented plane $\mathbf{R}^2$. We denote by $\mathrm{Conv}(\mathcal{C})$ the convex hull of $\mathcal{C}$ and by $\mathrm{Extr}(\mathcal{C})$ the set of all extremal elements in $\mathcal{C}$ (a point $P \in \mathcal{C}$ is *extremal* if $\mathrm{Conv}(\mathcal{C}\setminus\{P\}) \neq \mathrm{Conv}(\mathcal{C})$). The configuration $\mathcal{C}$ is said to be *strictly convex* if $\mathrm{Extr}(\mathcal{C}) = \mathcal{C}$. More generally, $\mathrm{Extr}(\mathcal{C})$ is the set of vertices of the strictly convex polygon formed by the convex hull of $\mathcal{C}$.

A *triangulation* of a configuration $\mathcal{C}$ is a triangulation of its convex hull with vertices in $\mathcal{C}$, *i.e.* a finite set $\mathcal{T} = \{\Delta_1,\ldots,\Delta_q\}$ of Euclidean triangles with vertices in $\mathcal{C}$ such that $\mathrm{Conv}(\mathcal{C}) = \cup_{i=1}^q \Delta_i$ and non-trivial intersections $\Delta_i \cap \Delta_j$ consist of a common vertex or a common edge. A triangulation of $\mathcal{C}$ is *maximal* if it involves all vertices of $\mathcal{C}$ (*i.e.* each point of $\mathcal{C}$ is a vertex of at least one triangle). The number of maximal triangulations of $\mathcal{C}$ is denoted by $\tau_{\max}(\mathcal{C})$. If $\mathcal{C}$ is strictly convex, it is well known that $\tau_{\max}(\mathcal{C})$ is a *Catalan number*:

Theorem 2.1 *All triangulations of a strictly convex $n-$gon are maximal and their number is $C_{n-2} = \binom{2(n-2)}{n-2}/(n-1)$.*

Sketch of proof: A deformation argument shows that combinatorial properties of triangulations of a strictly convex $n-$gon depend only on n. We denote by τ_n the number of triangulations of such a convex $n-$gon P. The choice of a marked edge E in P selects in every triangulation a unique triangle Δ containing E. The two remaining edges of Δ determine two triangulated convex polygons having respectively k and $n+1-k$ edges for some integer k such that $2 \leq k \leq n-1$. This decomposition amounts to the recurrence relation

$$\tau_n = \sum_{k=2}^{n-1} \tau_k \tau_{n+1-k}$$

holding for $n \geq 3$, using the convention $\tau_2 = 1$. Therefore, the generating function $\sum_{n=2} \tau_n x^n$ satisfies a quadratic equation. The classical resolution gives a formula, whose development in power series yields the result. □

Triangulations of a general configuration $\mathcal{C}$ are not necessarily maximal and enumerative properties are encoded by the triangulation polynomial $p_\tau(\mathcal{C}) = \sum \tau_k(\mathcal{C}) s^k$, where $\tau_k(\mathcal{C})$ counts the number of triangulations using exactly k points. The polynomial $p_\tau(\mathcal{C})$ has degree $n = \#\mathcal{C}$, with leading coefficient $\tau_n(\mathcal{C}) = \tau_{\max}(\mathcal{C})$ counting the number of maximal triangulations. Its monomial of lowest degree $m = \#\mathrm{Extr}(\mathcal{C})$ corresponds to the C_{m-2} triangulations of the convex hull $\mathrm{Conv}(\mathcal{C})$ involving only extremal vertices. Remark also that the average number of points of a triangulation is given by the logarithmic derivative $f'(1)/f(1)$ of the triangulation polynomial $f(s) = p_\tau(\mathcal{C})$.

Two configurations are *isotopic* if they are related by a continuous deformation which preserves collinearity and non-collinearity of triplets. Isotopic configurations have the same triangulation polynomial.

3 Weakly convex polygons

3.1 Definition and notations

A configuration $\mathcal{C}$ is *weakly convex* if it is contained in the boundary $\partial\mathrm{Conv}(\mathcal{C})$ of its convex hull. We also call $\mathcal{C}$ a *weakly convex polygon*. We are not aware of a published formula giving the number of triangulations of such polygons.

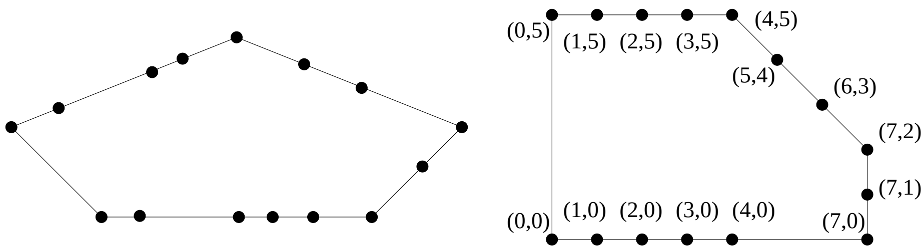

Figure 1: Two weakly convex polygons with edge-weights $1, 5, 2, 3, 4$.

Weakly convex polygons can be seen as strictly convex polygons with additional vertices subdividing their edges. We thus call *edges* the segments joining two consecutive extremal vertices of the underlying strictly convex polygon. An edge has *weight* a if it involves $a+1$ points of $\mathcal{C}$. The weights of consecutive edges, in counterclockwise order, define, up to cyclic permutations, a finite sequence a_1, a_2... a_l of total sum $n = \#\mathcal{C}$ (see Figure 1). This sequence characterizes $\mathcal{C}$ up to isotopy. Thus all combinatorial properties of triangulations depend only on the sequence a_1, a_2... a_l (up to cyclic permutations). We denote by $\tau_k(a_1, a_2, \ldots, a_l)$ the number of corresponding triangulations using exactly k points of $\mathcal{C}$. This number is non-zero only for $l \le k \le n$. We denote the triangulation polynomial of $\mathcal{C}$ by $p_\tau(a_1, a_2, \ldots, a_l)$.

The following notations will be useful. The number $\tau_n(a_1, a_2, \ldots, a_l)$ of maximal triangulations is also denoted by $\tau_{\max}(a_1, a_2, \ldots, a_l)$. We denote by $P(a_1, a_2, \ldots, a_l)$ an arbitrary weakly convex polygon with l edges of successive weights a_1, a_2... a_l. Moreover, we use an exponential notation for indicating several consecutive edges of weight 1: we denote for instance by $P(1^3, 5, 2) = P(1, 1, 1, 5, 2)$ a decagon with 5 edges: three consecutive edges of weight 1, followed by an edge of weight 5 and a final edge of weight 2. We use the same notation for the number of triangulations: $\tau_6(1^3, 5, 2)$ is the number of triangulations of $P(1^3, 5, 2)$ involving 6 vertices.

3.2 Inclusion-exclusion principle

Our first aim is the determination of the number of maximal triangulations for weakly convex polygons. This can be achieved by reducing the problem to the case of strictly convex polygons where formulae are known. Replacing the first edge of weight a_1 by edges of weight 1 leads to the following proposition.

Proposition 3.1 *Given an integer $l \geq 3$ and l strictly positive integers a_1, a_2... a_l, we have*

$$\tau_{\max}(a_1, a_2, \ldots, a_l) = \sum_{k=0}^{\lfloor a_1/2 \rfloor} (-1)^k \binom{a_1 - k}{k} \tau_{\max}(1^{a_1-k}, a_2, \ldots, a_l).$$

Proof. We consider the set $\mathcal{T}$ of maximal triangulations of $P(1^{a_1}, a_2, \ldots, a_3)$. A triangle of a triangulation is *exterior* if it involves two (necessarily adjacent) edges among the a_1 first edges (of weight 1) of $P(1^{a_1}, a_2, \ldots, a_l)$. There is an obvious one-to-one correspondence between the set $\mathcal{R}$ of maximal triangulations of $P(1^{a_1}, a_2, \ldots, a_l)$ without exterior triangles and the set of maximal triangulations of $P(a_1, a_2, \ldots, a_l)$, by continuously straightening the set formed by the first a_1 $1-$edges (see Figure 2). It is thus sufficient to enumerate $\mathcal{R}$. It is easier to enumerate the complementary set $\mathcal{T} \setminus \mathcal{R}$.

