Cones of Weighted and Partial Metrics

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Abstract

A *partial semimetric* on $V_n = \{1, \dots, n\}$ is a function $f = ((f_{ij})) : V_n^2 \longrightarrow \mathbb{R}_{\geq 0}$ satisfying $f_{ij} = f_{ji} \geq f_{ii}$ and $f_{ij} + f_{ik} - f_{jk} - f_{ii} \geq 0$ for all $i, j, k \in V_n$. The function f is a *weak partial semimetric* if $f_{ij} \geq f_{ii}$ is dropped, and it is a *strong partial semimetric* if $f_{ij} \geq f_{ii}$ is complemented by $f_{ij} \leq f_{ii} + f_{jj}$.

We describe the cones of weak and strong partial semimetrics via corresponding weighted semimetrics and list their $0, 1$-valued elements, identifying when they belong to extreme rays. We consider also related cones, including those of *partial hypermetrics*, *weighted hypermetrics*, ℓ_1*-quasi semimetrics* and weighted/partial cuts.

Key Words and Phrases: weighted metrics; partial metrics; hypermetrics; cuts; convex cones; computational experiments.

1 Convex cones under consideration

There are following two main motivations for this study. One is to extend the rich theory of metric, cut and hypermetric cones on weighted, partial and non-symmetric generalizations of metrics. Another is a new appoach to partial semimetrics (having important applications in Computer Science) via cones formed by them.

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A *convex cone in* $\mathbb{R}^m$ (see, for example, [Sc86]) is defined either by *generators* $v_1, \ldots, v_N$, as $\{\sum \lambda_i v_i : \lambda_i \geq 0\}$, or by *linear inequalities* $f^1, \ldots, f^M$, as $\cap_{j=1}^{M}\{x \in \mathbb{R}^m : f^j(x) = \sum_{i=1}^{m} f_i^j x_i \geq 0\}$.

Let C be an m'-dimensional convex cone in $\mathbb{R}^m$. Given $f \in \mathbb{R}^m$, the linear inequality $f(x) = \sum_{i=1}^{m} f_i x_i = \langle f, x\rangle \geq 0$ is said to be *valid* for C if it holds for all $x \in C$. Then the set $\{x \in C : \langle f, x\rangle = 0\}$ is called the *face* of C, *induced by* F. A face of dimension $m'-1$, $m'-2$, 1 is called a *facet*, *ridge*, *extreme ray* of C, respectively (a *ray* is a set $\mathbb{R}_{\geq 0}x$ with $x \in C$). Denote by $F(C)$ the set of facets of C and by $R(C)$ the set of its extreme rays. We consider only *polyhedral* (i.e., $R(C)$ and, alternatively, $F(C)$ is finite) *pointed* (i.e., $(0) \in C$) convex cones. Each ray $r \subset C$ below contains a unique *good representative*, i.e., an integer-valued vector $v(r)$ with g.c.d. 1 of its entries; so, by abuse of language, we will identify r with $v(r)$.

For a ray $r \subset C$ denote by $F(r)$ the set $\{f \in F(C) : r \subset f\}$. For a face $f \subset C$ denote by $R(f)$ the set $\{r \in R(C) : r \subset f\}$. The *incidence number* $\mathrm{Inc}(f)$ of a face f (or $\mathrm{Inc}(r)$ of a ray r) is the number $|\{r \in R(C) : r \subset f\}|$ (or, respectively, $|\{f \in F(C) : r \subset f\}|$). The $\mathrm{rank}(f)$ of a face f (or $\mathrm{rank}(r)$ of a ray r) is the dimension of $\{r \in R(C) : r \subset f\}$ (or of $\{f \in F(C) : r \subset f\}$).

Two extreme rays (or facets) of C are *adjacent on* C if they span a 2-dimensional face (or, respectively, their intersection has dimension $m'-2$). The *skeleton* $\mathrm{Sk}(C)$ is the graph whose vertices are the extreme rays of C and with an edge between two vertices if the corresponding rays are adjacent on C. The *ridge graph* $\mathrm{Ri}(C)$ is the graph whose vertices are facets of C and with an edge between two vertices if the corresponding facets are adjacent on C. Let $D(G)$ denote the diameter of the graph G

Given a cone C_n of some functions, say, $d = ((d_{ij})) : V_n^2 \longrightarrow \mathbb{R}_{\geq 0}$ the 0-*extension* of the inequality $\sum_{1 \leq i \neq j \leq n-1} F_{ij} d_{ij} \geq 0$ is the inequality

$$\sum_{1 \leq i \neq j \leq n} F'_{ij} d_{ij} \geq 0 \text{ with } F'_{ni} = F'_{in} = 0 \text{ and } F'_{ij} = F_{ij}, \text{ otherwise.}$$

Clearly, the 0-extension of any facet-defining inequality of a cone C_n is a valid inequality (usually, facet-defining) of C_{n+1}. The 0-extension of an extreme ray is defined similarly. For any cone C denote by $0,1$-C the cone generated by all extreme rays of C containing a non-zero $0,1$-valued point.

The cones C considered here will be symmetric under permutations and usually $\mathrm{Aut}(C) = \mathrm{Sym}(n)$. All orbits below are under $\mathrm{Sym}(n)$.

Set $V_n = \{1, \dots, n\}$. The function $f = ((f_{ij})) : V_n^2 \longrightarrow \mathbb{R}$ is called *weak partial semimetric* if the following holds:

(1) $f_{ij} = f_{ji}$ (*symmetry*) for all $i, j \in V_n$,

(2) $L_{ij} : f_{ij} \geq 0$ (*non-negativity*) for all $i, j \in V_n$, and

(3) $Tr_{ij,k} : f_{ik} + f_{kj} - f_{ij} - f_{kk} \geq 0$ (*triangle inequality*) for all $i, j, k \in V_n$.

Weak partial semimetrics were introduced in [He99] as a generalization of *partial semimetrics* introduced in [Ma92]. Clearly, all $Tr_{ij,i} = 0$ and $Tr_{ii,k} = 2f_{ik} - f_{ii} - f_{kk} = Tr_{ij,k} + Tr_{kj,i}$. So, it is sufficient to require (2) only for $i = j$ and (3) only for different i, j, k. The weak partial semimetrics on V_n form a $\binom{n+1}{2}$-dimensional convex cone with n facets L_{ii} and $3\binom{n}{3}$ facets $Tr_{ij,k}$. Denote this cone by *wPMET*$_n$.

A weak partial semimetric f is called *partial semimetric* if it holds that

(4) $M_{ij} : f_{ij} - f_{ii} \geq 0$ (*small self-distances*) for all different $i, j \in V_n$.

The partial semimetrics on V_n form a $\binom{n+1}{2}$-dimensional subcone, denote it by *PMET*$_n$, of *wPMET*$_n$. This cone has n facets L_{ii}, $2\binom{n}{2}$ facets $M_{ij,i}$ and $3\binom{n}{3}$ facets $Tr_{ij,k}$. Partial metrics were introduced by Matthews in [Ma92] for treatment of partially defined objects in Computer Science; see also [Ma08, Hi01, Se97]. The cone *PMET*$_n$ was considered in [DeDe10].

A partial semimetric f is called *strong partial semimetric* if it holds that

(5) $N_{ij} : f_{ii} + f_{jj} - f_{ij} \geq 0$ (*large self-distances*) for all $i, j \in V_n$.

So, $f_{ii} = N_{ij} + M_{ji} \geq 0$, i.e., (5) and (4) imply L_{ii} for all. i. The strong partial semimetrics on V_n form a $\binom{n+1}{2}$-dimensional subcone, denote it by *sPMET*$_n$, of *PMET*$_n$. This cone has $3\binom{n+1}{3}$ facets: $2\binom{n}{2}$ facets M_{ij}, $\binom{n}{2}$ facets N_{ij} and $3\binom{n}{3}$ facets $Tr_{ij,k}$.

A partial semimetric f is called *semimetric* if it holds that

(6) $f_{ii} = 0$ (*reflexivity*) for all $i \in V_n$.

The semimetrics on V_n form a $\binom{n}{2}$-dimensional convex cone, denoted by *MET*$_n$, which has $3\binom{n}{3}$ facets $Tr_{ij,k}$ (clearly, $f_{ij} = \frac{Tr_{ij,k} + Tr_{jk,i}}{2} \geq 0$). This cone is well-known; see, for example, [DeLa97] and references there.

The function f is *quasi-semimetric* if only (2), (3), (6) are required. The quasi-semimetrics on V_n form a $n(n-1)$-dimensional convex cone, denoted by *QMET*$_n$, which has $2\binom{n}{2}$ facets L_{ij} and $6\binom{n}{3}$ facets $OTr_{ij,k} : f_{ik} + f_{kj} - f_{ij} \geq 0$ (*oriented triangle inequality*). But other oriented versions of $Tr_{ij,k}$ (for example, $f_{ik} + f_{kj} - f_{ji}$) are not valid on *QMET*$_n$.

A quasi-semimetric f is *weightable* if there exist a (weight) function $w = (w_i) : V_n \longrightarrow \mathbb{R}_{\geq 0}$ such that $f_{ij} + w_i = f_{ji} + w_j$ for all $i, j \in V_n$. Such quasi-semimetrics f (or, equivalently, pairs (f, w)) on V_n form a $\binom{n+1}{2}$-dimensional cone, denote it by $WQMET_n$, with $2\binom{n}{2}$ facets L_{ij} and $3\binom{n}{3}$ facets $OTr_{ij,k}$ since, for a quasi-semimetric, $OTr_{ij,k} = OTr_{ji,k}$ if it is weightable.

