CHROMATIC STATISTICS FOR CATALAN AND FUSS–CATALAN NUMBERS

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Abstract. We refine Catalan numbers and Fuß–Catalan numbers by introducing colour statistics for triangulations of polygons and d-dimensional generalisations thereof which we call Fuß–Catalan complexes. Our refinements consist in showing that the number of triangulations, respectively Fuß–Catalan complexes, with a given colour distribution of its vertices is given by closed product formulae. The crucial ingredient in the proof is the Lagrange–Good inversion formula.

1. Introduction

1.1. **Catalan and Fuß–Catalan numbers.** The sequence $(C_n)_{n\ge0}$ of Catalan numbers

$$1, 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, 58786, \dots,$$

see [11, sequence A108], defined by

$$C_n := \frac{1}{n+1}\binom{2n}{n} = \frac{1}{n}\binom{2n}{n-1}, \tag{1.1}$$

is ubiquitous in enumerative combinatorics. Exercise 6.19 in [13] contains a list of 66 sequences of sets enumerated by Catalan numbers, with many more in the addendum [14]. In particular, there are $\frac{1}{n+1}\binom{2n}{n}$ triangulations of a convex polygon[1] with $n+2$ vertices (see [13, Ex. 6.19.a]).

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[1]As is common, when we speak of a "convex polygon," we always tacitly assume that all its angles are less than 180 degrees.

Even many years before Catalan's paper [4], Fuß [6] enumerated the dissections of a convex $((d-1)n+2)$-gon into $(d+1)$-gons (obviously, any such dissection will consist of n $(d+1)$-gons) and found that there are

$$\frac{1}{n}\binom{dn}{n-1} \tag{1.2}$$

of those. These numbers are now commonly known as *Fuß–Catalan numbers* (cf. [1, pp. 59–60]). A perhaps (geometrically) more natural generalisation of triangulations of a convex polygon (albeit more difficult to visualise) is to consider d-dimensional simplicial complexes on $n+d$ vertices homeomorphic to a d-ball that consist of n maximal faces all of dimension d, with the additional property that all faces of dimension up to $d-2$ lie in the boundary of the complex. (See Section 2 for the precise definition. We call these complexes *Fuß–Catalan complexes.*) Again, the number of these complexes is given by the Fuß–Catalan number (1.2). The reader is referred to [5, paragraph after (8.9)] for further combinatorial occurrences of the Fuß–Catalan numbers.

1.2. **Coloured refinements: short outline of this paper.** The main theorems of our paper present "coloured" refinements of the above classical results. More precisely, to each triangulation (respectively, more generally, Fuß–Catalan complex) we shall associate a colouring of its vertices. In a certain sense, this colouring measures whether or not a large number of triangles (respectively maximal faces) meets in single vertices. We show that the number of triangulations of a convex $(n+2)$-gon (respectively of d-dimensional Fuß–Catalan complexes on $n+d$ vertices) with a fixed distribution of colours of its vertices is given by closed formulae (see Theorems 1.1, 1.2, 2.1, and 2.2), thus refining the Catalan numbers (1.1) (respectively the Fuß–Catalan numbers (1.2)).

In order to give a clearer idea of what we have in mind, we shall use the remainder of this introduction to define precisely the colouring scheme for the case of triangulations, and we shall present the corresponding refined enumeration results (see Theorems 1.1 and 1.2). Subsequently, in Section 2 we generalise this setting by introducing d-dimensional Fuß–Catalan complexes for arbitrary positive integers d. The corresponding enumeration results generalising Theorems 1.1 and 1.2 are presented in Theorems 2.1 and 2.2. Section 3 is then devoted to the proof of Theorem 2.1, thus also establishing Theorem 1.1. Crucial in this proof is the Lagrange–Good inversion formula [7]. Finally, Section 4 is devoted to the proof of Theorem 2.2, and thus also of Theorem 1.2, which it generalises.

1.3. 3-**Coloured triangulations.** In the sequel, P_n stands for a convex polygon with n vertices. Since we are only interested in the combinatorics of triangulations of P_{n+2}, we can consider a unique polygon P_{n+2} for each integer $n \geq 0$. A triangulation of P_{n+2} has exactly n triangles. We shall always use the Greek letter τ to denote triangulations. We call a triangulation τ of P_{n+2} 3-*coloured* if the $n+2$ vertices of P_{n+2} are coloured with 3 colours in such a way that the three vertices of every triangle in τ have different colours. (Using a graph-theoretic term, we call a colouring with the latter property a *proper* colouring.) An easy induction on n shows the existence of such a colouring, and that it is unique up to permutations of all three colours.

A *rooted* polygon is, by definition, a (convex) polygon containing a marked oriented edge $\overrightarrow{e}$, the "root edge" (borrowing terminology from the theory of combinatorial maps;

cf. [16]) in its boundary. In the illustrations in Figure 1, the marked oriented edge is always indicated by an arrow. We write $P^{\to}_{n+2}$ for a rooted polygon with $n+2$ vertices. In the sequel, we omit a separate discussion of the degenerate case $n=0$, where the rooted "polygon" $P^{\to}_{2}$ essentially only consists of the marked oriented edge $\overrightarrow{e}$. We agree once and for all that there is one triangulation in this case.

