Generalized Compositions of Natural Numbers

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Abstract

We consider compositions of natural numbers when there are different types of each natural number. Several recursions as well as some closed formulas for the number of compositions is derived. We also find its relationships with some known classes of integers such as Fibonacci, Catalan, Pell, Pell-Lucas, and Jacobsthal numbers.

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1 Introduction

Let $\mathbf{b} = (b_1, b_2, \ldots)$ be a sequence of nonnegative integers. Compositions of n in which there are b_1 different type of 1's, b_2 different type of 2's, and so on, will be called generalized compositions of n. We let $c(n, \mathbf{b})$ denote its number. We may considered these compositions as colored compositions in which each number i may be colored by one of b_i colors. If all b_i are equal 1 then the standard compositions are obtained.

It is clear that the following recursion for $c(n, \mathbf{b})$ holds

$$c(n, \mathbf{b}) = b_1 c(n-1, \mathbf{b}) + b_2 c(n-2, \mathbf{b}) + \cdots + b_{n-1} c(1, \mathbf{b}) + \cdots + b_n,$$

having b_1 generalized compositions ending by one of 1's, b_2 generalized compositions ending by one of 2's, and so on. At the end, there are b_n generalized compositions consisting of one of n's.

We define the sequence $(a_1, a_2, \ldots)$ such that $a_1 = 1$, and

$$a_{n+1} = \sum_{i=1}^{n} b_{n+1-i} a_i. \tag{1}$$

It is clear that

$$a_{n+1} = c(n, \mathbf{b}),\ (n = 1, 2, \ldots).$$

Equation (1) connects two sequences of nonnegative integers

$$(b_1, b_2, \ldots),\ \text{and}\ (c_1, c_2, \ldots),$$

where $c_i = a_{i+1},\ (i = 1, 2, \ldots)$.

Obviously, for each sequence $(b_1, b_2, \ldots)$ we may form the sequence $(c_1, c_2, \ldots)$. Conversely is not true. Namely, equation (1) may be regarded as a recurrence relation with respect to b's, but it does not ultimately produce nonnegative integers.

The paper is organized as follows. In this section we find a simple but interesting connection of generalized compositions with Catalan numbers.

In Section 2 we consider the case when b's make an arithmetical progression. We shall prove that then the numbers $c(n, \mathbf{b})$ satisfy a three terms homogenous recursion with constant coefficients. This means that a close formula for generalized compositions may be obtained. In a particular case the number of generalized composition is a Pell-Lucas number.

In Section 3 we consider the case when b_i is a square function of i. Then the numbers $c(n, \mathbf{b})$ satisfy a four terms homogenous recursion with constant coefficients. Thus, in this case also we may derive an explicit formula for generalized compositions. Special attention is put on triangular numbers. Several results will be obtained in the case when b's are triangular numbers. Then the a's are sums of binomial coefficients. Some identities, concerning sums of binomial coefficients, will be derived by the the use of Zeilberger's algorithm, which is described by Petkovsek and all., in [3].

In Section 4 we investigate the case when b_i is an exponential function of i. In Section 5 two result concerning the floor and the ceil functions will be proved.

We shall see, in Section 6, that the generalized compositions are closely related with Fibonacci numbers, as is the case with the standard compositions. Several recurrence relations as well as some closed formulas for generalized compositions will be proved. New relationships of Fibonacci numbers with Pell, Jacobsthal and other classes of numbers are derived.

Note that there is a significant number of sequences in Sloane's OEIS, [4], which terms equal the number of generalized compositions. Comment of these sequences in OEIS offer other interpretations of compositions. Sequence A145839 connects generalized compositions with so called matrix compositions. Also A020729, A008776, A020698, A007484 connect them with Pisot sequences.

Proposition 1. *If* $\mathbf{b} = (p, p, \ldots)$, *then*

$$c(n, \mathbf{b}) = p(1 + p)^{n-1}.$$

Particulary, 2^{n-1} *is the number of all compositions of* n.

Proof. In this case the recurrence (1) becomes.

$$a_{n+1} = p \sum_{i=1}^{n} a_i.$$

Replacing n by $n + 1$ yields

$$a_{n+2} = p \sum_{i=1}^{n+1} a_i.$$

By substraction we obtain

$$a_{n+2} = (1+p)a_{n+1}.$$

From this equation the assertion follows easily.

$\square$

The next result shows that Catalan numbers give an example when b's produce b's again.

Proposition 2. *If* $\mathbf{b} = (C_0, C_1, \ldots)$, *where* $C_i,\ (i = 0, 1, \ldots)$ *are Catalan numbers, then*

$$c(n, \mathbf{b}) = C_n,\ (n \geq 1).$$

Proof. In this case equation (1) has the form:

$$a_{n+1} = \sum_{i=1}^{n} C_{n-i} a_i = \sum_{i=0}^{n-1} C_{n-i-1} a_{i+1}.$$

The equation $a_{n+1} = C_n$ follows by induction, using the well-known Segner's recurrence formula for Catalan numbers. $\square$

2 Arithmetic Progressions

In this section we consider the case when b_i is a linear function of i, that is, when b's make an arithmetic progression.