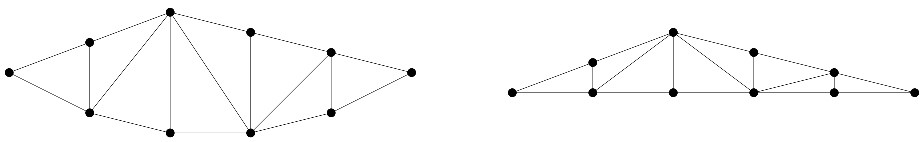

Figure 2: Straightening a maximal triangulation without exterior triangles.

Denoting by $\mathcal{E}$ the set of all $a_1 - 1$ possible exterior triangles and, for $\Delta \in \mathcal{E}$, $\mathcal{T}_\Delta$ the set of triangulations containing Δ, the set $\mathcal{T} \setminus \mathcal{R}$ is the union of the sets $\mathcal{T}_\Delta$. We enumerate this set by the inclusion-exclusion principle:

$$\#(\mathcal{T} \setminus \mathcal{R}) = \sum_{k=1}^{\lfloor a_1/2 \rfloor} (-1)^{k-1} \sum_{\{\Delta_1, \Delta_2, \ldots, \Delta_k\}} \#(\mathcal{T}_{\Delta_1} \cap \mathcal{T}_{\Delta_2} \cap \cdots \cap \mathcal{T}_{\Delta_k}).$$

The upper bound $\lfloor a_1/2 \rfloor$ in the summation is due to the fact that a triangulation contains at most $\lfloor a_1/2 \rfloor$ exterior triangles.

It remains to enumerate the intersections. Fix $k \leq a_1/2$. If some triangles among Δ_1, Δ_2... Δ_k have non-disjoint interiors, the intersection is empty. Otherwise, we associate to each element of $\mathcal{T}_{\Delta_1} \cap \mathcal{T}_{\Delta_2} \cap \cdots \cap \mathcal{T}_{\Delta_k}$ a maximal triangulation of the polygon $P(1^{a_1-k}, a_2, \ldots, a_l)$ by erasing triangles Δ_1, Δ_2... Δ_k. We also keep track of erased triangles by marking the remaining edge for each of these triangles (see Figure 3). This defines a map $\varphi_{\Delta_1, \Delta_2, \ldots, \Delta_k}$ from the set $\mathcal{T}_{\Delta_1} \cap \mathcal{T}_{\Delta_2} \cap \cdots \cap \mathcal{T}_{\Delta_k}$ to the set $\mathcal{M}_k$ of all triangulations of $P(1^{a_1-k}, a_2, \ldots, a_l)$ with k marked edges among the first $a_1 - k$ edges of weight 1. This map is obviously injective: we can reconstruct the initial triangulation by gluing triangles onto the marked edges. Moreover, the union of images of $\varphi_{\Delta_1, \Delta_2, \ldots, \Delta_k}$, for all "admissible" $k-$tuples of exterior triangles is clearly a disjoint union, by the same remark as for injectivity, and fills $\mathcal{M}_k$ for exactly the same reason: the reconstruction is unique and is always possible. Thus,

$$\sum_{\{\Delta_1, \Delta_2, \ldots, \Delta_k\}} \#(\mathcal{T}_{\Delta_1} \cap \mathcal{T}_{\Delta_2} \cap \cdots \cap \mathcal{T}_{\Delta_k}) = \#\mathcal{M}_k = \binom{a_1 - k}{k} \tau_{\max}(1^{a_1-k}, a_2, \ldots, a_l).$$

As $\#\mathcal{T} = \tau_{\max}(1^{a_1}, a_2, \ldots, a_l)$, the proof is achieved. □

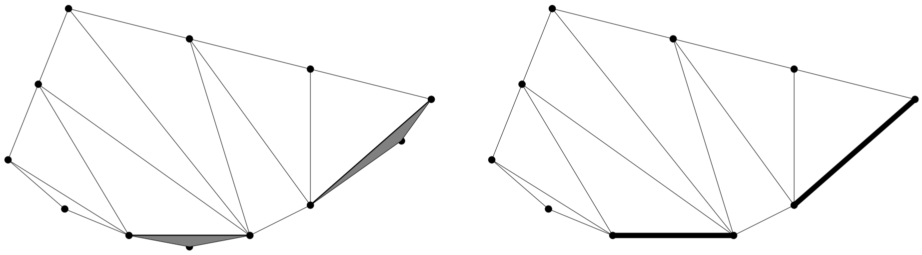

Figure 3: Erasing some exterior triangles and keeping marks.

3.3 Maximal triangulations

Since combinatorial properties of triangulations of weakly convex polygons are invariant under cyclic permutations of edges, we can "break up" all edges by iterating Proposition 3.1. Hence, $\tau_{\max}(a_1, a_2, \ldots, a_l)$ can be successively written as

$$\begin{aligned}
& \sum_{k_1=0}^{\lfloor a_1/2 \rfloor} (-1)^{k_1} \binom{a_1-k_1}{k_1} \tau_{\max}(1^{a_1-k_1}, a_2, \ldots, a_l) \\
= & \sum_{k_1=0}^{\lfloor a_1/2 \rfloor} \sum_{k_2=0}^{\lfloor a_2/2 \rfloor} (-1)^{k_1+k_2} \binom{a_1-k_1}{k_1} \binom{a_2-k_2}{k_2} \tau_{\max}(1^{(a_1-k_1)+(a_2-k_2)}, a_3, \ldots, a_l) \\
= & \ldots \\
= & \sum_{k_1,k_2,\ldots,k_l} \left((-1)^{\sum_{i=1}^l k_i} \prod_{i=1}^{l} \binom{a_i-k_i}{k_i} \right) \tau_{\max}(1^{\sum_{i=1}^l a_i-k_i}) \\
= & \sum_{k_1,k_2,\ldots,k_l} \left((-1)^{\sum_{i=1}^l k_i} \prod_{i=1}^{l} \binom{a_i-k_i}{k_i} \right) C_{(\sum_{i=1}^l a_i-k_i)-2}.
\end{aligned}$$

We thus obtain $\tau_{\max}(a_1, a_2, \ldots, a_l)$ as a linear combination $\sum_{j\geq 2} b_j C_{j-2}$ of Catalan numbers. The coefficients b_j are given by

$$\sum_{j\geq 2} b_j t^j = \sum_{k_1=0}^{\lfloor a_1/2 \rfloor} \cdots \sum_{k_l=0}^{\lfloor a_l/2 \rfloor} \left((-1)^{\sum_{i=1}^l k_i} \prod_{i=1}^{l} \binom{a_i-k_i}{k_i} \right) t^{\sum_{i=1}^l a_i-k_i} = \prod_{i=1}^{l} p_{a_i}$$

with

$$p_m = \sum_{k=0}^{\lfloor m/2 \rfloor} (-1)^k \binom{m-k}{k} t^{m-k} \tag{2}$$

defining the sequence $(p_m)m \geq 1$ of *maximal edge-polynomials*.

