DEGENERATE FLAG VARIETIES AND THE MEDIAN GENOCCHI NUMBERS

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Abstract. We study the $\mathbb{G}_a^M$ degenerations $\mathcal{F}_\lambda^a$ of the type A flag varieties $\mathcal{F}_\lambda$. We describe these degenerations explicitly as subvarieties in the products of Grassmanians. We construct cell decompositions of $\mathcal{F}_\lambda^a$ and show that for complete flags the number of cells is equal to the normalized median Genocchi numbers h_n. This leads to a new combinatorial definition of the numbers h_n. We also compute the Poincaré polynomials of the complete degenerate flag varieties via a natural statistics on the set of Dellac's configurations, similar to the length statistics on the set of permutations. We thus obtain a natural q-version of the normalized median Genocchi numbers.

Introduction

Let $\mathfrak{g} = \mathfrak{sl}_n$, $G = SL_n$. Fix the Cartan decomposition $\mathfrak{g} = \mathfrak{b} \oplus \mathfrak{n}^-$, where $\mathfrak{b}$ is a Borel subalgebra, $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$. In [Fe3] we considered the degenerate algebra $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is an abelian Lie algebra isomorphic to $\mathfrak{n}^-$ as a vector space. The corresponding Lie group is a semi-direct product $G^a = B \ltimes \mathbb{G}_a^M$, where $\mathbb{G}_a$ is the additive group of the field and $M = \dim \mathfrak{n}$. For a dominant integral weight λ let V_λ be the highest weight λ irreducible $\mathfrak{g}$-module with a highest weight vector v_λ. The increasing PBW filtration $F_\bullet$ on V_λ is defined as follows:

$$F_0 = \mathbb{C}v_\lambda,\ F_{s+1} = \mathrm{span}\{xv :\ x \in \mathfrak{g}, v \in F_s\}, s \geq 0$$

(see [Fe1], [Fe2], [FFoL1], [FFoL2], [K2]). The associated graded space $V_\lambda^a = F_0 \oplus F_1/F_0 \oplus F_2/F_1 \oplus \dots$ can be naturally endowed with the structure of a $\mathfrak{g}^a$- and G^a-module. A degenerate flag variety $\mathcal{F}_\lambda^a$ is a subvariety in $\mathbb{P}(V_\lambda^a)$ defined by $\mathcal{F}_\lambda^a = \overline{\mathbb{G}_a^M \cdot \mathbb{C}v_\lambda}$. These are the $\mathbb{G}_a^M$-degenerations of the classical (generalized) flag varieties $\mathcal{F}_\lambda$ (see [A], [AS], [Fe3], [HT]). For example, $\mathcal{F}_{\omega_d}^a \simeq Gr(d,n)$ for all fundamental weights. Recall also that in the classical case (for $\mathfrak{g} = \mathfrak{sl}_n$) the varieties $\mathcal{F}_\lambda = G \cdot \mathbb{C}v_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ are the usual flag varieties (maybe partial). In particular, if λ is regular, i.e. $(\lambda, \omega_d) > 0$ for all d, then $\mathcal{F}_\lambda$ is isomorphic to the variety $\mathcal{F}_n$ of complete flags in n-dimensional space V. Fix a basis $v_1, \dots, v_n$ of V.

For all weights λ, μ there exists an embedding of G^a-modules $V_{\lambda+\mu}^a \hookrightarrow V_\lambda^a \otimes V_\mu^a$ sending $v_{\lambda+\mu}$ to $v_\lambda \otimes v_\mu$ (see [FFoL1], [FFoL2]). This induces the embedding of varieties $\mathcal{F}_{\lambda+\mu}^a \hookrightarrow \mathcal{F}_\lambda^a \times \mathcal{F}_\mu^a$. Thus for any λ we obtain an embedding of $\mathcal{F}_\lambda^a$ into the product of Grassmanians. Our first result is an explicit description of this embedding. We state the theorem here for complete flag varieties $\mathcal{F}_n^a$. For this we need one more piece of notations. Let $pr_d : V \to V$ be the projection along the space $\mathbb{C}v_d$ to the linear span of the vectors v_i, $i \neq d$.

Theorem 0.1. *The image of the embedding of the variety $\mathcal{F}_n^a$ in the product $\prod_{d=1}^{n-1} Gr(d,n)$ is equal to the set of chains of subspaces $(V_1, \dots, V_{n-1})$, $V_d \in Gr(d,n)$ such that*

$$pr_{d+1}(V_d) \hookrightarrow V_{d+1}, \quad 1 \le d \le n-2.$$

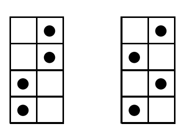
Our next goal is to compute the Poincaré polynomial of $\mathcal{F}_n^a$. Recall that in the classical case the flag variety $\mathcal{F}_n$ can be written as a disjoint union of $n!$ cells, each cell being associated with a torus fixed point. The fixed points are labeled by permutations from S_n. The length statistics $\sigma \to l(\sigma)$ gives the complex dimension of the cells. Therefore, the Poincaré polynomial $P_{\mathcal{F}_n}(t)$ of $\mathcal{F}_n$ is equal to $P_{\mathcal{F}_n}(t) = \sum_{\sigma \in S_n} t^{2l(\sigma)}$.

As an immediate corollary of Theorem 0.1 we obtain that the fixed points of the torus $T \subset G^a$ action on $\mathcal{F}_n^a$ are labeled by the sequences $I^1, \dots, I^{n-1}$, $I^d \subset \{1, \dots, n\}$, $\#I^d = d$, satisfying

$$I^d \setminus \{d+1\} \hookrightarrow I^{d+1}, \qquad d = 1, \dots, n-2. \tag{0.1}$$

(Note that this set of sequences has a subset with $I^d \hookrightarrow I^{d+1}$, which can be naturally identified with the permutations S_n). Our first task is to compute the number of such fixed points. To this end, recall the normalized median Genocchi numbers h_n, $n = 1, 2, \dots$ (also referred to as the normalized Genocchi numbers of second kind). These numbers have several definitions [De], [Du], [DR], [DZ], [G], [Kr], [Vien] (see section 3 for a review). Here we give the Dellac definition, which is the earliest one and which fits our construction in the best way.

