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ON SUMS RELATED TO CENTRAL BINOMIAL AND TRINOMIAL COEFFICIENTS

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Abstract. A generalized central trinomial coefficient $T_n(b,c)$ is the coefficient of x^n in the expansion of $(x^2+bx+c)^n$ with $b,c\in\mathbb{Z}$. In this paper we investigate congruences and series for sums of terms related to both central binomial coefficients and generalized central trinomial coefficients. For example, for any odd prime p we show that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}$$
$$\equiv\begin{cases}(\frac{-1}{p})4x^2\pmod p&\text{if }(\frac p7)=1\text{ and }p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if }(\frac p7)=-1,\text{i.e., }\ p\equiv 3,5,6\pmod 7\end{cases}$$

(with $(-)$ the Jacobi symbol), and conjecture the congruence

$$\sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}\equiv 7p\left(\frac{-1}{p}\right)\pmod{p^2}$$

as well as the following identity

$$\sum_{k=0}^{\infty}(30k+7)\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}=\frac{24}{\pi}.$$

The paper contains many conjectures on congruences and 48 proposed new series for $1/\pi$ motivated by congruences and related dualities.

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1. Introduction

Let p be an odd prime. Clearly

$$\binom{2k}{k}=\frac{(2k)!}{(k!)^2}\equiv 0 \pmod p \quad \text{for each } k=\frac{p+1}{2},\ldots,p-1.$$

The author [Su1] determined $\sum_{k=0}^{p-1}\binom{2k}{k}/m^k \bmod p^2$ for any integer $m\not\equiv 0 \pmod p$ in terms of Lucas sequences. In [Su2] the author made a conjecture on $\sum_{k=0}^{p-1}\binom{2k}{k}^2/m^k \bmod p^2$ with $m=8,-16,32$ and this was confirmed by the author's twin brother Z. H. Sun [S1]. Conjecture 5.4 of the author [Su2] states that

$$\sum_{k=0}^{p-1}\binom{2k}{k}^3\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } (\frac{p}{7})=1 \ \&\ p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0 \pmod{p^2} & \text{if } (\frac{p}{7})=-1,\ \text{i.e., } p\equiv 3,5,6 \pmod 7.\end{cases}$$

(As usual, $(-)$ denotes the Jacobi symbol.) To attack this conjecture and the author's other similar conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k \bmod p^2$ (with m a suitable integer not divisible by p) given in [Su4], Z. H. Sun [S2] found the useful combinatorial identity

$$\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}^2x^k=P_n(\sqrt{1+4x})^2 \tag{1.1}$$

where $P_n(x)$ is the Legendre polynomial of degree n given by

$$P_n(x):=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{x-1}{2}\right)^k.$$

We can rewrite this in the form

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\binom{2k}{k}(x(x+1))^k=D_n(x)^2 \tag{1.2}$$

where $D_n(x)$ is the Delannoy polynomial of degree n given by

$$D_n(x):=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}x^k.$$

Note that those $D_n=D_n(1)$ $(n=0,1,2,\ldots)$ are central Delannoy numbers (see, e.g., [CHV], [Su3] and [St, p. 178]). It is well known that $P_n(-x)=$

$(-1)^nP_n(x)$, i.e., $(-1)^nD_n(x) = D_n(-x-1)$ (cf. [Su3, Remark 1.2]). As observed by Z. H. Sun [S1, Lemma 2.2], if $0 \leqslant k \leqslant n = (p-1)/2$ then

$$\binom{n+k}{2k} \equiv \frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}$$

and hence

$$\binom{n}{k}\binom{n+k}{k} = \binom{n+k}{2k}\binom{2k}{k} \equiv \frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p^2}.$$

This simple trick was also realized by van Hamme [vH, p. 231]. Combining this useful trick with the identity (1.2), we see that

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^3}{(-16)^k}(x(x+1))^k \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(-16)^k}x^k\right)^2 \pmod{p^2}. \tag{1.3}$$

To study the author's conjectures on

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{m^k},\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{m^k},\ \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{m^k}$$

modulo p^2 given in [Su4] and [Su6], Z. H. Sun [S3,S4,S5] managed to prove the following congruences similar to (1.3):

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{4k}{2k}}{(-64)^k}(x(x+1))^k \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}x^k\right)^2 \pmod{p^2}, \tag{1.4}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}(x(x+1))^k \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}x^k\right)^2 \pmod{p^2}\ (p>3), \tag{1.5}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}\binom{6k}{3k}}{(-432)^k}(x(x+1))^k \equiv \left(\sum_{k=0}^{p-1} \frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}x^k\right)^2 \pmod{p^2}\ (p>3). \tag{1.6}$$

Let $n \in \mathbb{N} = \{0,1,2,\ldots\}$. Clearly $\binom{2n}{n}$ is the coefficient of x^n in the expansion of $(x^2+2x+1)^n = (x+1)^{2n}$. The nth central trinomial coefficient

$$T_n = [x^n](x^2+x+1)^n$$

is the coefficient of x^n in the expansion of $(x^2+x+1)^n$. Since T_n is the constant term of $(1+x+x^{-1})^n$, by the multi-nomial theorem we see that

$$T_n = \sum_{k=0}^{\lfloor n/2\rfloor} \frac{n!}{k!k!(n-2k)!} = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k} = \sum_{k=0}^{n}\binom{n}{k}\binom{n-k}{k}.$$

Central trinomial coefficients arise naturally in enumerative combinatorics (cf. [Sl]), e.g., T_n is the number of lattice paths from the point $(0,0)$ to $(n,0)$ with only allowed steps $(1,1)$, $(1,-1)$ and $(1,0)$.

Given $b,c\in\mathbb{Z}$, we define the *generalized central trinomial coefficients*

$$\begin{aligned}T_n(b,c):=&[x^n](x^2+bx+c)^n=[x^0](b+x+cx^{-1})^n\\ =&\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}\binom{2k}{k}b^{n-2k}c^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}b^{n-2k}c^k.\end{aligned}\tag{1.7}$$

Clearly $T_n(2,1)=\binom{2n}{n}$ and $T_n(1,1)=T_n$. An efficient way to compute $T_n(b,c)$ is to use the initial values $T_0(b,c)=1$ and $T_1(b,c)=b$, and the recursion

$$(n+1)T_{n+1}(b,c)=(2n+1)bT_n(b,c)-n(b^2-4c)T_{n-1}(b,c)\ \ (n=1,2,\ldots).$$

Note that the recursion is rather simple if $b^2-4c=0$.

Let $b,c\in\mathbb{Z}$ and $d=b^2-4c$. It is known that $T_n(b,c)=\sqrt{d}^nP_n(b/\sqrt{d})$ if $d\neq0$ (see, e.g., [N] and [Su5]). Thus

$$T_n(b,c)=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}\left(\frac{b-\sqrt{d}}{2}\right)^k\sqrt{d}^{n-k}.\tag{1.8}$$

(In the case $d=0$, (1.8) holds trivially since $x^2+bx+c=(x+b/2)^2$.) By the Laplace-Heine formula (cf. [Sz, p. 194]), for any complex number $x\notin[-1,1]$ we have

$$P_n(x)\sim\frac{(x+\sqrt{x^2-1})^{n+1/2}}{\sqrt{2n\pi}\sqrt[4]{x^2-1}}\quad\text{ as } n\to+\infty.$$

It follows that if $b>0$ and $c>0$ then

$$T_n(b,c)\sim f_n(b,c):=\frac{(b+2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}\quad\text{ as } n\to+\infty.\tag{1.9}$$

Note that $T_n(-b,c)=(-1)^nT_n(b,c)$.

We consider generalized central trinomial coefficients as natural extensions of central binomial coefficients. As

$$T_k(2,1)=\binom{2k}{k},\ T_{2k}(2,1)=\binom{4k}{2k}\text{ and }T_{3k}(2,1)=\binom{6k}{3k},$$

we are led to investigate more general sums

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{m^k}T_k(b,c),\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{m^k}T_k(b,c),\ \sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{m^k}T_k(b,c)$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{m^k}T_{2k}(b,c),\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{m^k}T_{3k}(b,c)$$

modulo p^2, where p is an odd prime, $b,c,m\in\mathbb{Z}$ and $m\not\equiv0\pmod p$. For this purpose, we need to extend those congruences (1.3)-(1.6).

Theorem 1.1. *Let p be a prime and let $a \in \mathbb{Z}^+ = \{1, 2, 3, \ldots\}$. Let h be a p-adic integer and set $w_k(h) = \binom{h}{k}\binom{h+k}{k}$ for $k \in \mathbb{N}$. Then*

$$\begin{aligned}&\bigg(\sum_{k=0}^{p^a-1} w_k(h)x^k\bigg)\bigg(\sum_{k=0}^{p-1} w_k(h)y^k\bigg)\\ &\equiv\sum_{k=0}^{p^a-1} w_k(h)\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j} \pmod{p^2}.\end{aligned}\tag{1.10}$$

In particular, if $p \neq 2$ then

$$\begin{aligned}&\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{(-16)^k}x^k\bigg)\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{(-16)^k}y^k\bigg)\\ &\equiv\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{(-16)^k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j} \pmod{p^2}\end{aligned}\tag{1.11}$$

and

$$\begin{aligned}&\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}x^k\bigg)\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}y^k\bigg)\\ &\equiv\sum_{k=0}^{p^a-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j} \pmod{p^2};\end{aligned}\tag{1.12}$$

provided $p > 3$ we have

$$\begin{aligned}&\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}x^k\bigg)\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}y^k\bigg)\\ &\equiv\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j} \pmod{p^2}\end{aligned}\tag{1.13}$$

and

$$\begin{aligned}&\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}x^k\bigg)\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}y^k\bigg)\\ &\equiv\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}\sum_{j=0}^{k}\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j} \pmod{p^2}.\end{aligned}\tag{1.14}$$

Remark 1.1. Note that

$$w_k\left(-\frac{1}{2}\right)=\frac{\binom{2k}{k}^2}{(-16)^k},\quad w_k\left(-\frac{1}{4}\right)=\frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k},$$
$$w_k\left(-\frac{1}{3}\right)=\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k},\quad w_k\left(-\frac{1}{6}\right)=\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}.$$

Also, (1.11)-(1.14) in the case $x=y$ and $a=1$ yield (1.3)-(1.6) respectively.

The reader may wonder how we found Theorem 1.1. In fact, we first discovered the identity

$$D_n(x)D_n(y)=\sum_{k=0}^n\binom{n+k}{2k}\binom{2k}{k}\sum_{j=0}^k\binom{k+j}{2j}\binom{2j}{j}(xy+y)^j(x-y)^{k-j},\ \ (1.15)$$

which is a natural extension of (1.2) and the main clue to the congruence (1.11). By refining our proof of (1.11)-(1.14) we found (1.10).