We consider the generating function

$$G_C(t) = \sum_{k\geq 0} C_k t^k = \sum_{k\geq 0} \binom{2k}{k} \frac{t^k}{k+1} \tag{3}$$

for the sequence of Catalan numbers (corresponding to the analytic expression $\frac{1-\sqrt{1-4t}}{2t}$). Given a polynomial $p(t) = \sum_{k\geq 2} \alpha_k t^k$, we define

$$\left\langle p(t), t^2 G_C(t) \right\rangle_t = \sum_{k\geq 2} \alpha_k C_{k-2}. \tag{4}$$

This "umbral" notation is suggested by the fact that polynomials and formal power series are mutually dual.

We have obtained the following result concerning the number of maximal triangulations:

Theorem 3.2 *Given natural numbers $l \geq 3$ and a_1, a_2... $a_l \geq 1$, we have*

$$\tau_{\max}(a_1, a_2, \ldots, a_l) = \Big\langle \prod_{i=1}^{l} p_{a_i}(t), t^2 G_C(t) \Big\rangle_t,$$

using the notations of formulae 2, 3 and 4 above.

3.4 Examples and remarks

The first few maximal edge-polynomials are

$$\begin{array}{ll} p_1 = t & p_5 = t^5 - 4t^4 + 3t^3 \\ p_2 = t^2 - t & p_6 = t^6 - 5t^5 + 6t^4 - t^3 \\ p_3 = t^3 - 2t^2 & p_7 = t^7 - 6t^6 + 10t^5 - 4t^3 \\ p_4 = t^4 - 3t^3 + t^2 & p_8 = t^8 - 7t^7 + 15t^6 - 10t^5 + t^4 \end{array}$$

Example. The two weakly convex polygons $P(1,5,2,3,4)$ of Figure 1 have

$$\begin{array}{rl} & \langle p_1\ p_5\ p_2\ p_3\ p_4, t^2 G_C(t)\rangle_t \\ = & \langle t(t^5 - 4t^4 + 3t^3)(t^2 - t)(t^3 - 2t^2)(t^4 - 3t^3 + t^2), t^2 G_C(t)\rangle_t \\ = & \langle t^{15} - 10t^{14} + 39t^{13} - 75t^{12} + 74t^{11} - 35t^{10} + 6t^9, t^2 G_C(t)\rangle_t \\ = & C_{13} - 10C_{12} + 39C_{11} - 75C_{10} + 74C_9 - 35C_8 + 6C_7 \\ = & 7429000 - 10 \cdot 208012 + 39 \cdot 58786 - 75 \cdot 16796 \\ & \quad +74 \cdot 4863 - 35 \cdot 1430 + 6 \cdot 429 \\ = & 8046 \end{array}$$

maximal triangulations.

Remark. We have $\tau_{max}(n, m, 1^2) = \binom{n+m}{n}$ for all $n, m \geq 1$.

Indeed, triangles in a maximal triangulation of $P(1,n,1,m)$ are linearly ordered and in one-to-one correspondence with the $n+m$ "segments" of the two opposite "long" edges. Gluing a triangle onto one of the two edges of length one, we have the formula $\tau_{\max}(n+1, m+1, 1) = \binom{n+m}{n}$.

Remark. The function $n \longmapsto f(k,n) = \tau_{\max}(1^{2+k}, n)$ is polynomial of degree k.

First, a classification of the triangulations of $P(1^{2+k}, n)$ according to the third vertex of the last triangle based on the edge of length n gives the formula

$$f(k,n) = \sum_{l=0}^{k} C_{k-l}\, f(l, n-1).$$

The result follows then from an induction on k. It is obvious that $f(0,n) = 1$ for all n. Suppose that $f(l,n)$ is polynomial in n of degree l for every $l \leq k$. Using $C_0 = 1$, the difference

$$f(k+1,n) - f(k+1,n-1) = \sum_{l=0}^{k} C_{k+1-l}\, f(l, n-1)$$

is then a polynomial of degree k and a sum over n implies the result.

Remark. The sequence of numbers of maximal triangulations of the weakly convex polygons $P(2,2,2)$, $P(2,2,2,2)$... starts as :

4, 30, 250, 2236, 20979, 203748, 2031054, 20662980, 213679114, 2239507936...
(see sequence A86452 of [12]).

Remark. Maximal edge polynomials can also be defined recursively by $p_0 = 1$, $p_1 = t$ and $p_m = t(p_{m-1} - p_{m-2})$ and are related to Fibonacci numbers (the closely related polynomials $\sum_k \binom{m-k}{k} x^k$ are also called "Fibonacci polynomials").

3.5 Non-maximal triangulations

An arbitrary (*i.e.* not necessarily maximal) triangulation of a weakly convex polygon $P(a_1,\ldots,a_l)$ is a maximal triangulation of a subset involving all extremal vertices of the weakly convex configuration $P(a_1,\ldots,a_l)$. It amounts thus to the choice, for every $1 \le i \le l$, of a number $1 \le b_i \le a_i$ and of $b_i - 1$ points among the $a_i - 1$ interior points of the $i-$th edge, followed by the choice of a triangulation of $P(b_1,\ldots,b_l)$.

The triangulation polynomial of $P(a_1,\ldots,a_l)$ is thus given by

$$\sum_{1\le b_i\le a_i} \left(\prod_{j=1}^{l} \binom{a_i-1}{b_i-1}\right) \langle \prod_{j=1}^{l} p_{b_j}(t), t^2 G_C(t)\rangle_t \, s^{\sum_{j=1}^{l} b_j}$$
$$= \langle \prod_{j=1}^{l} \sum_{b_j=1}^{a_j} \binom{a_j-1}{b_j-1} p_{b_j}(t) s^{b_j}, t^2 G_C(t)\rangle_t$$
$$= \langle \prod_{j=1}^{l} \overline{p}_{a_j}, t^2 G_C(t)\rangle_t,$$

where the *complete edge-polynomials* $\overline{p}_m \in \mathbf{Z}[s,t]$ are defined as

$$\overline{p}_m = \sum_{k=1}^{m} \binom{m-1}{k-1} p_k(t) s^k. \tag{5}$$

We have proved

Theorem 3.3 *Given natural numbers $l \ge 3$ and a_1, a_2... $a_l \ge 1$, the triangulation polynomial of $P(a_1,a_2,\ldots,a_l)$ is*

$$p_\tau(a_1,a_2,\ldots,a_l) = \sum_k \tau_k(a_1,a_2,\ldots,a_l) s^k = \langle \prod_{i=1}^{l} \overline{p}_{a_i}(t), t^2 G_C(t)\rangle_t,$$

using the notations of formulae 3, 4 and 5 above.

An immediate consequence is the following slightly surprising fact:

Corollary 3.4 *Enumerative properties of triangulations for weakly convex polygons do not depend on the particular cyclic order of edge weights.*

The first few complete edge-polynomials are

$$\begin{aligned}
\overline{p}_1 &= p_1\, s = t\, s\,,\\
\overline{p}_2 &= p_2\, s^2 + p_1\, s = (t^2 - t)\, s^2 + t\, s\,,\\
\overline{p}_3 &= p_3\, s^3 + 2p_2\, s^2 + p_1\, s = (t^3 - 2t^2)\, s^3 + 2(t^2 - t)\, s^2 + t\, s\,,\\
\overline{p}_4 &= p_4\, s^4 + 3p_3\, s^3 + 3p_2\, s^2 + p_1\, s\\
&= (t^4 - 3t^3 + t^2)\, s^4 + 3(t^3 - 2t^2)\, s^3 + 3(t^2 - t)\, s^2 + t\, s\,.
\end{aligned}$$

Example. The triangulation polynomial $p_\tau(1,5,2,3,4)$ of the two weakly convex polygons of Figure 1 equals

$$\begin{aligned}&\langle \overline{p}_1\,\overline{p}_5\,\overline{p}_2\,\overline{p}_3\,\overline{p}_4, t^2 G_C(t)\rangle_t\\ = \; & 8046\,s^{15}+37250\,s^{14}+77467\,s^{13}+95364\,s^{12}+77048\,s^{11}\\ & \quad +42776\,s^{10}+16584\,s^{9}+4460\,s^{8}+805\,s^{7}+90\,s^{6}+5\,s^{5}\ .\end{aligned}$$

4 Nearly convex polygons

Nearly convex polygons are small perturbations of weakly convex polygons and form the correct framework for generalizing Theorem 3.2 and Theorem 3.3 We give first a definition of near-edges, which are small deformations of edges, and introduce nearly convex polygons and, using the formalism of "roofs", the associated near-edge polynomials. Then we state and prove the main theorem. A last subsection describes factorization properties of near-edges and gives a classification of small near-edges.

