

# ON THE SEMIGROUP OF PARTIAL ISOMETRIES OF A FINITE CHAIN

R. Kehinde and A. Umar

## Abstract

Let  $\mathcal{I}_n$  be the symmetric inverse semigroup on  $X_n = \{1, 2, \dots, n\}$  and let  $\mathcal{DP}_n$  and  $\mathcal{ODP}_n$  be its subsemigroups of partial isometries and of order-preserving partial isometries of  $X_n$ , respectively. In this paper we investigate the cycle structure of a partial isometry and characterize the Green's relations on  $\mathcal{DP}_n$  and  $\mathcal{ODP}_n$ . We show that  $\mathcal{ODP}_n$  is a 0 –  $E$  – unitary inverse semigroup. We also investigate the cardinalities of some equivalences on  $\mathcal{DP}_n$  and  $\mathcal{ODP}_n$  which lead naturally to obtaining the order of the semigroups.<sup>1 2</sup>

*MSC2010*: 20M18, 20M20, 05A10, 05A15.

## 1 Introduction and Preliminaries

Let  $X_n = \{1, 2, \dots, n\}$  and  $\mathcal{I}_n$  be the partial one-to-one transformation semigroup on  $X_n$  under composition of mappings. Then  $\mathcal{I}_n$  is an *inverse* semigroup (that is, for all  $\alpha \in \mathcal{I}_n$  there exists a unique  $\alpha' \in \mathcal{I}_n$  such that  $\alpha = \alpha'\alpha$  and  $\alpha' = \alpha'\alpha'$ ). The importance of  $\mathcal{I}_n$  (more commonly known as the symmetric inverse semigroup or monoid) to inverse semigroup theory may be likened to that of the symmetric group  $\mathcal{S}_n$  to group theory. Every finite inverse semigroup  $S$  is embeddable in  $\mathcal{I}_n$ , the analogue of Cayley's theorem for finite groups, and to the regular representation of finite semigroups. Thus, just as the study of symmetric, alternating and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of  $\mathcal{I}_n$ , see for example [1, 3, 4, 6, 17].

A transformation  $\alpha \in \mathcal{I}_n$  is said to be *order-preserving* (*order-reversing*) if  $(\forall x, y \in \text{Dom } \alpha) x \leq y \implies x\alpha \leq y\alpha$  ( $x\alpha \geq y\alpha$ ) and, is said to be an *isometry* (*or distance-preserving*) if  $(\forall x, y \in \text{Dom } \alpha) |x - y| = |x\alpha - y\alpha|$ . Semigroups of partial isometries on more restrictive but richer mathematical structures have been studied [2, 19]. This paper investigates the algebraic and combinatorial properties of  $\mathcal{DP}_n$  and  $\mathcal{ODP}_n$ , the semigroups of partial isometries and of partial order-preserving isometries of an  $n$ -chain, respectively.

In this section we introduce basic terminologies and some preliminary results concerning the cycle structure of a partial isometry of  $X_n$ . In the

<sup>1</sup>*Key Words*: partial one-one transformation, partial isometries, height, right (left) waist, right (left) shoulder and fix of a transformation, idempotents and nilpotents.

<sup>2</sup>This work was carried out when the first named author was visiting Sultan Qaboos University for a 3-month research visit in Fall 2010.

next section, (Section 2) we characterize the classical Green's relations and show that  $\mathcal{ODP}_n$  is a 0-E-unitary inverse semigroup. We also show that certain Rees factor semigroups of  $\mathcal{ODP}_n$  are 0-E-unitary and categorical inverse semigroups. In Section 3 we obtain the cardinalities of two equivalences defined on  $\mathcal{DP}_n$  and  $\mathcal{ODP}_n$ . These equivalences lead to formulae for the order of  $\mathcal{DP}_n$  and  $\mathcal{ODP}_n$  as well as new triangles of numbers not yet recorded in [15].

For standard concepts in semigroup and symmetric inverse semigroup theory, see for example [9, 13, 11]. Let

$$(1) \quad \mathcal{DP}_n = \{\alpha \in \mathcal{I}_n : (\forall x, y \in X_n) \mid x - y \mid = \mid x\alpha - y\alpha \mid\}$$

be the subsemigroup of  $\mathcal{I}_n$  consisting of all partial isometries of  $X_n$ . Also let

$$(2) \quad \mathcal{ODP}_n = \{\alpha \in \mathcal{DP}_n : (\forall x, y \in X_n) x \leq y \implies x\alpha \leq y\alpha\}$$

be the subsemigroup of  $\mathcal{DP}_n$  consisting of all order-preserving partial isometries of  $X_n$ . It is clear that if  $\alpha \in \mathcal{DP}_n$  ( $\alpha \in \mathcal{ODP}_n$ ) then  $\alpha^{-1} \in \mathcal{DP}_n$  ( $\alpha^{-1} \in \mathcal{ODP}_n$ ) also. Thus we have the following result.