A weightable quasi-semimetric (f, w) with all $f_{ij} \leq w_j$ is a *weightable strong quasi-semimetric*. But if, on the contrary, (2) is weakened to $f_{ij} + f_{ji} \geq 0$ (so, $f_{ij} < 0$ is allowed), (f, w) is a *weightable weak quasi-semimetric*. Denote by $sWQMET_n$ and $wWQMET_n$ the corresponding cones.

Let us denote the function f by p, d, or q if it is a weak partial semimetric, semimetric, or weightable weak quasi-semimetric, respectively.

A *weighted semimetric* $(d; w)$ is a semimetric d with a weight function $w : V_n \to \mathbb{R}_{\geq 0}$ on its points. Denote by $(d; w)$ the matrix $((d'_{ij}))$, $0 \leq i, j \leq n$, with $d'_{00} = 0$, $d'_{0i} = d'_{i0} = w_i$ for $i \in V_n$ and $d'_{ij} = d_{ij}$ for $i, j \in V_n$. The weighted semimetrics $(d; w)$ on V_n form a $\binom{n+1}{2}$-dimensional convex cone with n facets $w_i \geq 0$ and $3\binom{n}{3}$ facets $Tr_{ij,k}$. Denote this cone by $WMET_n$. So, $MET_n \simeq \{(d; (k, \ldots, k)) : d \in WMET_n\}$. Also, $MET_n = QMET_n \cap PMET_n$.

Call a weighted semimetric $(d; w)$ *down-* or *up-weighted* if

(4') $d_{ij} \geq w_i - w_j$, or

(5') $d_{ij} \leq w_i + w_j$

holds (for all distinct $i, j \in V_n$), respectively. Denote by $dWMET_n$ the cone of down-weighted semimetrics on V_n and by $sWMET_n$ the cone of *strongly*, i.e., both, down- and up-, weighted semimetrics. So, $sWMET_n = MET_{n+1}$.

2 Maps P, Q and semimetrics

Given a weighted semimetric $(d; w)$, define the map P by the function $p = P(d; w)$ with $p_{ij} = \frac{d_{ij}+w_i+w_j}{2}$. Clearly, P is an *automorphism* (invertible linear operator) of the vector space $\mathbb{R}^{\binom{n+1}{2}}$, and $(d; w) = P^{-1}(p)$, where the inverse map P^{-1} is defined by $d_{ij} = 2p_{ij} - p_{ii} - p_{jj}$, $w_i = p_{ii}$.

Define the map Q by the function $(q, w) = Q(d; w)$ with $q_{ij} = \frac{d_{ij}-w_i+w_j}{2}$. So, $Q(d; w) = P(d; w) - ((1))w$ (i.e., $q_{ij} = p_{ij} - p_{ii}$) and $d_{ij} = q_{ij} + q_{ji}$, is the *symmetrization semimetric* of q.

Example. Below are given: the semimetric $d = 2\delta(\{56\}, \{1\}, \{23\}, \{4\}) - \delta(\{56\}) \in MET_6$, and, taking weight $w = (1_{i\in\{56\}}) = (0, 0, 0, 0, 1, 1)$, the partial semimetric $P(d; w) = J(\{56\}) + \delta(\{56\}, \{1\}, \{23\}, \{4\})$ (its ray is extreme in $PMET_6$) and the weightable quasi-semimetric $Q(d; w) = \delta'(\{1\}) + \delta'(\{23\}) + \delta'(\{4\})$ (its ray is not extreme in $WQMET_6$).

$$
\begin{array}{ccc}
\mathbf{0}\,2\,2\,2\,1\,1 & \mathbf{0}\,1\,1\,1\,1\,1 & \mathbf{0}\,1\,1\,1\,1\,1 \\
2\,\mathbf{0}\,0\,2\,1\,1 & 1\,\mathbf{0}\,0\,1\,1\,1 & 1\,\mathbf{0}\,0\,1\,1\,1 \\
2\,0\,\mathbf{0}\,2\,1\,1 & 1\,0\,\mathbf{0}\,1\,1\,1 & 1\,0\,\mathbf{0}\,1\,1\,1 \\
2\,2\,2\,\mathbf{0}\,1\,1 & 1\,1\,1\,\mathbf{0}\,1\,1 & 1\,1\,1\,\mathbf{0}\,1\,1 \\
1\,1\,1\,1\,\mathbf{0}\,0 & 1\,1\,1\,1\,\mathbf{1}\,1 & 0\,0\,0\,0\,\mathbf{0}\,0 \\
1\,1\,1\,1\,0\,\mathbf{0} & 1\,1\,1\,1\,1\,\mathbf{1} & 0\,0\,0\,0\,0\,\mathbf{0}
\end{array}
$$

Clearly, $d_{ij}+d_{ik}-d_{jk}=p_{ij}+p_{ik}-p_{jk}-p_{ii}=q_{ji}+q_{ik}-q_{jk}$, i.e., the triangle inequalities are equivalent on all three levels: d - of semimetrics, p - of would-be partial semimetrics and q - of would-be quasi-semimetrics.

Now, $p_{ij}\geq p_{ii}$ iff $d_{ij}\geq w_i-w_j$ iff $q_{ij}\geq 0$; so, (4) is equivalent to (4'),
$p_{ij}\leq p_{ii}+p_{jj}$ iff $d_{ij}\leq w_i+w_j$ iff $q_{ij}\leq w_j$; so, (5) is equivalent to (5'),
and $2p_{ij}\geq p_{ii}+p_{jj}$ iff $d_{ij}\geq 0$ iff $q_{ij}+q_{ji}\geq 0$. This implies

Lemma 1 *The following statements hold.*
(i) $wPMET_n=P(WMET_n)$, $PMET_n=P(dWMET_n)$ *and* $sPMET_n=P(sWMET_n)$,
(ii) $wWQMET_n=Q(WMET_n)$, $WQMET_n=Q(dWMET_n)$ *and* $sWQMET_n=Q(sWMET_n)$.

The metric cone $MET_n\in\mathbb{R}^{\binom{n}{2}}$ has a unique orbit of $3\binom{n}{3}$ facets $Tr_{ij,k}$. Its symmetry group Aut(MET_n) is Sym(n) $n\neq 4$. The number of extreme rays (orbits) of MET_n is 3 (1), 7 (2), 25 (3), 296 (7), 55226 (46) for $3\leq n\leq 7$. $D(\mathrm{Ri}(MET_n))=2$ for $n>3$, while $\mathrm{Ri}(MET_3)=\mathrm{Sk}(MET_3)=K_3$. $D(\mathrm{Sk}(MET_n))$ is 1 for $n=4$, 2 for $5\leq n\leq 6$ and 3 for $n=7$.

For a partition $\mathcal{S}=\{S_1,\dots,S_t\}$ of V_n, the *multicut* $\delta(\mathcal{S})\in MET_n$ has $\delta_{ij}(\mathcal{S})=1$ if $|\{i,j\}|=2>|\{i,j\}\cap S_h|, 1\leq h\leq t$ and $\delta_{ij}(\mathcal{S})=0$, otherwise. Call $\delta(\mathcal{S})$ a *t-cut* if $S_h\neq\emptyset$ for $1\leq h\leq t$. Clearly,

$$\delta(S_1,\dots,S_t)=\frac{1}{2}\sum_{h=1}^{t}\delta(S_h,\overline{S_h}).$$

Denote by CUT_n the cone generated by all $2^{n-1}-1$ 2-cuts $\delta(S,\overline{S})=\delta(S)$. $CUT_n=MET_n$ holds for $n\leq 4$ and $\mathrm{Aut}(CUT_n)=\mathrm{Aut}(MET_n)$. The number of facets (orbits) of CUT_n is 3 (1), 12 (1), 40 (2), 210 (4), 38780 (36) for $3\leq n\leq 7$. $D(\mathrm{Sk}(CUT_n))=1$ and $D(\mathrm{Ri}(CUT_n))=2,3,3$ for $n=5,6,7$. See, for example, [DeLa97, DDF96, Du08] for details on MET_n and CUT_n.

The number of t-cuts of V_n is the number of ways to partition a set of n objects into t groups, i.e., the *Stirling number of the second kind* $S(n,t) = \frac{1}{t!}\sum_{j=0}^{t}(-1)^j\binom{t}{j}(t-j)^n$. So, $S(n,2) = 2^{n-1}-1$ and $S(n,n-1) = \binom{n}{2}$. The number of multicuts of V_n is the *Bell number* $B(n) = \sum_{t=0}^{n} S(n,t) = \sum_{t=0}^{n-1}(t+1)S(n,t) = \sum_{t=0}^{n-1}\binom{n-1}{t}B(t)$ (the sequence A000110 $= 1,1,2,5,15,52,203,877,\dots$ in [Sl10]). The number of ways to write i as a sum of positive integers is i-th *partition number* Q_i (the sequence $A000041$ in [Sl10]).

The $0,1$-valued elements $d \in MET_n$ are all $B(n)$ multicuts $\delta(\{S_1,\dots,S_t\})$ of V_n. It follows by induction using that $d_{1i} = d_{1j} = 0$ implies $d_{ij} = 0$ and $d_{1i} \neq d_{1j}$ implies $d_{ij} = 1$. In fact, $S_1,\dots,S_t$ are the equivalence classes of the equivalence $\sim$ on V_n, defined by $i \sim j$ if $d_{ij} = 0$.

$R(0,1\text{-}MET_n)$ consists of all $S(n,2)$ 2-cuts; so, $0,1\text{-}MET_n = CUT_n$.