Consider a rectangle with n columns and $2n$ rows. It contains $n \times 2n$ boxes labeled by pairs (l,j), with $l = 1, \dots, n$ and $j = 1, \dots, 2n$. A Dellac configuration D is a subset of boxes, subject to the following conditions: first, each column contains exactly two boxes from D and each row contains exactly one box from D, and, second, if the (l,j)-th box is in D, then $l \le j \le n+l$. Let DC_n be the set of such configurations. Then h_n is the number of elements in DC_n. The first several median Genocchi numbers (starting from h_1) are as follows: $1, 2, 7, 38, 295$. For instance, the two Dellac configurations for $n = 2$ are as follows: (we specify boxes in a configuration by putting fat dots inside)



We prove the following theorem:

Theorem 0.2. *The number of sequences $I^1, \dots, I^{n-1}$ as above, satisfying* (0.1) *is equal to h_n.*

We also prove that the Dellac definition [De] is equivalent to the Dumont-Kreweras definition [Du], [Kr] (this fact is known to experts [G],[S] but we were unable to find the proof in the literature).

Recall that the length of a permutation $\sigma \in S_n$ can be defined as the number of pairs $1 \le l_1 < l_2 \le n$ satisfying $\sigma(l_1) > \sigma(l_2)$. We define a length $l(D)$ of a Dellac configuration D as the number of squares $(l_1, j_1), (l_2, j_2) \in D$ such that $l_1 < l_2$ and $j_1 > j_2$. We prove the following theorem:

Theorem 0.3. *The Poincaré polynomial* $P_{\mathcal{F}^a_n}(t)$ *is given by* $\sum_{D \in DC_n} t^{2l(D)}$.

Our paper is organized in the following way:
In Section 1 we recall main definitions and theorems from [Fe3],
In Section 2 we describe explicitly the image of the embedding of the varieties $\mathcal{F}^a_\lambda$ into the product of Grassmanians and construct the cell decomposition of $\mathcal{F}^a_\lambda$.
In Section 3 we study the combinatorics of the median Genocchi numbers and compute the Poincaré polynomials of the complete degenerate flag varieties.

1. PBW deformation

1.1. **Definitions.** We first recall basic definitions and constructions from [FFoL1] and [Fe3]. Let $\mathfrak{g}$ be a simple Lie algebra with the Cartan decomposition $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h} \oplus \mathfrak{n}^-$. We denote by M the number of positive roots of $\mathfrak{g}$, i.e. $M = \dim \mathfrak{n}$. Let $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{h}$ be a Borel subalgebra. Then the deformed algebra $\mathfrak{g}^a$ is defined as a sum of two subalgebras $\mathfrak{g}^a = \mathfrak{b} \oplus (\mathfrak{n}^-)^a$, where $(\mathfrak{n}^-)^a$ is an abelian Lie algebra isomorphic to $\mathfrak{n}^-$ as a vector space. The subalgebra $(\mathfrak{n}^-)^a \hookrightarrow \mathfrak{g}^a$ is an abelian ideal and the action of $\mathfrak{b}$ on $(\mathfrak{n}^-)^a$ is induced from the identification $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$.

Let G be the Lie group of the Lie algebra $\mathfrak{g}$. Let N, T, N^-, B be the Lie groups of the Lie algebras $\mathfrak{n}$, $\mathfrak{h}$, $\mathfrak{n}^-$, $\mathfrak{b}$. The deformed Lie group G^a is defined as a semi-direct product of B and the normal subgroup $\mathbb{G}_a^M$, where $\mathbb{G}_a$ is the additive group of the field (thus $\mathbb{G}_a^M$ is the Lie group of the Lie algebra $(\mathfrak{n}^-)^a$). The Borel group B acts on the vector space $(\mathfrak{n}^-)^a \simeq \mathfrak{g}/\mathfrak{b}$ via the restriction of the adjoint action and therefore there exists a natural homomorphism from B to $Aut(\mathbb{G}_a^M)$, defining the semi-direct product $G^a = B \ltimes \mathbb{G}_a^M$.

For a dominant integral weight λ we denote by V_λ the corresponding irreducible highest weight $\mathfrak{g}$-module with a highest weight vector v_λ. The Lie algebra $\mathfrak{g}^a$ and the Lie group G^a act on the deformed representations V_λ^a, where λ are dominant integral weights of $\mathfrak{g}$. The representations V_λ^a are defined as associated graded $gr_\bullet V_\lambda$ of the representation V_λ with respect to the PBW filtration F_s:

$$F_s = \mathrm{span}\{x_1 \dots x_l v_\lambda : \ x_i \in \mathfrak{g}, l \le s\}.$$

So $V_\lambda^a = \bigoplus_{s \ge 0} V_\lambda^a(s)$, where $V_\lambda^a(0) = \mathbb{C} v_\lambda$ and $V_\lambda^a(s) = F_s/F_{s-1}$ for $s > 0$. It is easy to see that the action of $\mathfrak{n}^-$ on V_λ becomes abelian on V_λ^a (i.e. it induces the action of $(\mathfrak{n}^-)^a$) and the action of the Borel subalgebra induces the action of (the same algebra) $\mathfrak{b}$. The actions of $(\mathfrak{n}^-)^a$ and $\mathfrak{b}$ glue together to the action of $\mathfrak{g}^a$.

Remark 1.1. Let $\tilde{\mathfrak{g}}^a = \mathfrak{g}^a \oplus \mathbb{C}p$ be the central, extension of $\mathfrak{g}^a$ with a single element p subject to the relations $[p, \mathfrak{b}] = 0$, $[p, f_\alpha] = f_\alpha$ for any positive root α and the corresponding weight element $f_\alpha \in (\mathfrak{n}^-)^a$. Thus the Cartan subalgebra of $\tilde{\mathfrak{g}}^a$ has one extra dimension. We note that the $\mathfrak{g}^a$-module structure of V_λ^a naturally lifts to the structure of representation of $\tilde{\mathfrak{g}}^a$ by setting $pv_\lambda = 0$ (in general, $p|_{V_\lambda^a(s)} = s$). An eigenvalue of the operator p is sometimes referred to as a PBW degree. The character of V_λ^a with respect to $\mathfrak{h} \oplus \mathbb{C}p$ was computed in [FFoL1] for $\mathfrak{sl}_n$ and in [FFoL2] for symplectic Lie algebras. We denote the Lie group of $\tilde{\mathfrak{g}}^a$ by $\tilde{G}^a$, which differs from G^a by an additional $\mathbb{C}^*$.