4.1 Near-edges

In order to describe deformations of an edge E in a weakly convex polygon $P = P(n,\dots)$, we choose coordinates such that P is contained in the upper half-plane $y \geq 0$ and E is a subset of the boundary $y = 0$. This leads to the following definition.

A *near-edge* of *weight* n (or an $n-$*near-edge*) is a sequence E of $n+1$ points

$$(P_0, P_1, \dots, P_n) = \left(\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}, \dots, \begin{pmatrix} x_n \\ y_n \end{pmatrix} \right) \in \left(\mathbf{R}^2\right)^{n+1} \tag{6}$$

such that $x_0 < x_1 < \cdots < x_n$ and $y_0 = y_n = 0$. We consider the complete order $P_0 < P_1 < P_2 < \cdots < P_n$ on E and call P_0, respectively P_n, the *initial*, respectively *final*, vertex of the near-edge E. We denote a near-edge E either by the sequence $E = (P_0, \dots, P_n)$ of its $n+1$ points or by the real matrix

$$E = \begin{pmatrix} x_0 & x_1 & \dots & x_{n-1} & x_n \\ 0 & y_1 & \dots & y_{n-1} & 0 \end{pmatrix}$$

of size $2 \times (n+1)$.

A continuous deformation of near-edges, which preserves collinearity and non-collinearity of all triplets of points, is called an *isotopy*. Two near-edges joined by an isotopy are *isotopic*.

Given a near-edge E with points $P_i \in \mathbf{R}^2$ as above, we denote by E^ε the near-edge with points

$$P_k(\varepsilon) = \begin{pmatrix} x_k \\ \varepsilon\, y_k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \varepsilon \end{pmatrix} P_k,$$

for $0 \leq k \leq n$.

In particular $E = E^1$ and all near-edges E^ε are isotopic for $\varepsilon > 0$.

4.2 Nearly convex polygons

Let P be a strictly convex polygon with l extremal vertices V_0, V_1, $V_2 \dots V_l = V_0$, appearing in counterclockwise order around the boundary ∂P of P.

Given a sequence $E_1,\dots,E_l$ where $E_i = (P_{i,0}, P_{i,1}, \dots, P_{i,n_i})$ is an n_i-near-edge, we denote by $G(E_1,\dots,E_l|P)$ the unique configuration obtained by gluing the n_i-near-edge E_i, rescaled suitably by an orientation-preserving similitude, along the oriented edge of P which starts at V_{i-1} and ends at V_i.

More precisely, the gluing map φ_i is the unique orientation-preserving similitude of $\mathbf{R}^2$ such that

$$\varphi_i(P_{i,0}) = V_{i-1} \text{ and } \varphi_i(P_{i,n_i}) = V_i \ .$$

The configuration $G(E_1,\dots,E_l|P)$ is the set of points $\cup_{i=1}^{l}\varphi_i(E_i) \subset \mathbf{R}^2$.

We have the following result which we state without proof.

Proposition 4.1 *Consider a strictly convex $l-$gon P and l near-edges $E_1,\dots,E_l$,*

1. *The configurations $G(E_1^{\varepsilon_1},\dots,E_l^{\varepsilon_l}|P)$ are all isotopic for all $\varepsilon_i > 0$ small enough. This defines an isotopy class associated to the near-edges and the polygon.*
2. *Given a second strictly convex $l-$gon Q, the configurations $G(E_1^{\varepsilon_1},\dots,E_l^{\varepsilon_l}|P)$ and $G(E_1^{\varepsilon_1},\dots,E_l^{\varepsilon_l}|Q)$ are isotopic for all $\varepsilon_i > 0$ small enough. Hence, the isotopy class above does not depend on P.*
3. *The isotopy class defined in this way depends only on the isotopy classes of the near-edges.*

A *nearly convex polygon* is a configuration in the isotopy class associated by Proposition 4.1 to a sequence $E_1,\dots,E_l$ of near-edges. For the sake of convenience, the isotopy class itself is also called a *nearly convex polygon*. As far as combinatorial properties of triangulations are concerned, all the configurations of the class are equivalent and we denote by $P(E_1,\dots,E_l)$ any such configuration. This notation is a natural extension of the notation already used for weakly convex polygons, with integers n representing weighted edges of weakly convex polygons.

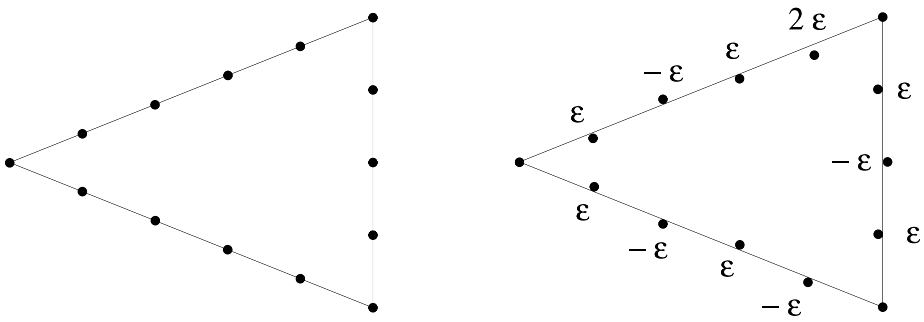

Figure 4: Perturbating $P(5,4,5)$ into a nearly convex polygon.

One can think of a nearly convex polygon as a small perturbation of the configuration associated to a weakly convex polygon. Figure 4 shows the weakly convex polygon $P(5,4,5)$ and a nearly convex polygon obtained by moving slightly non-extremal vertices perpendicularly to the three corresponding edges (with a hopefully evident notation indicating the perturbation). This nearly convex polygon is isotopic to

$P(E_a, E_b, E_c)$ where

$$E_a = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$
$$E_b = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix}$$
$$E_c = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{pmatrix}.$$

4.3 Edge-type triangles

When considering a triangulation $\mathcal{T}$ of a nearly convex polygon P, and a near-edge E of P, some triangles of $\mathcal{T}$ have their three vertices in E (we identify here the near-edge and its realization in P). We call such a triangle Δ an *edge-type triangle* of E and we write $\Delta \subset E$. The following technical lemma is the key of our main result.

Lemma 4.2 *Given a triangulation $\mathcal{T}$ of a nearly convex polygon P and a near-edge E of P, each point of E that is a vertex of $\mathcal{T}$ belongs to $\partial\mathrm{Conv}(P)$ or to at least one edge-type triangle of E.*

Proof. Recall that nearly convex polygons are "small" perturbations of weakly convex polygons. For the fixed particular realization of the near-edge E in the current polygon, denote by E^ε the realization obtained by setting all points of E closer to the edge $[P_0, P_n]$ by multiplying the distances to this edge by $\varepsilon \in (0,1)$. We can then replace E by E^ε ($\varepsilon \in (0,1)$) whithout changing the structure of the triangulation (Proposition 4.1).

