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ON SUMS INVOLVING PRODUCTS OF THREE BINOMIAL COEFFICIENTS

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Abstract. In this paper we mainly employ the Zeilberger algorithm to study congruences for sums of terms involving products of three binomial coefficients. Let $p > 3$ be a prime. We prove that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+d}}{64^k}\equiv 0 \pmod{p^2}$$

for all $d\in\{0,\ldots,p-1\}$ with $d\equiv (p+1)/2 \pmod 2$. If $p\equiv 1 \pmod 4$ and $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$, then we show

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k}\equiv 2p-2x^2 \pmod{p^2} \text{ and } \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k}\equiv -2p \pmod{p^2}$$

by means of determining x mod p^2 via

$$(-1)^{(p-1)/4}\,x\equiv\sum_{k=0}^{(p-1)/2}\frac{k+1}{8^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2 \pmod{p^2}.$$

We also solve the remaining open cases of Rodriguez-Villegas' conjectural congruences on

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k},\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}$$

modulo p^2.

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1. Introduction

Let p be an odd prime. It is known that (see, e.g., S. Ahlgren [A], L. van Hammer [vH] and T. Ishikawa [I])

$$\sum_{k=0}^{(p-1)/2}(-1)^k\binom{-1/2}{k}^3$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } p=x^2+y^2\ (4\mid x-1\ \&\ 2\mid y),\\ 0\pmod{p^2}&\text{if } p\equiv 3\pmod 4.\end{cases}$$

Clearly,

$$\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\quad\text{for all } k\in\mathbb{N}=\{0,1,2,\ldots\},$$

and

$$\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\equiv 0\pmod p\quad\text{for any } k=\frac{p+1}{2},\ldots,p-1.$$

After his determination of $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k \bmod p^2$ (where $m\in\mathbb{Z}$ and $m\not\equiv 0\pmod p$) in [Su1], the author [Su2, Su3] raised some conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k \bmod p^2$ with $m\in\{1,-8,16,-64,256,-512,4096\}$; for example, the author [Su2] conjectured that

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6\pmod 7,\end{cases}\tag{1.1}$$

where $(-)$ denotes the Legendre symbol. (It is known that if $(\frac{p}{7})=1$ then $p=x^2+7y^2$ for some $x,y\in\mathbb{Z}$; see, e.g., [C, p. 31].) Quite recently the author's twin brother Zhi-Hong Sun [S2] made remarkable progress on those conjectures; in particular, he proved (1.1) in the case $(\frac{p}{7})=-1$ and confirm the author's conjecture on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/(-8)^k \bmod p^2$.

Let $p=2n+1$ be an odd prime. It is easy to see that for any $k=0,\ldots,n$ we have

$$\binom{n+k}{2k}=\frac{\prod_{j=1}^{k}(-(2j-1)^2)}{4^k(2k)!}\prod_{j=1}^{k}\left(1-\frac{p^2}{(2j-1)^2}\right)\equiv\frac{\binom{2k}{k}}{(-16)^k}\pmod{p^2}.\tag{1.2}$$

Based on this observation Z. H. Sun [S2] studied the polynomial

$$f_n(x)=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2x^k$$

and found the key identity

$$f_n(x(x+1)) = D_n(x)^2 \tag{1.3}$$

in his approach to (1.1), where

$$D_n(x) := \sum_{k=0}^{n} \binom{n+k}{2k}\binom{2k}{k} x^k = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} x^k.$$

Note that those numbers $D_n = D_n(1)$ ($n \in \mathbb{N}$) are the so-called central Delannoy numbers and $P_n(x) := D_n((x-1)/2)$ is the Legendre polynomial of degree n.

Recall that Catalan numbers are those integers

$$C_n = \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n} - \binom{2n}{n+1} \ (n \in \mathbb{N})$$

while Schröder numbers are given by

$$S_n := \sum_{k=0}^{n} \binom{n+k}{2k} C_k = \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k} \frac{1}{k+1}.$$

We define the Schröder polynomial of degree n by

$$S_n(x) := \sum_{k=0}^{n} \binom{n+k}{2k} C_k x^k. \tag{1.4}$$

For basic information about D_n and S_n, the reader may consult [CHV], [Sl], and p. 178 and p. 185 of [St].

Via Schröder polynomials and the Zeilberger algorithm (cf. [PWZ]), we obtain the following result.

Theorem 1.1. *Let p be an odd prime. We have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{2k}{k+d}}{64^k} \equiv 0 \pmod{p^2} \tag{1.5}$$

for all $d \in \{0, 1, \ldots, p-1\}$ with $d \equiv (p+1)/2 \pmod 2$. If $p \equiv 3 \pmod 4$, then

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{2k}{k+1}}{64^k} \equiv (2p + 2 - 2^{p-1}) \binom{(p-1)/2}{(p+1)/4}^2 \pmod{p^2} \tag{1.6}$$

Now we state our second theorem the first part of which plays a key role in our proof of the second part.