Theorem 1.1 implies the following useful result on congruences for sums of central binomial coefficients and generalized central trinomial coefficients.

Theorem 1.2. *Let p be an odd prime and let x be a p-adic integer. Let $a\in\mathbb{Z}^+$, $b,c\in\mathbb{Z}$ and $d=b^2-4c$. Set $D:=1+2bx+dx^2$. Then we have*

$$\begin{aligned}&\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{(-16)^k}T_k(b,c)x^k\\ \equiv&\left(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{32^k}(1-\sqrt{D}+\sqrt{d}\,x)^k\right)\left(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}^2}{32^k}(1-\sqrt{D}-\sqrt{d}\,x)^k\right)\\ \equiv&P_{(p^a-1)/2}(\sqrt{D}+\sqrt{d}\,x)P_{(p^a-1)/2}(\sqrt{D}-\sqrt{d}\,x)\pmod{p^2}\end{aligned}\qquad(1.16)$$

and

$$\begin{aligned}\sum_{k=0}^{p^a-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(-64)^k}T_k(b,c)x^k\equiv&\left(\sum_{k=0}^{p^a-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{128^k}(1-\sqrt{D}+\sqrt{d}\,x)^k\right)\\ &\times\sum_{k=0}^{p^a-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{128^k}(1-\sqrt{D}-\sqrt{d}\,x)^k\pmod{p^2}.\end{aligned}\qquad(1.17)$$

If $p>3$, then

$$\begin{aligned}\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-27)^k}T_k(b,c)x^k\equiv&\left(\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{54^k}(1-\sqrt{D}+\sqrt{d}\,x)^k\right)\\ &\times\sum_{k=0}^{p^a-1}\frac{\binom{2k}{k}\binom{3k}{k}}{54^k}(1-\sqrt{D}-\sqrt{d}\,x)^k\pmod{p^2}\end{aligned}\qquad(1.18)$$

and

$$\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{(-432)^k}T_k(b,c)x^k \equiv\bigg(\sum_{k=0}^{p^a-1}\frac{\binom{6k}{3k}\binom{3k}{k}}{864^k}(1-\sqrt{D}+\sqrt{d}\,x)^k\bigg) \times\sum_{k=0}^{p^a-1}\frac{\binom{6k}{k}\binom{3k}{k}}{864^k}(1-\sqrt{D}-\sqrt{d}\,x)^k \pmod{p^2}. \tag{1.19}$$

Remark 1.2. Note that $\sqrt{d}$ and $\sqrt{D}$ in Theorem 1.2 are viewed as algebraic p-adic integers.

For $d\in\{2,3,4\}$, it is well known that an odd prime p can be written in the form x^2+dy^2 with $x,y\in\mathbb{Z}$ if and only if $(\frac{-d}{p})=1$ (see, e.g., [BEW] and [Co]). For a prime $p=x^2+4y^2$ with $x,y\in\mathbb{Z}$ and $x\equiv 1 \pmod 4$, Gauss' congruence $\binom{(p-1)/2}{(p-1)/4}\equiv 2x \pmod p$ was further refined by S. Chowla, B. Dwork and R. J. Evans [CDE] in 1986 who used Gauss and Jacobi sums to prove that

$$\binom{(p-1)/2}{(p-1)/4}\equiv\frac{2^{p-1}+1}{2}\left(2x-\frac{p}{2x}\right)\pmod{p^2}$$

(which was first conjectured by F. Beukers), and this implies that

$$\binom{(p-1)/2}{(p-1)/4}^2\equiv 2^{p-1}(4x^2-2p)\pmod{p^2}.$$

For results alone this line the reader may consult the survey [HW] by R. H. Hudson and K. S. Williams.

Applying (1.16) we get the following new results.

Theorem 1.3. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(1,-2)}{32^k} \equiv\begin{cases}(\frac{2}{p})(4x^2-2p)\pmod{p^2} & \text{if } 4\mid p-1 \text{ and } p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases} \tag{1.20}$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(2,-1)}{8^k} \equiv\begin{cases}(\frac{-1}{p})4x^2\pmod{p} & \text{if } (\frac{-2}{p})=1 \text{ and } p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } (\frac{-2}{p})=-1,\ \text{i.e., } p\equiv 5,7\pmod 8;\end{cases} \tag{1.21}$$

$$\begin{aligned}&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(4,1)}{(-4)^k}\\ \equiv&\begin{cases}4x^2\pmod p&\text{if } p\equiv 1\pmod 3 \text{ and } p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } p\equiv 2\pmod 3;\end{cases}\end{aligned}\tag{1.22}$$

and

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k(16,1)\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}\\ \equiv&\begin{cases}4x^2\pmod p&\text{if } (\frac{p}{7})=1 \text{ and } p=x^2+7y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2}&\text{if } (\frac{p}{7})=-1, \text{i.e.},\ p\equiv 3,5,6\pmod 7.\end{cases}\end{aligned}\tag{1.23}$$

Remark 1.3. Let p be an odd prime. We guess that $4x^2 \pmod p$ in (1.21)-(1.23) can be replaced by $4x^2-2p \pmod{p^2}$. Motivated by Theorem 1.3 and the congruence

$$\sum_{k=0}^{p-1}(21k+8)\binom{2k}{k}^3\equiv 8p+16p^4B_{p-3}\pmod{p^5}$$

(where $B_0,B_1,B_2,\ldots$ are Bernoulli numbers) proved in [Su4], we conjecture that

$$\begin{aligned}\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^2T_k(1,-2)}{32^k}&\equiv\left(\frac{-2}{p}\right)\frac{2p}{3-(\frac{-1}{p})}\pmod{p^2},\\ \sum_{k=0}^{p-1}(5k+2)\frac{\binom{2k}{k}^2T_k(2,-1)}{8^k}&\equiv p+p\left(\frac{-1}{p}\right)\pmod{p^2},\\ \sum_{k=0}^{p-1}(5k+2)\frac{\binom{2k}{k}^2T_k(4,1)}{(-4)^k}&\equiv\frac{2}{3}p\left(2\left(\frac{-1}{p}\right)+1\right)\pmod{p^2},\\ \sum_{k=0}^{p-1}(255k+112)(-1)^k\binom{2k}{k}^2T_k(16,1)&\equiv 16p\left(3+4\left(\frac{-1}{p}\right)\right)\pmod{p^2},\\ \sum_{k=0}^{p-1}(30k+7)\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}&\equiv 7p\left(\frac{-1}{p}\right)\pmod{p^2}.\end{aligned}$$

The last congruence led the author to find the conjectural identity

$$\sum_{k=0}^{p-1}\frac{30k+7}{(-256)^k}\binom{2k}{k}^2T_k(1,16)=\frac{24}{\pi}$$

in Jan 2011 which was the starting point of the discovery of over 40 series for $1/\pi$ of new types given in Section 5.

Recall that for given numbers A and B the Lucas sequence $u_n = u_n(A,B)$ ($n \in \mathbb{N}$) and its companion $v_n = v_n(A,B)$ ($n \in \mathbb{N}$) are defined by

$$u_0 = 0,\ u_1 = 1,\ u_{n+1} = Au_n - Bu_{n-1}\ (n = 1,2,3,\ldots),$$

and

$$v_0 = 2,\ v_1 = A,\ v_{n+1} = Av_n - Bv_{n-1}\ (n = 1,2,3,\ldots).$$

It is well-known that

$$(\alpha-\beta)u_n = \alpha^n - \beta^n \quad \text{and} \quad v_n = \alpha^n + \beta^n \quad \text{for all } n \in \mathbb{N},$$

where $\alpha = (A+\sqrt{\Delta})/2$ and $\beta = (A-\sqrt{\Delta})/2$ with $\Delta = A^2 - 4B$.

Our following conjecture implies that for any prime $p = x^2+7y^2$ with $x, y \in \mathbb{Z}$ we have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 T_k(1,16)}{(-256)^k} \equiv \left(\frac{-1}{p}\right)(4x^2 - 2p) \pmod{p^2}.$$

Conjecture 1.1. *Let p be an odd prime with $(\frac{p}{7}) = 1$. Write $p = x^2 + 7y^2$ with $x, y \in \mathbb{Z}$ such that $x \equiv 1 \pmod 4$ if $p \equiv 1 \pmod 4$, and $y \equiv 1 \pmod 4$ if $p \equiv 3 \pmod 4$. Then*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{256^k} u_k(1,16) \equiv \begin{cases} 0 \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ \frac{1}{3}(\frac{2}{p})(\frac{p}{7y} - 4y) \pmod{p^2} & \text{if } p \equiv 3 \pmod 4; \end{cases}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{256^k} v_k(1,16) \equiv \begin{cases} 2(\frac{2}{p})(2x - \frac{p}{2x}) \pmod{p^2} & \text{if } p \equiv 1 \pmod 4, \\ 0 \pmod{p^2} & \text{if } p \equiv 3 \pmod 4. \end{cases}$$

When $p \equiv 1 \pmod 4$, we have

$$\sum_{k=0}^{p-1} \frac{k\binom{2k}{k}^2}{16^k} u_k(1,16) \equiv \sum_{k=0}^{p-1} \frac{k\binom{2k}{k}^2}{256^k} u_k(1,16) \equiv \frac{1}{42}\left(\frac{2}{p}\right)\left(x - \frac{p}{2x}\right) \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(4k+3)\frac{\binom{2k}{k}^2}{16^k} v_k(1,16) \equiv 3\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}^2}{256^k} v_k(1,16) \equiv 6\left(\frac{2}{p}\right)x \pmod{p^2}.$$

When $p \equiv 3 \pmod 4$, we can determine y mod p^2 in the following way:

$$\sum_{k=0}^{p-1} \frac{k\binom{2k}{k}^2}{16^k} u_k(1,16) \equiv \sum_{k=0}^{p-1} \frac{k\binom{2k}{k}^2}{16^k} v_k(1,16) \equiv -\left(\frac{2}{p}\right)\frac{y}{2} \pmod{p^2}$$

and

$$3\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{256^k}u_k(1,16)\equiv\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}^2}{256^k}v_k(1,16)\equiv\left(\frac{2}{p}\right)\frac{y}{2}\pmod{p^2}.$$

Just like $\mathbb{Q}(\sqrt{-7})$, the imaginary quadratic field $\mathbb{Q}(\sqrt{-11})$ also has class number one. Let p be an odd prime. We guess that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(46,1)}{512^k}\equiv\begin{cases}x^2-2p\pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{11})=-1.\end{cases}$$

To attack this we note that (1.18) with $b=46$, $c=1$ and $x=-27/512$ yields

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{512^k}T_k(46,1)\equiv\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}\alpha^k\right)\times\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}\beta^k\pmod{p^2},$$

where $\alpha=(1+\sqrt{33})/2$ and $\beta=1-\sqrt{33})/2$. Note that $2\alpha^k=v_k(1,-8)+(\alpha-\beta)u_k(1,-8)$ and $2\beta^k=v_k(1,-8)-(\alpha-\beta)u_k(1,-8)$. So we have

$$\begin{aligned}&4\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(46,1)}{512^k}\\ \equiv&\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\right)^2-33\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\right)^2\pmod{p^2}.\end{aligned}$$

This, together with the author's conjecture on $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/64^k \bmod p^2$ (cf. [Su2, Conjecture 5.4]) leads us to raise the following conjecture.