For $n \geq 1$, a triangulation τ of $P^{\to}_{n+2}$ has a unique triangle Δ_* that contains the marked oriented edge $\overrightarrow{e}$. We consider this "root triangle" as a triangle with totally ordered vertices $v_0 < v_1 < v_2$, where $\overrightarrow{e}$ starts at v_1 and ends at v_2. The $n+2$ vertices of a triangulation τ of $P^{\to}_{n+2}$ can then be uniquely coloured with three colours $\{\mathtt{a},\mathtt{b},\mathtt{c}\}$ such that $\overrightarrow{e}$ starts at a vertex of colour $\mathtt{b}$, ends at a vertex of colour $\mathtt{c}$, and vertices of every triangle $\Delta \in \tau$ have different colours. Figure 1 shows all such 3-coloured triangulations of $P^{\to}_{n+2}$ for $n = 0, 1, 2, 3$.

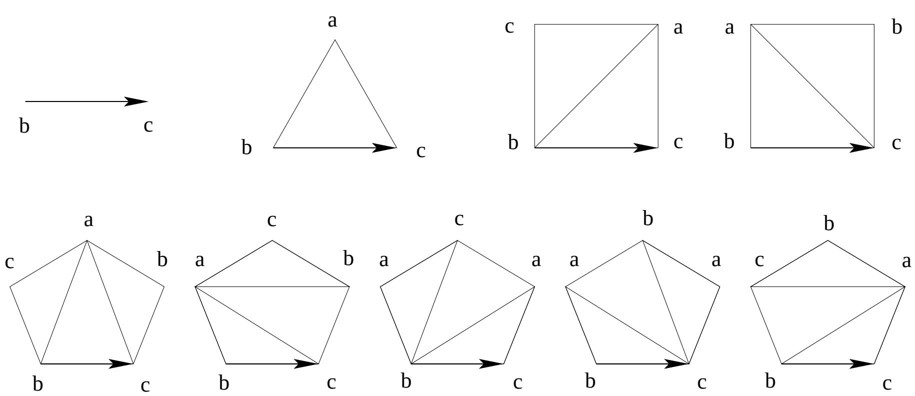

FIGURE 1. All 3-coloured triangulations for $n = 0, 1, 2, 3$.

Our first result provides a closed formula for the number of triangulations with a fixed colour distribution of its vertices.

Theorem 1.1. *Let n be a non-negative integer and α, β, γ non-negative integers with $\alpha + \beta + \gamma = n + 2$. Then the number of triangulations of the rooted polygon $P^{\to}_{n+2}$ with α vertices of colour* $\mathtt{a}$*, β vertices of colour* $\mathtt{b}$*, and γ vertices of colour* $\mathtt{c}$ *in the uniquely determined colouring induced by a triangulation, in which the starting vertex of the marked oriented edge $\overrightarrow{e}$ has colour* $\mathtt{b}$*, its ending vertex has colour* $\mathtt{c}$*, and the three vertices in each triangle have different colours, is equal to*

$$\frac{\alpha(\alpha+\beta+\gamma-2)}{(\beta+\gamma-1)(\alpha+\gamma-1)(\alpha+\beta-1)}\binom{\beta+\gamma-1}{\alpha}\binom{\alpha+\gamma-1}{\beta-1}\binom{\alpha+\beta-1}{\gamma-1}. \tag{1.3}$$

In the case where $\alpha = 0$, this has to be interpreted as the limit $\alpha \to 0$, that is, it is 1 if $(\alpha, \beta, \gamma) = (0, 1, 1)$ and 0 otherwise.

As we already announced, we shall generalise this theorem in Theorem 2.1 from triangulations to simplicial complexes. Its proof (given in Section 3) shows that the corresponding generating function, that is, the series

$$C = C(a, b, c) = \sum_{\alpha,\beta,\gamma \geq 0} C_{\alpha,\beta,\gamma} a^{\alpha} b^{\beta} c^{\gamma},$$

where $C_{\alpha,\beta,\gamma}$ is the number of triangulations in Theorem 1.1, is algebraic. To be precise, from the equations given in Section 3 (specialised to $d = 2$), one can extract that

$$(bc)^3(1+a) + (bc)^2((b+c)a-1)C + (bc)^2(a-2)C^2 + 2bcC^3 + bcC^4 - C^5 = 0 . \quad (1.4)$$

Next we identify two of the three colours. In other words, we now consider *improper* colourings of triangulations of $P_{n+2}^{\rightarrow}$ by *two* colours, say black and white, such that every triangle has exactly one black vertex and two white vertices. There are then two possibilities to colour the marked oriented edge $\overrightarrow{e}$: either both of its incident vertices are coloured white, or one is coloured white and the other black (for the purpose of enumeration, it does not matter which of the two is white respectively black in the latter case). Remarkably, in both cases there exist again closed enumeration formulae for the number of triangulations with a given colour distribution.