Let us prove the lemma by contradiction: suppose that there exists a point M of E which is an interior vertex of $\mathcal{T}$ and which belongs to no edge-type triangle of E. Hence, there are at least three edges of the triangulation starting at M and forming adjacent angles less than π. If no such edge joins two vertices of E, a substitution of E by E^ε, for appropriately small ε, moves M so close to the segment $[P_0, P_n]$ that one of these angles becomes greater than π. Therefore, one at least of these edges has its other endpoint $N \in E$. Suppose for convenience, but without loss of generality, that $M < N$. Then consider the next segment of the triangulation starting from M, according to the clockwise order around M, and denote by Q its other endpoint. The asumptions on M imply that $Q \notin E$. Replacing now E by E^ε, for ε sufficently small, gives an angle $\widehat{QMN} > \pi$ for the triangle (QMN), which is a contradiction. □

4.4 Roofs

Let us define the set

$$R(\mathcal{T}, E) = \partial\left(\mathrm{Conv}(P) \setminus \bigcup_{\Delta \subset E} \Delta\right) \cap \mathrm{Conv}(E),$$

which will be proved to be a piecewise linear path separating the edge-type triangles contained in E from the remaining triangles of $\mathcal{T}$.

The idea for counting triangulations of a nearly convex polygon is to classify triangulations according to the paths thus obtained from all near-edges and to enumerate all triangulations giving rise to such a set of paths. The following definition is useful for the description of all possibilities.

A *partial roof* with *length* $\mathrm{Len}(R) = k$ of E is a piecewise linear path R starting at the initial vertex and ending at the final vertex of E, whose $k+1$ vertices are elements of E in increasing order: $P_{j_0} = P_0, P_{j_1}, \ldots, P_{j_{k-1}}, P_{j_k} = P_n$. This partial roof is denoted by

$$R = [P_{j_0} = P_0, P_{j_1}, \ldots, P_{j_{k-1}}, P_{j_k} = P_n]$$

Lemma 4.3 *Given a triangulation $\mathcal{T}$ of a nearly convex polygon P and a near-edge E of P, the set $R(\mathcal{T}, E)$ is a partial roof of E.*

Proof. Let us introduce the set $P' = \overline{\mathrm{Conv}(P) \setminus \bigcup_{\Delta \subset E} \Delta} = \bigcup_{\Delta \not\subset E} \Delta$. Since $\mathrm{Conv}(P) \setminus \mathrm{Conv}(E)$ is a connected subset of P' which intersects all triangles $\Delta \not\subset E$, P' is connected. Hence, it is a (generally non convex) polygon with vertices in $\mathcal{T}$.

Considering the inclusions $\mathrm{Conv}(P) \setminus \mathrm{Conv}(E) \subset P' \subset \mathrm{Conv}(P)$, the boundary of P' coincides with the boundary of $\mathrm{Conv}(P)$ outside of $\mathrm{Conv}(E)$. Hence, $R(\mathcal{T}, E)$ is a piecewise linear path with vertices $P_{j_0}, P_{j_1}, \ldots, P_{j_{k-1}}, P_{j_k}$ in E, whose orientation can be chosen such that $P_{j_0} = P_0$ and $P_{j_k} = P_n$ (where P_0 and P_n are the initial and final vertices of E).

Let us now prove by contradiction that this sequence of points of E is increasing. Otherwise, consider the first decreasing step: $P_{j_{i+1}} < P_{j_i}$. There are two cases.

Suppose first that the point $P_{j_{i+1}}$ is above the line L defined by $P_{j_{i-1}}$ and P_{j_i}, according to the standard coordinates of the near-edge E. By definition of $R(\mathcal{T}, E)$, the point P_{j_i} is linked by an edge of $\mathcal{T}$ to a point $Q \in P \setminus E$ which crosses the segment $[P_{j_{i-1}}, P_{j_{i+1}}]$. As in the proof of Lemma 4.2, a substitution of E by E^{ε}, for ε sufficiently small, removes this crossing, which is in contradiction with Proposition 4.1.

Suppose now that the point $P_{j_{i+1}}$ is below the line L. The point $P_{j_{i+1}}$ is linked by an edge of $\mathcal{T}$ to a point $Q' \in P \setminus E$. A substitution of E by E^{ε}, for ε sufficiently small, creates a crossing between this edge and the edge $[P_{j_{i-1}}, P_{j_i}]$, which is in contradiction with Proposition 4.1. □

The definition of a partial roof refers to the coordinate representation of E: the "sky" (which corresponds to the interior of the nearly convex polygon) is "above" E. We are interested in points and triangles "sheltered" by a partial roof.

In order to define properly the region sheltered by a partial roof, we consider again coordinates (formula 6) of the near-edge E: we define the *lower boundary* $\partial^- E$ of E as the piecewise-linear path $\partial^- E = (\partial \mathrm{Conv}(E)) \cap \{(x,y) \mid y \leq 0\}$, which is the "lowest" possible partial roof. Remark that each partial roof R, and in particular the lower boundary, is the graph of a piecewise-affine function $f_R : [x_0, x_n] \longrightarrow \mathbf{R}$. The graph $\partial^- E$ is below each other partial roof: we have $f_{\partial^- E}(x) \leq f_R(x)$, for all x in $[x_0, x_n]$. The region *sheltered* by the partial roof R is then the (generally non convex) subset $S(E, R)$ enclosed by $\partial^- E$ and R :

$$S(E,R) = \{(x,y) | x \in [x_0, x_n],\, f_{\partial^- E}(x) \leq y \leq f_R(x)\}.$$

We use the same notations $\partial^- E$ and $S(E,R)$ for denoting the corresponding subsets of a realization of E in a nearly convex polygon P.

Lemma 4.4 *Given a triangulation $\mathcal{T}$ of a nearly convex polygon P and a near-edge E of P, $\mathcal{T}$ induces a triangulation on the sheltered region:*

$$S(E, R(\mathcal{T}, E)) = \bigcup_{\Delta \subset E} \Delta.$$

Proof. It is a consequence of the proof of Lemma 4.3. $\square$

A partial roof R is a *roof* if $E \subset S(E,R)$. If $\mathcal{T}$ is a maximal triangulation of P, $R = R(\mathcal{T},E)$ is a roof by Lemma 4.2 and $\mathcal{T}$ induces a triangulation on $S(E,R)$ whose vertices are exactly the points of E by Lemma 4.4. Enumerating induced triangulations on $S(E,R)$ leads to the definition of maximal near-edges polynomials.

4.5 Near-edge polynomials

We define the *maximal polynomial* of the near-edge E by

$$p_E = \sum_{R \text{ roof of } E} \tau_{\max}(E,R)\ p_{\mathrm{Len}(R)} \quad \in \mathbf{Z}[t], \tag{7}$$

where $\tau_{\max}(E,R)$ denotes the number of triangulations of $S(E,R)$ involving exactly all points of $E \cap S(E,R)$ and the polynomials p_m are the maximal edge polynomials given by formula 2 of subsection 3.3.

Starting from a triangulation $\mathcal{T}$ which is not maximal, the path R induced on a near-edge E is in general only a partial roof and $\mathcal{T}$ induces not necessarily a maximal triangulation of the sheltered region $S(E,R)$. This suggest to introduce "sub-near-edges" in order to deal with these difficulties.

Let $V^-(E) = \mathrm{Extr}(E) \cap \partial^- E$ denote the set of extremal points of the lower boundary. A $k-$*sub-near-edge* of E is an increasing subsequence $E' \subset E$ of $k+1 \leq n+1$ elements containing the set $V^-(E)$. In particular, any sub-near-edge E' of E has initial vertex P_0, final vertex P_n and verifies $\partial^- E' = \partial^- E$. We note $E' < E$.