Theorem 1.2. *Let $p\equiv 1\pmod 4$ be a prime and write $p=x^2+y^2$ with $x\equiv 1\pmod 4$ and $y\equiv 0\pmod 2$.*

(i) *We can determine x mod p^2 in the following way:*

$$(-1)^{(p-1)/4}x\equiv\sum_{k=0}^{(p-1)/2}\frac{k+1}{8^k}\binom{2k}{k}^2\equiv\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2\pmod{p^2}. \tag{1.7}$$

Also,

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{8^k}\equiv-2\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{8^k}\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{x}\right)\pmod{p^2}, \tag{1.8}$$

$$\begin{aligned}S_{(p-1)/2}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{(-16)^k}\equiv-8\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^2}{(-16)^k}\\ \equiv(-1)^{(p-1)/4}2\left(2x-\frac{p}{x}\right)\pmod{p^2},\end{aligned} \tag{1.9}$$

$$\sum_{k=0}^{(p-1)/2}\frac{k^2\binom{2k}{k}^2}{8^k}\equiv(-1)^{(p-1)/4}\left(x-\frac{3p}{4x}\right)\pmod{p^2}, \tag{1.10}$$

and

$$\sum_{k=0}^{(p-1)/2}\frac{k^2\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p+3)/4}\frac{p}{16x}\pmod{p^2}. \tag{1.11}$$

(ii) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k}\equiv 2p-2x^2\pmod{p^2} \tag{1.12}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k}\equiv-2p\pmod{p^2}. \tag{1.13}$$

Remark 1.1. Let p be an odd prime. We conjecture that

$$\sum_{k=0}^{p-1}\frac{k+1}{8^k}\binom{2k}{k}^2+\sum_{k=0}^{(p-1)/2}\frac{2k+1}{(-16)^k}\binom{2k}{k}^2$$
$$\equiv\begin{cases}2(\frac{2}{p})x\pmod{p^3} & \text{if } p=x^2+y^2\ (4\mid x-1\ \&\ 2\mid y),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}$$

Motivated by his study of Gaussian hypergeometric series and Calabi-Yau manifolds, in 2003 Rodriguez-Villegas [RV] raised some conjectures on congruences. In particular, he conjectured that for any prime $p > 3$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k} \equiv b(p) \pmod{p^2}, \quad \sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \equiv c(p) \pmod{p^2}, \tag{1.14}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}} \equiv \left(\frac{p}{3}\right) a(p) \pmod{p^2}, \tag{1.15}$$

where

$$\sum_{n=1}^{\infty} a(n)q^n = q\prod_{n=1}^{\infty}(1-q^{4n})^6 = \eta(4z)^6,$$

$$\sum_{n=1}^{\infty} b(n)q^n = q\prod_{n=1}^{\infty}(1-q^{6n})^3(1-q^{2n})^3 = \eta^3(6z)\eta^3(2z),$$

$$\sum_{n=1}^{\infty} c(n)q^n = q\prod_{n=1}^{\infty}(1-q^{n})^2(1-q^{2n})(1-q^{4n})(1-q^{8n})^2 = \eta^2(8z)\eta(4z)\eta(2z)\eta^2(z),$$

and the Dedekind η-function is given by

$$\eta(z) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n) \quad (\mathrm{Im}(z) > 0 \text{ and } q = e^{2\pi i z}).$$

In 1892 F. Klein and R. Fricke proved that (see also [SB])

$$a(p) = \begin{cases} 4x^2 - 2p \pmod{p^2} & \text{if } p \equiv 1 \pmod 4 \text{ and } p = x^2 + y^2\ (2 \nmid x), \\ 0 & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

By [SB] we also have

$$b(p) = \begin{cases} 4x^2 - 2p & \text{if } p \equiv 1 \pmod 3 \text{ and } p = x^2 + 3y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 & \text{if } p \equiv 2 \pmod 3; \end{cases}$$

and

$$c(p) = \begin{cases} 4x^2 - 2p & \text{if } (\frac{-2}{p}) = 1 \text{ and } p = x^2 + 2y^2 \text{ with } x, y \in \mathbb{Z}, \\ 0 & \text{if } (\frac{-2}{p}) = -1, \text{ i.e., } p \equiv 5, 7 \pmod 8. \end{cases}$$

Via an advanced approach involving the p-adic Gamma function and Gauss and Jacobi sums, E. Mortenson [M] managed to provide a partial solution of (1.14) and (1.15), with the following things open:

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{108^k}\equiv b(p)=0\pmod{p^2}\quad\text{if } p\equiv 5\pmod 6,\tag{1.16}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv c(p)\pmod{p^2}\quad\text{if } p\equiv 3\pmod 4,\tag{1.17}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv -a(p)\pmod{p^2}\quad\text{if } p\equiv 5\pmod 6.\tag{1.18}$$

(See also K. Ono [O, Chapter 11] for an introduction to this method.) Concerning (1.16)-(1.18), Mortenson's approach [M] only allowed him to show that for each of them the squares of both sides of the congruence are congruent modulo p^2.

Our following theorem confirms (1.16)-(1.18) and hence completes the proof of (1.14) and (1.15). So far, all conjectures of Rodriguez-Villegas [RV] involving at most three products of binomial coefficients have been proved!