**Lemma 1.1**  *$\mathcal{DP}_n$  and  $\mathcal{ODP}_n$  are inverse subsemigroups of  $\mathcal{I}_n$ .*

Next we prove a sequence of lemmas that help us understand the cycle structure of partial isometries. These lemmas also seem to be useful in investigating the combinatorial questions in Section 3. First, let  $\alpha$  be in  $\mathcal{I}_n$ . Then the *height* of  $\alpha$  is  $h(\alpha) = \mid \text{Im } \alpha \mid$ , the *right [left] waist* of  $\alpha$  is  $w^+(\alpha) = \max(\text{Im } \alpha)$  [ $w^-(\alpha) = \min(\text{Im } \alpha)$ ], the *right [left] shoulder* of  $\alpha$  is  $\varpi^+(\alpha) = \max(\text{Dom } \alpha)$  [ $\varpi^-(\alpha) = \min(\text{Dom } \alpha)$ ], and *fix* of  $\alpha$  is denoted by  $f(\alpha)$ , and defined by  $f(\alpha) = \mid F(\alpha) \mid$ , where

$$F(\alpha) = \{x \in X_n : x\alpha = x\}.$$

**Lemma 1.2** *Let  $\alpha \in \mathcal{DP}_n$  be such that  $h(\alpha) = p$ . Then  $f(\alpha) = 0$  or  $1$  or  $p$ .*

*Proof.* Suppose  $x, y \in F(\alpha)$ . Then  $x = x\alpha$  and  $y = y\alpha$ . Let  $z \in \text{Dom } \alpha$  where we may without loss of generality assume that  $x < y < z$ . Essentially, we consider two cases:  $y < z\alpha$  and  $x < z\alpha < y$ . In the former, we see that

$$z - y = \mid z\alpha - y\alpha \mid = \mid z\alpha - y \mid = z\alpha - y \implies z = z\alpha.$$

In the latter, we see that

$$z - x = \mid z\alpha - x\alpha \mid = \mid z\alpha - x \mid = z\alpha - x \implies z = z\alpha.$$

However, note that

$$\alpha = \begin{pmatrix} 2 & 3 & \dots & p+1 \\ 1 & 2 & \dots & p \end{pmatrix} \text{ and } \beta = \begin{pmatrix} \dots & i-1 & i & i+1 & \dots \\ \dots & i+1 & i & i-1 & \dots \end{pmatrix}$$

are nonidempotent partial isometries with  $f(\alpha) = 0$  and  $f(\beta) = 1$ . □

**Corollary 1.3** *Let  $\alpha \in \mathcal{DP}_n$ . If  $f(\alpha) = p > 1$  then  $f(\alpha) = h(\alpha)$ . Equivalently, if  $f(\alpha) > 1$  then  $\alpha$  is an idempotent.*

**Lemma 1.4** *Let  $\alpha \in \mathcal{DP}_n$ . If  $1 \in F(\alpha)$  or  $n \in F(\alpha)$  then for all  $x \in \text{Dom}\alpha$ , we have  $x\alpha = x$ . Equivalently, if  $1 \in F(\alpha)$  or  $n \in F(\alpha)$  then  $\alpha$  is a partial identity.*

*Proof.* Suppose  $1 \in F(\alpha)$ . Then for all  $x \in \text{Dom}\alpha$ ,  $x - 1 = x\alpha - 1\alpha = x\alpha - 1 \implies x = x\alpha$ . Similarly, if  $n \in F(\alpha)$ , then for all  $x \in \text{Dom}\alpha$ ,  $n - x = n\alpha - x\alpha = n - x\alpha \implies x = x\alpha$ .  $\square$

**Lemma 1.5** *Let  $\alpha \in \mathcal{ODP}_n$  and  $n \in \text{Dom}\alpha \cap \text{Im}\alpha$ . Then  $n\alpha = n$ .*

*Proof.* Since  $n = \max(\text{Dom}\alpha)$  and  $n = \max(\text{Im}\alpha)$ , and  $\alpha$  is order-preserving then  $n\alpha = n$ . However, note that in  $\mathcal{DP}_n$  we have  $\alpha = \begin{pmatrix} 1 & n \\ n & 1 \end{pmatrix}$ , where  $n \in \text{Dom}\alpha \cap \text{Im}\alpha$  but  $n\alpha \neq n$ .  $\square$

**Lemma 1.6** *Let  $\alpha \in \mathcal{ODP}_n$  and  $f(\alpha) \geq 1$ . Then  $\alpha$  is an idempotent.*

*Proof.* Let  $x$  be a fixed point of  $\alpha$  and suppose  $y \in \text{Dom}\alpha$ . If  $x < y$  then by the order-preserving and isometry properties we see that  $y - x = y\alpha - x\alpha = y\alpha - x \implies y = y\alpha$ . The case  $y < x$  is similar. However, note that in  $\mathcal{DP}_n$  we have  $\alpha = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix}$ , where  $f(\alpha) = 1$  but  $\alpha^2 \neq \alpha$ .  $\square$

## 2 Green's relations

For the definitions of Green's relations we refer the reader to Howie [9, Chapter 2]. It is now customary that when one encounters a new class of semigroups, one of the questions that is often asked concerns the characterization of Green's relations. By Lemma 1.1 and [9, Proposition 2.4.2 & Ex. 5.11.2] we deduce the following lemma.