3 Description of $wPMET_n$ and $sPMET_n$

Denote $MET_{n;0} = \{(d;(0)) : d \in MET_n\}$ and $CUT_{n;0} = \{(d;(0)) : d \in CUT_n\}$. So, $MET_n \simeq MET_{n;0} \simeq P(MET_{n;0}) \simeq Q(MET_{n;0})$ and $CUT_n \simeq CUT_{n;0} \simeq P(CUT_{n;0}) \simeq Q(CUT_{n;0})$. Denote by $WCUT_n$ the cone $\{d;w) \in WMET_n : d \in CUT_n\}$ of *weighted ℓ_1-semimetrics* on V_n.

Denote $e_j = (((0)); w = (w_i = 1_{i=j})) \in WMET_n$. So, $2P(e_j) = 2$ on the position (jj), 1 on $(ij),(ji)$ with $i \neq j$ and 0, else; $2Q(e_j) = -1$ on the positions (ji), 1 on (ij) (with $i \neq j$ again) and 0, else.

Theorem 1 *The following statements hold.*

(i) $R(WMET_n) = \{e_j : j \in V_n\} \cup R(MET_{n;0})$,
$R(wPMET_n) = \{2P(e_j) : j \in V_n\} \cup P(R(MET_{n;0}))$,
$R(wWQMET_n) = \{2Q(e_j) : j \in V_n\} \cup Q(R(MET_{n;0}))$.

(ii) $F(WMET_n) = \{w_j \geq 0 : j \in V_n\} \cup F(MET_{n;0})$,
$F(wPMET_n) = \{L_{jj} = p_{jj} \geq 0 : j \in V_n\} \cup F(P(MET_{n;0}))$,
$F(wWQMET_n) = \{w_j \geq 0 : j \in V_n\} \cup F(Q(MET_{n;0}))$.

(iii) $\mathrm{Inc}(2P(e_j)) = |F(wPMET_n)| - 1$ *and* $\mathrm{Inc}(L_{jj}) = |R(wPMET_n)| - 1$.

(iv) $\mathrm{Ri}(WMET_n) = \mathrm{Ri}(wPMET_n) = \mathrm{Ri}(wWQMET_n) = K_n \times \mathrm{Ri}(MET_n)$,
$\mathrm{Sk}(WMET_n) = \mathrm{Sk}(wPMET_n) = \mathrm{Sk}(wWQMET_n) = K_n \times \mathrm{Sk}(MET_n)$.

(v) $wPMET_n$ *has* Aut, $D(\mathrm{Sk})$, $D(\mathrm{Ri})$ *and edge-connectivity of* MET_n.

(vi) The $0,1$*-valued elements of* $wPMET_n$ *are the* $B(n+1)$ $0,1$*-valued elements of* $PMET_n$ *and* $0,1\text{-}wPMET_n = CUT_{n;0}$.

(vii) The $0,1$*-valued elements of* $WMET_n$ *are* $2^nB(n)$ $0,1$*-weighted multicuts of* V_n *and* $0,1$-$WMET_n = WCUT_n$.

$R(WCUT_n) = \{e_j : j \in V_n\} \cup R(CUT_{n;0})$ *and* $\mathrm{Sk}(WCUT_n) = K_{n+S(n,2)}$.

$F(WCUT_n) = \{w_j \geq 0 : j \in V_n\} \cup F(CUT_{n;0})$ *and* $\mathrm{Ri}(WCUT_n) = K_n \times \mathrm{Ri}(CUT_n)$ *has diameter* 2.

Proof.

(i). Let $p \in wPMET_n$. We will show that $p' = p - \frac{1}{2}\sum_{t=1}^{n} p_{tt} 2P(e_t) \in MET_{n;0}$. (For example, a well-known weak partial semimetric $i+j$ is the sum of $\sum_t t2P(e_t)$ and the all-zero semimetric $((0))$.)

In fact, $p'_{ii} = p_{ii} - \frac{1}{2}p_{ii}2P(e_i)_{ii} = 0$. Also, p' satisfies to all triangle inequalities (3), since for different $i,j,k \in V_n$, we have $Tr_{ij,k} = p'_{ij} + p'_{ik} - p'_{jk} =$

$$= \left(p_{ij} - \frac{p_{ii}+p_{jj}}{2}\right) + \left(p_{ik} - \frac{p_{ii}+p_{kk}}{2}\right) - \left(p_{jk} - \frac{p_{jj}+p_{kk}}{2}\right) =$$

$$= p_{ij} + p_{ik} - p_{jk} - p_{ii} \geq 0.$$

So, $2P(e_i)$, $1 \leq i \leq n$, and the generators of $P(MET_{n;0}) \simeq MET_{n;0}$ (i.e., the 0-extensions of the generators of MET_n) generate $wPMET_n$. They are, moreover, the generators of $wPMET_n$ since they belongs to all n (linearly independent) facets L_{ii}; so, their rank in $\mathbb{R}^{\binom{n+1}{2}}$is $(\binom{n}{2} - 1) + n = \binom{n+1}{2} - 1$.

Clearly, any $2P(e_i)$ belongs to all facets of $wPMET_n$ except L_{ii}, i.e., its incidence is $(n-1) + 3\binom{n}{3}$. So, its rank in $\mathbb{R}^{\binom{n+1}{2}}$ is $\binom{n+1}{2} - 1$. For $WMET_n$ and $wWQMET_n$, (i) follows similarly, as well as (ii).

(iii), (iv). The ray of $2P(e_i)$ is adjacent to any other extreme ray r, as the set of facets that contain r (with rank $\binom{n+1}{2} - 1$) only loses one element if we intersect it with the set of facets that contain $2P(e_i)$.

(v). The diameters of $\mathrm{Ri}(wPMET_n)$ and $\mathrm{Sk}(wPMET_n)$ being 2, their edge-connectivity is equal to their minimal degrees [Pl75]. But this degree is the same as of $\mathrm{Ri}(MET_n)$ (which is regular of degree $\frac{(n-3)(n^2-6)}{2}$ if $n > 3$) and of $\mathrm{Sk}(MET_n)$, respectively. $\mathrm{Aut}(wPMET_n)$ for $n \geq 5$ is $\mathrm{Sym}(n)$, because it contains $\mathrm{Sym}(n)$ but cannot be larger than $\mathrm{Aut}(MET_n) = \mathrm{Sym}(n)$.

(vi). If $p \in wPMET_n$ is $0,1$-valued, then $p_{ij} = 0 < p_{ii} = 1$ is impossible because $2p_{ij} \geq p_{ii} + p_{jj}$; so, $p \in PMET_n$. (vii) is implied by (i), (ii). $\square$

Any partial semimetric $p \in PMET_n$ induces the partial order on V_n by defining $i \preceq j$ if $p_{ii} = p_{ij}$. This *specialization order* is important in Computer Science applications, where the partial metrics act on certain posets called

Scott domains. In particular, $i_0 \in V_n$ is a *p-maximal* element in V_n if $p_{ii} = p_{ii_0}$ for all $i \neq i_0$. It is a *p-minimal* element in V_n if $p_{i_0i_0} = p_{ii_0}$ for all $i \neq i_0$. The *lifting* of $p \in PMET_n$ is the function $p^+ = ((p^+_{ij}))$, $i,j \in V_n$, with $p^+_{00} = 0$, $p^+_{0i} = p^+_{i0} = p^+_{ii}$ for $i \in V_n$ and $p^+_{ij} = p_{ij}$ for $i,j \in V_n$. Clearly, 0 is a p^+-maximal element in the specialization order, induced on $\{0\} \cup V_n = \{0,1,\dots,n\}$ by p^+, since $p^+_{ii} = p_{ii}$ as well as $p^+_{i0} = p_{ii}$ for all $i \in V_n$.

Theorem 2 *The following statements hold.*

(i) $sPMET_n) = \{p \in PMET_n : p^+ \in PMET_n\}$.

(ii) $sWMET_n = MET_{n+1} \simeq P(MET_{n+1}) = sPMET_n$.

(iii) The $0,1$*-valued elements of* $sPMET_n$ *are* $((0))$ *and* $2^n - 1$ *partial* 2*-cuts* $\gamma(S \neq \emptyset; \overline{S})$ *generating* $0,1$-$sPMET_n = CUT_{n+1}$,

$CUT_{n+1} \simeq P(CUT_{n+1}) = 0,1$-$sPMET_n$ *and* $Q(CUT_{n+1}) = OCUT_n$.

Proof. We should check for p^+ only inequalities (2), (3), (4) involving the new point 0. $2n+1$ of the required inequalities hold as equalities: $p^+_{00} = 0$ and $p^+_{0i} = p^+_{i0} = p^+_{ii} = p_{ii}$ for $i \in V_n$. All $Tr_{0j,i} = p_{ij} - p_{ij} \geq 0$ hold since (4) is satisfied. All $Tr_{ij,0} = p_{ii} + p_{jj} - p_{ij} \geq 0$ hold whenever p satisfies (5), i.e, $p \in PMET_n$. □

Given $p \in sPMET_n$, the semimetric $P^{-1}(p^+) \in MET_{n+2}$ is $P^{-1}(p) \in MET_{n+1}$ with the first point split in two coinciding points. The cone $sPMET_n$ is nothing but the linear image $P(sWPMET_n = MET_{n+1})$. So, for $n \geq 4$, $\mathrm{Aut}(sWPMET_n) = \mathrm{Sym}(n+1)$ on $\{0,1,\dots,n\}$ acting as $p' = P(\tau(P^{-1}(p)))$ on $sPMET_n$ for any $\tau \in \mathrm{Sym}(n+1)$. If τ fixes 0, then $p' = \tau(p)$.