Theorem 1.3. *Let $p>3$ be a prime.*

(i) *Given $d\in\{0,\ldots,p-1\}$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k+d}\binom{2k}{k}\binom{3k}{k}}{108^k}\equiv 0\pmod{p^2}\quad\text{if } d\equiv\frac{1+(\frac{p}{3})}{2}\pmod 2,\tag{1.19}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k+d}\binom{2k}{k}\binom{4k}{2k}}{256^k}\equiv 0\pmod{p^2}\quad\text{if } d\equiv\frac{1+(\frac{-2}{p})}{2}\pmod 2,\tag{1.20}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k+d}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv 0\pmod{p^2}\quad\text{if } d\equiv\frac{1+(\frac{-1}{p})}{2}\pmod 2.\tag{1.21}$$

(ii) *If $p\equiv 3\pmod 8$ and $p=x^2+2y^2$ with $x,y\in\mathbb{Z}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k}\equiv 4x^2-2p\pmod{p^2}.\tag{1.22}$$

(iii) *If $p\equiv 5\pmod{12}$ and $p=x^2+y^2$ with $2\nmid x$ and $2\mid y$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv 2p-4x^2\pmod{p^2}.\tag{1.23}$$

In the case $d=1$, Theorem 1.3(i) yields the following new result. (Note that $\binom{2k}{k}\binom{3k}{k+1}=2\binom{2k}{k+1}\binom{3k}{k}$ for any $k\in\mathbb{N}$.)

Corollary 1.1. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k+1}}{108^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 1 \pmod 3, \tag{1.24}$$

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}\binom{2k}{k+1}}{256^k}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 1,3 \pmod 8, \tag{1.25}$$

$$\sum_{k=0}^{p-1}\frac{\binom{6k}{3k}\binom{3k}{k}\binom{2k}{k+1}}{12^{3k}}\equiv 0 \pmod{p^2}\quad \text{if } p\equiv 1 \pmod 4. \tag{1.26}$$

We will prove Theorems 1.1-1.3 in Sections 2-4 respectively.

2. Proof of Theorem 1.1

Lemma 2.1. *For any positive integer n we have*

$$\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}x^{k-1}(x+1)^{k+1}=n(n+1)S_n(x)^2. \tag{2.1}$$

Proof. Observe that

$$S_n(x)^2=\sum_{k=0}^{n}\binom{n+k}{2k}C_kx^k\sum_{l=0}^{n}\binom{n+l}{2k}C_lx^l=\sum_{m=0}^{2n}a_m(n)x^m,$$

where

$$a_m(n):=\sum_{k=0}^{m}\binom{n+k}{2k}C_k\binom{n+m-k}{2m-2k}C_{m-k}.$$

Also, the coefficient of x^m on the left-hand side of (2.1) coincides with

$$\begin{aligned}b_m(n):=&\sum_{k=1}^{m+1}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\binom{k+1}{m+1-k}\\=&\sum_{k=0}^{m}\binom{n+k+1}{2k+2}\binom{2k+2}{k+1}\binom{2k+2}{k}\binom{k+2}{m-k}.\end{aligned}$$

Thus, for the validity of (2.1) it suffices to show that $b_m(n)=n(n+1)a_m(n)$ for all $m=0,1,\ldots$. Obviously, $a_0(n)=1$ and $b_0(n)=n(n+1)$. Also, $a_1(n)=n(n+1)$ and $b_1(n)=n^2(n+1)^2$. By the Zeilberger algorithm via

Mathematica (version 7) we find that both $u_m = a_m(n)$ and $u_m = b_m(n)$ satisfy the following recursion:

$$\begin{aligned}&(m+2)(m+3)(m+4)u_{m+2}\\=&2(2mn^2+5n^2+2mn+5n-m^3-6m^2-11m-6)u_{m+1}\\&-(m+1)(m-2n)(m+2n+2)u_m.\end{aligned}$$

So $b_m(n) = n(n+1)a_m(n)$ for all $m \in \mathbb{N}$. This proves (2.1). $\square$

Proof of Theorem 1.1. (i) We first determine $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{2k}{k+1}/64^k$ mod p^2 via Lemma 2.1, which actually led the author to the study of (1.5).

Recall the following combinatorial identity (cf. [Su2, (4.3)]):

$$\sum_{k=0}^{n}\binom{n+k}{2k}\frac{C_k}{(-2)^k} = \begin{cases}(-1)^{(n-1)/2}C_{(n-1)/2}/2^n & \text{if } 2\nmid n,\\ 0 & \text{if } 2\mid n.\end{cases}$$

Set $n = (p-1)/2$. Applying (2.1) with $x = -1/2$ we get

$$\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\frac{1}{(-2)^{k-1}2^{k+1}} = n(n+1)S_n\left(-\frac{1}{2}\right)^2.$$

Thus, with the help of (1.2), we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{64^k} \equiv&\sum_{k=1}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}\frac{1}{(-4)^k}\\=&-n(n+1)S_n\left(-\frac{1}{2}\right)^2 \equiv \frac{1}{4}S_n\left(-\frac{1}{2}\right)^2\\\equiv&\begin{cases}0 \pmod{p^2} & \text{if } p\equiv 1 \pmod 4\\ C_{(n-1)/2}^2/2^{2n+2} \pmod{p^2} & \text{if } p\equiv 3 \pmod 4.\end{cases}\end{aligned}$$