Conjecture 1.2. *Let $p>3$ be a prime. If $(\frac{p}{11})=-1$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\equiv0\pmod{p}.$$

When $(\frac{p}{11})=1$, $p\equiv1\pmod 3$, $4p=x^2+11y^2$ $(x,y\in\mathbb{Z})$ and $x\equiv1\pmod 3$, we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)\equiv&0\pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)\equiv&\frac{114}{11}\left(\frac{2p}{x}-x\right)\pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{216^k}u_k(8,27)\equiv&\frac{4}{99}\left(\frac{2p}{x}-x\right)\pmod{p^2},\\ \sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\equiv&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{216^k}v_k(8,27)\equiv2\left(\frac{p}{x}-x\right)\pmod{p^2},\end{aligned}$$

and

$$\sum_{k=0}^{p-1}(k+60)\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\equiv -60x \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(9k+2)\frac{\binom{2k}{k}\binom{3k}{k}}{216^k}v_k(8,27)\equiv -2x \pmod{p^2}.$$

When $(\frac{p}{11})=1$, $p\equiv 2 \pmod 3$, $4p=x^2+11y^2$ $(x,y\in\mathbb{Z})$ *and* $y\equiv 1 \pmod 3$, *we have*

$$11\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)\equiv -3\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\equiv \frac{3}{2}\left(\frac{p}{y}-11y\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(2k-155)\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}u_k(1,-8)\equiv \frac{759}{2}y \pmod{p^2},$$
$$\sum_{k=0}^{p-1}(2k-243)\frac{\binom{2k}{k}\binom{3k}{k}}{(-64)^k}v_k(1,-8)\equiv -\frac{4359}{2}y \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{216^k}u_k(8,27)\equiv y-\frac{p}{11y} \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{216^k}u_k(8,27)\equiv \frac{1}{8}\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{216^k}v_k(8,27)\equiv -\frac{y}{9} \pmod{p^2}.$$

Motivated by the author's investigation of $\sum_{k=0}^{p-1}\binom{2k}{k}\binom{3k}{k}T_k(3,1)/27^k \bmod p^2$ (with $p>3$ a prime) and the congruence (1.18), we pose the following conjecture which involves the well-known Fibonacci numbers $F_k=u_k(1,-1)$ $(k\in\mathbb{N})$ and Lucas numbers $L_k=v_k(1,-1)$ $(k\in\mathbb{N})$. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-15})$ has class number 2.

Conjecture 1.3. *Let* $p>5$ *be a prime. If* $p\equiv 1,4 \pmod{15}$ *and* $p=x^2+15y^2$ $(x,y\in\mathbb{Z})$ *with* $x\equiv 1 \pmod 3$, *then*

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}F_k\equiv \frac{2}{15}\left(\frac{p}{x}-2x\right) \pmod{p^2},$$
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}L_k\equiv 4x-\frac{p}{x} \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(3k+2)\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}L_k\equiv 4x \pmod{p^2}.$$

If $p\equiv 2,8 \pmod{15}$ *and* $p=3x^2+5y^2$ $(x,y\in\mathbb{Z})$ *with* $y\equiv 1 \pmod 3$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}F_k\equiv\frac{p}{5y}-4y \pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}F_k\equiv\sum_{k=0}^{p-1}\frac{k\binom{2k}{k}\binom{3k}{k}}{27^k}L_k\equiv\frac{4}{3}y \pmod{p^2}.$$

Remark 1.4. By (5.3) in Section 5, for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}F_k\equiv 0 \pmod{p^2}\ \text{ if }\ p\equiv 1 \pmod 3,$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{27^k}L_k\equiv 0 \pmod{p^2}\ \text{ if }\ p\equiv 2 \pmod 3.$$

In fact, we have many other conjectures similar to Conjectures 1.1-1.3; for the sake of brevity we don't include them in this paper.

We are going to prove Theorems 1.1-1.2 and (1.15) in the next section. In Section 3 we will show Theorem 1.3. Section 4 contains more conjectural congruences and they offer backgrounds for those conjectural series for $1/\pi$ in Section 5. In Section 5 we first show a theorem on dualities and then propose 48 conjectural series for $1/\pi$ based on our investigations of congruences.

2. Proofs of Theorems 1.1-1.2 and (1.15)

Lemma 2.1. *For* $m,n\in\mathbb{N}$ *we have*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{k+m}{n}w_{k+m}(h)=\frac{w_m(h)w_n(h)}{\binom{m+n}{n}}. \tag{2.1}$$

Proof. Let u_n denote the left-hand side of (2.1). By applying the Zeilberger algorithm (cf. [PWZ]) via `Mathematica`, we find the recursion:

$$(n+1)(m+n+1)u_{n+1}=(h-n)(h+n+1)u_n\ (n=0,1,2,\ldots).$$

Thus (2.1) can be easily proved by induction on n. □

Lemma 2.2. *For $k, m, n \in \mathbb{N}$ we have the combinatorial identity*

$$\begin{aligned}\sum_{j=0}^{m}(-1)^{m-j}\binom{m+j}{2j}\binom{2j}{j}\binom{j+k+m}{k}\binom{j}{n}\\=\binom{k+m+n}{m}\binom{k+m}{m}\binom{m}{n}.\end{aligned}\tag{2.2}$$

Proof. If $m < n$ then both sides of (2.2) vanish. (2.2) in the case $m = n$ can be directly verified. Let s_m denote the left-hand side of (2.2). By the Zeilberger algorithm we find the recursion

$$(m+1)(m-n+1)s_{m+1} = (k+m+1)(k+m+n+1)s_m \quad (m = n, n+1, \ldots).$$

So we can show (2.2) by induction. □

Proof of Theorem 1.1. In view of Remark 1.1 it suffices to prove (1.10). Note that both sides of (1.10) are polynomials in x and y and the degrees with respect to x or y are all smaller than p^a.

Fix $m, n \in \{0, \ldots, p^a - 1\}$ and let $c(m, n)$ denote the coefficient of $x^n y^m$ in the right-hand side of (1.10). Then

$$\begin{aligned}c(m,n) =&[x^n]\sum_{0\leqslant j\leqslant k<p^a} w_k(h)\binom{k+j}{2j}\binom{2j}{j}(x+1)^j\binom{k-j}{m-j}(-1)^{m-j}x^{k-m}\\
=&\sum_{k=m}^{p^a-1} w_k(h)\sum_{j=0}^{m}(-1)^{m-j}\binom{k+j}{2j}\binom{2j}{j}\binom{k-j}{m-j}\binom{j}{m+n-k}\\
=&\sum_{k=0}^{p^a-1-m} w_{k+m}(h)\sum_{j=0}^{m}(-1)^{m-j}\binom{k+m+j}{2j}\binom{2j}{j}\binom{k+m-j}{k}\binom{j}{n-k}\\
=&\sum_{k=0}^{p^a-1-m} w_{k+m}(h)\sum_{j=0}^{m}(-1)^{m-j}\binom{m+j}{2j}\binom{2j}{j}\binom{k+m+j}{k}\binom{j}{n-k}.\end{aligned}$$

Applying Lemma 2.2 we get

$$\begin{aligned}c(m,n) =&\binom{m+n}{m}\sum_{k=0}^{p^a-1-m} w_{k+m}(h)\binom{k+m}{m}\binom{m}{n-k}\\
=&\binom{m+n}{m}\sum_{k=0}^{p^a-1-m} w_{k+m}(h)\binom{k+m}{n}\binom{n}{k}.\end{aligned}$$

By Lemma 2.1,

$$\sum_{k=0}^{p^a-1} w_{k+m}(h)\binom{k+m}{n}\binom{n}{k} = \sum_{k=0}^{n} w_{k+m}(h)\binom{k+m}{n}\binom{n}{k} = \frac{w_m(h)w_n(h)}{\binom{m+n}{m}}.$$

So, it remains to show

$$\binom{m+n}{m}\sum_{k=p^a-m}^{p^a-1} w_{k+m}(h)\binom{k+m}{n}\binom{n}{k}\equiv 0 \pmod{p^2}. \tag{2.3}$$

To prove (2.3) we only need to show

$$\binom{m+n}{m}\equiv\binom{k+m}{n}\equiv 0 \pmod{p}$$

under the supposition $n\geqslant k\geqslant p^a-m$. Note that $m+n\geqslant k+m\geqslant p^a$ and $0<p^a-n\leqslant k+m-n\leqslant m<p^a$. As the addition of m and n in base p has at least one carry, we have $p\mid\binom{m+n}{m}$ by Kummer's theorem (cf. [R, p. 24]). Similarly, $p\mid\binom{k+m}{n}$.

So far we have completed the proof of Theorem 1.1. $\square$

Proof of (1.15). Let a_n denote the left hand side or the right-hand side of (1.15). It is easy to see that

$$a_0=1,\ a_1=(2x+1)(2y+1),\ a_2=(6x^2+6x+1)(6y^2+6y+1)$$

and

$$a_3=(20x^3+30x^2+12x+1)(20y^3+30y^2+12y+1).$$

Via the Zeilberger algorithm we find the recursion for $n\geqslant 3$:

$$\begin{aligned}&(n+1)^2(2n-3)a_{n+1}-(2n-3)(2n+1)^2(2x+1)(2y+1)a_n\\&+(2n-1)A(n,x,y)a_{n-1}-(2n-3)^2(2n+1)(2x+1)(2y+1)a_{n-2}\\&+(n-2)^2(2n+1)a_{n-3}\\&=0,\end{aligned}$$

where

$$A(n,x,y):=6n^2-6n-5+(16n^2-16n-12)(x+y-x^2-y^2).$$

Thus (1.15) holds by induction. $\square$

Proof of Theorem 1.2. Let $n=(p^a-1)/2$. For $k=0,\ldots,n$ we have

$$\binom{n+k}{2k}=\frac{\binom{2k}{k}}{(-16)^k}\prod_{0<j\leqslant k}\left(1-\frac{p^{2a}}{(2j-1)^2}\right)\equiv\frac{\binom{2k}{k}}{(-16)^k} \pmod{p^2}$$

and hence

$$\binom{n}{k}\binom{n+k}{k}=\binom{n+k}{2k}\binom{2k}{k}\equiv\frac{\binom{2k}{k}^2}{(-16)^k} \pmod{p^2}.$$

Note also that $p \mid \binom{2k}{k}$ for $k = n+1, \ldots, p^a - 1$ by Kummer's theorem. Thus

$$\begin{aligned} P_n(t) &= \sum_{k=0}^{n} \binom{n}{k}\binom{n+k}{k}\left(\frac{t-1}{2}\right)^k \\ &\equiv \sum_{k=0}^{n} \frac{\binom{2k}{k}^2}{(-16)^k}\left(\frac{t-1}{2}\right)^k \equiv \sum_{k=0}^{p^a-1} \frac{\binom{2k}{k}^2}{32^k}(1-t)^k \pmod{p^2}, \end{aligned}$$

and hence the second congruence in (1.16) follows.