4 $0,1$-valued elements of $PMET_n$ and $dWMET_n$

For a partition $\mathcal{S} = \{S_1,\dots,S_t\}$ of V_n and $A \subseteq \{1,\dots,t\}$, let us denote $\hat{A} = \bigcup_{h \in A} S_h$ and $w(\hat{A}) = (w_i = 1_{i \in \hat{A}})$. So, weight is constant on each S_h.

For any $S \subset V_n$, denote by $J(S)$ the $0,1$-valued function with $J(S)_{ij} = 1$ exactly when $i,j \in S_0$. So, $J(V_n)$ and $J(\emptyset)$ are all-ones and all-zeros partial semimetrics, respectively.

For any $S_0 \subset V_n$ and partition $\mathcal{S} = \{S_1,\dots,S_t\}$ of $\overline{S_0}$, denote $J(S_0) + \delta(S_0,S_1,\dots,S_t)$ by $\gamma(S_0;S_1,\dots,S_t)$ and call it a *partial multicut* or, specifically, a *partial t-cut*. Clearly, $\gamma(S_0;S_1,\dots,S_t) \in PMET_n$ and it is $P(d;w)$, where $d = 2\delta(S_0,S_1,\dots,S_t) - \delta(S_0) = \sum_{i=1}\delta(S_i)$ and $w = (w_i = 1_{i \in S_0})$.

Theorem 3 *The following statements hold.*

(i) The $0,1$*-valued elements of* $\mathrm{d}WMET_n$ *are* $\sum_{t=1}^{n} 2^t S(n,t)$ $(\delta(\mathcal{S}); w(\hat{A}))$.

$R(0,1\text{-}dWMET_n)$ *consists of all such elements with* $(|A|, t-|A|) = (1,0)$, $(0,2)$ *or* $(1,1)$*, i.e.,* $(((0)); (1))$ *and* 2*-cuts* $(\delta(S); w)$ *with weight* $(0), w' = (1_{i\in S})$ *or* $w'' = (1_{i\notin S})$*. There are* $1 + 3(2^{n-1} - 1)$ *of them, in* $\lfloor \frac{3n}{2} \rfloor$ *orbits.*

(ii) The $0,1$*-valued elements of* $WQMET_n$ *are* $Q(2\delta(\mathcal{S}) - \delta(\hat{A}; w(A)) = \delta(\mathcal{S}) - \delta'(\hat{A})$.

$R(0,1\text{-}dWMET_n)$ *consists of all such elements with either* $|A| = t-|A| = 1$ *(o-2-cuts), or* $2 \le |A|, t-|A| \le n-2$.

(iii) The $0,1$*-valued elements of* $PMET_n$ *are the partial multicuts*

$P((2\delta(\mathcal{S}) - \delta(\hat{A}); w(A)) = \delta(\mathcal{S}) + J(\hat{A})$ *with* $|A| \le 1$.

There are $B(n+1) = \sum_{i=0}^{n} \binom{n}{i} B(i)$ *of them, in* $\sum_{i=0}^{n} Q(i)$ *orbits.*

$R(0,1\text{-}PMET_n)$ *consists of all such elements except* $B(n) - (2^{n-1} - 1)$*, in* $Q(n) - \lfloor \frac{n}{2} \rfloor$ *orbits, those (*partial t-cuts*) with* $|A| = 0, t \neq 2$.

Proof.

(i). The $0,1$-valued elements of $WMET_n$ are $0,1$-weighted multicuts. Now, the inequality (4') $d_{ij} \ge w_i - w_j$, valid on $dWMET_n$, implies that w is constant on each S_h. $(((0)); (1))$ belongs to $R(0,1\text{-}dWMET_n)$ since its rank is $\binom{n}{2}$ plus $n-1$, the rank of the set of equalities $d_{ij} = w_i - w_j$. The same holds for $(\delta(S); w)$ with weight $(0), w' = (1_{i\in S})$ or $w'' = (1_{i\notin S})$, since their rank is $\binom{n}{2} - 1$ plus $k \ge 1$ equalities $w_i = 0$ plus, if $k < n$, $n-k$ equalities $d_{ij} = w_j - w_{i'} = 1$, where $w_{i'} = 0$ and $w_j = 1$. But the all-ones-weighted 2-cut δ is equal to $\frac{1}{2}((\delta; w') + (\delta; w'') + (((0))); (1))$. No other $(\delta(S_1, \dots, S_t); w)$ belongs to $R(0,1\text{-}dWMET_n)$ since t should be 2 (otherwise, the rank will be $< \binom{n}{2} - 1 + n$) and the weight should be constant on each S_h, $1 \le h \le t$.

(ii). Let $q \in WQMET$ be $0,1$-valued. Without loss of generality, let $\min_{i=1}^{n}(w_i) = w_1 = 0$. But $q_{1i} + w_1 = q_{i1} + w_i$ for any $i > 1$. So, $w_i = 1$ if and only if $q_{1i} \neq q_{i1}$. The quasi-semimetrics q, restricted on the sets $\{i : w_i = 0\}$ and $\{i : w_i = 1\}$, should be $0,1$-valued semimetrics, i.e., multicuts.

(iii) is proven in [DeDe10]. For example, there are $52 = 1\times1 + 4\times1 + 6\times2 + 4\times5 + 1\times15$ (1+1+2+3+5 orbits) $0,1$-valued elements of $PMET_4$. Among them, only $\delta(S_1, \dots, S_t)$ with $t = 1, 3, 4$, i.e., $((0))$, $\delta(\{1\}, \{2\}, \{3\}, \{4\})$ and 6 elements of the orbit with $t = 3$ are not representatives of extreme rays. □

5 Two generalized hypermetric cones

For a sequence $b = (b_1, \dots, b_n)$ of integers, where Σ_b denotes $\sum_{i=1}^n b_i$, and a symmetric $n \times n$ matrix $((a_{ij}))$, denote by $H_b(a)$ the sum $-\sum_{1 \le i,j \le n} b_i b_j a_{ij}$ of the entries of the matrix $-b^T ab$.

The cone HYP_n of all *hypermetrics*, i.e., semimetrics $d \in MET_n$ with $H_b(d) \ge 0$, whenever $\Sigma_b = 1$, was introduced in [De60].

This cone is polyhedral [DGL93]; $HYP_n \subseteq MET_n$ with equality for $n \le 4$ and $CUT_n \subseteq HYP_n$ with equality for $n \le 6$. HYP_7 was described in [DeDu04].

The hypermetrics have deep connections with Geometry of Numbers and Analysis; see, for example, [DeTe87, DeGr93, DGL95] and Chapters 13-17, 28 in [DeLa97]. So, generalizations of HYP_n can put those connections in a more general setting.

For a weighted semimetric $(d; w) \in WMET_n$, we will use the notation:
$\mathrm{Hyp}_b(d; w) = \frac{1}{2} H_b(d) + (1 - \Sigma_b)\langle b, w\rangle \ge 0$ and
$\mathrm{Hyp}'_b(d; w) = \frac{1}{2} H_b(d) + (1 + \Sigma_b)\langle b, w\rangle \ge 0$.

Denote by $WHYP_n$ the cone of all *weighted hypermetrics*, i.e., $(d; w) \in WMET_n$ with $\mathrm{Hyp}_b(d; w) \ge 0$ and $\mathrm{Hyp}'_b(d; w) \ge 0$ for all b with $\Sigma_b = 1$ or 0. Denote by $PHYP_n$ the cone of all *partial hypermetrics*, i.e., $p \in wPMET_n$ with $\mathrm{Hyp}_b(P^{-1}(p)) \ge 0$ for all b with $\Sigma_b = 1$ or 0. For $p = P(d; w)$ and $(q, w) = Q(d; w)$, we have

$$\mathrm{Hyp}_b(d; w) = H_b(p) + \sum_{i=1}^n b_i p_{ii} = H_b(q) + (1 - \Sigma_b)\langle b, w\rangle.$$

$WHYP_n \subset dWMET_n$ and $PMET_n \supset PHYP_n$ hold since the needed inequalities $w_i \ge 0$, (4') (and (4)) are provided by permutations of $\mathrm{Hyp}'_{(1,0,\dots,0)}(d; w) \ge 0$ and $\mathrm{Hyp}_{(1,-1,0,\dots,0)}(d; w) \ge 0$.

Lemma 2 *Besides the cases* $PMET_3 = PHYP_3 = 0,1\text{-}PMET_3$ *and* $0,1\text{-}dWMET_n = WHYP_n$ *for* $n = 3, 4$, $0,1\text{-}dWMET_n \subset WHYP_n \subset dWMET_n \simeq PMET_n \supset PHYP_n \supset 0,1\text{-}PMET_n$ *holds.*

Proof.
Denoting $\langle b, (1_{i \in S_h})\rangle$ by r_h, we have $r_0 = \Sigma_b - \sum_{h=1}^t r_h$ and

$$H_b(\delta(S_0, S_1, \dots, S_t)) = \frac{1}{2}\sum_{h=0}^t H_b(\delta(S_h, \overline{S_h})) = \sum_{h=0}^t r_h(r_h - \Sigma_b).$$

Let $(d = 2\delta(S_0, S_1, \dots, S_t) - \delta(S_0); w = (1_{i \in S_0}))$ be a generic $P^{-1}(p)$, where p is $0,1$-valued element of *PMET* belonging to its extreme ray. Then $\frac{1}{2}H_b(d) = \frac{1}{2}(2\sum_{h=0}^{t} r_h(r_h - \Sigma_b) - 2r_0(r_0 - \Sigma_b) = \sum_{h=1}^{t} r_h(r_h - \Sigma_b)$ implies

$\mathrm{Hyp}_b(d; w) = \sum_{h=1}^{t} r_h(r_h - 1) - \Sigma_b(\Sigma_b - 1) \geq 0$ for our $\Sigma = 0, 1$.