**Lemma 2.1** *Let  $\alpha, \beta \in \mathcal{DP}_n$ . Then*

- (1)  $\alpha \leq_{\mathcal{R}} \beta$  if and only if  $\text{Dom}\alpha \subseteq \text{Dom}\beta$ ;
- (2)  $\alpha \leq_{\mathcal{L}} \beta$  if and only if  $\text{Im}\alpha \subseteq \text{Im}\beta$ ;
- (3)  $\alpha \leq_{\mathcal{H}} \beta$  if and only if  $\text{Dom}\alpha \subseteq \text{Dom}\beta$  and  $\text{Im}\alpha \subseteq \text{Im}\beta$ .

**Theorem 2.2** *Let  $S = \mathcal{DP}_n$  be as defined in (1). Then  $\alpha \leq_{\mathcal{D}} \beta$  if and only if there exists an isometry  $\theta : \text{Dom}\alpha \rightarrow \text{Dom}\beta$ .*

*Proof.* Let  $\alpha \leq_{\mathcal{D}} \beta$ , then there exists  $\delta$  in  $\mathcal{DP}_n$  such that  $\alpha \leq_{\mathcal{R}} \delta \iff \alpha = \delta\eta_1$  and  $\delta \leq_{\mathcal{L}} \beta \iff \delta = \eta_2\beta$ . Thus  $\alpha = \delta\eta_1 = \eta_2\beta\eta_1$  and so  $Dom \alpha \subseteq Dom \eta_2$ . It is clear that  $\eta_2|_{Dom \alpha}$  is an isometry from  $Dom \alpha$  into  $Dom \beta$ .

Conversely, suppose  $\theta$  is an isometry from  $Dom \alpha$  into  $Dom \beta$ . Define  $\eta_1$  by  $x\eta_1 = x\theta^{-1}\alpha$  ( $x \in Dom \beta$ ). Then  $\eta_1 \in \mathcal{DP}_n$  and  $\theta\eta_1 = \theta(\theta^{-1}\alpha) = \alpha$ . Hence  $\alpha \leq_{\mathcal{R}} \theta$ . Similarly, define  $\eta_2$  by  $x\eta_2 = x\theta\beta^{-1}$  ( $x \in Dom \alpha$ ). Then  $\eta_2 \in \mathcal{DP}_n$  and  $\eta_2\beta = (\theta\beta^{-1})\beta = \theta$ . Hence  $\theta \leq_{\mathcal{L}} \beta$ , as required.  $\square$

The corresponding result for  $\mathcal{ODP}_n$  can be proved similarly.

**Theorem 2.3** *Let  $S = \mathcal{ODP}_n$  be as defined in (2). Then  $\alpha \leq_{\mathcal{D}} \beta$  if and only if there exists an order-preserving isometry  $\theta : Dom \alpha \rightarrow Dom \beta$ .*

Let  $E' = E \setminus 0$ . A semigroup  $S$  is said to be *0-E-unitary* if  $(\forall e \in E')(\forall s \in S) es \in E' \implies s \in E'$ . The structure theorem for 0-E-unitary inverse semigroup was given by Lawson [12], see also Szendrei [16] and Gomes and Howie [7]. The next result came as a pleasant surprise to us in the sense that we get a natural class of 0-E-unitary inverse semigroups.

**Theorem 2.4**  *$\mathcal{ODP}_n$  is a 0-E-unitary inverse subsemigroup of  $\mathcal{I}_n$ .*

*Proof.* Let  $\epsilon \in E(\mathcal{ODP}_n) \setminus \{0\}$ . Then  $Dom \epsilon \neq \emptyset$  and for  $\beta \in \mathcal{ODP}_n$  such that  $\epsilon\beta$  is a nonzero idempotent we see that  $Dom \epsilon\beta = \{x, \dots\} \neq \emptyset$ . Now, since  $Dom \epsilon\beta \subseteq Dom \epsilon$  it follows that  $x\epsilon = x \implies x\beta = x\epsilon\beta = x$ . Thus, for any  $y \in Dom \beta$ , (i) if  $x < y$ , we see that  $y - x = y\beta - x\beta = y\beta - x$ . Hence  $y = y\beta$ , showing that  $\beta$  is idempotent, as required. The case (ii)  $y < x$  is similar.  $\square$