Set

$$u = \frac{\sqrt{D}+\sqrt{d}\,x-1}{2} \quad \text{and} \quad v = \frac{\sqrt{D}-\sqrt{d}\,x-1}{2}.$$

Then

$$uv + v = \frac{D-(\sqrt{d}\,x+1)^2}{4} = \frac{b-\sqrt{d}}{2}x \quad \text{and} \quad u - v = \sqrt{d}\,x.$$

In view of (1.8), for any $k \in \mathbb{N}$ we have

$$\begin{aligned} &\sum_{j=0}^{k} \binom{k+j}{2j}\binom{2j}{j}(uv+v)^j(u-v)^{k-j} \\ =&x^k \sum_{j=0}^{k} \binom{k+j}{2j}\binom{2j}{j}\left(\frac{b-\sqrt{d}}{2}\right)^k \sqrt{d}^{k-j} = x^k T_k(b,c). \end{aligned}$$

So the first congruence in (1.16) follows from (1.11). Similarly, (1.17)-(1.19) are consequences of (1.12)-(1.14) respectively. We are done. □

3. Proof of Theorem 1.3

Lemma 3.1. *Let $p = 2n+1$ be an odd prime. Then*

$$\binom{2k}{k} \equiv (-1)^n 16^k \binom{2(n-k)}{n-k} \pmod{p} \quad \textit{for } k = 0, \ldots, n, \tag{3.1}$$

and

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{32^k} x^k \equiv \left(\frac{2}{p}\right) x^n P_n\left(1-\frac{4}{x}\right) \pmod{p}. \tag{3.2}$$

Proof. For any $k \in \{0, \ldots, n\}$ we have

$$\frac{\binom{2k}{k}}{(-4)^k} = \binom{-1/2}{k} \equiv \binom{n}{k} = \binom{n}{n-k} \equiv \binom{-1/2}{n-k} = \frac{\binom{2(n-k)}{n-k}}{(-4)^{n-k}} \pmod{p}.$$

Also, $4^n = 2^{p-1} \equiv 1 \pmod p$. So (3.1) holds.

With the help of (3.1), we get

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}x^k &\equiv \sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}x^k \equiv \sum_{k=0}^{n}\frac{256^k\binom{2(n-k)}{n-k}^2}{32^k}x^k = \sum_{k=0}^{n}\binom{2k}{k}^2(8x)^{n-k}\\ &\equiv \left(\frac{8}{p}\right)x^n\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{(-16)^k}\left(-\frac{2}{x}\right)^k\\ &\equiv \left(\frac{2}{p}\right)x^n\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\left(\frac{(1-4/x)-1}{2}\right)\\ &= \left(\frac{2}{p}\right)x^nP_n\left(1-\frac{4}{x}\right) \pmod p.\end{aligned}$$

This proves (3.2). □

Lemma 3.2. *Let $p = 2n+1$ be an odd prime. Then*

$$P_n(x) \equiv (2x+2)^nP_n\left(\frac{3-x}{1+x}\right) \pmod p. \tag{3.3}$$

Remark 3.1. (3.3) follows from [S1, Theorem 2.6] and its proof. It also appeared as [S2, (5.2)].

Proof of Theorem 1.3. For convenience we set $n = (p-1)/2$.

(i) Applying (1.16) with $b = 1$, $c = -2$ and $x = -1/2$, we obtain that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}T_k(1,-2) \equiv \left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\right)\left(\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k}\right) \pmod{p^2}.$$

The author [Su2, Conjecture 5.5] conjectured that $\sum_{k=0}^{p-1}\binom{2k}{k}^2/32^k \equiv 0 \pmod{p^2}$ if $p \equiv 3 \pmod 4$, and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k} \equiv \left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k} \equiv 2x - \frac{p}{2x} \pmod{p^2}$$

if $p \equiv 1 \pmod 4$ and $p = x^2+y^2$ with $x \equiv 1 \pmod 4$ and $y \equiv 0 \pmod 2$. This was confirmed by Z. H. Sun [S1]. So the desired (1.20) follows.

(ii) Applying (1.16) with $b = 2$, $c = -1$ and $x = -2$ we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(2,-1)}{8^k} \equiv \sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\alpha^k \times \sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\beta^k \pmod{p^2}.$$

where $\alpha=-4(1+\sqrt2)$ and $\beta=-4(1-\sqrt2)$. Clearly $\alpha\beta=-16$. By Lemma 3.1,

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\alpha^k\equiv\left(\frac{2}{p}\right)\alpha^nP_n(\sqrt2)\pmod p$$

and

$$\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\beta^k\equiv\left(\frac{2}{p}\right)\beta^nP_n(-\sqrt2)=\left(\frac{-2}{p}\right)\beta^nP_n(\sqrt2)\pmod p.$$

By [S2, Theorem 2.9], $P_n(\sqrt2)\equiv0\pmod p$ if $(\frac{2}{p})=-1$, and $P_n(\sqrt2)^2\equiv(\frac{-1}{p})4x^2\pmod p$ if $(\frac{2}{p})=1$ and $p=x^2+2y^2$ $(x,y\in\mathbb{Z})$. So (1.21) holds.

(iii) (1.16) with $b=x=4$ and $c=1$ yields that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-4)^k}T_k(4,1)\equiv P_n(15+8\sqrt3)P_n(15-8\sqrt3)\pmod{p^2}.$$

By Lemma 3.2,

$$\begin{aligned}(\pm1)^nP_n\left(\frac{\sqrt3}{2}\right)&=P_n\left(\pm\frac{\sqrt3}{2}\right)\\ \equiv&(2\pm\sqrt3)^nP_n\left(\frac{3\mp\sqrt3/2}{1\pm\sqrt3/2}\right)=(2\pm\sqrt3)^nP_n(15\mp8\sqrt3)\pmod p.\end{aligned}$$

By [S2, Theorem 2.10], $P_n(\sqrt3/2)\equiv0\pmod p$ if $p\equiv2\pmod 3$, and $P_n(\sqrt3/2)^2\equiv(-1)^n4x^2\pmod p$ if $p\equiv1\pmod3$ and $p=x^2+3y^2$ $(x,y\in\mathbb{Z})$. Therefore (1.22) is valid.

(iv) Applying (1.16) with $b=1$, $c=16$ and $x=1/16$ we obtain that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(1,16)}{(-256)^k}\equiv\left(\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\alpha^k\right)\times\sum_{k=0}^{n}\frac{\binom{2k}{k}^2}{32^k}\beta^k\pmod{p^2},$$

where $\alpha=(1+3\sqrt{-7})/16$ and $\beta=(1-3\sqrt{-7})/16$. Note that $\alpha\beta=1/4$. By Lemma 3.1,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\alpha^k\equiv\left(\frac{2}{p}\right)\alpha^nP_n(\sqrt{-63})\pmod p$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{32^k}\beta^k\equiv\left(\frac{2}{p}\right)\beta^nP_n(-\sqrt{-63})=\left(\frac{-2}{p}\right)\beta^nP_n(\sqrt{-63})\pmod p.$$

(1.16) with $b=16$, $c=1$ and $x=16$ yields that

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k(16,1)\equiv P_n(255+96\sqrt7)P_n(255-96\sqrt7)\pmod{p^2}.$$

By Lemma 3.2,

$$(\pm1)^nP_n\left(\frac{3\sqrt7}{8}\right)\equiv(8\pm3\sqrt7)^nP_n(255\mp96\sqrt7)\pmod p.$$

Therefore

$$(8^2-9\times7)^n\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k(16,1)\equiv(-1)^nP_n\left(\frac{3\sqrt7}{8}\right)^2\pmod p.$$

By [S2, Theorem 2.7], $P_n(\sqrt{-63})\equiv P_n(3\sqrt7/8)\equiv0\pmod p$ if $(\frac p7)=-1$, and

$$P_n(\sqrt{-63})^2\equiv(-1)^nP_n\left(\frac{3\sqrt7}{8}\right)^2\equiv4x^2\pmod p$$

if $(\frac p7)=1$ and $p=x^2+7y^2$ $(x,y\in\mathbb Z)$. Therefore (1.23) holds. □

4. More conjectural congruences

Conjecture 4.1. *Let $p>3$ be a prime.*

(i) *If $p\equiv1,4\pmod{15}$ and $p=x^2+15y^2$ with $x,y\in\mathbb Z$, then*

$$P_{(p-1)/2}(7\sqrt{-15}\pm16\sqrt{-3})\equiv\left(\frac{-\sqrt{-15}}{p}\right)\left(\frac{x}{15}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

(ii) *Suppose that $(\frac p5)=(\frac p7)=1$ and write $4p=x^2+35y^2$ with $x,y\in\mathbb Z$. If $p\equiv1\pmod3$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{3456^k}(64+27\sqrt5\pm\sqrt{-35})^k\equiv\left(\frac x3\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

If $p\equiv2\pmod3$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{3456^k}(64+27\sqrt5\pm\sqrt{-35})^k\equiv\pm\sqrt{-35}\left(\frac y3\right)\left(y-\frac{p}{35y}\right)\pmod{p^2}.$$

(iii) *If* $(\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1$ *and* $p=x^2+30y^2$ *with* $x,y\in\mathbb{Z}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{2916^k}(54-35\sqrt{2}\pm\sqrt{5})^k\equiv\left(\frac{x}{3}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

(iv) *If* $(\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{7})=1$, *and* $p=x^2+42y^2$ *with* $x,y\in\mathbb{Z}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{13500^k}(250-99\sqrt{6}\pm2\sqrt{14})^k\equiv\left(\frac{x}{3}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

(v) *If* $(\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1$ *and* $p=x^2+78y^2$ *with* $x,y\in\mathbb{Z}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{530604^k}(9826-6930\sqrt{2}\pm5\sqrt{26})^k\equiv\left(\frac{x}{3}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

(vi) *If* $(\frac{2}{p})=(\frac{p}{3})=(\frac{p}{17})=1$ *and* $p=x^2+102y^2$ *with* $x,y\in\mathbb{Z}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}}{3881196^k}(71874-17420\sqrt{17}\pm35\sqrt{2})^k\equiv\left(\frac{x}{3}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

(vii) *If* $(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{11})=1$ *and* $p=x^2+33y^2$ *with* $x,y\in\mathbb{Z}$, *then*

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}}{(2^{12}3)^k}(96-5\sqrt{11}\pm65\sqrt{3})^k\equiv\left(\frac{x}{3}\right)\left(2x-\frac{p}{2x}\right)\pmod{p^2}.$$

Remark 4.1. Let $p\equiv1,4\pmod{15}$ be a prime with $p=x^2+15y^2$ $(x,y\in\mathbb{Z})$. Applying (1.16) we see that

$$\begin{aligned}&\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k\\ \equiv&P_{(p-1)/2}(7\sqrt{-15}+16\sqrt{-3})P_{(p-1)/2}(7\sqrt{-15}-16\sqrt{-3})\pmod{p^2}.\end{aligned}$$

Thus part (i) of Conjecture 4.1 implies that

$$\sum_{k=0}^{p-1}(-1)^k\binom{2k}{k}^2T_k\equiv\left(2x-\frac{p}{2x}\right)^2\equiv4x^2-2p\pmod{p^2}.$$

We omit here similar comments on parts (ii)-(vii) of Conjecture 4.1. We also have many other conjectures similar to Conjecture 4.1.