We shall prove that then the numbers of generalized compositions satisfy a three terms recursion with constant coefficients. In this way the explicit formula for the number of compositions may be obtained.

Proposition 3. *Let* n *be a positive integer, let* $m,\ k$ *be nonnegative integers, and let* $b_i = m(i-1) + k,\ (i = 1, 2, \ldots)$. *Then*

$$c(1, \mathbf{b}) = k,\ c(2, \mathbf{b}) = m + k + k^2,$$

$$c(n+1, \mathbf{b}) = (k+2)c(n, \mathbf{b}) + (m-k-1)c(n-1, \mathbf{b}).$$

Proof. Equation (1) takes the form:

$$a_{n+1} = \sum_{i=1}^{n} [m(n-i) + k] a_i.$$

It is easy to see that

$$a_2 = k,\ a_3 = m + k + k^2.$$

Further, for $n > 2$ we have

$$a_{n+1} = ka_n + \sum_{i=1}^{n-1} [m(n-i) + k] a_i = ka_n + (m+k)a_{n-1} + \sum_{i=1}^{n-2} [m(n-1-i) + k] a_i + m \sum_{i=1}^{n-2} a_i.$$

We conclude that

$$a_{n+1} = (k+1)a_n + ma_{n-1} + m\sum_{i=1}^{n-2} a_i. \tag{2}$$

Replacing n by $n+1$ yields

$$a_{n+2} = (k+1)a_{n+1} + ma_n + m\sum_{i=1}^{n-1} a_i. \tag{3}$$

Subtracting equation (2) from (3) we obtain

$$a_{n+2} = (2+k)a_{n+1} + (m-k-1)a_n.$$

□

In the next corollary we give some particular cases.

Corollary 4. *(i) If* $m = 1,\ k = 0$ *then*

$$c(1,\mathbf{b}) = 0,\ c(n,\mathbf{b}) = 2^{n-2},\ (n > 1).$$

(ii) If $m = 2,\ k = 0$ *then*

$$c(n,\mathbf{b}) = 2\sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2i} 2^i.$$

(iii) If $m = 1,\ k = 1$ *then*

$$c(n,\mathbf{b}) = F_{2n}.$$

(iv) If $k = m-1$ *then*

$$c(1,\mathbf{b}) = m-1,\ c(n,\mathbf{b}) = m^2 \cdot (m+1)^{n-2},\ (n > 1).$$

Proof. (i) is obvious.

In the case (ii) the recurrence equation takes the form

$$c(1,\mathbf{b}) = 0,\ c(2,\mathbf{b}) = 2,$$

$$c(n+1,\mathbf{b}) = 2c(n,\mathbf{b}) + c(n-1,\mathbf{b}),$$

which is the recurrence for Pell-Lucas numbers.

In the case (iii) the recurrence becomes

$$c(1,\mathbf{b}) = 1,\ c(2,\mathbf{b}) = 3,$$

$$c(n+1,\mathbf{b}) = 3c(n,\mathbf{b}) - c(n-1,\mathbf{b}).$$

This is the recurrence equation for Fibonacci numbers with even indices by Identity 7 from [1].

Finally, for $k=m-1$ we have

$$c(1,\mathbf{b})=m-1,\ c(2,\mathbf{b})=m^2,$$

$$c(n+1,\mathbf{b})=(m+1)c(n,\mathbf{b}),\ (n>2),$$

and (iv) is true. □

Remark 5. The formulas from the preceding corollary generate the following sequences from OEIS.

$m=1,\ k=0$, A000079; $m=2,\ k=0$, Pell-Lucas numbers, A052542;
$m=1,\ k=1$, A001906.

In the case $k=m-1$ we have
$m=2$, A003946; $m=3$, A055841; $m=4$, A055842;
$m=5$, A055846; $m=6$, A055270; $m=7$, A055847;
$m=8$, A055995; $m=9$, A055996; $m=10$, A056002;
$m=11$, A056116.

3 Square Functions

In the case that b_i is a square function od i we obtain the four terms recurrence relation for $c(n,\mathbf{b})$. This means that we may obtained a closed formula for generalized compositions.