All $0,1$-valued elements $(d; w)$ of $dWMET_n$ belonging to its extreme rays are $(((0)); (1))$, $(\delta(S); (0))$, $(\delta(S,); w' = (1_{i \in S}))$ and $(\delta(S); w'' = (1_{i \notin S}))$. For them, $\mathrm{Hyp}'_b(d; w) = (\Sigma_b + 1)\Sigma_b$, $r_S(r_S - \Sigma)$, $r_S(r_S + 1)$ and $(\Sigma_b - r_S)(\Sigma_b - r_S + 1)$ hold, respectively, and so, $\mathrm{Hyp}'_b(d; w) \geq 0$ for our $\Sigma = 0, 1$.

Assuming polyhedrality of $WHYP_n$, the cases $n = 3, 4$ were checked by computation; see Lemma below. □

Lemma 3 *The following statements hold.*

(i) All facets of $WHYP_n$, $n \leq 4$, up to $\mathrm{Sym}(n)$ *and* 0*-extensions, are* Hyp_b *with* $b = (1, -1), (1, 1, -1), (1, 1, -1, -1)$ *and* Hyp'_b *with* $b = (1)$, $(1, 1, -1)$, $(1, 1, 1, -2)$, $(2, 1, -1, -1)$.

(ii) Besides $w_i \geq 0$, among the facets of $P^{-1}(PHYP_n)$, $n \leq 5$, up to $\mathrm{Sym}(n)$ *and* 0*-extensions, are:* Hyp_b *with* $b = (1, -1)$, $(1, 1, -1)$, $(1, 1, -1, -1)$, $(1, 1, 1, -1, -1)$, $(1, 1, 1, -1, -2)$, $(2, 1, 1, -1, -1)$.

Proof.

It was obtained by direct computation. The equality $WHYP_n = 0,1$-$WMET_n$ for $n = 3, 4$ holds, because only inequalities which are requested in $WHYP_n$ appeared among those of $0,1$-$WMET_n$.

The facets of $PHYP_4$ were deduced by computation using the tightness of the inclusions $0,1$-$PMET_4 \subset PHYP_4 \subset PMET_4$ (see Table 1): $0,1$-$PMET_4$ contained exactly one facet (orbit F_5) different from Hyp_b and $p_{ii} \geq 0$, and $PMET_4$ contained exactly two (orbits R_{10} and R_{11}) non-$0,1$-valued extreme ray representatives. The 6 rays from R_{10} are removed by 6 respective Hyp_b with $b = (1, 1, -1, -1)$, while the 12 rays from R_{11} are removed by 12 F_5. □

6 Oriented multicuts and quasi-semimetrics

For an ordered partition $(S_1, \dots, S_t)$ of V_n into non-empty subsets, the *oriented multicut* (or *o-multicut*, *o-t-cut*) $\delta'(S_1, \dots, S_t)$ on V_n is defined by:

$$\delta'_{ij}(S_1, \dots, S_t) = \begin{cases} 1, & \text{if } \; i \in S_h, j \in S_m, m > h, \\ 0, & \text{otherwise.} \end{cases}$$

R_i	Representative p	11	21	22	31	32	33	41	42	43	44	Inc.	Adj.	$\lvert O_i \rvert$
R_1	$\gamma(\{1,2,3,4\};)$	1	1	1	1	1	1	1	1	1	1	24	20	1
R_2	$\gamma(\{2\};\overline{\{2\}})$	0	1	1	0	1	0	0	1	0	0	21	38	4
R_3	$\gamma(\overline{\{3\}};\{3\})$	1	1	1	1	1	0	1	1	1	1	19	17	4
R_4	$\gamma(\emptyset;\overline{\{3\}},\{3\})$	0	0	0	1	1	0	0	0	1	0	19	32	4
R_5	$\gamma(\{1,2\};\overline{\{1,2\}})$	1	1	1	1	1	0	1	1	0	0	18	31	6
R_6	$\gamma(\emptyset;\{1,2\},\overline{\{1,2\}})$	0	0	0	1	1	0	1	1	0	0	16	32	3
R_7	$\gamma(\{1,4\};\{2\},\{3\})$	1	1	0	1	1	0	1	1	1	1	14	14	6
R_8	$\gamma(\{1\};\{2\},\{3,4\})$	1	1	0	1	1	0	1	1	0	0	14	20	12
R_9	$\gamma(\{4\};\{1\},\{2\},\{3\})$	0	1	0	1	1	0	1	1	1	1	9	9	4
R_{10}		1	1	0	1	1	0	2	1	1	1	10	18	6
R_{11}		0	2	0	1	1	0	2	2	3	2	9	9	12
F_1	$L_{11}: p_{11} \geq 0$	1	0	0	0	0	0	0	0	0	0	29	36	4
F_2	$\mathrm{Hyp}_{(-1,1,1,0)} \geq 0$	-1	1	0	1	-1	0	0	0	0	0	26	24	12
F_3	$M_{12} = \mathrm{Hyp}_{(-1,1,0,0)} \geq 0$	-1	1	0	0	0	0	0	0	0	0	23	23	12
F_4	$\mathrm{Hyp}_{(1,1,-1,-1)} \geq 0$	0	-1	0	1	1	-1	1	1	-1	-1	16	12	6
F_5	$H_{(2,1,-1,-1)} - 2p_{11} \geq 0$	-2	-2	0	2	1	0	2	1	-1	0	9	9	12

Table 1: The orbits of extreme rays in $PMET_4$ and facets in $0,1$-$PMET_4$

The o-2-multicuts $\delta^{'}(S,\overline{S})$ are called *o-cuts* and denoted by $\delta^{'}(S)$. Clearly,

$$\delta(S_1,\dots,S_t) = \sum_{i=1}^{t} \delta^{'}(S_i) = \sum_{i=1}^{t} \delta^{'}(\overline{S_i}) = \frac{1}{2}\sum_{i=1}^{t} \delta(S_i).$$

Denote by $OCUT_n$ and $OMCUT_n$ the cones generated by $2^n - 2$ non-zero o-cuts and $Bo(n) - 1$ non-zero o-multicuts, respectively. (Here, $Bo(n)$ are the *ordered Bell numbers* given by the sequence $A000670$ in [Sl10].) So, $CUT_n = \{q + q^T : q \in OMCUT_n\}$. In general, $Z_2 \times \mathrm{Sym}(n)$ is a symmetry group of $QMET_n$, $OMCUT_n$, $WQMET_n$, $OCUT_n$; Dutour, 2002, proved that it is the full group of those cones. The cones $QMET_n$ and $OMCUT_n$ were studied in [DDD03]. Clearly, $\delta^{'}_{ij}(S_1,\dots,S_t) \in WQMET_n$ if and only if $t = 2$ and then $w = (1_{i \notin S_1})$. So, $OCUT_n = OMCUT_n \cap WQMET_n$.

Theorem 4 *The following statement holds.*
$OCUT_n = Q(CUT_{n+1} = 0,1\text{-}sWMET_n) = Q(0,1\text{-}dWMET_n) = 0,1\text{-}Q(dWMET_n)$.

Proof. Given a representative $(d;w) = (\delta(S);w')$, $(\delta(S);w'')$, $(\delta(S);(0))$, $(\delta(\emptyset);(1))$ of an extreme ray of $0,1$-$dWMET_n$, we have $Q(q;w) = (\delta'(S),w')$, $(\delta'(\overline{S}),w'')$, $(\delta(S),(0))$, $(\delta(\emptyset),(1))$, respectively. But $\delta(S) = \delta'(S) + \delta'(\overline{S})$ and $(((0)),t(1))$ are not extreme rays. □

The above equality $OCUT_n = Q(0,1\text{-}sWMET_n)$ means that $q \in OCUT_n$ are $Q(d;w)$, where (d,w) is a semimetric $d' \in CUT_{n+1}$ on $V_n \cup \{0\}$. So,

$q_{ij} = \frac{1}{2}(d'_{ij} - d'_{0i} + d'_{0j})$. But CUT_n is the set of ℓ_1-*semimetrics* on V_n, see [DeLa97]. So, $q \in OCUT_n$ can be seen as ℓ_1-*quasi-semimetrics*; it was realized in [DDD03, CMM06]. In fact, $OCUT_n$ is the set of quasi-semimetrics q on V_n, for which there is some $x_1, \dots, x_m \in \mathbb{R}^m$ with all $q_{ij} = ||x_i - x_j||_{1.or}$, where the *oriented* ℓ_1-*norm* is defined as $||x-y||_{1.or} = \sum_{k=1}^{m} \max(x_k - y_k, 0)$; the proof is the same as in Proposition 4.2.2 of [DeLa97].