Consider the action of G^a on the projective space $\mathbb{P}(V_\lambda^a)$. Recall that in the classical situation the (generalized) flag varieties are defined as $\mathcal{F}_\lambda = G \cdot \mathbb{C}v_\lambda \hookrightarrow \mathbb{P}(V_\lambda)$ (see [K1]). The degenerate flag varieties $\mathcal{F}_\lambda^a \hookrightarrow \mathbb{P}(V_\lambda^a)$ are defined as the closures of the G^a orbit (or, equivalently, of the $\mathbb{G}_a^M$ orbit) of the line $\mathbb{C}v_\lambda$. We note that in the classical case the orbit $G \cdot \mathbb{C}v_\lambda$ already covers the whole flag variety. This is not true in the degenerate case: the orbit $G^a \cdot \mathbb{C}v_\lambda$ is an affine cell, whose closure gives a projective singular variety $\mathcal{F}_\lambda^a$.

1.2. **The type A case.** From now on we assume that $\mathfrak{g} = \mathfrak{sl}_n$ and $G = SL_n$. Then all positive roots are of the form

$$\alpha_{i,j} = \alpha_i + \dots + \alpha_j,\ 1 \le i \le j \le n-1$$

(for instance, $\alpha_{i,i} = \alpha_i$ are the simple roots). We denote by $f_{i,j} = f_{\alpha_{i,j}} \in \mathfrak{n}^-$ and $e_{i,j} = e_{\alpha_{i,j}} \in \mathfrak{n}$ the corresponding root elements. We have $\mathcal{F}_{\omega_d}^a \simeq \mathcal{F}_{\omega_d} \simeq Gr(d,n)$. The reason why the degenerate flag varieties are isomorphic to the non-degenerate ones for fundamental weights is that the radicals in $\mathfrak{sl}_n$, corresponding to ω_d, are abelian. In other words, define the set of positive roots

$$R_d = \{\alpha_{i,j} :\ 1 \le i \le d \le j \le n-1\}.$$

Define the subalgebra $\mathfrak{u}_d^- = \mathrm{span}\{f_\alpha :\ \alpha \in R_d\}$. Then $\mathfrak{u}_d^-$ is abelian and $V_{\omega_d} = U(\mathfrak{u}_d^-) \cdot v_\lambda$.

Remark 1.2. Let us explain the difference between the structure of $\mathfrak{g}$-module on V_{ω_d} and the structure of $\mathfrak{g}^a$-module on $V_{\omega_d}^a$. The operators f_α act trivially on $V_{\omega_d}^a$ unless $\alpha \in R_d$. Also, e_α act trivially on $V_{\omega_d}^a$ if $\alpha \in R_d$. Therefore, $\mathfrak{g}^a$ acts on $V_{\omega_d}^a$ via the projection to the subalgebra

$$\mathfrak{g}_d^a = \mathfrak{u}_d^- \oplus \mathfrak{h} \oplus \mathrm{span}\{e_\alpha :\ \alpha \notin R_d\}. \tag{1.1}$$

Similarly, the group G^a acts on $Gr(d,n)$ via the surjection to the Lie group of $\mathfrak{g}_d^a$. In particular, the group G^a does not act transitively on the deformed flag varieties even in the case of Grassmanians.

Remark 1.3. We note that though $\mathcal{F}^a_{\omega_d} \simeq \mathcal{F}_{\omega_d} \simeq Gr(d,n)$, the actions of the Borel groups $B \subset G$ and $B \subset G^a$ are very different. Let us consider the case $G = SL_2$. Then $\mathfrak{g}^a$ is spanned by three elements e^a, h^a and f^a subject to the relations

$$[h^a, e^a] = 2e^a,\ [h^a, f^a] = -2f^a,\ [e^a, f^a] = 0.$$

Let λ be a dominant weight of $\mathfrak{sl}_2$, $\lambda \in \mathbb{Z}_{\geq 0}$. Then V^a_λ is the direct sum of one-dimensional subspaces spanned by vectors v_l, $l = \lambda, \lambda-2, \dots, -\lambda$ such that

$$h^a v_l = l v_l,\ f^a v_l = v_{l-2},\ e^a v_l = 0.$$

Therefore, the Borel subgroup B acts trivially on $\mathcal{F}^a_\lambda \simeq \mathbb{P}^1$. For instance, there exists one point of $\mathbb{P}^1$, which is fixed by the action of the whole group G^a.

Let us now recall the Plücker relations for $\mathcal{F}_\lambda$ [Fu] and the deformed Plücker relations for $\mathcal{F}^a_\lambda$ [Fe3].

Let $1 \leq d_1 < \dots < d_s \leq n-1$ be a sequence of increasing numbers. Then for any positive integers $a_1, \dots, a_s$ the variety $\mathcal{F}_{a_1\omega_{d_1}+\dots+a_s\omega_{d_s}}$ is isomorphic to the partial flag variety

$$\mathcal{F}(d_1, \dots, d_s) = \{V_1 \hookrightarrow V_2 \hookrightarrow \dots \hookrightarrow V_s \hookrightarrow \mathbb{C}^n :\ \dim V_i = d_i\}.$$

In particular, if $s = 1$, then $\mathcal{F}(d)$ is the Grassmanian $Gr(d,n)$ and for $s = n-1$ $\mathcal{F}(1, \dots, n-1)$ is the variety of the complete flags. We recall that

$$V_{\omega_d} = \Lambda^d(V_{\omega_1}) = \Lambda^d(\mathbb{C}^n)$$

and the embedding $Gr(d,n) \hookrightarrow \mathbb{P}(\Lambda^d V_{\omega_1})$ is defined as follows: a subspace with a basis $w_1, \dots, w_d$ maps to $\mathbb{C}w_1 \wedge \dots \wedge w_d$. For general sequence $d_1, \dots, d_s$ one has embeddings:

$$\mathcal{F}(d_1, \dots, d_s) \hookrightarrow Gr(d_1, n) \times \dots \times Gr(d_s, n) \hookrightarrow \mathbb{P}(V_{\omega_{d_1}}) \times \dots \times \mathbb{P}(V_{\omega_{d_s}}).$$