Theorem 1.2. *Let n, b, w be non-negative integers with $b + w = n + 2$.*

(i) *The number of triangulations of the rooted polygon $P_{n+2}^{\rightarrow}$ with b black vertices and w white vertices in the uniquely determined colouring induced by a triangulation, in which both vertices of the marked oriented edge $\overrightarrow{e}$ are coloured white, and, in each triangle, exactly two of the three vertices are coloured white, is equal to*

$$\frac{2b}{(w-1)(2b+w-2)}\binom{2b+w-2}{w-2}\binom{w-1}{b} .$$

(ii) *The number of triangulations of the rooted polygon $P_{n+2}^{\rightarrow}$ with b black vertices and w white vertices in the uniquely determined colouring induced by a triangulation, in which the starting vertex of the marked oriented edge $\overrightarrow{e}$ is coloured white, its ending vertex is coloured black, and, in each triangle, exactly two of the three vertices are coloured white, is equal to*

$$\frac{1}{2b+w-2}\binom{2b+w-2}{w-1}\binom{w-1}{b-1} .$$

Obviously, the generating functions corresponding to the numbers in the above theorem must be algebraic. To be precise, it follows from (1.4) that the series $Y = C(x, y, y)$ (the generating function for the numbers in item (i) of Theorem 1.2) and the series $Z = C(x, x, y)$ (the generating function for the numbers in item (ii) of Theorem 1.2) satisfy the algebraic equations

$$(1+x)y^4 - y^2(1+2y)Y + y(2+y)Y^2 - Y^3 = 0 \quad (1.5)$$

and

$$x^2y^2 + xy(x-1)Z + Z^3 = 0 , \quad (1.6)$$

respectively. As we announced, Theorem 1.2 will be generalised from triangulations to simplicial complexes in Theorem 2.2.

Clearly, if we identify all three colours, then we are back to counting all triangulations of the polygon P_{n+2}, of which there are $C_n = \frac{1}{n+1}\binom{2n}{n}$.

We end this introduction by mentioning that checkerboard colourings of triangulations (obtained by colouring adjacent triangles with different colours chosen in a set of two colours) encode winding properties of the corresponding 3-vertex colouring. Indeed, a 3-coloured triangulation τ of P_{n+2} induces a unique piecewise affine map φ from P_{n+2} onto a vertex-coloured triangle Δ such that φ is colour-preserving on vertices and induces affine bijections between triangles of τ and Δ. The map φ is orientation-preserving, respectively orientation reverting, on black, respectively white, triangles of τ endowed with a suitable black-white checkerboard colouring. Restricting φ to the oriented boundary of P_{n+2} we get a closed oriented path contained in the boundary of Δ. The winding number of this path with respect to an interior point of Δ is given by the difference of black and white triangles in the checkerboard colouring mentioned above. The resulting statistics for Catalan numbers (and the obvious generalization to Fuß–Catalan numbers obtained by replacing winding numbers with the corresponding homology classes) have been studied by Callan in [2].

2. Refinements of Fuss–Catalan numbers

2.1. **Fuß–Catalan complexes.** Given an integer $d \geq 2$, we define a *d-dimensional Fuß–Catalan complex of index* $n \geq 1$ to be a simplicial complex Σ such that:

(i) Σ is a d-dimensional simplicial complex homeomorphic to a closed d-dimensional ball having n simplices of maximal dimension d.

(ii) All simplices of dimension up to $d-2$ of Σ are contained in the boundary $\partial\Sigma$ (homeomorphic to a $(d-1)$-dimensional sphere) of Σ. (Equivalently, the $(d-2)$-skeleton of Σ is contained in its boundary $\partial\Sigma$).

Such a complex Σ is *rooted* if its boundary $\partial\Sigma$ contains a marked $(d-1)$-simplex, Δ_* say, with totally ordered vertices. We denote a rooted d-dimensional Fuß–Catalan complex by the pair (Σ, Δ_*). By convention, a rooted d-dimensional Fuß–Catalan complex of index 0 is given by (Δ_*, Δ_*), where Δ_* is a simplex of dimension $d-1$ with totally ordered vertices.

Rooted d-dimensional Fuß–Catalan complexes are generalisations of rooted triangulations of polygons. In particular, a rooted 2-dimensional Fuß–Catalan complex of index n is a triangulation of the rooted polygon $P^{\rightarrow}_{n+2}$ with $n+2$ vertices.

2.2. **$(d+1)$-colourings of d-dimensional Fuß–Catalan complexes.** Let $\mathcal{C}$ be a set of colours. A *proper colouring* of a simplicial complex Σ with vertex set $\mathcal{V}$ by colours from $\mathcal{C}$ is a map $\gamma : \mathcal{V} \longrightarrow \mathcal{C}$ such that $\gamma(v) \neq \gamma(w)$ for any pair of vertices v, w defining a 1-simplex of Σ. Equivalently, a proper colouring of a simplicial complex Σ is a proper colouring of the graph defined by the 1-skeleton of Σ.