**Remark 2.5** *Note that  $\mathcal{DP}_n$  is not 0-E-unitary:*

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \in E(\mathcal{DP}_n) \text{ but } \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \notin E(\mathcal{DP}_n).$$

For natural numbers  $n, p$  with  $n \geq p \geq 0$ , let

$$(3) \quad L(n, p) = \{\alpha \in \mathcal{ODP}_n : h(\alpha) \leq p\}$$

be a two-sided ideal of  $\mathcal{ODP}_n$ , and for  $p > 0$ , let

$$(4) \quad Q(n, p) = L(n, p)/L(n, p-1)$$

be its Rees quotient semigroup. Then  $Q(n, p)$  is a 0-E-unitary inverse semigroup whose nonzero elements may be thought of as the elements of  $\mathcal{ODP}_n$  of height  $p$ . The product of two elements of  $Q(n, p)$  is 0 whenever their product in  $\mathcal{ODP}_n$  is of height less than  $p$ .

A semigroup  $S$  is said to be *categorical* if

$$(\forall a, b, c \in S), abc = 0 \implies ab = 0 \text{ or } bc = 0.$$

The structure theorem for 0-E-unitary categorical inverse semigroup was given by Gomes and Howie [7]. Now we have

**Theorem 2.6** Let  $Q(n, p)$  be as defined in (4). Then  $Q(n, p)$  is a 0 –  $E$  – unitary categorical inverse semigroup.

*Proof.* Let  $\alpha, \beta$  and  $\gamma \in Q(n, p)$ . Note that it suffices to prove that if  $\alpha\beta \neq 0$  and  $\beta\gamma \neq 0$  then  $\alpha\beta\gamma \neq 0$ . Now suppose  $\alpha\beta \neq 0$  and  $\beta\gamma \neq 0$ . Then  $Im \alpha\beta = Im \beta = Dom \gamma$ . Hence  $\alpha\beta\gamma \neq 0$ , as required.  $\square$

**Remark 2.7** Note that  $\mathcal{ODP}_n$  is not categorical:

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = 0$$

but

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \neq 0 \quad \text{and} \quad \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} \begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \neq 0.$$

### 3 Combinatorial results

Enumerative problems of an essentially combinatorial nature arise naturally in the study of semigroups of transformations. Many numbers and triangle of numbers regarded as combinatorial gems like the Stirling numbers [9, pp. 42 & 96], the factorial [14, 17], the binomial [6], the Fibonacci number [8], Catalan numbers [5], Lah numbers [5, 10], etc., have all featured in these enumeration problems. For a nice survey article concerning combinatorial problems in the symmetric inverse semigroup and some of its subsemigroups we refer the reader to Umar [18]. These enumeration problems lead to many numbers in Sloane's encyclopaedia of integer sequences [15] but there are also others that are not yet or have just been recorded in [15].

Now recall the definitions of *height* and *fix* of  $\alpha \in \mathcal{I}_n$  from the paragraph after Lemma 1.1. As in Umar [18], for natural numbers  $n \geq p \geq m \geq 0$  we define

$$(5) \quad F(n; p) = | \{ \alpha \in S : h(\alpha) = | Im \alpha | = p \} |,$$

$$(6) \quad F(n; m) = | \{ \alpha \in S : f(\alpha) = m \} |$$

where  $S$  is any subsemigroup of  $\mathcal{I}_n$ . Also, let  $i = a_i = a$ , for all  $a \in \{p, m\}$ , and  $0 \leq i \leq n$ .

**Lemma 3.1** Let  $S = \mathcal{ODP}_n$ . Then  $F(n; p_1) = F(n; 1) = n^2$  and  $F(n; p_n) = F(n; n) = 1$ , for all  $n \geq 2$ .

*Proof.* Since all partial injections of height 1 are vacuously partial isometries, the first statement of the lemma follows immediately. For the second statement, it is not difficult to see that there is exactly one partial isometry of height  $n$ :  $\begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}$  (the identity).  $\square$

**Lemma 3.2** *Let  $S = \mathcal{ODP}_n$ . Then  $F(n; p_2) = F(n; 2) = \frac{1}{6}n(n-1)(2n-1)$ , for all  $n \geq 2$ .*

*Proof.* First, we say that 2-subsets of  $X_n$  (that is, subsets of size 2) say,  $A = \{a_1, a_2\}$  and  $B = \{b_1, b_2\}$  are of the same *type* if  $|a_1 - a_2| = |b_1 - b_2|$ . Now observe that if  $|a_1 - a_2| = i$  ( $1 \leq i \leq n-1$ ) then there are  $n-i$  subsets of this type. However, for partial order-preserving isometries once we choose a 2-subset as a domain then the possible image sets must be of the same type and there is only one possible order-preserving bijection between any two 2-subsets of the same type. It is now clear that  $F(n; p_2) = F(n; 2) = \sum_{i=1}^{n-1} (n-i)^2 = \frac{1}{6}n(n-1)(2n-1)$ , as required.  $\square$