Proposition 6. *Let n be a positive integer, and let k,m,p be arbitrary (rational) numbers such that $b_i=ki^2+mi+p,\ (i=1,2,\ldots)$ are nonnegative integers. Then*

$$c(1,\mathbf{b})=k+m+p,\ c(2,\mathbf{b})=k^2+m^2+p^2+2(km+mp+kp)+4k+2m+p,$$

$$c(3,\mathbf{b})=8k+3m+p+2(4k^2+2m^2+p^2+6km+5kp+3mp)+$$
$$+k^3+m^3+p^3+3(km^2+kp^2+mp^2+2kmp).$$

$$c(n+1,\mathbf{b})=(k+m+p+3)c(n,\mathbf{b})+(k-m-2p-3)c(n-1,\mathbf{b})+(p+1)c(n-2,\mathbf{b}),\ (n\geq 3).$$

Proof. We have

$$a_{n+1}=\sum_{i=1}^{n}\big[k(n-i+1)^2+m(n-i+1)+p\big]a_i.$$

It is easy to obtain the values a_2,a_3 and a_4. For $n>3$ we have

$$a_{n+1}=(k+p+m)a_n+\sum_{i=1}^{n-1}\big[k(n-i+1)^2+m(n-i+1)+p\big]a_i.$$

It follows that

$$a_{n+1}=(k+m+p+1)a_n+(3k+m)a_{n-1}+\sum_{i=1}^{n-2}\big[k(2n-2i+1)+m\big]a_i. \tag{4}$$

Replacing n by $n+1$ we obtain

$$a_{n+2} = (k+m+p+1)a_{n+1} + (3k+m)a_n + \sum_{i=1}^{n-1} \big[k(2n-2i+1)+m\big]a_i + 2k\sum_{i=1}^{n-1} a_i. \quad (5)$$

Subtracting (4) from (5) yields

$$a_{n+2} = (k+m+p+2)a_{n+1} + (2k-p-1)a_n + 2k\sum_{i=1}^{n-1} a_i. \quad (6)$$

Replacing n by $n+1$ we obtain

$$a_{n+3} = (k+m+p+2)a_{n+2} + (2k-p-1)a_{n+1} + 2k\sum_{i=1}^{n} a_i. \quad (7)$$

Finally, subtracting (6) from (7) yields

$$a_{n+3} = (k+m+p+3)a_{n+2} + (k-m-2p-3)a_{n+1} + (p+1)a_n.$$

□

In the next corollary we give two particular cases.

Corollary 7. *(i) If $k=1, m=0, p=-1$ then*

$$c(1,\mathbf{b}) = 0, c(2,\mathbf{b}) = 3,$$

$$c(n,\mathbf{b}) = 8\cdot 3^{n-3},\ (n\geq 3).$$

(ii) If $k=1, m=1, p=-1$ then

$$c(1,\mathbf{b}) = 1, c(2,\mathbf{b}) = 6, c(3,\mathbf{b}) = 22,$$

$$c(n,\mathbf{b}) = \frac{9-5\sqrt{3}}{6}(2+\sqrt{3})^n + \frac{9+5\sqrt{3}}{6}(2-\sqrt{3})^n.$$

Proof. The assertion (i) is true since, in this case, the recurrence equation becomes

$$c(n+1,\mathbf{b}) = 3c(n,\mathbf{b}),\ (n\geq 2).$$

In the case (ii) the recurrence takes the form:

$$c(n+1,\mathbf{b}) = 4c(n,\mathbf{b}) - c(n-1,\mathbf{b}),\ (n\geq 3).$$

Solving the characteristic equation of this three terms recurrence equation we conclude that the assertion is true. □

Remark 8. We state two sequences from OEIS generated by the preceding formulas. $k = 1,\ m = 0,\ p = -1$, A118264, $k = 1,\ m = 1,\ p = -1$, A003699.

Since $\binom{n}{2}$ is a square function of n we may derive from the preceding proposition some formulas which connect triangular numbers with generalized compositions.

Corollary 9. *(i) If* $b_i = \binom{i-2}{2},\ (i = 1, 2, \ldots)$ *then*

$$c(1, \mathbf{b}) = 1, c(2, \mathbf{b}) = 1, c(3, \mathbf{b}) = 1,$$

$$c(n + 1, \mathbf{b}) = 4c(n, \mathbf{b}) - 6c(n - 1, \mathbf{b}) + 4c(n - 2, \mathbf{b}),\ (n \geq 3).$$

Explicitly,

$$c(n, \mathbf{b}) = \sum_{i=0}^{n} \binom{n}{4n - 4i}.$$

(ii) If $b_i = \binom{i-1}{2},\ (i = 1, 2, \ldots)$ *then*

$$c(1, \mathbf{b}) = 0, c(2, \mathbf{b}) = 0, c(3, \mathbf{b}) = 1,$$

$$c(n + 1, \mathbf{b}) = 3c(n, \mathbf{b}) - 3c(n - 1, \mathbf{b}) + 2c(n - 2, \mathbf{b}),\ (n \geq 3).$$

Explicitly,

$$c(n, \mathbf{b}) = \sum_{i=0}^{\lfloor \frac{n-3}{3} \rfloor} \binom{n - 1}{3i + 2}.$$

(iii) If $b_i = \binom{i}{2},\ (i = 1, 2, \ldots)$ *then*

$$c(1, \mathbf{b}) = 0, c(2, \mathbf{b}) = 1, c(3, \mathbf{b}) = 3,$$

$$c(n + 1, \mathbf{b}) = 3c(n, \mathbf{b}) - 2c(n - 1, \mathbf{b}) + c(n - 2, \mathbf{b}),\ (n \geq 3).$$