The composition of these embeddings is called the Plücker embedding. The image is described explicitly in terms of Plücker relations. Namely, let $v_1, \dots, v_n$ be a basis of $\mathbb{C}^n = V_{\omega_1}$. Then one gets a basis v_J of V_{ω_d} $v_J = v_{j_1} \wedge \dots \wedge v_{j_d}$ labeled by sequences $J = (1 \leq j_1 < j_2 < \dots < j_d \leq n)$. Let $X_J \in V^*_{\omega_d}$ be the dual basis. We denote by the same symbols the coordinates of a vector $v \in V_{\omega_d}$: $X_J = X_J(v)$. The image of the embedding

$$\mathcal{F}(d_1, \dots, d_s) \hookrightarrow \times_{i=1}^s \mathbb{P}(V_{\omega_{d_i}})$$

is defined by the Plücker relations. These relations are labeled by a pair of numbers $p \geq q$, $p, q \in \{d_1, \dots, d_s\}$, by a number k, $1 \leq k \leq q$ and by a pair of sequences $L = (l_1, \dots, l_p)$, $J = (j_1, \dots, j_q)$, $1 \leq l_\alpha, j_\beta \leq n$. The corresponding relation is denoted by $R^k_{L,J}$ and is given by

$$R^k_{L,J} = X_L X_J - \sum_{1 \leq r_1 < \dots < r_k \leq p} X_{L'} X_{J'}, \tag{1.2}$$

where L',J' are obtained from L, J by interchanging k-tuples $(l_{r_1},\dots,l_{r_k})$ and $(j_1,\dots,j_k)$ in L and J respectively, i.e.

$$J' = (l_{r_1},\dots,l_{r_k},j_{k+1},\dots,j_q),$$
$$L' = (l_1,\dots,l_{r_1-1},j_1,l_{r_1+1},\dots,l_{r_2-1},j_2,\dots,l_p).$$

We note that for any $\sigma\in S_d$ the equality

$$X_{j_{\sigma(1)},\dots,j_{\sigma(d)}} = (-1)^\sigma X_{j_1,\dots,j_d}$$

is assumed in (1.2). We denote the ideal generated by all $R^k_{L,J}$ by $I(d_1,\dots,d_s)$.

We introduce the notation

$$\mathcal{F}^a(d_1,\dots,d_s) = \mathcal{F}^a_{\omega_{d_1}+\dots+\omega_{d_s}},\ 1\le d_1<\dots<d_s<n.$$

Definition 1.4. Let $I^a(d_1,\dots,d_s)$ be an ideal in the polynomial ring in variables $X^a_{j_1,\dots,j_d}$, $d=d_1,\dots,d_s$, $1\le j_1<\dots<j_d<n$, generated by the elements $R^{k;a}_{L,J}$ given below. These elements are labeled by a pair of numbers $p\ge q$, $p,q\in\{d_1,\dots,d_s\}$, by an integer k, $1\le k\le q$ and by sequences $L=(l_1,\dots,l_p)$, $J=(j_1,\dots,j_q)$, which are arbitrary subsets of the set $\{1,\dots,n\}$. The generating elements are given by the formulas

$$R^{k;a}_{L,J} = X^a_{l_1,\dots,l_p}X^a_{j_1,\dots,j_q} - \sum_{1\le r_1<\dots<r_k\le p} X^a_{l'_1,\dots,l'_p}X^a_{j'_1,\dots,j'_q}, \tag{1.3}$$

where the terms of $R^{k;a}_{L,J}$ are the terms of $R^k_{L,J}$ (1.2) (with a superscript a, to be precise) such that

$$\{l_{r_1},\dots,l_{r_k}\}\cap\{q+1,\dots,p\}=\emptyset. \tag{1.4}$$

Remark 1.5. The initial term $X^a_{l_1,\dots,l_p}X^a_{j_1,\dots,j_q}$ is also subject to the condition (1.4), i.e. it is not present in $R^{k;a}_{L,J}$ if $\{j_1,\dots,j_k\}\cap\{q+1,\dots,p\}\neq\emptyset$.

Example 1.6. Let $s=1$. Then $I^a(d)=I(d)$, since there are no numbers l such that $d+1\le l\le d$ and thus $R^{k;a}_{L,J}=R^k_{L,J}$ (up to a superscript a in the notations of variables X_J). Hence $\mathcal{F}^a_{\omega_d}\simeq\mathcal{F}_{\omega_d}$.

The following theorem is proved in [Fe3].

Theorem 1.7. *The variety $\mathcal{F}^a(d_1,\dots,d_s)\hookrightarrow \times_{i=1}^s\mathbb{P}(\Lambda^{d_i}\mathbb{C}^n)$ is defined by the ideal $I^a(d_1,\dots,d_s)$.*

Example 1.8. Let $s=2$, $d_1=1$, $d_2=n-1$. Then the classical flag variety $\mathcal{F}(1,n-1)$ is a subvariety in $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$ defined by a single relation

$$\sum_{i=1}^n(-1)^{i-1}X_iX_{1,\dots,i-1,i+1,\dots,n}=0.$$

The degenerate variety $\mathcal{F}(1,n-1)$ is also a subvariety in $\mathbb{P}^{n-1}\times\mathbb{P}^{n-1}$, defined by a "degenerate" relation

$$X^a_1X^a_{2,\dots,n}+(-1)^{n-1}X^a_nX^a_{1,\dots,n-1}=0.$$

2. Cell decomposition

In this section we describe explicitly the image of $\mathcal{F}^a_\lambda$ inside the product of Grassmanians and construct the cell decomposition of the degenerate flag varieties. We start with the case of $\lambda = \omega_d$.

2.1. **Cell decomposition for Grassmanians.** Recall that $\mathcal{F}^a_{\omega_d} \simeq \mathcal{F}_{\omega_d} \simeq Gr(d,n)$. Given an increasing tuple $L = (l_1 < \dots < l_d)$ we set

$$p_L = \mathrm{span}(v_{l_1}, \dots, v_{l_d}) \in Gr(d,n).$$

The subspace p_L is T-invariant. Let k be a number such that $l_k \le d < l_{k+1}$.