In the case $p \equiv 3 \pmod 4$, clearly

$$\begin{aligned}\frac{C_{(n-1)/2}^2}{2^{2n+2}} =&\frac{\left(\binom{(p-1)/2}{(p+1)/4}\frac{2}{p-1}\right)^2}{4\times 2^{p-1}}\\\equiv&\frac{1}{(1-2p)(1+p\,q_p(2))}\binom{(p-1)/2}{(p+1)/4}^2\\\equiv&(1+2p-p\,q_p(2))\binom{(p-1)/2}{(p+1)/4}^2 \pmod{p^2}\end{aligned}$$

where $q_p(2) = (2^{p-1}-1)/p$. Therefore (1.5) with $d = 1$ holds if $p \equiv 1 \pmod 4$, and (1.6) is valid when $p \equiv 3 \pmod 4$.

(ii) For $d = 0, 1, 2, \ldots$ set

$$u_d = \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 \binom{2k}{k+d}}{64^k} = \sum_{d \leqslant k < p} \frac{\binom{2k}{k}^2 \binom{2k}{k+d}}{64^k}.$$

By the Zeilberger algorithm we find the recursion

$$(2d+1)^2 u_d - (2d+3)^2 u_{d+2} = \frac{(2p-1)^2(d+1)}{64^{p-1}p} \binom{2p}{p+d+1} \binom{2p-2}{p-1}^2.$$

Note that

$$\binom{2p-2}{p-1} = pC_{p-1} \equiv 0 \pmod{p}.$$

If $0 \leqslant d < p-2$, then

$$\binom{2p}{p+d+1} = \frac{2p}{p+d+1} \binom{2p-1}{p+d} \equiv 0 \pmod{p}$$

and hence

$$(2d+1)^2 u_d \equiv (2d+3)^2 u_{d+2} \pmod{p^2}.$$

For $d \in \{0, \ldots, p-3\}$ with $d \equiv (p+1)/2 \pmod 2$, clearly $p \neq 2d+1 < 2p$ and hence

$$u_{d+2} \equiv 0 \pmod{p^2} \implies u_d \equiv 0 \pmod{p^2}.$$

If $p \equiv 3 \pmod 4$ then $p-1 \equiv (p+1)/2 \pmod 2$; if $p \equiv 1 \pmod 4$ then $p-2 \equiv (p+1)/2 \pmod 2$ and $p-2 \geqslant (p+1)/2$. Thus, if $d \in \{p-1, p-2\}$ and $d \equiv (p+1)/2 \pmod 2$, then $d \geqslant (p+1)/2$ and hence $u_d \equiv 0 \pmod{p^2}$. It follows that $u_d \equiv 0 \pmod{p^2}$ (i.e., (1.5) holds) for all $d \in \{0, \ldots, p-1\}$ with $d \equiv (p+1)/2 \pmod 2$.

By the above we have completed the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

Lemma 3.1. *For any $n \in \mathbb{N}$ we have*

$$\sum_{k=0}^{n} \binom{2k}{k}^3 \binom{k}{n-k} (-16)^{n-k} = \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2. \tag{3.1}$$

Proof. For $n = 0, 1$, both sides of (3.1) take the values 1 and 8 respectively. Let u_n denote the left-hand side of (3.1) or the right-hand side of (3.1). Via the Zeilberger algorithm for `Mathematica`, we obtain the recursion

$$(n+2)^3 u_{n+2} = 8(2n+3)(2n^2+6n+5)u_{n+1} - 256(n+1)^3 u_n \ (n \in \mathbb{N}).$$

So, by induction (3.1) holds for all $n = 0, 1, 2, \ldots$. □

Lemma 3.2. *Let p be an odd prime. Then*

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\ \equiv&\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\ \equiv&p\left(\frac{-1}{p}\right)\pmod{p^3}.\end{aligned}$$

Proof. In view of Lemma 3.1, we have

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^n\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\ =&\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^n\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}\\ =&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{8^k}\sum_{j=0}^{p-1-k}(k+j+1)\binom{k}{j}\frac{(-16)^j}{8^j}\\ \equiv&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{8^k}\left((k+1)\sum_{j=0}^{k}\binom{k}{j}(-2)^j-2k\sum_{j=1}^{k-1}\binom{k-1}{j-1}(-2)^{j-1}\right)\\ =&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{8^k}\left((k+1)(-1)^k-2k(-1)^{k-1}\right)\\ \equiv&\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\pmod{p^3}.\end{aligned}$$

In [Su3] the author conjectured that

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)+p^3E_{p-3}\pmod{p^4},$$

where $E_0,E_1,E_2,\ldots$ are Euler numbers given by

$$E_0=1\ \text{ and }\ \sum_{\substack{k=0\\ 2\mid k}}^n\binom{n}{k}E_{n-k}=0\ (n=1,2,3,\ldots).$$

The last congruence is still open but [GZ] confirmed that

$$\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)\pmod{p^3}.$$

So we have

$$\sum_{n=0}^{p-1}\frac{n+1}{8^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\equiv p\left(\frac{-1}{p}\right)\pmod{p^3}.$$