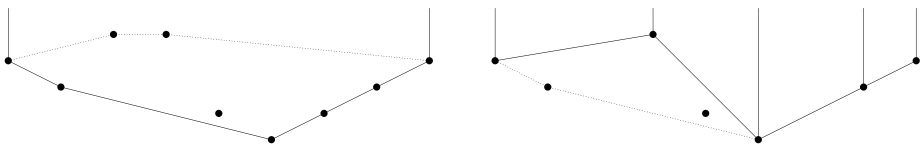

Figure 5: An $8-$near-edge and a roof of length 4 of a $6-$sub-near-edge.

Example. The left half of Figure 5 displays the $8-$near-edge

$$E = (P_0, \ldots, P_8) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 0 & -1 & 1 & 1 & -2 & -3 & -2 & -1 & 0 \end{pmatrix}.$$

We have $V^-(E) = (P_0, P_1, P_5, P_8)$ and E has 2^5 sub-near-edges obtained by removing any subset of vertices among $\{P_2, P_3, P_4, P_6, P_7\}$ from E. The right half of Figure 5 shows the roof $R = [P_0, P_3, P_5, P_7, P_8]$ of the sub-near-edge E' of E defined by $(P_0, P_1, P_3, P_4, P_5, P_7, P_8)$.

The *complete polynomial* $\overline{p}(E)$ of an $n-$near-edge $E = (P_0, \ldots, P_n)$ is defined as

$$\overline{p}_E = \sum_{m=1}^{n} \sum_{\substack{E' < E \\ \#E' = m}} p_{E'} s^m. \tag{8}$$

Example. We compute the complete polynomial $\overline{p}_{E_a}$ of the near-edge

$$E_a = (P_0, \ldots, P_5) = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$

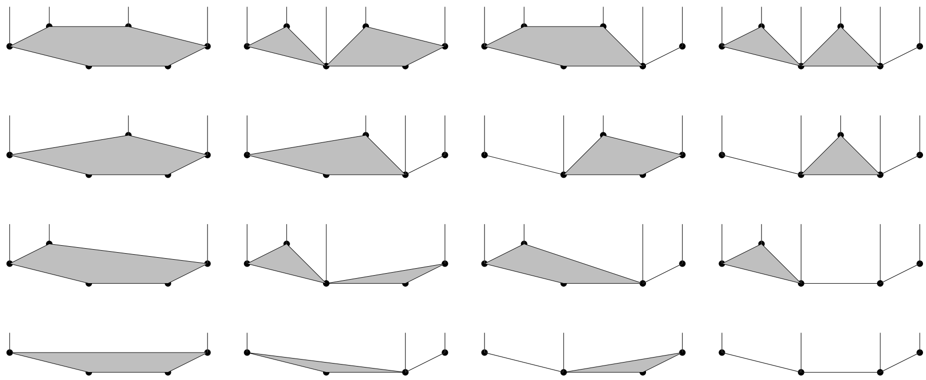

Figure 6: All roofs of sub-near-edges involved in $\overline{p}_{E_a}$

involved in the nearly convex polygon $P(E_a, E_b, E_c)$ of Figure 4.

Figure 6 contains all roofs of the four possible sub-near-edges of E_a obtained by removing any subset of points in $\{P_1, P_3\}$. Their contributions to $\overline{p}_{E_a}$ are given by

sub-near-edge				
$(P_0, P_1, P_2, P_3, P_4, P_5)$	$14p_3s^5$	$2p_4s^5$	$5p_4s^5$	p_5s^5
$(P_0, P_2, P_3, P_4, P_5)$	$5p_2s^4$	$2p_3s^4$	$2p_3s^4$	p_4s^4
$(P_0, P_1, P_2, P_4, P_5)$	$5p_2s^4$	p_3s^4	$2p_3s^4$	p_4s^4
(P_0, P_2, P_4, P_5)	$2p_1s^3$	p_2s^3	p_2s^3	p_3s^3

They sum up to the complete near-edge polynomial of E_a:

$$\overline{p}_{E_a} = (14p_3 + 7p_4 + p_5)s^5 + (10p_2 + 7p_3 + 2p_4)s^4 + (2p_1 + 2p_2 + p_3)s^3.$$

4.6 Main result

We can now state and prove the central theorem.

Theorem 4.5 *Given $l \geq 3$ near-edges E_1, E_2... E_l, the number of maximal triangulations of the nearly convex polygon $P(E_1, \ldots, E_l)$ is given by*

$$\tau_{\max}(P(E_1, \ldots, E_l)) = \Big\langle \prod_{i=1}^{l} p_{E_i}(t), t^2 G_C(t) \Big\rangle_t$$

and its triangulation polynomial is defined by

$$p_\tau(P(E_1, \ldots, E_l)) = \sum_k \tau_k(P(E_1, \ldots, E_l))s^k = \Big\langle \prod_{i=1}^{l} \overline{p}_{E_i}(t), t^2 G_C(t) \Big\rangle_t,$$

using the notations of formulae 3, 7 and 8 above.

Corollary 4.6 *The number of maximal triangulations and the triangulation polynomial of a nearly convex polygon $P(E_1, \ldots, E_l)$ does not depend on the cyclic order of the near-edges E_i.*

Proof of Theorem 4.5. Fix a triangulation $\mathcal{T}$ of $P = P(E_1, E_2, \ldots, E_l)$, and a near-edge $E = E_i$. We have seen in Lemma 4.3 and Lemma 4.4 that we can associate to E a partial roof $R = R(\mathcal{T}, E)$ and a triangulation of the sheltered region $S(E,R)$ by the edge-type triangles of E. We denote by E' the subset of all points of E occuring in $\mathcal{T}$. The set E' is a sub-near-edge since $\mathcal{T}$ triangulates $\mathrm{Conv}(P)$ and thus involves all extremal points. A crucial remark is that $E' \subset S(E,R)$ by Lemma 4.2. Therefore, R is a roof for E'. Moreover, by definition of E', the triangulation on $S(E,R) = S(E',R)$ is maximal with respect to E'.

Writing $R_i = R(\mathcal{T}, E_i)$, we obtain a triangulation of each $S(E_i,R_i)$. We get also a triangulation of the complement $I = \mathrm{Conv}(P) \setminus \cup_i S(E_i,R_i)$, whose boundary is the union of the roofs R_i. This triangulation is not arbitrary: each vertex is in a roof R_i and a triangle in I is never of edge-type and has thus not all three vertices in the same roof. The triangulation induced on I yields hence a triangulation of the weakly convex polygon $P(\mathrm{Len}(R_1), \ldots, \mathrm{Len}(R_l))$ as can be seen on Figure 7.

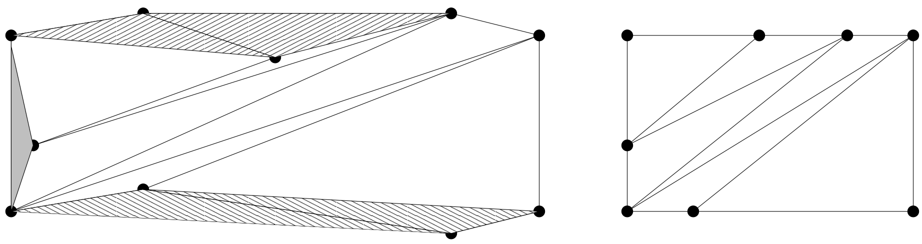

Figure 7: Triangulation of the interior region I.