Proposition 2.1. *The orbit $G^a \cdot p_L$ is an affine cell and $Gr(d,n)$ is the disjoint union of all such cells.*

Proof. Recall that G^a acts on $Gr(d,n)$ via the projection to the Lie group of $\mathfrak{g}_d$ (see (1.1)). Therefore the elements of $G^a \cdot p_L$ are exactly the subspaces of V having a basis $e_1, \dots, e_d$ of the form

$$e_j = v_{l_j} + \sum_{i=1}^{l_j-1} a_{i,j} v_i + \sum_{i=d+1}^{n} a_{i,j} v_i, \ j = 1, \dots, k \tag{2.1}$$

$$e_j = v_{l_j} + \sum_{i=d+1}^{l_j-1} a_{i,j} v_i, \ j = k+1, \dots, d. \tag{2.2}$$

Such elements in $Gr(d,n)$ obviously form an affine cell and one has a decomposition $Gr(d,n) = \sqcup_L G^a \cdot p_L$. □

Remark 2.2. Formulas (2.1) and (2.2) can be combined together as follows. Let $[k]_+ = k$ if $k > 0$ and $[k]_+ = k + n$ if $k \le 0$. Then each element of $G^a \cdot p_L$ has a basis $e_1, \dots, e_d$ of the form

$$e_j = v_{l_j} + \sum_{i=1}^{[l_j-d]_+-1} a_{i,j} v_{[l_j-i]_+}. \tag{2.3}$$

Remark 2.3. The orbit $G^a \cdot p_L$ can be identified with a certain cell $B \cdot p_J$ in the usual cell decomposition of $Gr(d,n)$. Namely, define J as follows:

$$J = (l_{k+1} - d, l_{k+2} - d, \dots, l_d - d, l_1 - d + n, l_2 - d + n, \dots, l_k - d + n).$$

Then the map

$$\psi : V \to V, \ \psi(v_i) = v_{[i-d]_+}, i = 1, \dots, n$$

sends $G^a \cdot p_L$ to $B \cdot p_j$ (this is clear from the explicit description (2.1), (2.2)).

Example 2.4. Let $n = 9$, $d = 4$ and $L = (2,3,6,7)$ (thus $k = 2$). Then the elements of $G^a \cdot p_L$ can be identified with the following matrices (the columns of a matrix form a basis of the corresponding subspace):

$$\begin{pmatrix} * & * & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ * & * & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ * & * & 0 & 0 \\ * & * & * & * \end{pmatrix}$$

Here $*$ denotes arbitrary entries and hence the number of stars coincides with the dimension of the cell.

2.2. **Chains of subspaces.** In this section we fix the numbers $d_1, \dots, d_s$ and write $\mathcal{F}^a$ for $\mathcal{F}^a(d_1, \dots, d_s)$. Let $v_1, \dots, v_n$ be some basis of $V \simeq \mathbb{C}^n$. For $1 \le i < j \le n$ we define the projections $pr_{i+1,j} : V \to V$ by the formula

$$pr_{i+1,j}(\sum_{l=1}^{n} c_l v_l) = \sum_{l=1}^{i} c_l v_l + \sum_{l=j+1}^{n} c_l v_l.$$

The goal of this subsection is to prove the following theorem.

Theorem 2.5. *The variety $\mathcal{F}^a \hookrightarrow Gr(d_1, n) \times \dots \times Gr(d_s, n)$ is formed by all sequences $V_1, \dots, V_s$, $V_l \in Gr(d_l, n)$ such that for all $1 \le l < m \le s$*

$$pr_{d_l+1,d_m} V_l \hookrightarrow V_m. \tag{2.4}$$

Remark 2.6. It is easy to see that the set of conditions (2.4) is equivalent to the subset with $m = l + 1$, i.e. to the set of conditions

$$pr_{d_l+1,d_{l+1}} V_l \hookrightarrow V_{l+1}, \quad l = 1, \dots, s-1. \tag{2.5}$$

Lemma 2.7. *Let $(V_1, \dots, V_s) \in \mathcal{F}^a$. Then conditions (2.4) are satisfied.*

Proof. Let us first look at the big cell $\mathbb{G}_a^M \cdot \mathbb{C}v_\lambda \subset \mathcal{F}^a$. Note that the line $\mathbb{C}v_\lambda$ is represented by the point

$$\times_{i=1}^{s} \mathrm{span}(v_1, \dots, v_{d_i}) \in \times_{i=1}^{s} Gr(d_i, n).$$

Take an element $g = \exp(\sum s_{i,j} f_{i,j}) \in \mathbb{G}_a^M \subset G^a$. Then one has

$$g \cdot \mathrm{span}(v_1, \dots, v_d) = \mathrm{span}(v_1 + \sum_{j=d}^{n-1} s_{1,j} v_{j+1}, \dots, v_d + \sum_{j=d}^{n-1} s_{d,j} v_{j+1}).$$

Therefore conditions (2.4) hold for all points from the big cell of the degenerate flag varieties. Since $\mathcal{F}^a_\lambda$ is the closure of the big cell, the lemma is proved. □

Proposition 2.8. *Let $V_1, \dots, V_s$ be a set of subspaces of V satisfying (2.4) with $\dim V_l = d_l$. Then $(V_1, \dots, V_s) \in \mathcal{F}^a$.*

Proof. We know that the image of the embedding

$$\mathcal{F}^a \hookrightarrow \times_{i=1}^s Gr(d_i, n) \hookrightarrow \times_{i=1}^s \mathbb{P}(\Lambda^{d_i} V)$$

is defined by the set of relations $R^{k;a}_{J,I} = 0$. Our goal is to prove that (2.4) implies that all the relations $R^{k;a}_{J,I}$ vanish. Fix a pair $1 \le l \le m \le s$. In what follows we denote the projection pr_{d_l+1,d_m} simply by pr.

Let $(V_1, \dots, V_s)$ be a collection of subspaces satisfying (2.4). Fix tuples $I = (i_1, \dots, i_l)$ and $J = (j_1, \dots, j_m)$ and a number k. We prove that the relation $R^{k;a}_{J,I}$ vanishes on $(V_1, \dots, V_s)$. Without loss of generality we assume that $i_1, \dots, i_k \notin [d_l + 1, d_m]$. We also rearrange the entries of I in such a way that the elements from $I \cap [d_l + 1, d_m]$ are concentrated at the end of I, i.e. there exists a number b such that

$$i_1, \dots, i_b \notin [d_l + 1, d_m], \quad i_{b+1}, \dots, i_l \in [d_l + 1, d_m].$$

Obviously, $b \ge k$. Let $l - c = \dim(\ker pr \cap V_l)$. We fix a basis $e_1, \dots, e_l$ of V_l such that $pre_1, \dots, pre_c$ is a basis of prV_l and $e_{c+1}, \dots, e_l$ form a basis of $\ker pr \cap V_l$. We denote by $a_{s,t}$ the coefficients of the expansion of e_s in terms of v_t:

$$e_q = \sum_{r=1}^{l} a_{r,q} v_r.$$

The idea of the proof is to use the following decomposition of a Plücker coordinate X_I:

$$X_I = \sum_{1 \le \alpha_1 < \dots < \alpha_{l-b} \le l} \pm a_{i_{b+1},\alpha_1} \dots a_{i_l,\alpha_{l-b}} X_{i_1,\dots,i_b}. \tag{2.6}$$

Here $X_{i_1,\dots,i_b}$ is the $(i_1, \dots, i_b)$-th Plücker coordinate of the vector space $\mathrm{span}(e_{\beta_1}, \dots, e_{\beta_b})$, where the set of β's is complementary to the set of α's, i.e.

$$\{\beta_1, \dots, \beta_b\} \cup \{\alpha_1, \dots, \alpha_{l-b}\} = \{i_1, \dots, i_l\}.$$

The decomposition (2.6) induces the decomposition of the relation $R^{k;a}_{J,I}$, such that each term can be shown to vanish. Note that if $b > c$ then X_I vanishes on V_l. We thus assume that $b \le c$.