Explicitly,

$$c(n, \mathbf{b}) = \sum_{i=0}^{n} \binom{n + i}{3i + 2}.$$

(iv) If $b_i = \binom{i+1}{2},\ (i = 1, 2, \ldots)$ *then*

$$c(1, \mathbf{b}) = 1, c(2, \mathbf{b}) = 4, c(3, \mathbf{b}) = 13,$$

$$c(n + 1, \mathbf{b}) = 4c(n, \mathbf{b}) - 3c(n - 1, \mathbf{b}) + c(n - 2, \mathbf{b}),\ (n \geq 3).$$

Explicitly,

$$c(n, \mathbf{b}) = \sum_{i=0}^{n} \binom{n + 2i - 1}{n - i}.$$

Proof. The assertion (i) is obtained for $k = \frac{1}{2},\ m = -\frac{5}{2},\ p = 3$.

The assertion (ii) is obtained for $k = \frac{1}{2},\ m = -\frac{3}{2},\ p = 1$.

The assertion (iii) is obtained for $k = \frac{1}{2},\ m = -\frac{1}{2},\ p = 0$.

The assertion (iv) is obtained for $k = \frac{1}{2},\ m = \frac{1}{2},\ p = 0$.

The explicit formulas are obtained by the use of Zeilberger's algorithm, [3]. □

Remark 10. In the case $b_i = \binom{i+2}{2},\ (i = 1, 2, \ldots)$ we obtain A145839 which counts the number of 3-compositions of n. This connects our compositions with the so called matrix compositions.

The following sequences in OEIS are generated by the preceding formulas.

(i), A038503; (ii), A024495; (iii), A095263; (iv), A095263.

4 Exponential Functions

The following result concerns the case when b_i is an exponential functions of i. Then, again, the numbers $c(n, \mathbf{b})$ satisfy a three terms homogeneous recurrence relation with constant coefficients.

Proposition 11. *Let n be a positive integer. If $b_i = k + pm^{i-1},\ (i = 1, 2, \ldots)$, then*

$$c(1, \mathbf{b}) = k + p,\ c(2, \mathbf{b}) = k + pm + (k + p)^2,$$

$$c(n, \mathbf{b}) = (k + m + p + 1)c(n - 1, \mathbf{b}) - (km + m + p)c(n - 1, \mathbf{b}),\ (n > 2).$$

Proof. Equation (1) has the form:

$$a_{n+1} = \sum_{i=1}^{n} (k + pm^{n-i})a_i.$$

It follows that

$$a_2 = k + p,\ a_3 = k + pm + (k + p)^2.$$

Further we have

$$a_{n+1} = (k + p)a_n + \sum_{i=1}^{n-1} (k + pm^{n-i})a_i = (k + p)a_n + k \sum_{i=1}^{n-1} a_i + p \sum_{i=1}^{n-1} m^{n-i} a_i.$$

Hence,

$$a_{n+1} = (k + p)a_n + (k + p)a_{n-1} + k \sum_{i=1}^{n-2} a_i + pm \sum_{i=1}^{n-2} m^{n-1-i} a_i + p(m - 1)a_{n-1},$$

that is,

$$a_{n+1} = (k + p + 1)a_n + p(m - 1)a_{n-1} + p(m - 1) \sum_{i=1}^{n-2} m^{n-1-i} a_i. \tag{8}$$

Replacing n by $n+1$ we obtain

$$a_{n+2} = (k+p+1)a_{n+1} + p(m-1)a_n + pm(m-1)\sum_{i=1}^{n-1} m^{n-1-i}a_i. \tag{9}$$

Subtracting (8) multiplied by m from (9) we obtain

$$a_{n+2} = (k+p+m+1)a_{n+1} - (p+m+mk)a_n.$$

□

Some particular cases of the preceding proposition follow.

Corollary 12. *(i) If $k=0$ then*

$$c(n,\mathbf{b}) = p(m+p)^{n-1}.$$

(ii) If $k=1, m=2, p=1$ then

$$c(1,\mathbf{b}),\ c(2,\mathbf{b}) = 7,$$

$$c(n,\mathbf{b}) = 5c(n-1,\mathbf{b}) - 5c(n-2,\mathbf{b}),\ (n>2).$$

(iii) If $k=-1, m=2, p=1$ then

$$c(n,\mathbf{b}) = F_{2n-2}.$$

Proof. In the case $k=0$ we have

$$c(1,\mathbf{b}) = p,\ c(2,\mathbf{b}) = p(m+p),$$

$$c(n,\mathbf{b}) = (m+p+1)c(n-1,\mathbf{b}) - (m+p)c(n-1,\mathbf{b}),\ (n>2).$$

The roots of the characteristic equation are $\alpha=1,\ \beta=m+p$. Solving the system

$$c_1\alpha + c_2\beta = p,\ c_1\alpha^2 + c_2\beta^2 = p(m+p),$$

yields $c_1 = 0,\ c_2 = \frac{p}{m+p}$, and the assertion (i) is true.
In the case (ii) we have