In the following conjectures, when we write a multiple of a prime in the form ax^2+by^2, we always assume that x and y are integers.

Conjecture 4.2. *Let $p>5$ be a prime. Then*

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(62,1)}{(-128^2)^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(62,1)}{(-480^2)^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,9\pmod{20}\ \&\ p=x^2+5y^2,\\ 2x^2-2p\pmod{p^2} & \text{if } p\equiv 3,7\pmod{20}\ \&\ 2p=x^2+5y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-5}{p})=-1,\ \text{i.e.},\ p\equiv 11,13,17,19\pmod{20}.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(340k+111)\frac{\binom{2k}{k}^2T_{2k}(62,1)}{(-128^2)^k}\equiv 3p\left(\frac{-1}{p}\right)\left(22+15\left(\frac{p}{15}\right)\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}(340k+59)\frac{\binom{2k}{k}^2T_{2k}(62,1)}{(-480^2)^k}\equiv p\left(\frac{-1}{p}\right)\left(51+8\left(\frac{p}{15}\right)\right)\pmod{p^2}.$$

Conjecture 4.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k^2}{4^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(4,1)^2}{16^k}$$

$$\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(10,1)}{(-64)^k}\equiv\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(6,1)}{256^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(6,1)}{1024^k}$$

$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24}\ \&\ p=x^2+6y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24}\ \&\ p=2x^2+3y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1,\ \text{i.e.},\ p\equiv 13,17,19,23\pmod{24};\end{cases}$$

and

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(6,1)^2}{192^k}$$

$$\equiv\begin{cases}(\frac{-1}{p})(4x^2-2p)\pmod{p^2} & \text{if } p\equiv 1,7\pmod{24}\ \&\ p=x^2+6y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } p\equiv 5,11\pmod{24}\ \&\ p=2x^2+3y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-6}{p})=-1,\ \text{i.e.},\ p\equiv 13,17,19,23\pmod{24}.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(3k+1)\frac{\binom{2k}{k}^2T_k(10,1)}{(-64)^k}\equiv\frac{p}{4}\left(3\left(\frac{p}{3}\right)+1\right)\pmod{p^2},$$

$$\sum_{k=0}^{p-1}(4k+1)\frac{\binom{2k}{k}T_k(6,1)^2}{192^k}\equiv p\left(\frac{-6}{p}\right)\left(4-3\left(\frac{2}{p}\right)\right)\pmod{p^2}.$$

Conjecture 4.4. *Let $p>5$ be a prime.*

(i) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}T_k(3,1)^2}{36^k}\equiv\left(\frac{2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(34,1)}{(-64)^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(18,1)}{4096^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } p\equiv 1,9,11,19\pmod{40}\ \&\ p=x^2+10y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } p\equiv 7,13,23,37\pmod{40}\ \&\ p=2x^2+5y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-10}{p})=-1,\ i.e.,\ p\equiv 3,17,21,27,29,31,33,39\pmod{40}.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(16k+5)\frac{\binom{2k}{k}T_k(3,1)^2}{36^k}\equiv 5p\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(60k+23)\frac{\binom{2k}{k}^2T_k(34,1)}{(-64)^k}\equiv p\left(8\left(\frac{2}{p}\right)+15\left(\frac{-1}{p}\right)\right)\pmod{p^2}.$$

Conjecture 4.5. *Let $p>7$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(18,1)}{512^k}\equiv\left(\frac{10}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(6,1)}{(-512)^k}$$
$$\equiv\begin{cases}x^2-2p\pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=1\ \&\ 4p=x^2+35y^2,\\ 2p-5x^2\pmod{p^2} & \text{if } (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ 4p=5x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{35})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(35k+9)\frac{\binom{2k}{k}\binom{3k}{k}T_k(18,1)}{512^k}\equiv\frac{9p}{2}\left(7-5\left(\frac{p}{5}\right)\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(35k+9)\frac{\binom{2k}{k}^2T_{3k}(6,1)}{(-512)^k}\equiv\frac{9p}{32}\left(\frac{2}{p}\right)\left(25+7\left(\frac{p}{7}\right)\right)\pmod{p^2}.$$

Conjecture 4.6. *Let $p\neq 2,29$ be a prime. When $p\neq 5,7$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(19602,1)}{78400^{2k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{29}{p})=1\ \&\ p=x^2+58y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{29}{p})=-1\ \&\ p=2x^2+29y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-58}{p})=-1.\end{cases}$$

Provided $p\neq 13$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(19602,1)}{78416^{2k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{29}{p})=1\ \&\ p=x^2+58y^2,\\ 2p-8x^2\pmod{p^2}&\text{if }(\frac{-2}{p})=(\frac{29}{p})=-1\ \&\ p=2x^2+29y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-58}{p})=-1.\end{cases}$$

Conjecture 4.7. *Let $p>5$ be a prime. Then*

$$\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(26,1)}{(-24)^{3k}}\equiv\left(\frac{15}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}$$
$$\equiv\begin{cases}x^2-2p\pmod{p^2}&\text{if }(\frac{p}{7})=(\frac{p}{13})=1\ \&\ 4p=x^2+91y^2,\\ 2p-7x^2\pmod{p^2}&\text{if }(\frac{p}{7})=(\frac{p}{13})=-1\ \&\ 4p=7x^2+13y^2,\\ 0\pmod{p^2}&\text{if }(\frac{p}{91})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}(819k+239)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(26,1)}{(-24)^{3k}}\equiv\frac{p}{32}\left(\frac{-6}{p}\right)\left(949+6699\left(\frac{p}{7}\right)\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(1638k+277)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)}{(-240)^{3k}}\equiv\frac{p}{40}\left(\frac{-105}{p}\right)\left(8701+2379\left(\frac{p}{7}\right)\right)\pmod{p^2}.$$

Remark 4.2. Note that the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number two for $d=5,6,10,15,35,58,91$.

Conjecture 4.8. *Let $p>3$ be a prime. We have*

$$\left(\frac{-6}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}T_k(110,1)}{(-96^2)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+21y^2,\\ 12x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{7})=-1,\ (\frac{p}{3})=1\ \&\ p=3x^2+7y^2,\\ 2x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=(\frac{p}{3})=-1,\ (\frac{p}{7})=1\ \&\ 2p=x^2+21y^2,\\ 6x^2-2p\pmod{p^2}&\text{if }(\frac{-1}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1,\ \&\ 2p=3x^2+7y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-21}{p})=-1,\end{cases}$$

and

$$\sum_{k=0}^{p-1}(28k+5)\frac{\binom{4k}{2k}\binom{2k}{k}T_k(110,1)}{(-96^2)^k}\equiv\frac{p}{8}\left(\frac{-6}{p}\right)\left(33+7\left(\frac{p}{7}\right)\right)\pmod{p^2}.$$

Conjecture 4.9. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(18,1)}{256^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{5})=1\ \&\ p=x^2+30y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=3x^2+10y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{5})=-1\ \&\ p=2x^2+15y^2,\\ 2p-6x^2\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ 2p=3x^2+10y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-30}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(30,1)}{256^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{3})=(\frac{p}{7})=1\ \&\ p=x^2+42y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=1,\ (\frac{p}{3})=(\frac{p}{7})=-1\ \&\ p=3x^2+14y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{-2}{p})=(\frac{p}{3})=-1\ \&\ p=2x^2+21y^2,\\ 2p-6x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{-2}{p})=(\frac{p}{7})=-1\ \&\ 2p=3x^2+14y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-42}{p})=-1.\end{cases}$$

Conjecture 4.10. *Let $p>3$ be a prime. When $p\neq 13,17$, we have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(102,1)}{102^{3k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{3})=(\frac{p}{13})=1\ \&\ p=x^2+78y^2,\\ 2p-8x^2\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{3})=(\frac{p}{13})=-1\ \&\ p=2x^2+39y^2,\\ 12x^2-2p\pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{2}{p})=(\frac{p}{3})=-1\ \&\ p=3x^2+26y^2,\\ 2p-24x^2\pmod{p^2} & \text{if } (\frac{p}{3})=1,\ (\frac{2}{p})=(\frac{p}{13})=-1\ \&\ p=6x^2+13y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-78}{p})=-1.\end{cases}$$

Provided $p\neq 11,17$, we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(198,1)}{198^{3k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p3)=(\frac p{17})=1\ \&\ p=x^2+102y^2,\\ 2p-8x^2\pmod{p^2}&\text{if }(\frac p{17})=1,\ (\frac 2p)=(\frac p3)=-1\ \&\ p=2x^2+51y^2,\\ 12x^2-2p\pmod{p^2}&\text{if }(\frac p3)=1,\ (\frac 2p)=(\frac p{17})=-1\ \&\ p=3x^2+34y^2,\\ 2p-24x^2\pmod{p^2}&\text{if }(\frac 2p)=1,\ (\frac p3)=(\frac p{17})=-1\ \&\ p=6x^2+17y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-102}p)=-1.\end{cases}$$

Conjecture 4.11. *Let p be an odd prime and let $m\in\{2,3,6,10,18,30,102,198\}$. If $p\nmid m$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(m,1)}{m^{3k}}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(m,1)}{256^k}\pmod{p^2}.\tag{4.1}$$

If $m^2\not\equiv-12\pmod p$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(m,1)}{256^k}\equiv\left(\frac{m^2+12}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{2k}\binom{2k}{k}T_k(m^2-2,1)}{(m^2+12)^{2k}}\pmod{p^2}.\tag{4.2}$$

Remark 4.3. We observe that (4.1) holds mod p for any integer $m\not\equiv0\pmod p$, and (4.2) holds mod p for any $m\in\mathbb{Z}$ with $m^2\not\equiv-12\pmod p$.