Similarly,

$$\begin{aligned}&\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^2\binom{2(n-k)}{n-k}^2\\ =&\sum_{n=0}^{p-1}\frac{2n+1}{(-16)^n}\sum_{k=0}^{n}\binom{2k}{k}^3\binom{k}{n-k}(-16)^{n-k}\\ \equiv&\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^3}{(-16)^k}\left((2k+1)\sum_{j=0}^{k}\binom{k}{j}+2k\sum_{j=1}^{k}\binom{k-1}{j-1}\right)\\ \equiv&\sum_{k=0}^{p-1}\frac{3k+1}{(-8)^k}\binom{2k}{k}^3\equiv p\left(\frac{-1}{p}\right)\pmod{p^3}.\end{aligned}$$

This concludes the proof. □

Lemma 3.3. *Let p be an odd prime. Then*

$$\begin{aligned}2\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^3}{8^k}+\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{8^k}\equiv&2p^2\left(\frac{2}{p}\right)\pmod{p^3},\\ 8\sum_{k=0}^{(p-1)/2}\frac{k\binom{2k}{k}^3}{(-16)^k}+\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{(-16)^k}\equiv&2p^2\left(\frac{-1}{p}\right)\pmod{p^3},\\ \sum_{k=0}^{(p-1)/2}(2k^2+4k+1)\frac{\binom{2k}{k}^2}{8^k}\equiv&p^2\left(\frac{2}{p}\right)\pmod{p^3},\\ \sum_{k=0}^{(p-1)/2}(8k^2+4k+1)\frac{\binom{2k}{k}^2}{(-16)^k}\equiv&p^2\left(\frac{-1}{p}\right)\pmod{p^3}.\end{aligned}$$

Proof. By induction, for every $n=0,1,2,\ldots$ we have

$$\begin{aligned}\sum_{k=0}^n\left(2k+\frac{1}{k+1}\right)\frac{\binom{2k}{k}^2}{8^k}&=\frac{(2n+1)^2}{(n+1)8^n}\binom{2n}{n}^2,\\ \sum_{k=0}^n\left(8k+\frac{1}{k+1}\right)\frac{\binom{2k}{k}^2}{(-16)^k}&=\frac{(2n+1)^2}{(n+1)(-16)^n}\binom{2n}{n}^2,\\ \sum_{k=0}^n(2k^2+4k+1)\frac{\binom{2k}{k}^2}{8^k}&=\frac{(2n+1)^2}{8^n}\binom{2n}{n}^2,\\ \sum_{k=0}^n(8k^2+4k+1)\frac{\binom{2k}{k}^2}{(-16)^k}&=\frac{(2n+1)^2}{(-16)^n}\binom{2n}{n}^2.\end{aligned}$$

Applying these identities with $n=(p-1)/2$ we immediately get the desired congruences. □

Let $p\equiv 1\pmod 4$ be a prime and write $p=x^2+y^2$ with $x\equiv 1\pmod 4$ and $y\equiv 0\pmod 2$. In 1828 Gauss showed the congruence $\binom{(p-1)/2}{(p-1)/4}\equiv 2x\pmod p$. In 1986, S. Chowla, B. Dwork and R. J. Evans [CDE] used Gauss and Jacobi sums to prove that

$$\binom{(p-1)/2}{(p-1)/4}\equiv\frac{2^{p-1}+1}{2}\left(2x-\frac{p}{2x}\right)\pmod{p^2},\tag{3.2}$$

which was first conjectured by F. Beukers. (See also [BEW] and [HW] for further related results.) In 2009, the author (see [Su2]) conjectured that

$$\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{8^k}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}^2}{(-16)^k}\equiv(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right)\pmod{p^2},\tag{3.3}$$

and this was confirmed by Z. H. Sun [S1] with helps of (3.2) and Legendre polynomials.

Proof of Theorem 1.2(i). By (1.2),

$$S_{(p-1)/2}\equiv\sum_{k=0}^{(p-1)/2}\frac{\binom{2k}{k}C_k}{(-16)^k}\pmod{p^2}.$$

In view of this and Lemma 3.3 and (3.3), it suffices to show (1.7).

As $p \mid \binom{2k}{k}$ for all $k = (p+1)/2, \ldots, p-1$, we have

$$\begin{aligned}
&\sum_{n=0}^{p-1} \frac{n+1}{8^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2 \\
=&\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \sum_{n=k}^{p-1} \frac{n+1}{8^{n-k}} \binom{2(n-k)}{n-k}^2 \\
=&\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k} \sum_{j=0}^{p-1-k} \frac{k+j+1}{8^j} \binom{2j}{j}^2 \\
\equiv&\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{8^k} \sum_{j=0}^{(p-1)/2} \frac{(k+1)+(j+1)-1}{8^j} \binom{2j}{j}^2 \\
=&2\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{8^k} \sum_{j=0}^{(p-1)/2} \frac{(j+1)\binom{2j}{j}^2}{8^j} - \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{8^k}\right)^2 \pmod{p^2}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\sum_{n=0}^{p-1} \frac{2n+1}{(-16)^n} \sum_{k=0}^{n} \binom{2k}{k}^2 \binom{2(n-k)}{n-k}^2 \\
\equiv&2\sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(-16)^k} \sum_{j=0}^{(p-1)/2} \frac{(2j+1)\binom{2j}{j}^2}{(-16)^j} - \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k}\right)^2 \pmod{p^2}.
\end{aligned}$$