**Lemma 3.3** *Let  $S = \mathcal{ODP}_n$ . Then  $F(n; p) = F(n-1; p-1) + F(n-1; p)$ , for all  $n \geq p \geq 3$ .*

*Proof.* Let  $\alpha \in \mathcal{ODP}_n$  and  $h(\alpha) = p$ . Then it is clear that  $F(n; p) = |A| + |B|$ , where  $A = \{\alpha \in \mathcal{ODP}_n : h(\alpha) = p \text{ and } n \notin \text{Dom } \alpha \cup \text{Im } \alpha\}$  and  $B = \{\alpha \in \mathcal{ODP}_n : h(\alpha) = p \text{ and } n \in \text{Dom } \alpha \cup \text{Im } \alpha\}$ . Define a map  $\theta : \{\alpha \in \mathcal{ODP}_{n-1} : h(\alpha) = p\} \rightarrow A$  by  $(\alpha)\theta = \alpha'$  where  $x\alpha' = x\alpha$  ( $x \in \text{Dom } \alpha$ ). This is clearly a bijection since  $n \notin \text{Dom } \alpha \cup \text{Im } \alpha$ . Next, recall the definitions of  $\varpi^+(\alpha)$  and  $w^+(\alpha)$  from the paragraph after Lemma 1.1. Now, define a map  $\Phi : \{\alpha \in \mathcal{ODP}_{n-1} : h(\alpha) = p-1\} \rightarrow B$  by  $(\alpha)\Phi = \alpha'$  where

- (i)  $x\alpha' = x\alpha$  ( $x \in \text{Dom } \alpha$ ) and  $n\alpha' = n$  (if  $\varpi^+(\alpha) = w^+(\alpha)$ );
- (ii)  $x\alpha' = x\alpha$  ( $x \in \text{Dom } \alpha$ ) and  $n\alpha' = n - \varpi^+(\alpha) + w^+(\alpha) < n$  (if  $\varpi^+(\alpha) > w^+(\alpha)$ );
- (iii)  $x(\alpha')^{-1} = x\alpha^{-1}$  ( $x \in \text{Im } \alpha$ ) and  $n(\alpha')^{-1} = n - \varpi^+(\alpha)^{-1} + w^+(\alpha^{-1}) < n$  (if  $\varpi^+(\alpha) < w^+(\alpha)$ ).

In all cases  $h(\alpha') = p$ , and case (i) coincides with  $n \in \text{Dom } \alpha' \cap \text{Im } \alpha'$ ; case (ii) coincides with  $n \in \text{Dom } \alpha' \setminus \text{Im } \alpha'$ ; case (iii) coincides with  $n \in \text{Im } \alpha' \setminus \text{Dom } \alpha'$ . Thus  $\Phi$  is onto. Moreover, it is not difficult to see that  $\Phi$  is one-to-one. Hence  $\Phi$  is a bijection, as required. This establishes the statement of the lemma.  $\square$

**Proposition 3.4** *Let  $S = \mathcal{ODP}_n$  and  $F(n; p)$  be as defined in (2) and (5), respectively. Then  $F(n; p) = \frac{(2n-p+1)}{p+1} \binom{n}{p}$ , where  $n \geq p \geq 2$ .*

*Proof.* (The proof is by induction).

Basis step: First, note that  $F(n; 1)$ ,  $F(n; n)$  and  $F(n; 2)$  are true by Lemmas 3.1 and 3.2.

Inductive step: Suppose  $F(n - 1; p)$  is true for all  $n - 1 \geq p$ . (This is the induction hypothesis.) Now using Lemma 3.3, we see that

$$\begin{aligned}
F(n; p) &= F(n - 1; p - 1) + F(n - 1; p) \\
&= \frac{(2n - p)}{p} \binom{n - 1}{p - 1} + \frac{(2n - p - 1)}{p + 1} \binom{n - 1}{p} \quad (\text{by ind. hyp.}) \\
&= \frac{(2n - p)}{p} \frac{p}{n} \binom{n}{p} + \frac{(2n - p - 1)}{p + 1} \frac{(n - p)}{n} \binom{n}{p} \\
&= \frac{(2n - p)(p + 1) + (2n - p - 1)(n - p)}{n(p + 1)} \binom{n}{p} \\
&= \frac{(2n^2 - np + n)}{n(p + 1)} \binom{n}{p} = \frac{(2n - p + 1)}{p + 1} \binom{n}{p},
\end{aligned}$$

as required.  $\square$

**Lemma 3.5** For integers  $n, p$  such that  $n \geq p \geq 2$ , we have  $\sum_{p=2}^n \frac{2n-p+1}{p+1} \binom{n}{p} = 3 \cdot 2^n - n^2 - 2n - 3$ .