Conjecture 4.12. *Let $p\neq 2,5,19$ be a prime. We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(5778,1)}{1216^{2k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=(\frac p5)=(\frac p{19})=1\ \&\ p=x^2+190y^2,\\ 8x^2-2p\pmod{p^2}&\text{if }(\frac 2p)=1,\ (\frac p5)=(\frac p{19})=-1\ \&\ p=2x^2+95y^2,\\ 2p-20x^2\pmod{p^2}&\text{if }(\frac 2p)=(\frac p5)=-1,\ (\frac p{19})=1\ \&\ p=5x^2+38y^2,\\ 2p-40x^2\pmod{p^2}&\text{if }(\frac 2p)=(\frac p{19})=-1,\ (\frac p5)=1\ \&\ p=10x^2+19y^2,\\ 0\pmod{p^2}&\text{if }(\frac{-190}p)=-1,\end{cases}$$

and

$$\sum_{k=0}^{p-1}(57720k+24893)\frac{\binom{2k}{k}^2T_{2k}(5778,1)}{1216^{2k}}\equiv p\left(11548+13345\left(\frac{p}{95}\right)\right)\pmod{p^2}.$$

Provided $p\neq 17$ we have

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(5778,1)}{439280^{2k}}\equiv\left(\frac{p}{5}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(5778,1)}{1216^{2k}}\pmod{p^2}$$

and

$$\sum_{k=0}^{p-1}(57720k+3967)\frac{\binom{2k}{k}^2T_{2k}(5778,1)}{439280^{2k}}\equiv p\left(\frac{p}{19}\right)\left(3983-16\left(\frac{p}{95}\right)\right)\pmod{p^2}.$$

Conjecture 4.13. *Let $p>5$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(198,1)}{224^{2k}}\equiv\left(\frac{p}{7}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(322,1)}{48^{4k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=(\frac{p}{5})=(\frac{p}{7})=1\ \&\ p=x^2+70y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{p}{7})=1,\ (\frac{2}{p})=(\frac{p}{5})=-1\ \&\ p=2x^2+35y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{2}{p})=(\frac{p}{7})=-1\ \&\ p=5x^2+14y^2,\\ 28x^2-2p\pmod{p^2} & \text{if } (\frac{2}{p})=1,\ (\frac{p}{5})=(\frac{p}{7})=-1\ \&\ p=7x^2+10y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-70}{p})=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(322,1)}{(-2^{10}3^4)^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-1}{p})=(\frac{p}{5})=(\frac{p}{17})=1\ \&\ p=x^2+85y^2,\\ 2p-2x^2\pmod{p^2} & \text{if } (\frac{p}{17})=1,\ (\frac{-1}{p})=(\frac{p}{5})=-1\ \&\ 2p=x^2+85y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{-1}{p})=1,\ (\frac{p}{5})=(\frac{p}{17})=-1\ \&\ p=5x^2+17y^2,\\ 10x^2-2p\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{-1}{p})=(\frac{p}{17})=-1\ \&\ 2p=5x^2+17y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-85}{p})=-1.\end{cases}$$

And

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_{2k}(1298,1)}{24^{4k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=(\frac{p}{5})=(\frac{p}{13})=1\ \&\ p=x^2+130y^2,\\ 8x^2-2p\pmod{p^2} & \text{if } (\frac{-2}{p})=1,\ (\frac{p}{5})=(\frac{p}{13})=-1\ \&\ p=2x^2+65y^2,\\ 2p-20x^2\pmod{p^2} & \text{if } (\frac{p}{5})=1,\ (\frac{-2}{p})=(\frac{p}{13})=-1\ \&\ p=5x^2+26y^2,\\ 2p-40x^2\pmod{p^2} & \text{if } (\frac{p}{13})=1,\ (\frac{-2}{p})=(\frac{p}{5})=-1\ \&\ p=10x^2+13y^2,\\ 0\pmod{p^2} & \text{if } (\frac{-130}{p})=-1.\end{cases}$$

Remark 4.4. The imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number four for $d = 21, 30, 42, 70, 78, 85, 102, 130, 190$.

Conjecture 4.14. *Let $p > 3$ be a prime. Then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}}{(-27)^k}\equiv\begin{cases}(\frac{p}{3})(4x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{7})=1\ \&\ p=x^2+7y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{7})=-1;\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(26,81)}{24^{3k}}\equiv\begin{cases}(\frac{6}{p})(x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{11})=1\ \&\ 4p=x^2+11y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{11})=-1;\end{cases}$$

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(10,1)}{24^{3k}}\equiv\begin{cases}(\frac{6}{p})(x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{19})=1\ \&\ 4p=x^2+19y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{19})=-1.\end{cases}$$

If $p \neq 13$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(106,1)}{312^{3k}}$$
$$\equiv\begin{cases}(\frac{78}{p})(x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{43})=1\ \&\ 4p=x^2+43y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{43})=-1.\end{cases}$$

If $p \neq 73$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(586,1)}{1752^{3k}}$$
$$\equiv\begin{cases}(\frac{438}{p})(x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{67})=1\ \&\ 4p=x^2+67y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{67})=-1.\end{cases}$$

If $p \neq 8893$, then

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(71146,1)}{213432^{3k}}$$
$$\equiv\begin{cases}(\frac{53358}{p})(x^2-2p)\pmod{p^2} & \text{if } (\frac{p}{163})=1\ \&\ 4p=x^2+163y^2,\\ 0\pmod{p^2} & \text{if } (\frac{p}{163})=-1.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(2,-1)}{(-3456)^k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(2,9)}{24^{3k}}$$
$$\equiv\begin{cases}(\frac{2}{p})(x^2-2p)\pmod{p^2} & \text{if } p\equiv 1\pmod 3\ \&\ 4p=x^2+27y^2,\\ 0\pmod{p^2} & \text{if } p\equiv 2\pmod 3).\end{cases}$$

And

$$\sum_{k=0}^{p-1}(15k+2)\frac{\binom{2k}{k}\binom{3k}{k}T_{3k}(2,-1)}{(-3456)^k}$$
$$\equiv\begin{cases}2p(\frac{2}{p})\pmod{p^2} & \text{if } 3\mid p-1 \text{ and } 2 \text{ is a cubis residue mod } p,\\ 0\pmod p & \text{otherwise.}\end{cases}$$

Remark 4.5. The imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$ has class number one for $d=7,11,19,43,67,163$.

Though we will not list many other conjectures similar to Conjectures 4.2-4.14, the above conjectures should convince the reader that our conjectural series for $1/\pi$ in the next section are indeed reasonable in view of the corresponding congruences.

5. Dualities and new series for $1/\pi$

As mentioned in Section 1, for $b>0$ and $c>0$ the main term of $T_n(b,c)$ as $n\to+\infty$ is
$$f_n(b,c):=\frac{(b+2\sqrt{c})^{n+1/2}}{2\sqrt[4]{c}\sqrt{n\pi}}.$$
Here we formulate a further refinement of this.

Conjecture 5.1. *For any positive real numbers b and c, we have*
$$T_n(b,c)=f_n(b,c)\left(1+\frac{b-4\sqrt{c}}{16n\sqrt{c}}+O\left(\frac{1}{n^2}\right)\right)$$
as $n\to+\infty$. If $c>0$ and $b=4\sqrt{c}$, then
$$\frac{T_n(b,c)}{\sqrt{c}^n}=T_n(4,1)=\frac{3\times 6^n}{\sqrt{6n\pi}}\left(1+\frac{1}{8n^2}+\frac{15}{64n^3}+\frac{21}{32n^4}+O\left(\frac{1}{n^5}\right)\right).$$
If $c<0$ and $b\in\mathbb{R}$ then
$$\lim_{n\to\infty}\sqrt[n]{|T_n(b,c)|}=\sqrt{b^2-4c}.$$

Let p be an odd prime. Z. H. Sun [S1] proved the congruence
$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}x^k\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k}(1-x)^k\pmod{p^2}\tag{5.1}$$

via Legendre polynomials; in fact this follows from the well-known identity $P_n(-x) = (-1)^n P_n(x)$ with $n = (p-1)/2$. In [Su7] the author managed to show the following congruences via the Zeilberger algorithm:

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} x^k \equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{64^k} (1-x)^k \pmod{p^2}, \tag{5.2}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} x^k \equiv \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{27^k} (1-x)^k \pmod{p^2} \quad (p \neq 3), \tag{5.3}$$

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k} x^k \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{432^k} (1-x)^k \pmod{p^2} \quad (p \neq 3). \tag{5.4}$$

The first part of our following result on dualities was motivated by (5.1)-(5.4).

Theorem 5.1. *Let p be an odd prime and let b, c and $m \not\equiv 0 \pmod p$ be rational p-adic integers.*

(i) *We have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(16m)^k} T_k(b,c) \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2}{(16m)^k} T_k(m-b,c) \pmod{p^2}, \tag{5.5}$$

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k} T_k(b,c) \equiv \left(\frac{-2}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k} T_k(m-b,c) \pmod{p^2}, \tag{5.6}$$

Provided $p > 3$ we also have

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{(27m)^k} T_k(b,c) \equiv \left(\frac{p}{3}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}\binom{3k}{k}}{(27m)^k} T_k(m-b,c) \pmod{p^2}, \tag{5.7}$$

$$\sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{(432m)^k} T_k(b,c) \equiv \left(\frac{-1}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{3k}{k}\binom{6k}{3k}}{(432m)^k} T_k(m-b,c) \pmod{p^2}. \tag{5.8}$$

(ii) *Suppose that $d = b^2 - 4c \not\equiv 0 \pmod p$. Then, for any $h \in \mathbb{Z}^+$ we have*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^h T_{2k}(b,c)}{m^k} \equiv \left(\frac{(-1)^h dm}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^h T_{2k}(b,c)}{(16^h d^2/m)^k} \pmod p. \tag{5.9}$$

Proof. (i) Since the proofs of (5.5)-(5.8) are very similar, we just show (5.6) in detail.