Every rooted d-dimensional Fuß–Catalan complex (Σ, Δ_*) has a unique colouring by $(d+1)$ totally ordered colours $c_0 < c_1 < \cdots < c_d$ such that the i-th vertex of Δ_* (in the given total order of the vertices of Δ_*) has colour c_i, $i = 1, 2, \ldots, d$. The following theorem presents a closed formula for the number of Fuß–Catalan complexes of index n with a given colour distribution.

Theorem 2.1. *Let d, n, $\gamma_0, \gamma_1, \ldots, \gamma_d$ be non-negative integers with $d \geq 2$ and $\gamma_0 + \gamma_1 + \cdots + \gamma_d = n + d$. Then the number of d-dimensional Fuß–Catalan complexes (Σ, Δ_*) of index n with γ_i vertices of colour c_i, $i = 0, 1, \ldots, d$, in the uniquely determined proper colouring by the colours $c_0, c_1, \ldots, c_d$ in which the i-th vertex of the root simplex Δ_* has colour c_i, $i = 1, 2, \ldots, d$, is equal to*

$$s^{d-1} \frac{\gamma_0}{s-\gamma_0+1} \binom{s-\gamma_0+1}{\gamma_0} \prod_{j=1}^{d} \frac{1}{s-\gamma_j+1} \binom{s-\gamma_j+1}{\gamma_j-1}, \tag{2.1}$$

where $s = -d + \sum_{j=0}^{d} \gamma_j$*. In the case where* $\gamma_0 = 0$*, this has to be interpreted as the limit* $\gamma_0 \to 0$*, that is, it is* 1 *if* $(\gamma_0, \gamma_1, \dots, \gamma_d) = (0, 1, 1, \dots, 1)$ *and* 0 *otherwise.*

Formula (2.1) generalises Formula (1.3), the latter corresponding to the case $d = 2$ of the former.

2.3. **Specialisations obtained by identifying colours.** Generalising the scenario in Theorem 1.2, we now identify some of the colours. Namely, given a non-negative integer k and $k+1$ positive integers $\beta_0, \beta_1, \beta_2, \dots, \beta_k$ with $\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k = d+1$, we set

$$\begin{aligned} c_0 = \cdots = c_{\beta_0 - 1} &= c'_0 \\ c_{\beta_0} = \cdots = c_{\beta_0 + \beta_1 - 1} &= c'_1 \\ &\vdots \\ c_{\beta_0 + \beta_1 + \cdots + \beta_{i-1}} = \cdots = c_{\beta_0 + \beta_1 + \cdots + \beta_i - 1} &= c'_i \\ &\vdots \\ c_{\beta_0 + \beta_1 + \cdots + \beta_{k-1}} = \cdots = c_d &= c'_k. \end{aligned}$$

Given a rooted Fuß–Catalan complex (Σ, Δ_*) with its uniquely determined colouring as in Theorem 2.1, after this identification we obtain a colouring of the simplices of (Σ, Δ_*) in which each d-dimensional simplex has β_i vertices of colour c'_i, $i = 0, 1, \dots, k$. Our next theorem presents a closed formula for the number of d-dimensional Fuß–Catalan complexes of index n with a given colour distribution *after* this identification of colours.

Theorem 2.2. *Let* d*,* k*,* n*,* $\beta_0, \beta_1, \dots, \beta_k$*,* $\gamma_0, \gamma_1, \dots, \gamma_k$ *be non-negative integers with* $d \ge 2$*,* $\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k = d + 1$*, and* $\gamma_0 + \gamma_1 + \cdots + \gamma_k = n + d$*. Then the number of* d*-dimensional Fuß–Catalan complexes* (Σ, Δ_*) *of index* n *with* γ_i *vertices of colour* c'_i*,* $i = 0, 1, \dots, k$*, in the uniquely determined colouring in which the first* $\beta_0 - 1$ *vertices of the root simplex* Δ_* *have colour* c'_0*, the next* β_1 *vertices have colour* c'_1*, the next* β_2 *vertices have colour* c'_2*, ..., the last* β_k *vertices have colour* c'_k*, and in which each* d*-dimensional simplex has* β_i *vertices of colour* c'_i*,* $i = 0, 1, \dots, k$*, is equal to*

$$s^{k-1} \frac{\gamma_0 - \beta_0 + 1}{\beta_0 s + \beta_0 - \gamma_0} \binom{\beta_0 s + \beta_0 - \gamma_0}{\gamma_0 - \beta_0 + 1} \prod_{j=1}^{k} \frac{\beta_j}{\beta_j s + \beta_j - \gamma_j} \binom{\beta_j s + \beta_j - \gamma_j}{\gamma_j - \beta_j},$$

where $s = -d + \sum_{j=0}^{k} \gamma_j$*.*

This theorem contains all the afore-mentioned results as special cases. Clearly, Theorem 2.1 is the special case of Theorem 2.2 where $k = d$ and $\beta_0 = \beta_1 = \cdots = \beta_d = 1$ (and Theorem 1.1 is the further special case in which $d = 2$). Item (i) of Theorem 1.2 results for $d = 2$, $k = 1$, $\beta_0 = 1$, $\beta_1 = 2$, while item (ii) results for $d = 2$, $k = 1$, $\beta_0 = 2$, $\beta_1 = 1$. Moreover, upon setting $k = 0$ and $\beta_0 = d+1$ in Theorem 2.2, we obtain Formula (1.2) (and (1.1) in the further special case where $d = 2$).