Let C be any cone closed under *reversal*, i.e., $q \in C$ implies $q^T \in C$. If the linear inequality $\sum_{1 \le i,j \le n} f_{ij} q_{ij} = \langle F, q \rangle \ge 0$ is valid on C, then F also defines a face of $\{q + q^T : q \in C\}$. Given a valid inequality $G : \sum_{1 \le i < j \le n} g_{ij} d_{ij}$ of $\{q + q^T : q \in C\}$ and an *oriented* K_n (i.e., exactly one arc connects any i and j) O, let $G^O = ((g^O_{ij}))$ where $g^O_{ij} = g_{ij}$ if the arc (ij) belongs to O and $= 0$, otherwise. Call G^O *standard* if there exists $\tau \in \mathrm{Sym}(n)$ with $(ij) \in O$ if and only if $\tau(i) < \tau(j)$, and *reversal-stable* (*rs* for short) if $\langle G^O, q \rangle = \langle G^O, q^T \rangle$. In general, G^O is not valid on C and does not preserve the rank of G.

For example, the standard $Tr_{12,3} : q_{13} + q_{23} - q_{12} \ge 0$ is not valid on $OCUT_n$, and the standard $L_{ij} : q_{ij} \ge 0$ defines a facet in $OCUT_n$, while $G : d_{ij} \ge 0$ only defines a face in MET_n. If $F = G^O$ is rs, then $\langle F, q \rangle = \frac{1}{2} \langle G, q + q^T \rangle$, i.e., F is valid on C if G is valid on $\{q + q^T : q \in C\}$.

Let E be an equality that holds on C, i.e., $\sum_{1 \le i,j \le n} e_{ij} q_{ij} = \langle E, q \rangle = 0$ holds for any $q \in C$. If the dimension of the subspace $\mathcal{E}$, spanned by all such equalities, is greater than zero, and $F \ge 0$ is a facet-defining inequality, then for any $E \in \mathcal{E}$, $F + E \ge 0$ defines the same facet. We call a facet standard or rs if one of its defining inequalities is standard or rs. Of all the defining inequalities we can choose one of them (up to a positive factor) to be the *canonical representative* – let it be such a $G = F + E$, $E \in \mathcal{E}$ that $\langle G, E \rangle = 0$ holds for all $E \in \mathcal{E}$, i.e., G is orthogonal to $\mathcal{E}$.

Lemma 4 *Let C be a cone closed under reversal, and $\mathcal{E}$ the subspace of its equalities. Then, the following statements hold.*

(i) If $C \subseteq WQMET_n$ is of the same dimension as $WQMET_n$, then $\mathcal{E}$ is spanned by the equalities $q_{ij} + q_{jk} + q_{ki} = q_{ji} + q_{kj} + q_{ik}$ for $i, j, k \in V_n$ and its dimension is $\binom{n-1}{2}$.

(ii) For each $E \in \mathcal{E}$, E is rs.

(iii) If a facet of C is rs, then all of its defining inequalities are rs.

(iv) A facet is rs iff its canonical representative G is symmetric*, e.g. $G = G^T$ holds.*

Proof.

(i). The equalities $E_{ijk} = q_{ij} + q_{jk} + q_{ki} - q_{ji} - q_{kj} - q_{ik} = 0$ for $i, j, k \in V_n$ follow directly from the weightability condition $q_{ij} + w_i = q_{ji} + w_j$. Since $E_{ijk} = E_{jki} = -E_{kji}$ and $E_{jk\ell} = E_{ijk} - E_{ij\ell} + E_{ik\ell}$ hold, we can choose a basis of $\mathcal{E}$ such that the indices (ijk) of the basis elements E_{ijk} are ordered triples that all contain a fixed element of V_n (say, n). There are $\binom{n-1}{2}$ such triples, and since all such E_{ijk} are linearly independent, the subspace $\mathcal{E}$ has dimension $\binom{n-1}{2}$.

(ii). As C is closed under reversal, each equality $E \in \mathcal{E}$ holds for both $q, q^T \in C$. Therefore, $0 = \langle E, q\rangle = \langle E, q^T\rangle$, so E is rs.

(iii). If F is a defining inequality of a facet and is rs, then for each $E \in \mathcal{E}$ and $q \in C$, $\langle F + E, q^T\rangle = \langle F + E, q\rangle$, so $F + E$, and by extension any defining inequality, is also rs.

(iv). Clearly, if G is symmetric, it is also rs and so is the facet it defines. If a facet is rs, then by (iii), so is its canonical representative G, for which $\langle G, q\rangle = \langle G, q^T\rangle = \langle G^T, q\rangle$ holds for all $q \in C$. Therefore, $G - G^T \in \mathcal{E}$, but as $G - G^T$ is also orthogonal to $\mathcal{E}$, $G = G^T$ follows. □

The facets $OTr_{ij,k}$ and L_{ij} (only 1st is rs) of $WQMET_n$ are standard and of the form Hyp_b where b is a permutation of $(1, 1, -1, 0, \dots, 0)$ or $(1, -1, 0, \dots, 0)$. $OCUT_4$ has one more orbit: six standard, non-rs facets of the form Hyp_b where b is a permutation of $(1, 1, -1, -1)$, or $q_{13} + q_{14} + q_{23} + q_{24} - (q_{12} + q_{34}) \geq 0$.

$OCUT_5$ has, up to $\mathrm{Sym}(n)$, 3 *new* (i.e., in addition to 0-extensions of the facets of $OCUT_4$) such orbits: one standard rs $(1, 1, 1, -1, -1)$ and two non-standard, non-rs orbits. $OCUT_6$ has, among its 56 new orbits, two non-standard rs orbits for $b = (2, 1, 1, -1, -1, -1)$ and $(1, 1, 1, 1, -1, -2)$.

The adjacencies of cuts in CUT_n are defined only by the facets $Tr_{ij,k}$, and adjacencies of those facets are defined only by cuts. It gives at once $\binom{n}{2} - 1$ linearly independent facets $OTr_{ij,k}$ containing any given pair $(\delta'(S_1), \delta'(S_2))$, using that $OTr_{ij,k}$ are rs facets. So, only n more facets are needed to get the adjacencies of o-cuts. It is a way to prove Conjecture 1 (i) below.

Call a *tournament* (K_n with unique arc between any i, j) *admissible* if its arcs can be partitioned into arc-disjoint directed cycles. It does not exists for even n, because then the number of arcs involving each vertex is odd, while each cycle provides 0 or 2 such arcs. But for odd n, there are at least $2^{\frac{n-3}{2}}$ admissible tournaments: take the decomposition of K_n into $\frac{n-1}{2}$ disjoint Hamiltonian cycles and, fixing the order on one them, all possible orders on remaining cycles. For odd n, denote by Oc the *canonic admissible tournament* consiting of all $(i, i+k)$ with $1 \leq i \leq n-1, 1 \leq k \leq \lceil \frac{n}{2} \rceil + 1 - i$ and $(i+k, i)$ with $1 \leq i \leq \lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil \leq k \leq n-i$, i.e., $0 = C_{1,2,3,4,5,6,7,\dots} + C_{1,3,5,7,\dots} + C_{1,4,7,\dots} + \dots$. The *Kelly conjecture* state that the arcs of every *regular* (i.e., the vertices have the same outdegree) tournament can be partitioned into arc-disjoint directed Hamiltonian cycles.

0-extensions of $q_{ij} \geq 0$ and $q_{13} + q_{14} + q_{23} + q_{24} - (q_{12} + q_{34}) \geq 0$ can be seen, as the first instances (for $b = (1, -1, 0, \dots, 0), (1, 1, -1, -1, 0, \dots, 0)$) of the *oriented negative type inequality* $\mathrm{ONeg}_{b,O}(q) = -\sum_{1 \leq i < j \leq n} b_i b_j q_{a(ij)} \geq 0$, where for a given $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$, $\Sigma_b = 0$, and the arcs $a(ij)$ on the edges (ij) by some rule defined by a given tournament O.

Denote by $OWHYP_n$ the cone consisting of all $q \in WQMET_n$, satisfying the two above orbits and all *oriented hypermetric inequalities*

$$\mathrm{OHyp}_{b,O}(q) = -\sum_{1 \leq i < j \leq n} b_i b_j q_{a(ij)} \geq 0,$$

where $b = (b_1, \dots, b_n) \in \mathbb{Z}^n$, $\Sigma_b = 1$, O is an admissible tournament, and the arc $a(ij)$ on the edge (ij) is the same as in O if $b_i b_j \geq 0$, or the opposite one otherwise. So, $OWHYP_n = OCUT_n$ for $n = 3, 4$.

Theorem 5 *$OCUT_n \subset OWHYP_n \subset WQMET_n$ holds for $n \geq 5$.*

Proof.

Without loss of generality, let $b_i = 1$ for $1 \leq i \leq \lfloor \frac{n}{2} \rfloor$ and $b_i = -1$, otherwise. The general case means only that we have sets of $|b_i|$ coinciding points. $\mathrm{OHyp}_{b,O}$ is rs, because Lemma 4 in [DeDe10] implies that any inequality on a $q \in WQMET_n$ is preserved by the reversal of q. So, $\mathrm{OHyp}_{b,O}(q) = \frac{1}{2}\,\mathrm{Hyp}_b(q + q^T)$. On an o-cut $\delta'(S)$ it gives, putting $r = \langle b, (1_{i \in S}) \rangle$,

$$\frac{1}{2}\,\mathrm{Hyp}_b(\delta'(S) + \delta'(\overline{S})) = \mathrm{Hyp}_b(\delta(S)) = r(r - \Sigma_b) \geq 0.$$

$OWHYP_5$ has, besides o-cuts, 40 extreme rays in two orbits: F_{ab}, F'_{ab}, having 2 on the position (ab), 1 on ba, 0 on three other (ka) for $k \neq b$ in F_{ab}, or on three other (bk) for $k \neq a$ in F'_{ab}, and ones on other non-diagonal positions. Also, $D(\mathrm{Sk}(OWHYP_5)) = D(\mathrm{Ri}(OWHYP_5)) = 2$.