For $d=0,\ldots,p-1$, by taking differentiations of both sides (5.2) d times we get

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\binom{k}{d}x^{k-d}\equiv\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}(-1)^d\binom{k}{d}(1-x)^{k-d}\pmod{p^2}.$$

In view of this, we have

$$\begin{aligned}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}T_k(b,c)=&\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}\binom{2j}{j}b^{k-2j}c^j\\=&\sum_{j=0}^{p-1}\binom{2j}{j}\frac{c^j}{m^{2j}}\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\binom{k}{2j}\left(\frac{b}{m}\right)^{k-2j}\\\equiv&\sum_{j=0}^{p-1}\binom{2j}{j}\frac{c^j}{m^{2j}}\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{64^k}\binom{k}{2j}\left(1-\frac{b}{m}\right)^{k-2j}\\=&\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}\sum_{j=0}^{\lfloor k/2\rfloor}\binom{k}{2j}\binom{2j}{j}(m-b)^{k-2j}c^j\\=&\left(\frac{-2}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{4k}{2k}}{(64m)^k}T_k(m-b,c)\pmod{p^2}.\end{aligned}$$

(ii) In view of (3.1) and the known result

$$d^kT_{p-1-k}(b,c)\equiv\left(\frac{d}{p}\right)T_k(b,c)\pmod p\quad\text{for }k=0,\ldots,p-1$$

(see [N, (14)] or [Su5, Lemma 2.1]), we have

$$\begin{aligned}\sum_{k=0}^{n}\frac{\binom{2k}{k}^hT_{2k}(b,c)}{m^k}\equiv&\sum_{k=0}^{n}\frac{((-1)^n16^k\binom{2(n-k)}{n-k})^h}{m^k}\left(\frac{d}{p}\right)d^{2k}T_{2(n-k)}(b,c)\\=&(-1)^{hn}\left(\frac{d}{p}\right)\sum_{j=0}^{n}\left(\frac{16^hd^2}{m}\right)^{n-j}\binom{2j}{j}^hT_{2j}(b,c)\\\equiv&\left(\frac{(-1)^hdm}{p}\right)\sum_{k=0}^{n}\frac{\binom{2k}{k}^hT_{2k}(b,c)}{(16^hd^2/m)^k}\pmod p.\end{aligned}$$

Recall that $p\mid\binom{2k}{k}$ for each $k=n+1,\ldots,p-1$. So (5.9) follows.

The proof of Theorem 5.1 is now complete. □

Example 5.1. Let p be an odd prime. By (5.5) we have

$$\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(5,4)}{16^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2T_k(-4,4)}{16^k}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\pmod{p^2}.$$

The author [Su4] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{(-8)^k}\equiv\begin{cases}4x^2-2p\pmod{p^2} & \text{if } 4\mid p-1\ \&\ p=x^2+4y^2\ (x,y\in\mathbb{Z}),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4,\end{cases}$$

and this was recently confirmed by Z. H. Sun [S2]. When $p>3$, by (5.7) we have

$$\left(\frac{p}{3}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(3,1)}{27^k}\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(-2,1)}{27^k}=\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2\binom{3k}{k}}{(-27)^k}\pmod{p^2},$$

the reader may consult [Su4, Conjecture 5.6] for $\sum_{k=0}^{p-1}\binom{2k}{k}^2\binom{3k}{k}/(-27)^k$ mod p^2.

Based on our investigations of congruences on sums of central binomial coefficients and central trinomial coefficients, and the author's philosophy about series for $1/\pi$ stated in [Su6], we raise many conjectural series for $1/\pi$ of the following five new types with a_0,a_1,b,c,m integers and $a_0a_1bc(b^2-4c)m$ nonzero.

`Type I`. $\sum_{k=0}^\infty(a_0+a_1k)\binom{2k}{k}^2T_k(b,c)/m^k$.
`Type II`. $\sum_{k=0}^\infty(a_0+a_1k)\binom{2k}{k}\binom{3k}{k}T_k(b,c)/m^k$.
`Type III`. $\sum_{k=0}^\infty(a_0+a_1k)\binom{4k}{2k}\binom{2k}{k}T_k(b,c)/m^k$.
`Type IV`. $\sum_{k=0}^\infty(a_0+a_1k)\binom{2k}{k}^2T_{2k}(b,c)/m^k$.
`Type V`. $\sum_{k=0}^\infty(a_0+a_1k)\binom{2k}{k}\binom{3k}{k}T_{3k}(b,c)/m^k$.

Recall that a series $\sum_{k=0}^\infty a_k$ is said to converge at a geometric rate with ratio r if $\lim_{k\to+\infty}a_{k+1}/a_k=r\in(0,1)$. All the series in Conjectures I-V below converge at geometrical rates, and most of them were found by the author during Jan.-Feb. 2011.

Conjecture I. *We have the following identities:*

$$\sum_{k=0}^\infty\frac{30k+7}{(-256)^k}\binom{2k}{k}^2T_k(1,16)=\frac{24}{\pi},\tag{I1}$$

$$\sum_{k=0}^\infty\frac{30k+7}{(-1024)^k}\binom{2k}{k}^2T_k(34,1)=\frac{12}{\pi},\tag{I2}$$

$$\sum_{k=0}^\infty\frac{30k-1}{4096^k}\binom{2k}{k}^2T_k(194,1)=\frac{80}{\pi},\tag{I3}$$

$$\sum_{k=0}^\infty\frac{42k+5}{4096^k}\binom{2k}{k}^2T_k(62,1)=\frac{16\sqrt3}{\pi}.\tag{I4}$$

Remark 5.1. (I1) was the first identify for $1/\pi$ involving generalized central trinomial coefficients; it was discovered on Jan. 2, 2011. Different from classical Ramanujan-type series for $1/\pi$ (cf. N. D. Baruah and B. C. Berndt [BB], and Berndt [Be, pp. 353-354]) and their known generalizations (see, e.g., S. Cooper [C]), the two numbers in the linear part $30k-1$ of (I3) have *different signs*, and also its corresponding p-adic congruence (with $p>3$ a prime) involves *two* Legendre symbols:

$$\sum_{k=0}^{p-1}(30k-1)\frac{\binom{2k}{k}^2T_k(194,1)}{4096^k}\equiv p\left(5\left(\frac{-1}{p}\right)-6\left(\frac{3}{p}\right)\right)\pmod{p^2}.$$

Conjecture II. *We have*

$$\sum_{k=0}^{\infty}\frac{15k+2}{972^k}\binom{2k}{k}\binom{3k}{k}T_k(18,6)=\frac{45\sqrt3}{4\pi},\tag{II1}$$

$$\sum_{k=0}^{\infty}\frac{91k+12}{10^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(10,1)=\frac{75\sqrt3}{2\pi},\tag{II2}$$

$$\sum_{k=0}^{\infty}\frac{15k-4}{18^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(198,1)=\frac{135\sqrt3}{2\pi},\tag{II3}$$

$$\sum_{k=0}^{\infty}\frac{42k-41}{30^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(970,1)=\frac{525\sqrt3}{\pi},\tag{II4}$$

$$\sum_{k=0}^{\infty}\frac{18k+1}{30^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(730,729)=\frac{25\sqrt3}{\pi},\tag{II5}$$

$$\sum_{k=0}^{\infty}\frac{6930k+559}{102^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(102,1)=\frac{1445\sqrt6}{2\pi},\tag{II6}$$

$$\sum_{k=0}^{\infty}\frac{222105k+15724}{198^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(198,1)=\frac{114345\sqrt3}{4\pi},\tag{II7}$$

$$\sum_{k=0}^{\infty}\frac{390k-3967}{102^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(39202,1)=\frac{56355\sqrt3}{\pi},\tag{II8}$$

$$\sum_{k=0}^{\infty}\frac{210k-7157}{198^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(287298,1)=\frac{114345\sqrt3}{\pi},\tag{II9}$$

and

$$\sum_{k=0}^{\infty}\frac{45k+7}{24^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(26,729)=\frac{8}{3\pi}(3\sqrt3+\sqrt{15}),\tag{II10}$$

$$\sum_{k=0}^{\infty}\frac{9k+2}{(-5400)^k}\binom{2k}{k}\binom{3k}{k}T_k(70,3645)=\frac{15\sqrt3+\sqrt{15}}{6\pi},\tag{II11}$$

$$\sum_{k=0}^{\infty}\frac{63k+11}{(-13500)^k}\binom{2k}{k}\binom{3k}{k}T_k(40,1458)=\frac{25}{12\pi}(3\sqrt3+4\sqrt6),\tag{II12}$$

Remark 5.2. In view of (5.7), we may view (II9) as the dual of (II7) since $198^3/27-198=187298$. The series in (II7) converges rapidly at a geometric rate with ratio $25/35937$, but the series in (II9) converges very slow at a geometric rate with ratio $71825/71874$. (II2), (II9) and (II10) were motivated by the following congruences (with $p>3$ a prime):

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}\binom{3k}{k}T_k(10,1)}{10^{3k}}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1,3\pmod 8\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }(\frac{-2}{p})=-1,\text{ i.e., }p\equiv5,7\pmod 8,\end{cases}$$
$$\sum_{k=0}^{p-1}(91k+12)\frac{\binom{2k}{k}\binom{3k}{k}T_k(10,1)}{10^{3k}}\equiv\frac{3p}{2}\left(9\left(\frac{-3}{p}\right)-1\right)\pmod{p^2}\quad(p\ne5);$$

$$\sum_{k=0}^{p-1}(210k-7157)\frac{\binom{2k}{k}\binom{3k}{k}T_k(287298,1)}{198^{3k}}$$
$$\equiv p\left(35\left(\frac{-3}{p}\right)-7192\right)\pmod{p^2}\quad(p\ne11);$$

$$\sum_{k=0}^{p-1}\frac{45k+7}{24^{3k}}\binom{2k}{k}\binom{3k}{k}T_k(26,729)\equiv\frac{p}{2}\left(9\left(\frac{-3}{p}\right)+5\left(\frac{-15}{p}\right)\right)\pmod{p^2}.$$

Conjecture III. *We have the following formulae:*

$$\sum_{k=0}^{\infty}\frac{85k+2}{66^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(52,1)=\frac{33\sqrt{33}}{\pi},\tag{III1}$$

$$\sum_{k=0}^{\infty}\frac{28k+5}{(-96^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(110,1)=\frac{3\sqrt{6}}{\pi},\tag{III2}$$

$$\sum_{k=0}^{\infty}\frac{40k+3}{112^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(98,1)=\frac{70\sqrt{21}}{9\pi},\tag{III3}$$

$$\sum_{k=0}^{\infty}\frac{80k+9}{264^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(257,256)=\frac{11\sqrt{66}}{2\pi},\tag{III4}$$

$$\sum_{k=0}^{\infty}\frac{80k+13}{(-168^2)^k}\binom{4k}{2k}\binom{2k}{k}T_k(7,4096)=\frac{14\sqrt{210}+21\sqrt{42}}{8\pi},\tag{III5}$$

$$\sum_{k=0}^{\infty}\frac{760k+71}{336^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(322,1)=\frac{126\sqrt{7}}{\pi},\tag{III6}$$

$$\sum_{k=0}^{\infty}\frac{10k-1}{336^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(1442,1)=\frac{7\sqrt{210}}{4\pi},\tag{III7}$$

$$\sum_{k=0}^{\infty}\frac{770k+69}{912^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(898,1)=\frac{95\sqrt{114}}{4\pi},\tag{III8}$$

$$\sum_{k=0}^{\infty}\frac{280k-139}{912^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(12098,1)=\frac{95\sqrt{399}}{\pi},\tag{III9}$$

$$\sum_{k=0}^{\infty}\frac{84370k+6011}{10416^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(10402,1)=\frac{3689\sqrt{434}}{4\pi},\tag{III10}$$

$$\sum_{k=0}^{\infty}\frac{8840k-50087}{10416^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(1684802,1)=\frac{7378\sqrt{8463}}{\pi},\tag{III11}$$

$$\sum_{k=0}^{\infty}\frac{11657240k+732103}{39216^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(39202,1)=\frac{80883\sqrt{817}}{\pi},\tag{III12}$$

$$\sum_{k=0}^{\infty}\frac{3080k-58871}{39216^{2k}}\binom{4k}{2k}\binom{2k}{k}T_k(23990402,1)=\frac{17974\sqrt{2451}}{\pi}.\tag{III13}$$