$$c(1,\mathbf{b}) = 2,\ c(2,\mathbf{b}) = 7,$$

$$c(n,\mathbf{b}) = 5c(n-1,\mathbf{b}) - 5c(n-2,\mathbf{b}),\ (n>2).$$

Finally, in the case (iii) we have

$$c(1,\mathbf{b}) = 0,\ c(2,\mathbf{b}) = 1,$$

$$c(n,\mathbf{b}) = 3c(n-1,\mathbf{b}) - c(n-2,\mathbf{b}),\ (n>2).$$

The assertion follows from Identity 7 in [1]. □

Remark 13. We state sequences in OEIS defined with $k = 0$, $p = 1$, and m ranges from 2 to 39.
A000244, A000302, A000351, A000400, A000420, A001018, A001019, A011557, A001020, A001021, A001022. A001023, A001024, A001025, A001026, A001027, A001029, A009964, A009965, A009966, A009967, A009968, A009969, A009970, A009971, A009972, A009973, A009974, A009975, A009976, A009977, A009978, A009979, A009980, A009981, A009982, A009983, A009984.

More sequences follow

k=0, m=2, p=3, A005053;	k=0, m=2, p=2, A081294;	k=1, m=2, p=1, A052936;
k=1, m=3, p=1, A034999;	k=0, m=2, p=4, A067411;	k=0, m=3, p=2, A020729;
k=0, m=4, p=2, A167747;	k=0, m=4, p=3, A169634;	k=0, m=4, p=5, A067403;
k=0, m=4, p=6, A090019;	k=0, m=5, p=2, A109808;	k=0, m=5, p=3, A103333;
k=0, m=6, p=2, A013730;	k=0, m=6, p=3, A013708;	k=0, m=6, p=4, A093141;
k=2, m=2, p=1, A163606;	k=-1, m=2, p=1, A001906;	k=-2, m=2, p=1, A001333;
k=-1, m=3, p=1, A052530.		

5 Floor and Ceil Functions

In this section we derive two results when b_i is a floor, and a ceil function of i.

Proposition 14. *Let n be a positive integer, and $b_i = \lfloor \frac{i}{2} \rfloor$, $(i = 1, 2, \ldots)$. Then*

$$c(1, \mathbf{b}) = 0,\ c(2, \mathbf{b}) = 1,\ c(3, \mathbf{b}) = 1,$$

$$c(n, \mathbf{b}) = c(n-1, \mathbf{b}) + 2c(n-2, \mathbf{b}) - c(n-3, \mathbf{b}),\ (n > 3).$$

Proof. It is easy to see that

$$a_2 = 0,\ a_3 = a_4 = 1.$$

For $n > 3$ we have

$$a_{n+1} = \sum_{i=1}^{n-1} \left\lfloor \frac{n-i+1}{2} \right\rfloor a_i = a_{n-1} + \sum_{i=1}^{n-2} \left\lfloor \frac{n-i+1}{2} \right\rfloor a_i =$$

$$= a_n + a_{n-1} + \sum_{i=1}^{n-2} \left\{ \left\lfloor \frac{n-i+1}{2} \right\rfloor - \left\lfloor \frac{n-1-i+1}{2} \right\rfloor \right\} a_i.$$

It follows that

$$a_{n+1} = a_n + a_{n-1} + \sum_{i=1}^{n-2} \frac{(-1)^{n-i+1}+1}{2} a_i. \tag{10}$$

Replacing n by $n+1$ we obtain

$$a_{n+2} = a_{n+1} + a_n - \sum_{i=1}^{n-1} \frac{(-1)^{n-i+1}+1}{2} a_i + \sum_{i=1}^{n-1} a_i. \tag{11}$$

Substituting (10) into (11) yields

$$a_{n+2} = 2a_n + \sum_{i=1}^{n-1} a_i. \tag{12}$$

Replacing n by $n+1$ we have

$$a_{n+3} = 2a_{n+1} + \sum_{i=1}^{n} a_i. \tag{13}$$

Subtracting (12) from (13) yields

$$a_{n+3} = a_{n+2} + 2a_{n+1} - a_n.$$

$\square$

Remark 15. Sequence A006053 is generated by this function.

In a similar way the following proposition may be proved:

Proposition 16. *Let n be a positive integer, and $b_i = \lceil \frac{i}{2} \rceil$, $(i = 1, 2, \ldots)$. Then*

$$c(1, \mathbf{b}) = 1,\ c(2, \mathbf{b}) = 2,\ c(3, \mathbf{b}) = 5,$$

$$c(n, \mathbf{b}) = 2c(n-1, \mathbf{b}) + c(n-2, \mathbf{b}) - c(n-3, \mathbf{b}),\ (n > 3).$$

Remark 17. Sequence A006054 is generated by this function.

6 Fibonacci numbers

In this section we prove several formulas in which the number of compositions is related with Fibonacci numbers.

Our first result extends the result from [2], where compositions with two different types of 1 are considered, as well as some other known results about standard compositions.