*Proof.* It is enough to observe that  $2n - p + 1 = (2n - 2p) + (p + 1)$ .  $\square$

**Theorem 3.6** Let  $\mathcal{ODP}_n$  be as defined in (2). Then

$$|\mathcal{ODP}_n| = 3 \cdot 2^n - 2(n + 1).$$

*Proof.* It follows from Proposition 3.4 and Lemma 3.5, and some algebraic manipulation.  $\square$

**Lemma 3.7** Let  $S = \mathcal{ODP}_n$ . Then  $F(n; m) = \binom{n}{m}$ , for all  $n \geq m \geq 1$ .

*Proof.* It follows directly from Lemma 1.6.  $\square$

**Proposition 3.8** Let  $S = \mathcal{ODP}_n$  and  $F(n; m)$  be as defined in (2) and (6), respectively. Then  $F(n; 0) = 2^{n+1} - (2n + 1)$ .

*Proof.* It follows from Theorem 3.6, Lemma 3.7 and the fact that  $|\mathcal{ODP}_n| = \sum_{m=0}^n F(n; m)$ .  $\square$

**Remark 3.9** The triangles of numbers  $F(n; p)$  and  $F(n; m)$ , the sequence  $F(n; m_0)$  are as at the time of submitting this paper not in Sloane [15]. However,  $|\mathcal{ODP}_n|$  is [15, A097813]. For some computed values of  $F(n; p)$  and  $F(n; m)$  in  $\mathcal{ODP}_n$ , see Tables 3.1 and 3.2.

$n \setminus p$	0	1	2	3	4	5	6	7	$\sum F(n; p) =  \mathcal{ODP}_n $
0	1								1
1	1	1							2
2	1	4	1						6
3	1	9	5	1					16
4	1	16	14	6	1				38
5	1	25	30	20	7	1			84
6	1	36	55	50	27	8	1		178
7	1	49	91	105	77	35	9	1	368

Table 3.1

$n \setminus m$	0	1	2	3	4	5	6	7	$\sum F(n; m) =  \mathcal{ODP}_n $
0	1								1
1	1	1							2
2	3	2	1						6
3	9	3	3	1					16
4	23	4	6	4	1				38
5	53	5	10	10	5	1			84
6	115	6	15	20	15	6	1		178
7	241	7	21	35	35	21	7	1	368

Table 3.2

**Remark 3.10** For  $p = 0, 1$  the concepts of order-preserving and order-reversing coincide but distinct otherwise. However, there is a bijection between the two sets for  $p \geq 2$ , see [4, page 2, last paragraph].

**Lemma 3.11** Let  $\alpha \in \mathcal{DP}_n$ . Then  $\alpha$  is either order-preserving or order-reversing.

*Proof.* If  $h(\alpha) = 2$  then the result is obvious. However, if  $h(\alpha) > 2$  we must consider cases. First suppose that  $\{a_1, a_2, a_3\} \subseteq \text{Dom}\alpha$ , where  $a_i\alpha = b_i$  ( $i = 1, 2, 3$ ) and  $1 \leq a_1 < a_2 < a_3 \leq n$ . There are four cases to consider if  $\alpha$  is neither order-preserving or order-reversing:  $b_1 < b_3 < b_2$ ,  $b_2 < b_1 < b_3$ ,  $b_2 < b_3 < b_1$  and  $b_3 < b_1 < b_2$ . In the first case, note that  $b_2 - b_1 = (b_2 - b_3) + (b_3 - b_1)$ . But  $a_3 - a_1 = (a_3 - a_2) + (a_2 - a_1) = |a_3 - a_2| + |a_2 - a_1| = |b_3 - b_2| + |b_2 - b_1| = |b_3 - b_2| + |b_2 - b_3| + |b_3 - b_1| = 2|b_3 - b_2| + |b_3\alpha^{-1} - b_1\alpha^{-1}| = 2|b_3 - b_2| + |a_3 - a_1| = 2|b_3 - b_2| + a_3 - a_1$ , which implies that  $|b_3 - b_2| = 0 \Leftrightarrow b_3 = b_2$ . This is a contradiction. The other three cases are similar.  $\square$

We now use Remark 3.10 and Lemma 3.11 to deduce corresponding results for  $\mathcal{DP}_n$  from those of  $\mathcal{ODP}_n$  above.

**Lemma 3.12** Let  $S = \mathcal{DP}_n$ . Then  $F(n; p_1) = F(n; 1) = n^2$  and  $F(n; p_n) = F(n; n) = 2$ , for all  $n \geq 2$ .