Remark 5.3. (III12) and (III13) are dual in view of (5.6). Other dual pairs include (III6) and (III7), (III8) and (III9), (III10) and (III11). Below are the corresponding p-adic congruences for (III1) and (III13) (with $p>3$ a prime):

$$\sum_{k=0}^{p-1}(85k+2)\frac{\binom{4k}{2k}\binom{2k}{k}T_k(52,1)}{66^{2k}}$$
$$\equiv p\left(12\left(\frac{-33}{p}\right)-10\left(\frac{33}{p}\right)\right)\pmod{p^2},\quad (p\neq 11),$$

$$\sum_{k=0}^{p-1}(3080k-58871)\frac{\binom{4k}{2k}\binom{2k}{k}T_k(23990402,1)}{39216^{2k}}$$
$$\equiv p\left(385\left(\frac{-2451}{p}\right)-59256\left(\frac{1634}{p}\right)\right)\pmod{p^2}\quad (p\neq 19,43).$$

Conjecture IV. *We have*

$$\sum_{k=0}^{\infty}\frac{26k+5}{(-48^2)^k}\binom{2k}{k}^2T_{2k}(7,1)=\frac{48}{5\pi},\tag{IV1}$$

$$\sum_{k=0}^{\infty}\frac{340k+59}{(-480^2)^k}\binom{2k}{k}^2T_{2k}(62,1)=\frac{120}{\pi},\tag{IV2}$$

$$\sum_{k=0}^{\infty}\frac{13940k+1559}{(-5760^2)^k}\binom{2k}{k}^2T_{2k}(322,1)=\frac{4320}{\pi},\tag{IV3}$$

$$\sum_{k=0}^{\infty}\frac{8k+1}{96^{2k}}\binom{2k}{k}^2T_{2k}(10,1)=\frac{10\sqrt{2}}{3\pi},\tag{IV4}$$

$$\sum_{k=0}^{\infty}\frac{10k+1}{240^{2k}}\binom{2k}{k}^2T_{2k}(38,1)=\frac{15\sqrt{6}}{4\pi},\tag{IV5}$$

$$\sum_{k=0}^{\infty}\frac{14280k+899}{39200^{2k}}\binom{2k}{k}^2T_{2k}(198,1)=\frac{1155\sqrt{6}}{\pi},\tag{IV6}$$

$$\sum_{k=0}^{\infty}\frac{120k+13}{320^{2k}}\binom{2k}{k}^2T_{2k}(18,1)=\frac{12\sqrt{15}}{\pi},\tag{IV7}$$

$$\sum_{k=0}^{\infty}\frac{21k+2}{896^{2k}}\binom{2k}{k}^2T_{2k}(30,1)=\frac{5\sqrt{7}}{2\pi},\tag{IV8}$$

$$\sum_{k=0}^{\infty}\frac{56k+3}{24^{4k}}\binom{2k}{k}^2T_{2k}(110,1)=\frac{30\sqrt{7}}{\pi},\tag{IV9}$$

and

$$\sum_{k=0}^{\infty}\frac{56k+5}{48^{4k}}\binom{2k}{k}^2T_{2k}(322,1)=\frac{72\sqrt7}{5\pi},\tag{IV10}$$

$$\sum_{k=0}^{\infty}\frac{10k+1}{2800^{2k}}\binom{2k}{k}^2T_{2k}(198,1)=\frac{25\sqrt{14}}{24\pi},\tag{IV11}$$

$$\sum_{k=0}^{\infty}\frac{195k+14}{10400^{2k}}\binom{2k}{k}^2T_{2k}(102,1)=\frac{85\sqrt{39}}{12\pi},\tag{IV12}$$

$$\sum_{k=0}^{\infty}\frac{3230k+263}{46800^{2k}}\binom{2k}{k}^2T_{2k}(1298,1)=\frac{675\sqrt{26}}{4\pi},\tag{IV13}$$

$$\sum_{k=0}^{\infty}\frac{520k-111}{5616^{2k}}\binom{2k}{k}^2T_{2k}(1298,1)=\frac{1326\sqrt{3}}{\pi},\tag{IV14}$$

$$\sum_{k=0}^{\infty}\frac{280k-149}{20400^{2k}}\binom{2k}{k}^2T_{2k}(4898,1)=\frac{330\sqrt{51}}{\pi},\tag{IV15}$$

$$\sum_{k=0}^{\infty}\frac{78k-1}{28880^{2k}}\binom{2k}{k}^2T_{2k}(5778,1)=\frac{741\sqrt{10}}{20\pi},\tag{IV16}$$

$$\sum_{k=0}^{\infty}\frac{57720k+3967}{439280^{2k}}\binom{2k}{k}^2T_{2k}(5778,1)=\frac{2890\sqrt{19}}{\pi},\tag{IV17}$$

$$\sum_{k=0}^{\infty}\frac{1615k-314}{243360^{2k}}\binom{2k}{k}^2T_{2k}(54758,1)=\frac{1989\sqrt{95}}{4\pi}.\tag{IV18}$$

Remark 5.4. For (IV6), `Mathematica` indicates that if we set

$$s(n):=\sum_{k=0}^{n}\frac{14280k+899}{39200^{2k}}\binom{2k}{k}^2T_{2k}(198,1)$$

then

$$\left|s(15)\times\frac{\pi}{1155\sqrt6}-1\right|<\frac{1}{10^{50}}\quad\text{and}\quad\left|s(30)\times\frac{\pi}{1155\sqrt6}-1\right|<\frac{1}{10^{100}}.$$

Below are corresponding p-adic congruences of (IV9)-(IV11) and (IV18) with

$p>5$ a prime:

$$\sum_{k=0}^{p-1}(56k+3)\frac{\binom{2k}{k}^2T_{2k}(110,1)}{24^{4k}}\equiv\frac{p}{4}\left(35\left(\frac{p}{7}\right)-23\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(56k+5)\frac{\binom{2k}{k}^2T_{2k}(322,1)}{48^{4k}}\equiv\frac{p}{20}\left(147\left(\frac{p}{7}\right)-47\right)\pmod{p^2},$$
$$\sum_{k=0}^{p-1}(10k+1)\frac{\binom{2k}{k}^2T_{2k}(198,1)}{2800^{2k}}\equiv\frac{p}{12}\left(\frac{2}{p}\right)\left(13\left(\frac{p}{7}\right)-1\right)\pmod{p^2}\quad(p\neq 7),$$

and

$$\sum_{k=0}^{p-1}(1615k-314)\frac{\binom{2k}{k}^2T_{2k}(54758,1)}{243360^{2k}}$$
$$\equiv\frac{p}{26}\left(6137\left(\frac{p}{95}\right)-14301\right)\pmod{p^2}\quad(p\neq 13).$$

Conjecture V. *We have the formula*

$$\sum_{k=0}^{\infty}\frac{1638k+277}{(-240)^{3k}}\binom{2k}{k}\binom{3k}{k}T_{3k}(62,1)=\frac{44\sqrt{105}}{\pi}.\tag{V1}$$

Remark 5.5. (V1) was motivated by Conjecture 4.7; the series converges at a geometric rate with ratio $-64/125$.

We conjecture that (IV1)-(IV18) have exhausted all identities of the form

$$\sum_{k=0}^{\infty}(a_0+a_1k)\frac{\binom{2k}{k}^2T_{2k}(b,1)}{m^k}=\frac{C}{\pi}$$

with $a_0,a_1,m\in\mathbb{Z}$, $b\in\{1,3,4,\ldots\}$, $a_1>0$, and C^2 positive and rational. This comes from our following hypothesis motivated by (5.9) in the case $h=2$ and the author's philosophy about series for $1/\pi$ stated in [Su6]. We have applied the hypothesis to seek for series for $1/\pi$ of type IV and checked all those $b=1,\ldots,10^6$ via computer.

Hypothesis 5.1. (i) *Suppose that*

$$\sum_{k=0}^{\infty}\frac{a_0+a_1k}{m^k}\binom{2k}{k}^2T_{2k}(b,1)=\frac{C}{\pi}$$

with $a_0, a_1, m \in \mathbb{Z}$, $b \in \mathbb{Z}^+$ *and* $C^2 \in \mathbb{Q} \setminus \{0\}$. *Then* $\sqrt{|m|}$ *is an integer dividing* $16(b^2-4)$. *Also,* $b = 7$ *or* $b \equiv 2 \pmod 4$.

(ii) *Let* $\varepsilon \in \{\pm 1\}$, $b, m \in \mathbb{Z}^+$ *and* $m \mid 16(b^2-4)$. *Then, there are* $a_0, a_1 \in \mathbb{Z}$ *such that*

$$\sum_{k=0}^{\infty} \frac{a_0 + a_1 k}{(\varepsilon m^2)^k} \binom{2k}{k}^2 T_{2k}(b,1) = \frac{C}{\pi}$$

for some $C \neq 0$ *with* C^2 *rational, if and only if* $m > 4(b+2)$ *and*

$$\sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 T_{2k}(b,1)}{(\varepsilon m^2)^k} \equiv \left(\frac{\varepsilon(b^2-4)}{p}\right) \sum_{k=0}^{p-1} \frac{\binom{2k}{k}^2 T_{2k}(b,1)}{(\varepsilon \bar{m}^2)^k} \pmod{p^2}$$

for all odd primes $p \nmid b^2 - 4$, *where* $\bar{m} = 16(b^2-4)/m$.

Concerning the 48 new identities in Conjectures I-V, actually we first discovered congruences without linear parts related to binary quadratic forms (like many congruences in Section 4), then found corresponding p-adic congruences with linear parts, and finally figured out the series for $1/\pi$.

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