Combining these with Lemma 3.2 and (3.3), we immediately obtain (1.7). □

Lemma 3.4. *Let $p \equiv 1 \pmod 4$ be a prime. Write $p = x^2 + y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. Then*

$$D_{(p-1)/2} \equiv (-1)^{(p-1)/4} \left(2x - \frac{p}{2x}\right) \pmod{p^2}. \tag{3.4}$$

Proof. By (1.2),

$$D_{(p-1)/2} \equiv \sum_{k=0}^{(p-1)/2} \frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p^2}.$$

So (3.4) follows from (3.3). □

Remark 3.1. If p is a prime with $p \equiv 3 \pmod 4$, then $n = (p-1)/2$ is odd and hence

$$\begin{aligned}
D_n \equiv&\sum_{k=0}^{n} (-1)^k \frac{\binom{2k}{k}^2}{16^k} = \sum_{k=0}^{n} (-1)^k \binom{-1/2}{k}^2 \\
\equiv&\sum_{k=0}^{n} (-1)^k \binom{n}{k}^2 = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k}^2 = 0 \pmod{p}.
\end{aligned}$$

The following result was conjectured by the author [Su2] and confirmed by Z. H. Sun [S2].

Lemma 3.5. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } 4\mid p-1\ \&\ p=x^2+y^2\ (2\nmid x),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}\tag{3.5}$$

Remark 3.2. Fix an odd prime $p=2n+1$. By (1.2) and (1.3) we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2 2^k=D_n^2 \pmod{p^2}.$$

Hence (3.5) follows from Lemma 3.4 and Remark 3.1.

Lemma 3.6. *For any positive integer n we have*

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\frac{2k+1}{(k+1)^2}x^k(x+1)^{k+1}=\frac{S_n(x)}{2}(D_{n-1}(x)+D_{n+1}(x)).\tag{3.6}$$

Proof. Note that

$$S_n(x)(D_{n-1}(x)+D_{n+1}(x))=\sum_{m=0}^{2n+1}c_m(n)x^m$$

where

$$\begin{aligned}c_m(n)=&\sum_{k=0}^{m}\binom{n+k}{2k}C_k\binom{2m-2k}{m-k}\left(\binom{n-1+m-k}{2m-2k}+\binom{n+1+m-k}{2m-2k}\right)\\ =&2\sum_{k=0}^{m}\binom{n+k}{2k}C_k\binom{n+m-k}{2m-2k}\binom{2m-2k}{m-k}\frac{(m+n-k)^2-n(2m-2k-1)}{(m+n-k)(n-m+k+1)}.\end{aligned}$$

By the Zeilberger algorithm we find that $u_m=c_m(n)/2$ satisfies the recursion

$$\begin{aligned}&(m+2)(m+3)^2(m^2+5m+6+4n(n+1))u_{m+2}+2P(m,n)u_{m+1}\\ =&(m+2)((2n+1)^2-m^2)(m^2+7m+12+4n(n+1))u_m\end{aligned}\tag{3.7}$$

where $P(m,n)$ denotes the polynomial

$$\begin{aligned}&m^5+11m^4+45m^3+83m^2+64m+12+20n^4-40n^3-58n^2-38n\\ &-25mn+m^2n+2m^3n-33mn^2+m^2n^2+2m^3n^2-16mn^3-8mn^4.\end{aligned}$$

Clearly the coefficient of x^m on the left-hand side of (3.6) coincides with

$$d_m(n)=\sum_{k=0}^{m}\binom{n+k}{2k}\binom{2k}{k}^2\binom{k+1}{m-k}\frac{2k+1}{(k+1)^2}.$$

By the Zeilberger algorithm $u_m=d_m(n)$ also satisfies the recursion (3.7). Thus we have $d_m(n)=c_m(n)$ by induction on m. So (3.6) holds. □

Proof of Theorem 1.2(ii). Write $p=2n+1$. By (2.1),

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}2^k=\frac{n(n+1)}{2}S_n^2.$$

Thus, by (1.2) and (1.9) we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{2k}{k+1}}{(-8)^k}\equiv&\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}\binom{2k}{k+1}2^k\\ \equiv&\frac{p^2-1}{8}4(4x^2-4p)\pmod{p^2}\end{aligned}$$

and hence (1.12) holds.