**Lemma 3.13** Let  $S = \mathcal{DP}_n$ . Then  $F(n; p_2) = F(n; 2) = \frac{1}{3}n(n-1)(2n-1)$ , for all  $n \geq 2$ .

**Lemma 3.14** Let  $S = \mathcal{DP}_n$ . Then  $F(n; p) = F(n-1; p-1) + F(n-1; p)$ , for all  $n \geq p \geq 3$ .

**Proposition 3.15** Let  $S = \mathcal{DP}_n$  and  $F(n; p)$  be as defined in (1) and (5), respectively. Then  $F(n; p) = \frac{2(2n-p+1)}{p+1} \binom{n}{p}$ , where  $n \geq p \geq 2$ .

**Theorem 3.16** Let  $\mathcal{DP}_n$  be as defined in (1). Then

$$|\mathcal{DP}_n| = 3 \cdot 2^{n+1} - (n+2)^2 - 1.$$

*Proof.* It follows from Proposition 3.15, Lemma 3.5 and some algebraic manipulation.  $\square$

**Lemma 3.17** Let  $\alpha \in \mathcal{DP}_n$ . For  $1 < i < n$ , if  $F(\alpha) = \{i\}$  then for all  $x \in \text{Dom } \alpha$  we have that  $x + x\alpha = 2i$ .

*Proof.* Let  $F(\alpha) = \{i\}$  and suppose  $x \in \text{Dom } \alpha$ . Obviously,  $i + i\alpha = i + i = 2i$ . If  $x < i$  then  $x\alpha > i$ , for otherwise we would have  $i - x = |i\alpha - x\alpha| = |i - x\alpha| = i - x\alpha \implies x = x\alpha$ , which is a contradiction. Thus,  $i - x = |i\alpha - x\alpha| = |i - x\alpha| = |x\alpha - i| = x\alpha - i \implies x + x\alpha = 2i$ . The case  $x > i$  is similar.  $\square$

**Lemma 3.18** Let  $S = \mathcal{DP}_n$ . Then  $F(n; m) = \binom{n}{m}$ , for all  $n \geq m \geq 2$ .

*Proof.* It follows from Corollary 1.3.  $\square$

**Proposition 3.19** Let  $S = \mathcal{DP}_n$ . Then  $F(2n; m_1) = F(2n; 1) = \frac{2(2^{2n}-1)}{3}$  and  $F(2n-1; m_1) = F(2n-1; 1) = \frac{2(2^{2n-2}-1)}{3} + 2^{2n-2}$ , for all  $n \geq 1$ .

*Proof.* Let  $F(\alpha) = \{i\}$ . Then by Lemma 3.17, for any  $x \in \text{Dom } \alpha$  we have  $x + x\alpha = 2i$ . Thus there  $2i - 2$  possible elements for  $\text{Dom } \alpha : (x, x\alpha) \in \{(1, 2i-1), (2, 2i-2), \dots, (2i-1, 1)\}$ . However, (excluding  $(i, i)$ ) we see that there are  $\sum_{j=0}^{2i-2} \binom{2i-2}{j} = 2^{2i-2}$ , possible partial isometries with  $F(\alpha) = \{i\}$ , where  $2i-1 \leq n \iff i \leq (n+1)/2$ . Moreover, by symmetry we see that  $F(\alpha) = \{i\}$  and  $F(\alpha) = \{n-i+1\}$  give rise to equal number of partial isometries. Note that if  $n$  is odd the equation  $i = n-i+1$  has one solution. Hence, if  $n = 2a-1$  we have

$$2 \sum_{i=1}^{a-1} 2^{2i-2} + 2^{2a-2} = \frac{2(2^{2a-2}-1)}{3} + 2^{2a-2}$$

partial isometries with exactly one fixed point; if  $n = 2a$  we have

$$2 \sum_{i=1}^a 2^{2i-2} = \frac{2(2^{2a} - 1)}{3}$$

partial isometries with exactly one fixed point.  $\square$

**Proposition 3.20** *Let  $S = \mathcal{DP}_n$ . Then  $F(n; m_0) = F(n; 0) = \frac{13 \cdot 2^n - (3n^2 + 9n + 10)}{3}$ , ( $n \geq 0$ , if  $n$  is even) and  $F(n; m_0) = F(n; 0) = \frac{25 \cdot 2^{n-1} - (3n^2 + 9n + 10)}{3}$ , ( $n \geq 1$ , if  $n$  is odd).*

*Proof.* It follows from Theorem 3.16, Lemma 3.18, Proposition 3.19 and the fact that  $|\mathcal{DP}_n| = \sum_{m=0}^n F(n; m)$ .  $\square$