3. Generating functions and the Lagrange–Good inversion formula

In this section we provide the proof of Theorem 2.1. It makes use of generating function calculus, which serves to reach a situation in which the Lagrange–Good inversion formula [7] (see also [9, Sec. 5] and the references cited therein) can be applied to compute the numbers that we are interested in. The proof requires as well a determinant evaluation, which we state and establish separately at the end of this section.

Proof of Theorem 2.1. Let

$$C_d(x_0,x_1,\dots,x_d):=\sum_{(\Sigma,\Delta_*)} x_0^{\gamma_0(\Sigma,\Delta_*)}x_1^{\gamma_1(\Sigma,\Delta_*)}\cdots x_d^{\gamma_d(\Sigma,\Delta_*)},$$

where the sum is over all d-dimensional Fuß–Catalan complexes (Σ,Δ_*) (of any index, including the $(d-1)$-dimensional complex (Δ_*,Δ_*) of index 0), and where $\gamma_i(\Sigma,\Delta_*)$ denotes the number of vertices of colour c_i in the unique colouring of (Σ,Δ_*) described in the statement of Theorem 2.1. It is our task to compute the coefficient of $x_0^{\gamma_0}x_1^{\gamma_1}\cdots x_d^{\gamma_d}$ in the series $C_d(x_0,x_1,\dots,x_d)$.

Starting from our generating function $C_d(x_0,x_1,\dots,x_d)$, we define $d+1$ series by cyclically permuting the variables,

$$\begin{aligned}
C^{\{0\}}(x_0,x_1,\dots,x_d)&=C_d(x_0,x_1,\dots,x_d),\\
C^{\{1\}}(x_0,x_1,\dots,x_d)&=C_d(x_1,x_2,\dots,x_d,x_0),\\
&\ \ \vdots\\
C^{\{d\}}(x_0,x_1,\dots,x_d)&=C_d(x_d,x_0,x_1,\dots,x_{d-1}).
\end{aligned}$$

By the usual decomposition of rooted d-dimensional Fuß–Catalan complexes (Σ,Δ_*) determined by the unique d-dimensional simplex containing Δ_*, we shall set up a system of equations relating these $d+1$ series.

To be precise, let (Σ,Δ_*) be a rooted d-dimensional Fuß–Catalan complex of index $n\ge 1$, and let Δ_*^d be its unique d-dimensional simplex containing Δ_*. It intersects $\Sigma\backslash\Delta_*^d$ along d rooted sub-Fuß–Catalan complexes, with their marked $(d-1)$-dimensional simplices defined by their intersection with the boundary of Δ_*^d. These sub-complexes define a decomposition of (Σ,Δ_*). It shows that

$$C_d(x_0,x_1,\dots,x_d)=x_1\cdots x_d+\frac{1}{x_0(x_0x_1\cdots x_d)^{d-2}}\prod_{j=1}^{d}C^{\{j\}}(x_0,x_1,\dots,x_d),$$

and, more generally,

$$C^{\{i\}}(x_0,x_1,\dots,x_d)=\frac{x_0x_1\cdots x_d}{x_i}+\frac{1}{x_i(x_0x_1\cdots x_d)^{d-2}}\prod_{\substack{j=0\\ j\ne i}}^{d}C^{\{j\}}(x_0,x_1,\dots,x_d),$$

$$i=0,1,\dots,d.\quad(3.1)$$

In order to simplify this system of equations, we define $d+1$ series $g_0, g_1, \dots, g_d$ by the equations

$$C^{\{i\}}(x_0, x_1, \dots, x_d) = \frac{1}{x_i}\left(1 + g_i(x_0, x_1, \dots, x_d)\right) \prod_{j=0}^{d} x_j, \qquad i = 0, 1, \dots, d. \tag{3.2}$$

The reader should keep in mind that we want to compute the coefficient of $x_0^{\gamma_0} x_1^{\gamma_1} \cdots x_d^{\gamma_d}$ in the series $C_d(x_0, x_1, \dots, x_d)$, that is, in terms of the new series, the coefficient of $x_0^{\gamma_0} x_1^{\gamma_1 - 1} x_2^{\gamma_2 - 1} \cdots x_d^{\gamma_d - 1}$ in the series $g_0(x_0, x_1, \dots, x_d)$.