Define the subspace

$$E_\beta = pr(\mathrm{span}(e_{\beta_1}, \dots, e_{\beta_b})).$$

We know that $E_\beta \hookrightarrow V_m$. In addition, the coordinates $X_{(i_1,\dots,i_b)}$ of the space $\mathrm{span}(e_{\beta_1}, \dots, e_{\beta_b})$ coincide with the Plücker coordinates $Y_{(i_1,\dots,i_b)}$ of E_β, because $i_1, \dots, i_b \notin [d_l + 1, d_m]$ (we are using the notations Y_I to distinguish between Plücker coordinated of different spaces). Since $E_\beta \hookrightarrow V_m$, the classical relations $R^k_{J,(i_1,\dots,i_b)}$ vanish on the pair (E_β, V_m). Since

$$E_\beta \hookrightarrow \mathrm{span}(v_1, \dots, v_{d_l}, v_{d_m+1}, \dots, v_n),$$

a Plücker coordinate $Y_{q_1,\dots,q_b}$ of E_β vanishes unless non of the indices $q_\bullet$ are between $d_l + 1$ and d_m. Hence the degenerate Plücker relation $R^{k;a}_{J,(i_1,\dots,i_b)}$

also vanishes on (E_β, V_m). Note also that the decomposition (2.6) induces the decomposition

$$R^{k;a}_{J,I} = \sum_{1\le\alpha_1<\dots<\alpha_{l-b}\le l} \pm a_{i_{b+1},\alpha_1}\dots a_{i_l,\alpha_{l-b}} R^{k;a}_{J,(i_{\beta_1},\dots,i_{\beta_b})}.$$

But as we have shown above, each of the relations $R^{k;a}_{J,(i_{\beta_1},\dots,i_{\beta_b})}$ vanishes on (V_l, V_m). Hence so does $R^{k;a}_{J,I}$. □

Example 2.9. Let $\lambda = \omega_1 + \omega_{n-1}$, i.e. $s = 2$, $d_1 = 1$, $d_2 = n-1$. Then the image of $\mathcal{F}^a(1, n-1)$ inside $Gr(1, n) \times Gr(n-1, n)$ is formed by all pairs V_1, V_2 such that $pr_{2,n-1}V_1 \hookrightarrow V_2$. Since $pr_{2,n-1}V_1 \hookrightarrow \mathrm{span}(v_1, v_n)$, the image of the embedding $\mathcal{F}^a(1, n-1) \hookrightarrow \mathbb{P}^{n-1} \times \mathbb{P}^{n-1}$is defined by a single relation

$$X^a_1 X^a_{2,\dots,n} + (-1)^{n-1} X^a_n X^a_{1,\dots,n-1} = 0,$$

which agrees with Example 1.8.

Corollary 2.10. *Theorem 2.5 is true.*

Corollary 2.11. *Let $I^1, \dots, I^s$, $I^l \subset \{1, \dots, n\}$ be a collection of tuples such that the cardinality of I^l is d_l. Then a point $p_{I^1} \times \dots \times p_{I^s}$ belongs to $\mathcal{F}^a$ if and only if*

$$I^l \setminus \{d_l + 1, \dots, d_{l+1}\} \subset I^{l+1}. \quad (2.7)$$

Example 2.12. Consider the case of the complete flags: $s = n-1$, $d_l = l$. Set $pr_l = pr_{l,l}$. Then the embedding of $\mathcal{F}^a$ into the product of Grassmanians is defined by the conditions

$$pr_{l+1}V_l \hookrightarrow V_{l+1},\ l = 1, \dots, n-2 \quad (2.8)$$

and the conditions (2.7) read as $I^l \setminus \{l+1\} \subset I^{l+1}$ for $l = 1, \dots, n-2$.

2.3. **Cells for $\mathcal{F}^a$.** Recall that the cell decomposition for a Grassmanian is given by the G^a-orbits of the torus fixed points. However this is not true for the case of general $\mathcal{F}^a_\lambda$. Moreover, the number of G^a-orbits can be infinite. The simplest example is as follows.

Example 2.13. Let $n = 4$, $\lambda = \omega_1 + \omega_3$. Then $\mathcal{F}^a_\lambda$ is embedded into $\mathbb{P}^3 \times \mathbb{P}^3$ (two Grassmanians for $\mathfrak{sl}_4$) with the coordinates $(x_1 : x_2 : x_3 : x_4)$ and $(x_{123} : x_{124} : x_{133} : x_{234})$. The variety $\mathcal{F}^a_{\omega_1+\omega_3}$ is defined by a single relation $x_1x_{234} - x_4x_{123} = 0$. Therefore, $\mathcal{F}^a_{\omega_1+\omega_3}$ contains the product $\mathbb{P}^2 \times \mathbb{P}^2$ defined by $x_1 = x_{123} = 0$. We note that the subgroup $\mathbb{G}^6_a$ of G^a acts trivially on this $\mathbb{P}^2 \times \mathbb{P}^2$ (the PBW-degree in both V_{ω_1} and V_{ω_3} is at most one). Therefore, we are left with an action of the Borel subgroup. Let w_1, w_2, w_3, w_4 and $w_{123}, w_{124}, w_{134}, w_{234}$ be the standard bases for V_{ω_1} and V_{ω_3}. The group B acts on the span of w_2, w_3, w_4 (resp. on the span of $w_{124}, w_{134}, w_{234}$) as on the quotient of the vector representation (resp. the dual vector representation) by $\mathbb{C}w_1$ (resp. $\mathbb{C}w_{123}$). It is easy to see that the corresponding B-action on $\mathbb{P}^2 \times \mathbb{P}^2$ has infinitely many orbits.

In the following proposition we describe the cell decomposition for $\mathcal{F}^a = \mathcal{F}^a(d_1, \dots, d_s)$.