We associate thus to each triangulation of P a couple $((E'_i, R_i, \mathcal{T}_i)_i, \mathcal{T}_0)$, with the following specifications: for each $1 \le i \le l$, E'_i is a sub-near-edge of E_i, R_i is a roof of E'_i and $\mathcal{T}_i$ is a triangulation of $S(E'_i, R_i) = S(E_i, R_i)$, maximal relatively to $E'i$; $\mathcal{T}_0$ is a maximal triangulation of $P(\mathrm{Len}(R_1), \ldots, \mathrm{Len}(R_l))$. This correspondence is clearly one-to-one: it is easy to reconstruct a triangulation of P, given such data, and there is only one possible reconstruction.

We consider first the case of maximal triangulations. They satisfy $E'_i = E_i$ for all i and this property characterizes maximal triangulations. Theorem 3.2 shows that the number of corresponding triangulations equals

$$\left(\prod_{i=1}^{l} \tau_{\max}(E_i, R_i)\right) \tau_{\max}(\mathrm{Len}(R_1), \ldots, \mathrm{Len}(R_l))$$

$$= \Big\langle \prod_{i=1}^{l} \tau_{\max}(E_i, R_i) p_{\mathrm{Len}(R_i)}(t), t^2 G_C(t) \Big\rangle_t$$

for each choice of a family of roofs R_i. Summation over all possible choices of R_i and inversion of sum and product give the result.

For the triangulation polynomial, according to the result above, each choice of a family of sub-near-edges gives the contribution

$$\Big\langle \prod_{i=1}^{l} p_{E'_i}(t), t^2 G_C(t) \Big\rangle_t \cdot s^{\sum k_i},$$

where the k_i are the weights of the sub-near-edges $E'i$. Summation over all choices of sub-near-edges and inversion of sum and product achieve the proof. □

Example. Consider the nearly convex polygon $P(E_a, E_b, E_c)$ isotopic to the perturbation depicted on Figure 4. Its near-edges

$$E_a = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & -1 & 1 & -1 & 0 \end{pmatrix}$$
$$E_b = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 1 & 0 \end{pmatrix}$$
$$E_c = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{pmatrix}$$

have complete polynomials

$$\begin{aligned} \overline{p}_{E_a} &= (14p_3 + 7p_4 + p_5)s^5 + (10p_2 + 7p_3 + 2p_4)s^4 \\ &\quad +(2p_1 + 2p_2 + p_3)s^3 , \\ \overline{p}_{E_b} &= (5p_3 + p_4)s^4 + 2(2p_2 + p_3)s^3 + (p_1 + p_2)s^2 \\ \overline{p}_{E_c} &= (10p_3 + 7p_4 + 2p_5)s^5 + (3p_2 + 13p_3 + 4p_4)s^4 \\ &\quad +3(2p_2 + p_3)s^3 + (p_1 + p_2)s^2, \end{aligned}$$

and the triangulation polynomial $p_\tau(P(E_a, E_b, E_c))$ of $P(E_a, E_b, E_c)$ is given by

$$\begin{aligned} &\langle \overline{p}_{E_a}(t)\overline{p}_{E_b}(t)\overline{p}_{E_c}(t), t^2 G_C(t)\rangle_t \\ = \;& 194939\, s^{14} + 338669\, s^{13} + 263615\, s^{12} + 119944\, s^{11} \\ &\quad +34773\, s^{10} + 6522\, s^9 + 748\, s^8 + 42\, s^7 \ . \end{aligned}$$

4.7 Arithmetics of near-edges

4.7.1 Factorization

A near-edge $E = (P_0, \ldots, P_n)$ *factorizes* into near-edges E_1, E_2 if there exists a lower extremal vertex $P_k \in V^-(E)$ such that the two near-edges defined by

$$E_1 = (P_0, P_1, \ldots, P_{k-1}, P_k),\ E_2 = (P_k, P_{k+1}, \ldots, P_{n-1}, P_n)$$

have the property that all points of $E \setminus E_i$ lie strictly above every line defined by two distinct points of E_i for $i = 1, 2$. We write $E = E_1 \cdot E_2$ if the near-edge E factorizes with first factor E_1 and second factor E_2. A near-edge is *prime* if it has no non-trivial factorization. It is easy to show that every near-edge has a unique factorization into prime near-edges.

Proposition 4.7 *Given a factorization $E = E_1 \cdot E_2$ of a near-edge E we have*

$$p_E = p_{E_1}\ p_{E_2} \text{ and } \overline{p}_E = \overline{p}_{E_1}\ \overline{p}_{E_2} \ .$$

Proof. Since the nearly convex polygons $P(1^k, E)$ (with the same notation as before: k successive edges of weight 1) and $P(1^k, E_1, E_2)$ are isotopic for all $k = 2, 3, \ldots$ we have

$$\left\langle t^k\ \overline{p}_E, t^2 G_C(t)\right\rangle_t = \left\langle t^k\ \overline{p}_{E_1}\ \overline{p}_{E_2}, t^2 G_C(t)\right\rangle_t.$$

This implies the result since $\det(\left(C_{i+j+k}\right)_{0 \le i,j \le n}) > 0$ for all $k \ge 0$ and $n \ge 1$ (this follows for instance easily from Exercice 6.26.b in [14]). □

4.7.2 Polynomials for small near-edges

This subsection describes all $1-$, $2-$ and $3-$near-edges up to isotopy and gives their polynomials. We will use the following definition: a near-edge is *generic* if its underlying set of points is a generic configuration of $\mathbf{R}^2$, i.e. if three distinct points of E are never collinear.

We will also use the following obvious fact. If two $n-$near-edges $E = (P_0, \dots, P_n)$ and E' are *vertical mirrors, i.e.* if $E' = (\overline{P}_n, \overline{P}_{n-1}, \dots, \overline{P}_1, \overline{P}_0)$ where $\overline{P}_i = \begin{pmatrix} -x_i \\ y_i \end{pmatrix}$ is the Euclidean reflection of $P_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ with respect to the vertical line $x = 0$, then $\overline{p}_E = \overline{p}'_E$.

$1-$**near-edges** The unique $1-$near-edge can be represented by $E_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. It is generic and prime and has complete polynomial $\overline{p}_{E_1} = p_1 s = s\,t$.

$2-$**near-edges** There are two generic $2-$near-edges, represented by

$$E_{2,1} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 0 \end{pmatrix}, \quad E_{2,2} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & -1 & 0 \end{pmatrix} .$$

$E_{2,1}$ is prime while $E_{2,2} = E_1 \cdot E_1$. They have complete polynomials

$$\overline{p}_{E_{2,1}} = p_2 s^2 + p_1 s,\ \overline{p}_{E_{2,2}} = (p_2 + p_1) s^2 = \overline{p}_{E_1}^2 .$$

Moreover, there is also a unique non-generic $2-$near-edge represented for instance by

$$E_{2,3} = \begin{pmatrix} 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

with complete polynomial given by

$$\overline{p}_{E_{2,3}} = p_2 s^2 + p_1 s = \overline{p}_{E_{2,1}} .$$

$3-$**near-edges** There are eight generic $3-$near-edges represented by

$$\begin{array}{ll}
E_{3,1} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 3 & 0 \end{pmatrix} & E_{3,2} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 1 & 0 \end{pmatrix} \\
E_{3,3} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 3 & 1 & 0 \end{pmatrix} & E_{3,4} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & -1 & 0 \end{pmatrix} \\
E_{3,5} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & 1 & 0 \end{pmatrix} & E_{3,6} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -3 & -1 & 0 \end{pmatrix} \\
E_{3,7} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & -1 & 0 \end{pmatrix} & E_{3,8} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & -3 & 0 \end{pmatrix}
\end{array} .$$