Proposition 18. *Let n be a positive integer, let p, q be a nonnegative integers, and let $\mathbf{b} = (p, q, q, \ldots)$. Then*

$$c(1, \mathbf{b}) = p,\ c(2, \mathbf{b}) = p^2 + q,$$

$$c(n, \mathbf{b}) = (1+p)c(n-1, \mathbf{b}) + (q-p)c(n-2, \mathbf{b}),\ (n > 2). \tag{14}$$

Explicitly,

$$c(n, \mathbf{b}) = u\alpha^n + v\beta^n,$$

where

$$\alpha = \frac{1 + p + \sqrt{(p-1)^2 + 4q}}{2},\ \beta = \frac{1 + p - \sqrt{(p-1)^2 + 4q}}{2},$$

$$u = \frac{4q - (p-1)^2 + (p-1)\sqrt{(p-1)^2 + 4q}}{2\sqrt{(p-1)^2 + 4q}},$$
$$v = \frac{4q + (p-1)^2 - (p-1)\sqrt{(p-1)^2 + 4q}}{2\sqrt{(p-1)^2 + 4q}}.$$

Proof. In this case equation (1) has the form:

$$a_{n+1} = pa_n + q\sum_{i=1}^{n-1} a_i.$$

We easily obtain that

$$a_2 = p,\ a_3 = p^2 + q.$$

Next we have

$$a_{n+1} = pa_n + qa_{n-1} - pa_{n-1} + pa_{n-1} + q\sum_{i=1}^{n-2} a_i =$$
$$= (1+p)a_n + (q-p)a_{n-1}.$$

The characteristic equation for this recurrence is $x^2 - (p+1)x - q + p = 0$. Solving this equation we obtain the explicit formula. □

In the following corollary we shall state some particular cases of this proposition. The first is the well-known formula for the number of all standard compositions. In the rest Fibonacci numbers are produced.

Corollary 19. *(i) If* $\mathbf{b} = (0, 1, 1, \ldots)$ *then* $c(n, \mathbf{b}) = F_{n-1}$. *This is the well-known result that says that there are* F_{n-1} *compositions of* n *in which each part is* ≥ 2.

(ii) If $\mathbf{b} = (2, 1, 1, \ldots)$ *then* $c(n, \mathbf{b}) = F_{2n+1}$. *This is the result from [2].*

(iii) If $\mathbf{b} = (3, 4, 4, \ldots)$ *then*

$$c(n, \mathbf{b}) = F_{3n+1}.$$

Proof. (i). In this case we have $p = 0,\ q = 1$. It follows that

$$\alpha = \frac{1+\sqrt{5}}{2},\ \beta = \frac{1-\sqrt{5}}{2}, u = \frac{5-\sqrt{5}}{10},\ v = \frac{5+\sqrt{5}}{10},$$

and the assertion follows from Binet formula.

(ii). In this case we have $p = 2,\ q = 1$ and the recurrence relation has the form

$$a_2 = 2,\ a_3 = 5,\ a_{n+1} = 3a_n - a_{n-1}.$$

The assertion follows by induction using Identity 17 from [1].

(iii). The recurrence equation in this case has the form

$$a_{n+1} = 4a_n + a_{n-1}.$$

It is easy to prove that for Fibonacci numbers the following identity holds

$$F_{k+2} = 4F_{k-1} + F_{k-4}.$$

Using induction and this identity we conclude that the assertion holds. $\square$

Remark 20. We state several sequences from OEIS which are generated by (14).

p=2, q=1, A001519;	p=3, q=1 A007052;	p=4, q=1, A018902;
p=5, q=1, A018903;	p=6, q=1, A018904;	p=1, q=2, A001333;
p=1, q=3, A026150;	p=1, q=4, A046717;	p=1, q=5, A084057;
p=1, q=6, A002533;	p=1, q=7, A083098;	p=1, q=8, A083100;
p=1 ,q=9, A003665;	p=1, q=10, A002535;	p=1, q=11, A083101;
p=1, q=12, A090042;	p=1, q=13, A125816;	p=1, q=14, A133343;
p=1, q=15, A133345;	p=1, q=16, A120612;	p=1, q=17, A133356;
p=1, q=18, A125818;	p=2, q=3, A052924;	p=2, q=4, A104934;
p=2, q=6, A122117;	p=3, q=2, A001835;	p=3, q=4, A033887;
p=3, q=6, A122558;	p=3, q=8, A083217;	p=3, q=9, A147518;
p=4, q=2, A052913;	p=4, q=3, A004253;	p=4, q=5, A100237;
p=5, q=2, A158869;	p=5, q=4, A001653.	

The next result also generalizes a classical result for standard compositions.

Proposition 21. *If* $\mathbf{b} = (p, 1, 0, 0, \ldots)$ *where* m *is a positive integer then*

$$c(n, \mathbf{b}) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-i}{i} p^{n-2i}.$$

Proof. The recurrence relation for Fibonacci polynomials is

$$F_{n+1}(x) = xF_n(x) + F_{n-1}(x).$$

It follows that for a positive integer p we have $F_{n+1}(p) = c(n, \mathbf{b})$. The required equation follows from the well-known formula for Fibonacci polynomials.