Now we consider (1.13). Observe that

$$\binom{2k}{k+1}^2=\left(1-\frac{2k+1}{(k+1)^2}\right)\binom{2k}{k}^2\quad\text{for } k=0,1,2,\ldots,$$

and

$$\binom{2(p-1)}{p-1}\binom{2(p-1)}{(p-1)+1}^2=\frac{p}{2p-1}\binom{2p-1}{p-1}\binom{2p-2}{p-2}^2\equiv -p\pmod{p^2}.$$

Thus we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{2k}{k+1}^2}{(-8)^k}\equiv -p+\sum_{k=0}^{n}\frac{\binom{2k}{k}^3}{(-8)^k}-\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\pmod{p^2}.\quad(3.8)$$

By (1.2) and (3.6) with $x=1$,

$$\begin{aligned}\sum_{k=0}^{n}\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\equiv&\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2\frac{(2k+1)2^k}{(k+1)^2}\\ =&\frac{S_n}{4}(D_{n-1}+D_{n+1})\pmod{p^2}.\end{aligned}$$

It is known (cf. [Sl] and [St]) that

$$(n+1)D_{n+1}=3(2n+1)D_n-nD_{n-1}\quad\text{and}\quad D_{n+1}-3D_n=2nS_n.$$

Thus

$$\begin{aligned}n(D_{n-1}+D_{n+1})=&3(2n+1)D_n-D_{n+1}\\ =&3(2n+1)D_n-(3D_n+2nS_n)=2n(3D_n-S_n)\end{aligned}$$

and hence

$$\sum_{k=0}^n\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\equiv\frac{S_n}{2}(3D_n-S_n)\pmod{p^2}.$$

With helps of (1.9) and (3.4), we have

$$\frac{S_n}{2}(3D_n-S_n)\equiv\left(2x-\frac{p}{x}\right)\left(3\left(2x-\frac{p}{2x}\right)-\left(4x-\frac{2p}{x}\right)\right)\pmod{p^2}$$

and hence

$$\sum_{k=0}^n\frac{(2k+1)\binom{2k}{k}^3}{(k+1)^2(-8)^k}\equiv 4x^2-p\pmod{p^2}.$$

Combining this with (3.5) and (3.8), we immediately obtain (1.13). □

4. Proof of Theorem 1.3

Lemma 4.1. *Let p be an odd prime. Then, for any p-adic integer $x\not\equiv 0,-1\pmod p$ we have*

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\left(\frac{-x}{64}\right)^k\equiv\left(\frac{x+1}{p}\right)\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{4k}{2k}\left(\frac{x}{64(x+1)^2}\right)^k\pmod p. \tag{4.1}$$

Proof. Taking $n=(p-1)/2$ in the MacMahon identity (see, e.g., [G, (6.7)])

$$\sum_{k=0}^n\binom{n}{k}^3x^k=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}\binom{n-k}{k}x^k(1+x)^{n-2k}$$

and noting (1.2) and the basic facts

$$\binom{n}{k}\equiv\binom{-1/2}{k}=\frac{\binom{2k}{k}}{(-4)^k}\pmod p$$

and
$$\binom{n-k}{k} \equiv \binom{-1/2-k}{k} = \frac{\binom{4k}{2k}}{(-4)^k} \pmod p,$$
we immediately get (4.1). $\square$

Proof of Theorem 1.3. (i) For $d = 0, 1, 2, \ldots$, we define
$$f(d) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+d}\binom{2k}{k}\binom{3k}{k}}{108^k}, \quad g(d) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+d}\binom{2k}{k}\binom{4k}{2k}}{256^k},$$
and
$$h(d) = \sum_{k=0}^{p-1} \frac{\binom{2k}{k+d}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}.$$
By the Zeilberger algorithm, we find the recursive relations:
$$\begin{aligned}&(3d+1)(3d+2)f(d) - (3d+4)(3d+5)f(d+2)\\ =&\frac{(3p-1)(3p-2)(d+1)}{108^{p-1}p}\binom{2p}{p+d+1}\binom{2p-2}{p-1}\binom{3p-3}{p-1},\end{aligned}$$

$$\begin{aligned}&(4d+1)(4d+3)g(d) - (4d+5)(4d+7)g(d+2)\\ =&\frac{(4p-1)(4p-3)(d+1)}{256^{p-1}p}\binom{2p}{p+d+1}\binom{2p-2}{p-1}\binom{4p-4}{2p-2},\end{aligned}$$
and
$$\begin{aligned}&(6d+1)(6d+5)h(d) - (6d+7)(6d+11)h(d+2)\\ =&\frac{(6p-1)(6p-5)(d+1)}{1728^{p-1}p}\binom{2p}{p+d+1}\binom{3p-3}{p-1}\binom{6p-6}{3p-3}.\end{aligned}$$
Recall that $\binom{2p-2}{p-1} = pC_{p-1} \equiv 0 \pmod p$. Also,
$$\begin{aligned}(3p-2)\binom{3p-3}{p-1} =&p\binom{3p-2}{p} \equiv 0 \pmod p,\\ (4p-3)\binom{4p-4}{2p-2} =&p\binom{4p-2}{2p} \equiv 0 \pmod p,\\ (6p-5)\binom{6p-6}{3p-3} =&\frac{3p(3p-1)(3p-2)}{(6p-3)(6p-4)}\binom{6p-3}{3p} \equiv 0 \pmod p.\end{aligned}$$
If $0 \leqslant d < p-1$, then
$$\binom{2p}{p+d+1} = \binom{2p}{p-1-d} \equiv 0 \pmod p.$$