**Remark 3.21** *The triangles of numbers  $F(n; p)$  and  $F(n; m)$  and, the sequences  $|\mathcal{DP}_n|$  and  $F(n; m_0)$ , are as at the time of submitting this paper not in Sloane [15]. However,  $F(n; m_1)$  is [15, A061547]. For some computed values of  $F(n; p)$  and  $F(n; m)$  in  $\mathcal{DP}_n$ , see Tables 3.3 and 3.4.*

$n \setminus p$	0	1	2	3	4	5	6	7	$\sum F(n; p) =  \mathcal{DP}_n $
0	1								1
1	1	1							2
2	1	4	2						7
3	1	9	10	2					22
4	1	16	28	12	2				59
5	1	25	60	40	14	2			142
6	1	36	110	100	54	16	2		319
7	1	49	182	210	154	70	18	2	686

Table 3.3

$n \setminus m$	0	1	2	3	4	5	6	7	$\sum F(n; m) =  \mathcal{DP}_n $
0	1								1
1	1	1							2
2	4	2	1						7
3	12	6	3	1					22
4	38	10	6	4	1				59
5	90	26	10	10	5	1			142
6	220	42	15	20	15	6	1		319
7	460	106	21	35	35	21	7	1	686

Table 3.4

**Acknowledgements.** The first named author would like to thank Bowen University, Iwo and Sultan Qaboos University for their financial support and hospitality, respectively.

## References

- [1] BORWEIN, D., RANKIN, S. AND RENNER, L. Enumeration of injective partial transformations. *Discrete Math.* **73** (1989), 291–296.
- [2] BRACCI, L., AND PICASSO, L. E. Representations of semigroups of partial isometries. *Bull. Lond. Math. Soc.* **39** (2007), 792–802.
- [3] FERNANDES, V. H. The monoid of all injective orientation-preserving partial transformations on a finite chain. *Comm. Algebra* **32** (2000), 3401–3426.
- [4] FERNANDES, V. H., GOMES, G. M. S. AND JESUS, M. M. The cardinal and idempotent number of various monoids of transformations on a finite chain. Algebra Colloquium. (*To appear*).
- [5] GANYUSHKIN, O. AND MAZORCHUK, V. *Classical Finite Transformation Semigroups: An Introduction*, Springer, London, 2009.
- [6] GARBA, G. U. Nilpotents in semigroups of partial one-to-one order-preserving mappings. *Semigroup Forum*, **48** (1994), 37–49.
- [7] GOMES, G. M. S. AND HOWIE, J. M. A P-theorem for inverse semigroups with zero. *Portugaliae Math.* **53** (1996), 257–278.
- [8] HOWIE, J. M. Products of idempotents in certain semigroups of transformations. *Proc. Edinburgh Math. Soc.* **17** (1971), 223–236.
- [9] HOWIE, J. M. *Fundamentals of semigroup theory*. Oxford: Clarendon Press, 1995.
- [10] LARADJI, A. AND UMAR, A. Combinatorial results for the symmetric inverse semigroup. *Semigroup Forum*, **75** (2007), 221–236.
- [11] LAWSON, M. V. *Inverse semigroups: the theory of partial symmetries*, Singapore: World Scientific, 1998.
- [12] LAWSON, M. V. The structure of 0-E-unitary inverse semigroups I: the monoid case *Proc. Edinb. Math. Soc.*, **42** (1999), 497–520.
- [13] LIMPSCOMB, S. *Symmetric Inverse Semigroups*, Mathematical Surveys of The American mathematical Society, no. 46, Providence, R. I., 1996.
- [14] MUNN, W. D. The characters of the symmetric inverse semigroup. *Proc. Cambridge Philos. Soc.*, **53** (1957), 13–18.
- [15] SLOANE, N. J. A. – *The On-Line Encyclopedia of Integer Sequences*, @<http://www.research.att.com/~njas/sequences/>.
- [16] SZENDREI, M. B. A generalization of McAlister’s P-theorem for E-unitary regular semigroups, *Acta Sci Math (Szeged)* **57** (1987), 229–249.
- [17] UMAR, A. On the semigroups of partial one-to-one order-decreasing finite transformations, *Proc. Roy. Soc. Edinburgh*, **123A** (1993), 355–363.

- [18] UMAR, A. Some combinatorial problems in the theory of symmetric inverse semigroups, *Algebra and Discrete Math.* 9 (2010), 115–126.
- [19] WALLEN, LAWRENCE J. Semigroups of partial isometries. *Bull. Amer. Math. Soc.* **75** (1969), 763–764.

R. Kehinde  
Department of Mathematics and Statistics  
Bowen University  
P. M. B. 284, Iwo, Osun State  
Nigeria.  
E-mail: kennyrot2000@yahoo.com

A. Umar  
Department of Mathematics and Statistics  
Sultan Qaboos University  
Al-Khod, PC 123 – OMAN  
E-mail: aumarh@squ.edu.om