From now on, we suppress the arguments of series for the sake of better readability; that is, we write g_i instead of $g_i(x_0, x_1, \dots, x_d)$, etc., for short. With this notation, the system (3.1) becomes

$$g_i = \frac{x_i}{(1+g_i)} \prod_{j=0}^{d} (1+g_j), \qquad i = 0, 1, \dots, d,$$

or, equivalently,

$$x_i = \frac{g_i(1+g_i)}{\prod_{j=0}^{d}(1+g_j)}, \qquad i = 0, 1, \dots, d.$$

By a straightforward application of the Lagrange–Good inversion formula [7], we have

$$\left\langle \mathbf{x}^{\boldsymbol{\gamma}} \right\rangle g_0 = \left\langle \mathbf{x}^{-\mathbf{1}} \right\rangle x_0 \det(J_{d+1}) \prod_{j=0}^{d} \frac{(1+x_j)^{d+|\boldsymbol{\gamma}|-\gamma_j}}{x_j^{\gamma_j+1}},$$

where $\langle \mathbf{x}^{\boldsymbol{\gamma}} \rangle g_0$ denotes the coefficient of $x_0^{\gamma_0} x_1^{\gamma_1} \cdots x_d^{\gamma_d}$ in the series g_0, $\langle \mathbf{x}^{-\mathbf{1}} \rangle f$ denotes the coefficient of $x_0^{-1} x_1^{-1} \cdots x_d^{-1}$ in the series f, $|\boldsymbol{\gamma}|$ stands for $\sum_{j=0}^{d} \gamma_j$, and J_{d+1} is the Jacobian of the map $(x_0, x_1, \dots, x_d) \longmapsto (y_0, y_1, \dots, y_d)$ defined by

$$y_i = \frac{x_i(1+x_i)}{\prod_{j=0}^{d}(1+x_j)}, \qquad i = 0, 1, \dots, d.$$

A simple computation yields that the entries of J_{d+1} are given by

$$\begin{aligned}
(J_{d+1})_{i,j} &= -\frac{x_i(1+x_i)}{(1+x_j)\prod_{k=0}^{d}(1+x_k)}, \qquad \text{if } i \neq j,\\
(J_{d+1})_{i,i} &= \frac{1+x_i}{\prod_{k=0}^{d}(1+x_k)}.
\end{aligned}$$

By Proposition 3.1 at the end of this section, it follows that

$$\begin{aligned}
\left\langle \mathbf{x}^{\boldsymbol{\gamma}} \right\rangle g_0 &= \left\langle \mathbf{x}^{-\mathbf{1}} \right\rangle x_0 \left(\prod_{j=0}^{d} \frac{(1+x_j)^{|\boldsymbol{\gamma}|-\gamma_j-1}(1+2x_j)}{x_j^{\gamma_j+1}} \right) \left(1 - \sum_{k=0}^{d} \frac{x_k}{1+2x_k} \right)\\
&= \left\langle \mathbf{x}^{\boldsymbol{\gamma}} \right\rangle x_0 \left(\prod_{j=0}^{d} (1+x_j)^{|\boldsymbol{\gamma}|-\gamma_j-1}(1+2x_j) \right) \left(1 - \sum_{k=0}^{d} \frac{x_k}{1+2x_k} \right).
\end{aligned}$$

Consequently, we get

$$\begin{aligned}
&\langle \mathbf{x}^{\boldsymbol\gamma}\rangle\, g_0\\
&=\left(\binom{|\boldsymbol\gamma|-\gamma_0-1}{\gamma_0-1}+2\binom{|\boldsymbol\gamma|-\gamma_0-1}{\gamma_0-2}\right)\prod_{j=1}^{d}\left(\binom{|\boldsymbol\gamma|-\gamma_j-1}{\gamma_j}+2\binom{|\boldsymbol\gamma|-\gamma_j-1}{\gamma_j-1}\right)\\
&\qquad-\binom{|\boldsymbol\gamma|-\gamma_0-1}{\gamma_0-2}\prod_{j=1}^{d}\left(\binom{|\boldsymbol\gamma|-\gamma_j-1}{\gamma_j}+2\binom{|\boldsymbol\gamma|-\gamma_j-1}{\gamma_j-1}\right)\\
&\qquad-\left(\binom{|\boldsymbol\gamma|-\gamma_0-1}{\gamma_0-1}+2\binom{|\boldsymbol\gamma|-\gamma_0-1}{\gamma_0-2}\right)\\
&\qquad\qquad\times\sum_{k=1}^{d}\binom{|\boldsymbol\gamma|-\gamma_k-1}{\gamma_k-1}\prod_{\substack{j=1\\ j\neq k}}^{d}\left(\binom{|\boldsymbol\gamma|-\gamma_j-1}{\gamma_j}+2\binom{|\boldsymbol\gamma|-\gamma_j-1}{\gamma_j-1}\right)\ .
\end{aligned}$$