$\square$

As an immediate consequence we obtain the following well-known result.

Corollary 22. *The number of compositions of* n *in which each part is either* 1 *or* 2 *is* F_{n+1}.

Proof. Take $p = 1$ in the preceding proposition and apply Identity 4 in [1]. $\square$

Remark 23. As before, we state a few sequences from OEIS obtained for different values of p.

p=3, A000129. Pell numbers,	p=4, A006190,	p=5, A001076,
p=6, A052918,	p=7, A054413.	

The following result also extends a well-known result for standard compositions.

Proposition 24. *Let n be a positive integer, let $p,\ q$ be nonnegative integers, and let $\mathbf{b} = (p, q, p, q, \ldots)$. Then*

$$c(1, \mathbf{b}) = p,\ c(2, \mathbf{b}) = p^2 + q,$$

$$c(n, \mathbf{b}) = pc(n-1, \mathbf{b}) + (1+q)c(n-2, \mathbf{b}),\ (n > 2).$$

Proof. In this case we first have

$$a_{2n+1} = \sum_{i=1}^{2n} b_{2n-i+1}a_i.$$

Hence,

$$a_{2m+1} = q(a_1 + a_3 + \cdots + a_{2m-1}) + p(a_2 + a_4 + \cdots + a_{2m}) =$$

$$= qa_{2m-1} + pa_{2m} + a_{2m-1} = pa_{2m} + (1+q)a_{2m-1},$$

and the assertion is true for even n. Also,

$$a_{2m} = \sum_{i=1}^{2m-1} b_{2m-i}a_i,$$

that is

$$a_{2m} = p(a_1 + a_3 + \cdots + a_{2m-1}) + q(a_2 + a_4 + \cdots + a_{2m-2}) =$$

$$= pa_{2m-1} + qa_{2m-2} + a_{2m-2} = pa_{2m-1} + (1+q)a_{2m-2}.$$

Hence, the assertion is also true for odd n. □

Corollary 25. *Let n be a positive integer, and let $\mathbf{b} = (1, 0, 1, 0, \ldots)$. Then*

$$c(n, \mathbf{b}) = F_n.$$

In other word, F_n is the number of compositions of n in which all parts are odd.

Proof. Since $p = 1,\ q = 0$ the recurrence from the preceding propositions becomes recurrence relation for Fibonacci numbers. □

Remark 26. The following sequences from OEIS are generated by the formula from this proposition.

p=1, q=2, A105476, p=2,q=1, A052945, p=2,q=3, A162770.

In the rest of this section Fibonacci numbers play the role of the b's.

We shall prove that there are a closed formula for $c(n, \mathbf{b})$ in the case when

$$b_i = F_{m+k(i-1)},\ (i = 1, 2, \ldots, n)$$

where $m \geq -1$ and $k \geq 0$ are arbitrary integers. For this we need the following identities for Fibonacci numbers.

Lemma 27. *Let $m \geq -1,\ k \geq 0$ be integers. Then*

$$F_{m+2k} + F_{m-2k} = F_m(F_{2k-1} + F_{2k+1}). \tag{15}$$

Also,

$$F_{m+2k-1} - F_{m-2k+1} = F_m(F_{2k-2} + F_{2k}). \tag{16}$$

Proof. The assertion (15) is obviously true for $m = 0$. Since $F_{-(2k-1)} = F_{2k-1}$ it is also true for $m = 1$ and $m = -1$. Assume that it is true for m_1 such that $0 \leq m_1 < m$. Then, for $m \geq 2$ we have

$$F_{m+2k} + F_{m-2k} = F_{m-1+2k} + F_{m-1-2k} + F_{m-2+2k} + F_{m-2-2k}.$$

Using the induction hypothesis yields

$$F_{m+2k} + F_{m-2k} = (F_{m-1} + F_{m-2})(F_{2k-1} + F_{2k+1}) = F_m(F_{2k-1} + F_{2k+1}).$$

The assertion (16) may be proved in a similar way. $\square$

Proposition 28. *Let $m \geq -1$ be an integer, let k be a nonnegative integer, and let $b_i = F_{m+k(i-1)},\ (i = 1, 2, \ldots, n)$. Then,*

$$c(1, \mathbf{b}) = F_m,\ c(2, \mathbf{b}) = F_{m+k} + F_m^2,$$

$$c(n+1, \mathbf{b}) = (F_m + F_{k-1} + F_{k+1})c(n, \mathbf{b}) + (-1)^{k-1}(F_{m-k} + 1)c(n-1, \mathbf{b}),\ (n > 1).$$

Proof. We have

$$a_1 = 1,\ a_{n+1} = \sum_{i=1}^{n} F_{m+k(n-i)}a_i.$$

For $n = 1, 2$ we easily obtain

$$a_2 = F_m,\ a_3 = F_{m+k} + F_m^2.$$

Assume that k is even and denote $k = 2p$. Then for $n > 2$ we obtain

$$a_{n+1} = \sum_{i=1}^{n} F_{m+2p(n-i)}a_i.$$

Using (15) yields

$$(F_{2p-1} + F_{2p+1})a_{n+1} = \sum_{i=1}^{n} F_{m+2p(n+1-i)}a_i + \sum_{i=1}^{n} F_{m+2p(n-1-i)}a_i =$$

$$= a_{n+2} - F_m a_{n+1} + a_n + F_{m-2k}a_n,$$

and the assertion holds.