The cone $QHYP_n = \{q \in QMET_n : ((q_{ij} + q_{ji})) \in HYP_n\}$ was considered in [DDD03]. Clearly, it is polyhedral and coincides with $QMET_n$

for $n = 3, 4$; $QHYP_5$ has 90 facets ($20 + 60$ from $QMET_5$ and those with $b = (1, 1, 1, -1, -1)$) and 78810 extreme rays; $D(\mathrm{Ri}(QHYP_5)) = 2$.

Besides $OCUT_3 = 0,1\text{-}WQMET_3 = WQMET_3$ and $0,1\text{-}WQMET_4 = WQMET_4$, $OCUT_n \subset 0,1\text{-}WQMET_n \subset WQMET_n$ holds. We conjecture $\mathrm{Sk}(OCUT_n) \subset \mathrm{Sk}(0,1\text{-}WQMET_n) \subset \mathrm{Sk}(WQMET_n)$ and $\mathrm{Ri}(0,1\text{-}WQMET_n) \supset \mathrm{Ri}(WQMET_n) \supset \mathrm{Ri}(MET_n)$. $0,1\text{-}WQMET_5$ has $OTr_{ij,k}$, L_{ij} and 3 other, all standard, orbits. Those facets give, for permutations of $b = (1, -1, 1, -1, 1)$, the non-negativity of $-\sum_{1 \le i < j \le 5} b_i b_j q_{ij}$ plus q_{24}, q_{23} or $q_{12} + q_{45}$.

The cone $\{q + q^T : q \in 0,1\text{-}WQMET_n\}$ coincides with MET_n for $n \le 5$, but for $n = 6$ it has 7 orbits of extreme rays (all those of MET_6 except the one, good representatives of which are not $0,1,2$-valued as required); its skeleton, excluding another orbit of 90 rays, is an induced subgraph of $\mathrm{Sk}(MET_6)$. It has 3 orbits of facets including $Tr_{ij,k}$ (forming $\mathrm{Ri}(MET_6)$ in its ridge graph) and the orbit of $\sum_{(ij) \in C_{123456}} d_{ij} + d_{14} + d_{35} - d_{13} - d_{46} - 2d_{25} \ge 0$.

If $q \in QMET_n$ is $0,1$-valued with $S = \{i : q_{i1} = 1\}$, $S' = \{i : q_{1i} = 1\}$, then $q_{ij} = 0$ for $i, j \in \overline{S} \cap \overline{S'}$ (since $q_{i1} + q_{1j} \ge q_{ij}$) and $q_{ij} = q_{ji} = 1$ for $i \in S, j \in \overline{S'}$ (since $q_{ij} + q_{j1} \ge q_{i1}$); so, $|\overline{S} \cap \overline{S'}|(|\overline{S} \cap \overline{S'}| - 1) + |S|(|\overline{S}| - 1) + |S'|(|\overline{S'}| - 1) - |S \cap \overline{S'}||\overline{S} \cap S'|$ elements q_{ij} with $2 \le i \ne j \le n$ are defined.

7 The cases of $3, 4, 5, 6$ points

In Table 2 we summarize the most important numeric information on cones under consideration for $n \le 6$. The column 2 indicates the dimension of the cone, the columns 3 and 4 give the number of extreme rays and facets, respectively; in parentheses are given the numbers of their orbits. The columns 5 and 6 give the diameters of the skeleton and the ridge graph. The expanded version of the data can be found on the third author's homepage [Vi10].

In the simplest case $n = 3$ the numbers of extreme rays and facets are:

$0,1\text{-}WMET_3 = WHYP_3 = WMET_3 \simeq wPMET_3$: $(6, 6$, simplicial) and $0,1\text{-}wPMET_3$: $(3, 3$, simplicial);

$0,1\text{-}sWMET_3 = sWMET_3 = CUT_4 = HYP_4 = MET_4 \simeq 0,1\text{-}sPMET_3 = sPMET_3$: $(7, 12)$;

$0,1\text{-}PMET_3 = PHYP_3 = PMET_3$ $(13, 12)$ and $0,1\text{-}dWMET_3$: $(10, 15)$;

$0,1\text{-}QMET_3 = QHYP_3 = QMET_3$: $(12, 12$, simplicial) and $OCUT_3 = 0,1\text{-}WQMET_3 = WQMET_3$: $(6, 9)$.

cone	dim.	Nr. ext. rays (orbits)	Nr. facets (orbits)	diam.	diam. dual
$wPMET_3$	6	6 (2)	6 (2)	1	1
$wPMET_4$	10	11 (3)	16 (2)	1	2
$wPMET_5$	15	30 (4)	35 (2)	2	2
$wPMET_6$	21	302 (8)	66 (2)	2	2
$sPMET_3 = 0,1\text{-}sPMET_3$	6	7 (2)	12 (1)	1	2
$sPMET_4$	10	25 (3)	30 (1)	2	2
$sPMET_5$	15	296 (7)	60 (1)	2	2
$sPMET_6$	21	55226 (46)	105 (1)	3	2
$0,1\text{-}sPMET_4$	10	15 (2)	40 (2)	1	2
$0,1\text{-}sPMET_5$	15	31 (3)	210 (4)	1	3
$0,1\text{-}sPMET_6$	21	63 (3)	38780 (36)	1	3
$PMET_3 = 0,1\text{-}PMET_3$	6	13 (5)	12 (3)	3	2
$PMET_4$	10	62 (11)	28 (3)	3	2
$PMET_5$	15	1696 (44)	55 (3)	3	2
$PMET_6$	21	337092 (734)	96 (3)	3	2
$PHYP_4$	10	56 (10)	34 (4)	3	2
$0,1\text{-}PMET_4$	10	44 (9)	46 (5)	3	2
$0,1\text{-}PMET_5$	15	166 (14)	585 (15)	3	3
$0,1\text{-}PMET_6$	21	705 (23)		3	
$0,1\text{-}dWMET_3$	6	10 (4)	15 (4)	2	2
$0,1\text{-}dWMET_4$	10	22 (6)	62 (7)	2	3
$0,1\text{-}dWMET_5$	15	46 (7)	1165 (27)	2	3
$0,1\text{-}dWMET_6$	21	94 (9)	369401 (806)	2	
$WQMET_3 = OCUT_3$	5	6 (2)	9 (2)	1	2
$WQMET_4 = 0,1\text{-}WQMET_4$	9	20 (4)	24 (2)	2	2
$WQMET_5$	14	190 (11)	50 (2)	2	2
$WQMET_6$	20	18502 (77)	90 (2)		2
$0,1\text{-}WQMET_5$	14	110 (8)	250 (5)	2	2
$0,1\text{-}WQMET_6$	20	802 (17)			
$\{q + q^T : q \in 0,1\text{-}WQMET_6\}$	15	206 (7)	510 (3)	2	3
$OWHYP_5$	14	70 (6)	90 (4)	2	2
$OCUT_4$	9	14 (3)	30 (3)	1	2
$OCUT_5$	14	30 (4)	130 (6)	1	3
$OCUT_6$	20	62 (5)	16460 (62)	1	

Table 2: Main parameters of cones with $n \le 6$

$R(dWMET_3) \setminus R(0,1\text{-}dWMET_3)$ and $F(0,1\text{-}dWMET_3) \setminus F(dWMET_3)$ consist of 3 simplicial elements forming $\overline{K_3}$ in the graph. But only Ri$(0,1\text{-}dWMET_3)$ is an induced subgraph of Ri$(dWMET_3)$.

Recall that $2^{n-1}-1$ is the Stirling number $S(n,2)$, Sk$(CUT_n) = K_{S(n,2)}$, and [DeDe94] Ri(MET_n), $n \ge 4$, has diameter 2 with $Tr_{ij,k} \nsim Tr_{i'j',k'}$ whenever they are *conflicting*, i.e., have values of different sign on a position (p,q), $p,q \in \{i,j,k\} \cap \{i',j',k'\}$. Clearly, $|\{i,j,k\} \cap \{i',j',k'\}|$ should be 3 or 2, and $Tr_{ij,k}$ conflicts with 2 and $4(n-3)$ $Tr_{i'j',k'}$'s, respectively. The proofs of the conjectures below should be tedious but easy.