Setting

$$\begin{aligned}
P&=\prod_{j=1}^{d}\left(\binom{|\boldsymbol\gamma|-\gamma_j-1}{\gamma_j}+2\binom{|\boldsymbol\gamma|-\gamma_j-1}{\gamma_j-1}\right)\\
&=|\boldsymbol\gamma|^d\prod_{j=1}^{d}\frac{(|\boldsymbol\gamma|-\gamma_j-1)!}{\gamma_j!\,(|\boldsymbol\gamma|-2\gamma_j)!}\ ,
\end{aligned}$$

we can rewrite this as

$$\begin{aligned}
\langle \mathbf{x}^{\boldsymbol\gamma}\rangle\, g_0&=\left(\binom{|\boldsymbol\gamma|-\gamma_0-1}{\gamma_0-1}+\binom{|\boldsymbol\gamma|-\gamma_0-1}{\gamma_0-2}\right)P\\
&\qquad-\left(\binom{|\boldsymbol\gamma|-\gamma_0-1}{\gamma_0-1}+2\binom{|\boldsymbol\gamma|-\gamma_0-1}{\gamma_0-2}\right)P\sum_{k=1}^{d}\frac{\binom{|\boldsymbol\gamma|-\gamma_k-1}{\gamma_k-1}}{\binom{|\boldsymbol\gamma|-\gamma_k-1}{\gamma_k}+2\binom{|\boldsymbol\gamma|-\gamma_k-1}{\gamma_k-1}}\\
&=\binom{|\boldsymbol\gamma|-\gamma_0}{\gamma_0-1}P-\frac{|\boldsymbol\gamma|-1}{|\boldsymbol\gamma|-\gamma_0}\binom{|\boldsymbol\gamma|-\gamma_0}{\gamma_0-1}P\sum_{k=1}^{d}\frac{\gamma_k}{|\boldsymbol\gamma|}\\
&=\frac{1}{|\boldsymbol\gamma|}\binom{|\boldsymbol\gamma|-\gamma_0}{\gamma_0-1}P\ .
\end{aligned}$$

This shows that

$$\langle \mathbf{x}^{\boldsymbol\gamma}\rangle\, g_0=\frac{|\boldsymbol\gamma|^{d-1}(|\boldsymbol\gamma|-\gamma_0)!}{(\gamma_0-1)!\,(|\boldsymbol\gamma|+1-2\gamma_0)!}\prod_{j=1}^{d}\frac{(|\boldsymbol\gamma|-\gamma_j-1)!}{\gamma_j!\,(|\boldsymbol\gamma|-2\gamma_j)!}\ . \tag{3.3}$$

Now we should remember that we actually wanted to compute the coefficient of $x_0^{\gamma_0}x_1^{\gamma_1-1}x_2^{\gamma_2-1}\cdots x_d^{\gamma_d-1}$ in the series $g_0(x_0,x_1,\dots,x_d)$. So, we have to replace γ_i by γ_i-1 for $i=1,2,\dots,d$ and, thus, $|\boldsymbol\gamma|$ by $s=-d+\sum_{j=0}^{d}\gamma_j$ in (3.3). If we do this, then we arrive at (2.1) upon little rewriting. $\square$

Proposition 3.1. *Let d be a non-negative integer and J_{d+1} be the $(d+1)\times(d+1)$ matrix*

$$\left(\begin{cases}\frac{1+x_i}{\prod_{k=0}^{d}(1+x_k)} & i=j\\ -\frac{x_i(1+x_i)}{(1+x_j)\prod_{k=0}^{d}(1+x_k)} & i\neq j\end{cases}\right)_{0\le i,j\le d}.$$

Then we have

$$\det(J_{d+1})=\left(1-\sum_{k=0}^{d}\frac{x_k}{1+2x_k}\right)\prod_{j=0}^{d}\frac{1+2x_j}{(1+x_j)^{d+1}}. \tag{3.4}$$

Proof. By factoring terms that only depend on the row index or only on the column index, we see that

$$\det(J_{d+1})=\prod_{j=0}^{d}\frac{1}{(1+x_j)^{d+1}}\det\left(\begin{cases}1+x_i & i=j\\ -x_i & i\neq j\end{cases}\right)_{0\le i,j\le d}. \tag{3.5}$$

The above determinant equals the sum over all principal minors of the matrix

$$\left(\begin{cases}x_i & i=j\\ -x_i & i\neq j\end{cases}\right)_{0\le i,j\le d},$$

where, as usual, a principal minor is by definition the determinant of a submatrix with rows and columns indexed by a common subset of $\{0,1,\dots,d\}$. Again factoring terms that only depend on the row index, we may write the principal minor corresponding to the submatrix indexed by $i_1,i_2,\dots,i_k$ in the form

$$x_{i_1}x_{i_2}\cdots x_{i_k}\det\left(\begin{cases}1 & i=j\\ -1 & i\neq j\end{cases}\right)_{1\le i,j\le k}. \tag{3.6}$$