Proposition 2.14. *Let* $\mathbf{I} = (I^1, \dots, I^s)$ *be a set of sequences satisfying the condition* (2.7)*. Then there exists a cell decomposition* $\mathcal{F}^a = \sqcup_{\mathbf{I}} C_{\mathbf{I}}$*, where*

$$C_{\mathbf{I}} = (G^a \cdot p_{I^1} \times \cdots \times G^a \cdot p_{I^s}) \cap \mathcal{F}^a.$$

In other words, a cell is given by the intersection of the degenerate flag variety, embedded into the product of Grassmanians, with the product of the corresponding cells in $Gr(d_i, n)$*.*

Proof. In Theorem 3.6 we compute the dimensions of $C_{\mathbf{I}}$. In the proof we construct explicitly the coordinates on $C_{\mathbf{I}}$ thus showing that $C_{\mathbf{I}}$ is a cell. □

3. The median Genocchi numbers

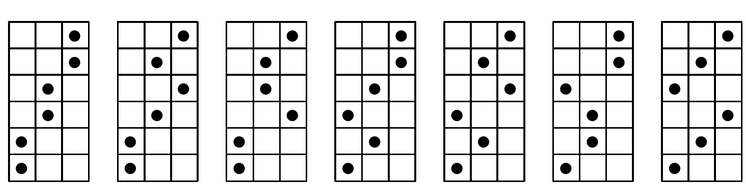
3.1. **Combinatorics.** Let h_n be the normalized Genocchi numbers of the second kind. They are also referred to as the normalized median Genocchi numbers. These numbers have several definitions (see [De], [Du], [Kr], [S]). The first several h_n's are as follows: $1, 2, 7, 38, 295, 3098$. We first briefly recall definitions of these numbers.

We start with the Dellac definition (see [De]). Consider a rectangle with n columns and $2n$ rows. It contains $n \times 2n$ boxes labeled by pairs (l, j), where $l = 1, \dots, n$ is the number of a column and $j = 1, \dots, 2n$ is the number of a row. A Dellac configuration D is a subset of boxes, subject to the following conditions:

- each column contains exactly two boxes from D,
- each row contains exactly one box from D,
- if the (l, j)-th box is in D, then $l \le j \le n + l$.

Let DC_n be the set of such configurations. Then the number of elements in DC_n is equal to h_n.

We list all Dellac's configurations for $n = 3$. We specify boxes in a configuration by putting fat dots inside.

(3.1)  .

The Dellac definition is the earliest one, but the most well-known definition is via the Seidel triangle. The Seidel triangle is of the form

1				
1				
1	1			
2	1			
2	3	3		
8	6	3		
8	14	17	17	
56	48	34	17	
56	104	138	155	155

By definition, the triangle is formed by the numbers $G_{k,n}$ (n is the number of a row and k is the number of a column) with $n = 1, 2, \ldots$ and $1 \le k \le \frac{n+1}{2}$, subject to the relations $G_{1,1} = 1$ and

$$G_{k,2n} = \sum_{i\ge k} G_{i,2n-1},\ G_{k,2n+1} = \sum_{i\le k} G_{i,2n}.$$

The numbers $G_{n,2n-1}$ are called the Genocchi numbers of the first kind and the numbers $G_{1,2n}$ are called the Genocchi numbers of the second kind (or the median Genocchi numbers). Barsky [Ba] and then Dumont [Du] proved that the number $G_{1,2n+2}$ is divisible by 2^n. The normalized median Genocchi numbers h_n are defined as the corresponding ratios: $h_n = G_{1,2n+2}/2^n$.

In [Kr] Kreweras suggested another description of the numbers h_n. Namely, a permutation $\sigma \in S_{2n+2}$ is called a normalized Dumont permutation of the second kind if the following conditions are satisfied:

- $\sigma(k) < k$ if k is even,
- $\sigma(k) > k$ if k is odd,
- $\sigma^{-1}(2k) < \sigma^{-1}(2k+1)$ for $k = 1, \ldots, n$.

The set of such permutations is denoted by $PD2N_n$ (P for permutations, D for Dumont, 2 for the second kind and N for normalized). According to Kreweras, the number of elements of $PD2N_n$ is equal to h_n. In Proposition 3.3 we show that the definitions of Dellac and Kreweras are equivalent (this seems to be known to expert – see [G], [S], but we were not able to find a proof in the literature).

In the following proposition we show that the conditions from Example 2.12 give rise to a new definition of the numbers h_n.

Proposition 3.1. *The number of tuples $I^1, \ldots, I^{n-1}$, with $I^l \subset \{1, \ldots, n\}$, $\#I^l = l$ subject to the condition*

$$I^{l-1} \setminus \{l\} \subset I^l,\ l = 2, \ldots, n-1 \tag{3.2}$$

is equal to h_n.

Proof. Let $\bar{h}_n$ be the number of tuples as above. We compare $\bar{h}_n$ with the Dellac definition of h_n. Given a set $I^1, \ldots, I^{n-1}$ subject to the condition

(3.2), we construct the corresponding Dellac's configuration D and then prove that this map is one-to-one. The rule is as follows. Let us explain what are the boxes of D in the l-th column.

First, suppose $l \notin I^{l-1}$. Then because of the condition (3.2) the difference $I^l \setminus I^{l-1}$ contains exactly one number j. There are two cases:

- If $j > l$, then D contains boxes (l, l) and (l, j).
- If $j \le l$, then D contains boxes (l, l) and $(l, j + n)$.

Now, suppose $l \in I^{l-1}$. Then either $l \in I^l$, or $L \notin I^l$. If $l \in I^l$, then $I^l \setminus I^{l-1}$ contains exactly one number j. There are two cases:

- If $j > l$, then D contains boxes $(l, l + n)$ and (l, j).
- If $j \le l$, then D contains boxes $(l, l + n)$ and $(l, j + n)$.

Finally, let $l \in I^{l-1}$ and $l \notin I^l$. Then $I^l \setminus I^{l-1}$ contains exactly two numbers j_1 and j_2. There are four variants:

- If $j_1 > l$ and $j_2 > l$, then D contains boxes (l, j_1) and (l, j_2).
- If $j_1 > l$ and $j_2 \le l$, then D contains boxes (l, j_1) and $(l, n + j_2)$.
- If $j_1 \le l$ and $j_2 > l$, then D contains boxes $(l, j_1 + n)$ and (l, j_2).
- If $j_1 \le l$ and $j_2 \le l$, then D contains boxes $(l, j_1 + n)$ and $(l, j_2 + n)$.