R_i	Representative	11	21	22	31	32	33	Inc.	Adj.	$\lvert R_i \rvert$
R_1 ▲	$\gamma(\{1,2,3\};)$	1	1	1	1	1	1	9	6	1
R_2 ○	$\gamma(\{1\};\{2,3\})$	1	1	0	1	0	0	8	9	3
R_3 ●	$\gamma(\{2,3\};\{1\})$	0	1	1	1	1	1	7	6	3
R_4 □	$\gamma(\emptyset;\{1\},\{2,3\})$	0	1	0	1	0	0	7	8	3
R_5 ■	$\gamma(\{3\};\{1\},\{2\})$	0	1	0	1	1	1	5	5	3
F_1 ○	$L_{11} : p_{11} \geq 0$	1	0	0	0	0	0	8	9	3
F_2 ▲	$Tr_{12,3} : p_{13} + p_{23} - p_{12} - p_{33} \geq 0$	0	-1	0	1	1	-1	8	7	3
F_3 ●	$M_{12} : p_{12} - p_{11} \geq 0$	-1	1	0	0	0	0	7	6	6

Table 3: The orbits of extreme rays and facets in $PMET_3 = 0,1\text{-}PMET_3$

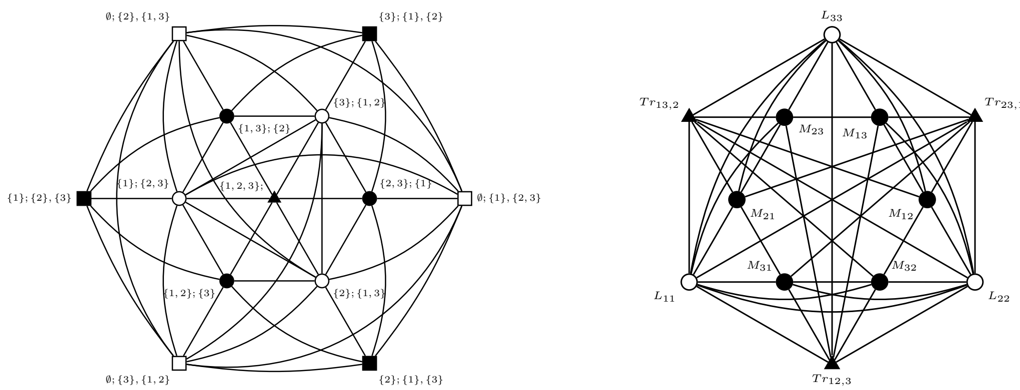

Figure 1: The skeleton and ridge graph of $PMET_3 = 0,1\text{-}PMET_3$

Conjecture 1 *(i)* $\mathrm{Sk}(OCUT_n) = K_{2S(n,2)}$ *and belongs to* $\mathrm{Sk}(WQMET_n)$.
(ii) $\overline{\mathrm{Sk}(0,1\text{-}dWMET_n)} = K_{1,S(n,2)} + S(n,2)K_2$;
$\mathrm{Sk}(0,1\text{-}dWMET_n)$ *has diameter* 2*, all non-adjacencies are of the form:*
$(((0));(1)) \nsim (\delta'(S);(0))$ *and* $(\delta'(S);w') \nsim (\delta'(S);w')$.

Conjecture 2 *(i)* $\mathrm{Ri}(PMET_n)$ *has diameter* 2*, all non-adjacencies are:*
$L_{ii} \nsim M_{ik}$; $M_{ij} \nsim M_{ji}, M_{ki}, M_{jk}, Tr_{ij,k}$; $Tr_{ij,k} \nsim Tr_{i'j',k'}$ *if they conflict.*
(ii) $\mathrm{Ri}(WQMET_n)$ *has diameter* 2*; it is* $\mathrm{Ri}(PMET_n)$ *without vertices* L_{ii}.

R_i	Representative	1	21	2	31	32	3	Inc.	Adj.	$\lvert R_i \rvert$
R_1 ▲	$(\delta(\emptyset);(1))$	1	0	1	0	0	1	9	6	1
R_2 ○	$(\delta(\{1\};w'')$	0	1	1	1	0	1	9	8	3
R_3 ●	$(\delta(\{1\};w')$	1	1	0	1	0	0	9	8	3
R_4 □	$(\delta(\{1\};(0))$	0	1	0	1	0	0	9	8	3
F_1 ○	$L_1 : w_1 \geq 0$	1	0	0	0	0	0	6	9	3
F_2 ▲	$Tr_{12,3} : d_{13} + d_{23} - d_{12} \geq 0$	0	-1	0	1	1	0	7	8	3
F_3 ●	$M'_{12} : d_{12} + (w_2 - w_1) \geq 0$	-1	1	1	0	0	0	6	6	6
F_4 △	$Tr'_{12,3} : (d_{13} + d_{23} - d_{12}) + 2(w_1 + w_2 - w_3) \geq 0$	2	-1	2	1	1	-2	5	5	3

Table 4: The orbits of extreme rays and facets in $0,1\text{-}dWMET_3$

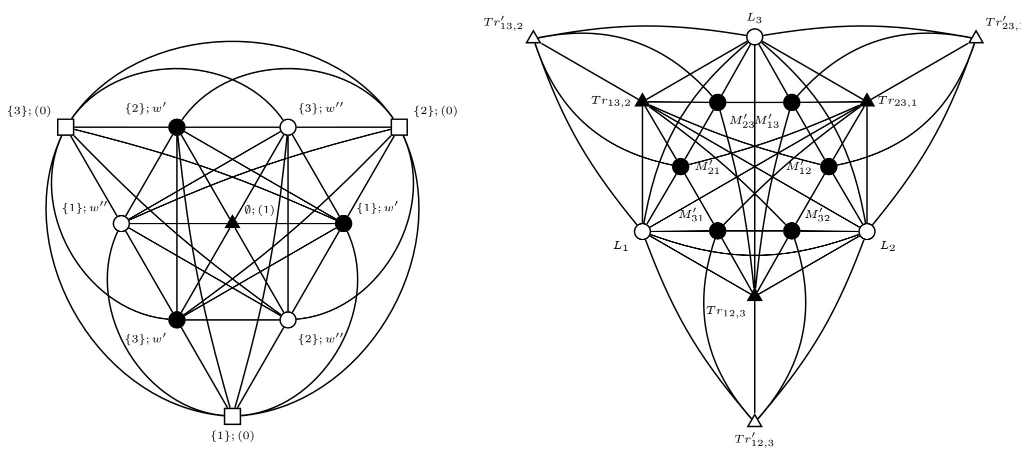

Figure 2: The skeleton and ridge graph of $0,1\text{-}dWMET_3$

References

[CMM06] M. Charikar, K. Makarychev and Y. Makarychev, *Directed Metrics and Directed Graph Partitioning Problems*, Proc. of 17th ACM-SIAM Symposium on Discrete Algorithms (2006) 51–60.

[DeDe94] A. Deza and M. Deza, *The ridge graph of the metric polytope and some relatives*, in T. Bisztriczky, P. McMullen, R. Schneider and A. Ivic Weiss eds. Polytopes: Abstract, Convex and Computational (1994) 359–372.

[DDF96] A. Deza, M. Deza and K. Fukuda, *On Skeletons, Diameters and Volumes of Metric Polyhedra*, in Combinatorics and Computer Science, Lecture Notes in Computer Science **1120**, Springer (1996) 112–127.

[De60] M. Tylkin (=M. Deza), *Hamming geometry of unitary cubes*, Doklady Akademii Nauk SSSR **134-5** (1960) 1037–1040. (English translation in *Cybernetics and Control Theory* **134-5** (1961) 940–943.

[DeDe10] M. Deza and E. Deza, *Cones of Partial Metrics*, Contributions in Discrete Mathematics, 2010.

[DDD03] M. Deza, M. Dutour and E. Deza, *Small cones of oriented semi-metrics*, American Journal of Mathematics and Management Science **22-3,4** (2003) 199–225.

[DeDu04] M. Deza and M. Dutour, *The hypermetric cone on seven vertices*, Experimental Mathematics **12** (2004) 433–440.

[DGL93] M. Deza, V. P. Grishukhin and M. Laurent, *The hypermetric cone is polyhedral*, Combinatorica 13 (1993) 397–411.

[DeGr93] M. Deza and V. P. Grishukhin, *Hypermetric graphs*, The Quarterly Journal of Mathematics Oxford, 2 (1993) 399–433.

[DGL95] M. Deza, V. P. Grishukhin and M. Laurent, *Hypermetrics in geometry of numbers*. In W. Cook, L. Lovász and P. Seymour, editors, *Combinatorial Optimization*, DIMACS Series in Discrete Mathematics and Theoretical Computer Science **20** AMS (1995) 1–109.

[DeLa97] M. Deza and M. Laurent, *Geometry of cuts and metrics*, Springer-Verlag, Berlin, 1997.

[DeTe87] M. Deza and P. Terwilliger, *The classification of finite connected hypermetric spaces*, Graphs and Combinatorics **3** (1987) 293–298.

[Du08] M. Dutour Sikirić, *Cut and Metric Cones*,
`http://www.liga.ens.fr/~dutour/Metric/CUT_MET/index.html`.

[Du10] M. Dutour Sikirić, *Polyhedral*,
`http://www.liga.ens.fr/~dutour/polyhedral`.

[Fu95] K. Fukuda, *The cdd program*,
`http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html`.

[Gr92] V. P. Grishukhin, *Computing extreme rays of the metric cone for seven points*, European Journal of Combinatorics **13** (1992) 153–165.

[He99] R. Heckmann, *Approximation of Metric Spaces by Partial Metric Spaces*, Applied Categorical Structures **7** (1999) 7–83.

[Hi01] P. Hitzler, *Generalized Metrics and Topology in Logic Programming Semantics*, PhD Thesis, Dept. Mathematics, National University of Ireland, University College Cork, 2001.

[Ma92] S. G. Matthews, *Partial metric topology*, Research Report 212, Dept. of Computer Science, University of Warwick, 1992.

[Ma08] S. G. Matthews, A collection of resources on partial metric spaces, available at `http://partialmetric.org`, 2008.

[Pl75] J. Plesník, *Critical graphs of given diameter*, Acta Math. Univ. Comenian **30** (1975) 71–93.

[Sc86] A. Schrijver, *Theory of Linear and Integer Programming*, Wiley, 1986.

[Se97] A. K. Seda, *Quasi-metrics and the semantic of logic programs*, Fundamenta Informaticae **29** (1997) 97–117.

[Sl10] N. Sloane, *The On-Line Encyclopedia of Integer Sequences*, published electronically at `http://oeis.org`, 2010.

[Vi10] J. Vidali, *Cones of Weighted and Partial Metrics*,
`http://lkrv.fri.uni-lj.si/~janos/cones/`.