So, by the above, for any $d \in \{0, \ldots, p-1\}$ we have

$$(3d+1)(3d+2)f(d) \equiv (3d+4)(3d+5)f(d+2) \pmod{p^2}, \tag{4.2}$$

$$(4d+1)(4d+3)g(d) \equiv (4d+5)(4d+7)g(d+2) \pmod{p^2}, \tag{4.3}$$

$$(6d+1)(6d+5)h(d) \equiv (6d+7)(6d+11)h(d+2) \pmod{p^2}. \tag{4.4}$$

Fix $0 \leqslant d \leqslant p-1$. If $d \equiv (1+(\frac{p}{3}))/2 \pmod 2$, then it is easy to verify that $\{3d+1, 3d+2\} \cap \{p, 2p\} = \emptyset$, hence $(3d+1)(3d+2) \not\equiv 0 \pmod p$ and thus by (4.2) we have

$$f(d+2) \equiv 0 \pmod{p^2} \implies f(d) \equiv 0 \pmod{p^2}.$$

If $d \equiv (1+(\frac{-2}{p}))/2 \pmod 2$, then $\{4d+1, 4d+3\} \cap \{p, 3p\} = \emptyset$, hence $(4d+1)(4d+3) \not\equiv 0 \pmod p$ and thus by (4.3) we have

$$g(d+2) \equiv 0 \pmod{p^2} \implies g(d) \equiv 0 \pmod{p^2}.$$

If $d \equiv (1+(\frac{-1}{p}))/2 \pmod 2$, then $\{6d+1, 6d+3\} \cap \{p, 3p, 5p\} = \emptyset$, hence $(6d+1)(6d+3) \not\equiv 0 \pmod p$ and thus (4.4) yields

$$h(d+2) \equiv 0 \pmod{p^2} \implies h(d) \equiv 0 \pmod{p^2}.$$

Since

$$f(p) = f(p+1) = g(p) = g(p+1) = h(p) = h(p+1) = 0,$$

by the above for every $d = p+1, p, \ldots, 0$ we have the desired (1.19)-(1.21).

(ii) Assume that $p \equiv 3 \pmod 8$ and $p = x^2 + 2y^2$ with $x, y \in \mathbb{Z}$. Since $4x^2 \not\equiv 0 \pmod p$ and Mortenson [M] already proved that the squares of both sides of (1.22) are congruent modulo p^2, (1.22) is reduced to its mod p form. Applying (4.1) with $x = 1$ we get

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-64)^k} \equiv \left(\frac{2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{256^k} \pmod p.$$

By [A, Theorem 5(3)], we have

$$\left(\frac{-1}{p}\right) \sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} (-1)^k \equiv 4x^2 - 2p \pmod p,$$

where $n=(p-1)/2$. For $k=0,\ldots,n$ clearly

$$\begin{aligned}\binom{n}{k}^2\binom{n+k}{k}(-1)^k=&\binom{(p-1)/2}{k}^2\binom{-(p+1)/2}{k}\\ \equiv&\binom{-1/2}{k}^3=\frac{\binom{2k}{k}^3}{(-64)^k}\pmod p,\end{aligned}$$

therefore

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-64)^k}\equiv\left(\frac{-1}{p}\right)(4x^2-2p)\pmod p$$

and hence (1.22) follows.

(iii) Finally we suppose $p\equiv 5\pmod{12}$ and write $p=x^2+y^2$ with x odd and y even. Once again it suffices to show the mod p form of (1.23) in view of Mortenson's work [M]. As Z. H. Sun observed,

$$\binom{(p-5)/6+k}{2k}\binom{2k}{k}\equiv\binom{k-5/6}{2k}\binom{2k}{k}=\frac{\binom{3k}{k}\binom{6k}{3k}}{(-432)^k}\pmod p$$

for all $k=0,1,2,\ldots$. If $p/6<k<p/3$ then $p\mid\binom{6k}{3k}$; if $p/3<k<p/2$ then $p\mid\binom{3k}{k}$; if $p/2<k<p$ then $p\mid\binom{2k}{k}$. Thus

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{12^{3k}}\equiv&\sum_{k=0}^{(p-5)/6}\binom{(p-5)/6+k}{2k}\binom{2k}{k}^2\left(-\frac14\right)^k\\ =&D_{2n}\left(-\frac12\right)^2\pmod p\quad(\text{by }(1.3)),\end{aligned}$$

where $n=(p-5)/12$. Note that

$$D_{2n}\left(-\frac12\right)=\frac{1}{(-4)^n}\binom{2n}{n}$$

by [G, (3.133) and (3.135)], and

$$\binom{(p-1)/2}{(p-1)/4}\equiv 12(-432)^n\binom{2n}{n}\pmod p$$

by P. Morton [Mo]. Therefore

$$D_{2n}\left(-\frac12\right)^2=\frac{1}{16^n}\binom{2n}{n}^2\equiv\frac{\binom{(p-1)/2}{(p-1)/4}^2}{12^{6n+2}}\equiv\left(\frac{12}{p}\right)\binom{(p-1)/2}{(p-1)/4}^2\pmod p.$$

Thus, by applying Gauss' congruence $\binom{(p-1)/2}{(p-1)/4}\equiv 2x\pmod p$ (cf. [BEW, (9.0.1)] or [HW]) we immediately get the mod p form of (1.23) from the above.

The proof of Theorem 1.3 is now complete. □

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