The first five are prime. The last three have factorizations

$$E_{3,6} = E_1 E_{2,1}\ , E_{3,7} = E_1^3,\ E_{3,8} = E_{2,1} E_1 .$$

The pairs $\{E_{3,1}, E_{3,3}\}$, $\{E_{3,4}, E_{3,5}\}$, $\{E_{3,6}, E_{3,8}\}$ are vertical mirrors. The prime near-edges have complete polynomials

$$\begin{aligned}
&\overline{p}_{E_{3,1}} = \overline{p}_{E_{3,3}} = (p_2+p_3)s^3 + 2p_2 s^2 + p_1 s \; , \\
&\overline{p}_{E_{3,2}} = 2p_3 s^3 + 2p_2 s^2 + p_1 s \; , \\
&\overline{p}_{E_{3,4}} = \overline{p}_{E_{3,5}} = (2p_2+p_3)s^3 + p_1^2 s^2 \; .
\end{aligned}$$

There are moreover nine more $3-$near-edges which are not generic. They are represented for instance by

$$\begin{aligned}
&E_{3,9} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 2 & 0 \end{pmatrix} && E_{3,10} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & -2 & 0 \end{pmatrix} \\
&E_{3,11} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 0 \end{pmatrix} && E_{3,12} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & 0 \end{pmatrix} \\
&E_{3,13} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 0 \end{pmatrix} && E_{3,14} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -1 & 0 & 0 \end{pmatrix} \\
&E_{3,15} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 2 & 1 & 0 \end{pmatrix} && E_{3,16} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & -2 & -1 & 0 \end{pmatrix} \\
&E_{3,17} = \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} && .
\end{aligned}$$

The following near-edges factorize:

$$E_{3,10} = E_{2,3}\, E_1, \qquad E_{3,16} = E_1\, E_{2,3}$$

The remaining near-edges are prime and have complete polynomials

$$\begin{aligned}
\overline{p}_{E_{3,9}} &= \overline{p}_{E_{3,15}} &&= p_3 s^3 + 2p_2 s^2 + p_1 s, \\
\overline{p}_{E_{3,11}} &= \overline{p}_{E_{3,13}} &&= (p_2+p_3)s^3 + 2p_2 s^2 + p_1 s, \\
\overline{p}_{E_{3,12}} &= \overline{p}_{E_{3,14}} &&= (p_2+p_3)s^3 + p_2 s^2, \\
\overline{p}_{E_{3,17}} &= \overline{p}_3 &&= p_3 s^3 + 2p_2 s^2 + p_1 s \; .
\end{aligned}$$

5 Remarks and questions

5.1 Choice of the triangulation polynomial

One can also consider the triangulation polynomial defined by

$$\sum_{k_0,k_1} \tau_{k_0,k_1}(\mathcal{C}) s_0^{k_0} s_1^{k_1}$$

counting the number of triangulations using k_0 vertices and k_1 edges. The number k_2 of triangles can then be recovered using the Euler characteristic $k_0 - k_1 + k_2 = 1$ of a compact, simply connected triangulated polygonal region in $\mathbf{R}^2$. This more general polynomial yields the same information as the complete polynomial considered above except if the boundary $\partial(\mathrm{Conv}(\mathcal{C}))$ contains points of $\mathcal{C}$ which are not extremal. Most of the results and algorithms can easily be modified in order to deal with this more general polynomial. For clarity and concision we described here the simpler version defined above.

5.2 General configurations and nearly convex polygons

Remark that every generic configuration is isotopic to a nearly convex polygon. Indeed, every extremal point Q of a generic configuration $\mathcal{C}$ yields a realization of $\mathcal{C}$ as a nearly convex polygon with two trivial near-edges (each consisting of Q and of a neighbouring extremal point) and a near-edge defined by $\mathcal{C} \setminus \{Q\}$, which is unique up to isotopy.

However, the framework of near-edges is not interesting for a general generic configuration. It speeds up computations only in the case where the configuration has a "non-trivial" factorization into near-edges.

5.3 Remarks on effectiveness

Near-edge polynomials are in general difficult to compute. We will present a few algorithms dealing with them in a further paper. One of these algorithms is a slightly more sophisticated version of an algorithm by Kaibel and Ziegler described in [9] and yields also a general purpose algorithm (unfortunately of exponential complexity), for computing arbitrary triangulation polynomials. This algorithm, based on a transfer matrix, is fairly simple and it would be interesting to compare its performance with existing algorithms, like for instance the algorithm of Aichholzer described in [1].

The next subsection describes a family of near-edges for which the computation of near-edge polynomials is much easier and can be achieved by an algorithm of polynomial time-complexity. A detailled description of the algorithm will be given in our planned future paper.

5.4 Convex near-edges

A near-edge $E = \{P_0, \dots, P_n\}$ is *convex* if $P_0, \dots, P_n$ are extremal points of $\mathrm{Conv}(E)$. Otherwise stated, the points of a convex near-edge are the vertices of a convex polygon with $(n+1)$ edges. There are thus exactly 2^{n-1} non-isotopic convex $n-$near-edges.

A convex near-edge can be represented by a sequence of points

$$P_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \dots, P_i = \begin{pmatrix} i \\ \varepsilon_i \ i(n-i) \end{pmatrix}, \dots, P_n = \begin{pmatrix} n \\ 0 \end{pmatrix}$$

where $\varepsilon_1, \dots, \varepsilon_{n-1} \in \{\pm 1\}$. There are 2^{n-1} equivalence classes of convex $n-$near-edges, encoded by $n-$tuples $(\varepsilon_1, \dots, \varepsilon_{n-1})$ in $\{\pm 1\}^{n-1}$, The convex near-edge with $\varepsilon_i = -1$, for all i, has the factorization E_1^n. All others are prime.

A future paper will describe an algorithm having polynomial time and memory requirements for computing maximal and complete edge-polynomials of convex near-edges. It provides an efficient method for counting triangulations of nearly convex polygons involving only convex near-edges. Completing each convex $n-1-$near-edge with two trivial near-edges, we get an exponentially large class of configurations for which the problem of counting triangulations can be solved in polynomial time.

5.5 Convex near-edges related to the Legendre symbol

We used the Legendre symbol to produce data for testing our algorithm. The surprising results lead to formulate the conjecture below.

Given an odd prime p, the Legendre symbol, denoted by $\left(\frac{x}{p}\right) \in \{\pm 1\}$ for $1 \leq x \leq p-1$ defines a non-trivial homomorphism between the multiplicative groups $(\mathbf{Z}/p\mathbf{Z})^*$

and $\{\pm 1\}$. It can be computed using quadratic reciprocity or the equality

$$\left(\frac{x}{p}\right) \equiv x^{(p-1)/2} \pmod{p} .$$

We consider two convex $(p+1)-$near-edges E_p^+, E_p^- associated to the sequences

$$\left(\frac{1}{p}\right), \left(\frac{2}{p}\right), \dots, \left(\frac{p-1}{p}\right) \text{ and } -\left(\frac{1}{p}\right), -\left(\frac{2}{p}\right), \dots, -\left(\frac{p-1}{p}\right)$$

of (negated) Legendre symbols. For $p \equiv 3 \pmod{4}$ the identity $\left(\frac{x}{p}\right) = -\left(\frac{-x}{p}\right)$ implies that E_p^+ and E_p^- have identical (complete) triangulation polynomials.

Computation of the maximal triangulation polynomials $p_{E_p^+}$ and $p_{E_p^-}$ for all odd primes $p < 200$ suggests:

Conjecture 5.1 *Using the notations of formulae 3 and 4, we have*

$$\langle P(t), t^2 G_C(t)\rangle_t \equiv \left(\frac{-1}{p}\right) \pmod{p} = \begin{cases} 1 & p \equiv 1 \pmod{4} \\ -1 & p \equiv 3 \pmod{4} \end{cases}$$

for P a polynomial of the form $t\ p_{E_p^+}^2$, $t\ p_{E_p^-}^2$ or $t\ p_{E_p^+}\ p_{E_p^-}$.

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