If k is odd and $k = 2p - 1$, then for $n > 2$ we have

$$a_{n+1} = \sum_{i=1}^{n} F_{m+(2p-1)(n-i)} a_i.$$

Using (16) yields

$$(F_{2p-2} + F_{2p})a_{n+1} = \sum_{i=1}^{n} F_{m+(2p-1)(n+1-i)} a_i - \sum_{i=1}^{n} F_{m+(2p-1)(n-1-i)} a_i =$$

$$= a_{n+2} - F_m a_{n+1} - a_n - F_{m-2p+1} a_n,$$

and the assertion also holds in this case. □

Particulary, for $k = 0$ we have

Corollary 29. *Let $m \geq -1$ be an integer, and let $\mathbf{b} = (F_m, F_m, \ldots)$. Then*

$$c(n, \mathbf{b}) = F_m(1 + F_m)^{n-1}, \ (n = 1, 2, \ldots).$$

The preceding equation generalizes the formula for the number of all standard composition of n which is obtained for $m = 1$.

Remark 30. The preceding formula generates the following sequences in OEIS
m=1 or m=2, A000079, m=3, A008776, m=4, A002001
m=5, A052934, m=6, A055275.

Remark 31. We again state some sequences from OEIS generated with the formula from the preceding proposition.
m=0, k=1 A001045 (Jacobsthal numbers) m=1, k=1 A000129, (Pell numbers),
m=2, k=1, A028859; m=3, k=1, A007484; m=-1, k=2, A007051;
m=0, k=2, A000244; m=1, k=2, A007052; m=2, k=2, A001353;
m=3, k=2, A020698; m=-1, k=3, A147722; m=1, k=3, A005054;
m=3, k=3, A078469.

For our last result we need the following forth terms recursion for squares of Fibonacci numbers:

Lemma 32. *The following equation holds*

$$F_{n+3}^2 = 2F_{n+2}^2 + 2F_{n+1}^2 - F_n^2.$$

Proof. The formula is easy to prove by squaring the expressions $F_{n+3} = 2F_{n+1} + F_n$ and $F_{n+2} = F_{n+1} + F_n$.

□

Proposition 33. *Let n be a positive integer, and let $b_i = F^2_{k+i-1}$, $(i = 1, 2, \ldots)$. Then*

$$c(1, \mathbf{b}) = F_k^2,\ c(2, \mathbf{b}) = F^2_{k+1} + F_k^4,\ c(3, \mathbf{b}) = F^2_{k+2} + 2F_k^2 F^2_{k+1} + F_k^6,$$

and, for $n > 3$,

$$c(n, \mathbf{b}) = (F_k^2 + 2)c(n-1, \mathbf{b}) + (2F^2_{k-1} - F^2_{k-2} + 2)c(n-2, \mathbf{b}) - (F^2_{k-1} + 1)c(n-3, \mathbf{b}).$$

Proof. In this case we have

$$a_{n+1} = \sum_{i=1}^{n} F^2_{k+n-i} a_i.$$

Using the preceding lemma yields

$$a_{n+1} = 2\sum_{i=1}^{n} F^2_{k+n-1-i} a_i + 2\sum_{i=1}^{n} F^2_{k+n-2-i} a_i - \sum_{i=1}^{n} F^2_{k+n-3-i} a_i =$$

$$= 2F^2_{k-1}a_n + 2a_n + 2F^2_{k-2}a_n + 2F^2_{k-1}a_{n-1} + 2a_{n-1} - F^2_{k-3}a_n -$$
$$-F^2_{k-2}a_{n-1} - F^2_{k-1}a_{n-2} - a_{n-2}.$$

Using the preceding lemma once more we obtain

$$a_{n+1} = (F_k^2 + 2)a_n + (2F^2_{k-1} - F^2_{k-2} + 2)a_{n-1} - (F^2_{k-1} + 1)a_{n-2},$$

and the assertion is proved. □

Remark 34. The following two sequences in OEIS are generated by the preceding formula: m=0, A054854, m=1, A030186.

References

[1] A. T. Benjamin and J. J. Quinn, *Proofs that Really Count,* MAA, 2003.

[2] E. Deutsch, Advanced exercise H-641, *Fibonacci Quart.* **44** (2006), 188.

[3] M. Petkovsek, H. S. Wilf and D. Zeilberger, *A=B*, Peters, 1996.

[4] N. J. A. Sloane, *Online Encyclopedia of Integer Sequences*,

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