The determinant in this expression occurs frequently. In fact, we have

$$\det(\lambda I_k-A_k)=\lambda^{k-1}(\lambda-k),$$

where I_k is the $k\times k$ identity matrix and A_k the $k\times k$ all-1's-matrix. (This is easily seen by observing that the matrix A_k has an eigenvector $(1,1,\dots,1)$ with eigenvalue k and that the space orthogonal to $(1,1,\dots,1)$ is the kernel of A_k.) By using this observation with $\lambda=2$, it follows that the expression (3.6) simplifies to

$$x_{i_1}x_{i_2}\cdots x_{i_k}2^{k-1}(2-k).$$

If this is substituted in (3.5), we obtain

$$\det(J_{d+1})=\prod_{j=0}^{d}\frac{1}{(1+x_j)^{d+1}}\sum_{k=0}^{d+1}2^{k-1}(2-k)e_k(x_0,x_1,\dots,x_d), \tag{3.7}$$

where $e_k(x_0,x_1,\dots,x_d)=\sum_{0\le i_1<\dots<i_k\le d}x_{i_1}x_{i_2}\cdots x_{i_k}$ denotes the k-th elementary symmetric function. As is well-known, these polynomials satisfy the generating function identity

$$\sum_{k=0}^{d+1}e_k(x_0,x_1,\dots,x_d)\,t^k=\prod_{j=0}^{d}(1+x_jt). \tag{3.8}$$

By differentiating this identity with respect to t, we obtain the further equation

$$\sum_{k=1}^{d+1} k\, e_k(x_0,x_1,\dots,x_d)\, t^{k-1} = \sum_{k=0}^{d} \frac{x_k}{1+x_k t} \prod_{j=0}^{d} (1+x_j t)\ .$$

Using both with $t=2$ in (3.7), we arrive exactly at the right-hand side of (3.4). □

4. Proof of Theorem 2.2

We perform a reverse induction on k. For the start of the induction, we remember that Theorem 2.2 is nothing but Theorem 2.1 (which we established in the previous section) if $k=d$ and $\beta_0=\beta_1=\beta_2=\cdots=\beta_d=1$.

For the induction step, we have to distinguish two cases. Suppose first that $\beta_0=1$ and that Theorem 2.2 holds for all (suitable) sequences $\beta_0=1,\beta_1,\beta_2,\dots,\beta_{k+1}$. Then Theorem 2.2 holds for $\beta_0=1,\beta_1+\beta_2,\beta_3,\dots,\beta_{k+1}$ if and only if

$$s\sum_{k=\beta_1}^{\gamma-\beta_2} \frac{\beta_1}{\beta_1 s+\beta_1-k}\binom{\beta_1 s+\beta_1-k}{k-\beta_1} \frac{\beta_2}{\beta_2 s+\beta_2-(\gamma-k)}\binom{\beta_2 s+\beta_2-(\gamma-k)}{\gamma-k-\beta_2} \\ = \frac{(\beta_1+\beta_2)}{(\beta_1+\beta_2)(s+1)-\gamma}\binom{(\beta_1+\beta_2)(s+1)-\gamma}{\gamma-\beta_1-\beta_2}$$

for all $\gamma\ge\beta_1+\beta_2$. (Without loss if generality, it suffices to consider the addition of β_1 and β_2, since all other combinations lead to analogous and equivalent statements.) This is a special case of an identity commonly attributed to Rothe [12] (to be precise, it is the case $\alpha\to\beta_1 s$, $\beta\to -1$, $\gamma\to\beta_2 s+\beta_1+\beta_2$, $n\to\gamma-\beta_1-\beta_2$ of [8, Eq. (4)]; see [15] for historical comments and more on this kind of identities, although, for some reason, it misses [3]), which establishes the induction step in this case.

Suppose now that Theorem 2.2 holds for all (suitable) sequences $\beta_0,\beta_1,\beta_2,\dots,\beta_{k+1}$. Then Theorem 2.2 holds for $\beta_0+\beta_1,\beta_2,\beta_3,\dots,\beta_{k+1}$ if and only if

$$\sum_{k=\beta_0}^{\gamma-\beta_1} \frac{\beta_1 s}{\beta_0 s+\beta_0-k}\binom{\beta_0 s+\beta_0-k}{k-\beta_0}\binom{\beta_1 s+\beta_1-1-(\gamma-k)}{\gamma-k-\beta_1} \\ = \binom{(\beta_0+\beta_1)(s+1)-1-\gamma}{\gamma-\beta_0-\beta_1}$$

for all $\gamma\ge\beta_0+\beta_1$. (Again, without loss if generality, it suffices to consider the addition of β_0 and β_1.) This is a special case of another identity commonly attributed to Rothe [12] (to be precise, it is the case $\alpha\to\beta_0 s$, $\beta\to -1$, $\gamma\to\beta_1 s+\beta_0+\beta_1-1$, $n\to\gamma-\beta_0-\beta_1$ of [8, Eq. (11)]), establishing the induction step in this case also. □

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