This rule explains how to pick boxes in columns from 1 to $n-1$. To complete the configuration we simply pick two boxes in the last column in the unique way to make D a configuration.

In order to prove that this map is a bijection, we construct the inverse map. Let D be a Dellac configuration. We define I^l inductively. First, let $l = 1$. Then the box $(1, 1)$ necessarily belongs to D. Let $j > 1$ and D contains $(1, j)$. Then if $j = n + 1$, then $I^1 = (1)$. Otherwise $I^1 = (j)$.

Now assume that I^{l-1} is already defined. First, suppose that the (l, l)-th box belongs to D. Then there exists one more box (l, j) in D with $n + l \ge j > l$. If $j \le n$ we set $I^l = I^{l-1} \cup \{j\}$. Otherwise, we set $I^l = I^{l-1} \cup \{j - n\}$. Second, suppose that the (l, l)-th box does not belong to D. Since the l-th row of D contains exactly one box, there exists $l_1 < l$ such that the (l_1, l)-th box belongs to D. Therefore, $l \subset I^{l-1}$. There exist exactly two boxes (l, j_1) and (l, j_2) in D in the l-th column. Then we set $I^l = I^{l-1} \setminus \{l\} \cup \{\bar{j}_1, \bar{j}_2\}$, where $\bar{j} = j$, if $j \le n$ and $\bar{j} = j - n$ otherwise. □

Example 3.2. Let $n = 3$. The pairs I^1, I^2, corresponding to the Dellac configurations (3.1) are as follows (the order is the same as on picture (3.1)):

$$\{(2), (13)\}, \quad \{(2), (23)\}, \quad \{(2), (12)\}, \quad \{(3), (13)\},$$
$$\{(3), (23)\}, \quad \{(1), (13)\}, \quad \{(1), (12)\}.$$

We now compare the definitions by Dellac and by Kreweras.

Proposition 3.3. *The number of elements in $PD2N_n$ is equal to the number of elements in DC_n.*

Proof. We construct a bijection $A : PD2N_n \to DC_n$. Let $\sigma \in PD2N_n$. We determine the boxes in the k-th column of $A(\sigma)$ using the values of $\sigma^{-1}(2k)$ and $\sigma^{-1}(2k + 1)$.

Let us start with $k = 1$. We note that $\sigma(2) = 1$, $\sigma(4)$ is equal to 2 or to 3. In addition, $\sigma^{-1}(2) = 1$ or 4 and the possible values of $\sigma^{-1}(3)$ are $4, 6, \dots, 2n+2$. Therefore, all possible values of the pair $(\sigma^{-1}(2), \sigma^{-1}(3))$ are as follows:

$$(1,4),\ (4,6),\ (4,8), \dots, (4, 2n+2).$$

If the first possibility occurs, then by definition the first column of $A(\sigma)$ contains boxes $(1,1)$ (as any Dellac's configuration) and $(1, n+1)$. If $\sigma^{-1}(2) = 4$ and $\sigma^{-1}(3) = 2l+2$, then the first column of $A(\sigma)$ contains boxes $(1,1)$ and $(1,l)$.

Now let us consider the case $k = n$. We note that $\sigma(2n+1) = 2n$, $\sigma(2n-1)$ is equal to $2n$ or to $2n+1$. In addition, $\sigma^{-1}(2n+1) = 2n+2$ or $2n-1$ and the possible values of $\sigma^{-1}(2n)$ are $1, 3, \dots, 2n-1$. Therefore, all possible values of the pair $(\sigma^{-1}(2n), \sigma^{-1}(2n+1))$ are as follows:

$$(2n-1, 2n+2),\ (1, 2n-1),\ (3, 2n-1), \dots, (2n-3, 2n-1).$$

If the first possibility occurs, then by definition the n-th column of $A(\sigma)$ contains boxes $(n, 2n)$ (as any Dellac's configuration) and (n,n). If

$$(\sigma^{-1}(2n), \sigma^{-1}(2n+1)) = (2l-1, 2n-1),$$

then the first column of $A(\sigma)$ contains boxes $(n, 2n)$ and $(n, n+l)$.

Finally, take $k = 2, \dots, n-1$. We note that the possible values of $\sigma^{-1}(2k)$ are $1, 3, \dots, 2k-1, 2k+2, \dots, 2n$. Also, the possible values of $\sigma^{-1}(2k+1)$ are $3, 5, \dots, 2k-1, 2k+2, \dots, 2n, 2n+2$. We now define the k-th column of $A(\sigma)$ as follows:

(i) If the pair $(\sigma^{-1}(2k), \sigma^{-1}(2k+1))$ contains $2l-1$, $l = 1, \dots, k$, then the k-th column of $A(\sigma)$ contains a box $(k, n+l)$.
(ii) If the pair $(\sigma^{-1}(2k), \sigma^{-1}(2k+1))$ contains $2l+2$, $l = k, \dots, n$, then the k-th column of $A(\sigma)$ contains a box (k, l).

We note that $A(\sigma) \in DC_n$. In fact, by definition any column of $A(\sigma)$ contains exactly two boxes and every row contains exactly one box (this follows from the definition above and because σ is one-to-one). In order to prove that A is a bijection it suffices to note that formulas (i) and (ii) allow to construct explicitly the map A^{-1}. □

Example 3.4. Let $n = 3$. The elements of $PD2N_3$ corresponding to the Dellac configurations on picture (3.1) are as follows (the order is the same as on picture (3.1)):

$$(41627385),\quad (61427385),\quad (41526387),\quad (41627583),$$
$$(61427583),\quad (21637485),\quad (21436587).$$

We recall that the main ingredient for the Kreweras construction of $PD2N_n$ is the following triangle:

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 1 &&&& \\ &&& 2 && 3 && 2 &&& \\ && 7 && 12 && 12 && 7 && \\ & 38 && 69 && 81 && 69 && 38 & \\ 295 && 552 && 702 && 702 && 552 && 295 \end{array}$$

The rule is as follows: denote the numbers in the n-th line by $h_{n,1},\dots,h_{n,n}$. For example, $h_{4,2}=12$. Then the Kreweras triangle is defined by

$$h_{n,1}=h_{n-1,1}+\dots+h_{n-1,n-1},\ h_{n,2}=2h_{n,1}-h_{n-1,1},$$
$$h_{n,k}=2h_{n,k-1}-h_{n,k-2}-h_{n-1,k-2}-h_{n-1,k-1},\ k\ge 3.$$

Kreweras proved that $h_{n+1,1}$ is the n-th Genocchi number h_n and in general $h_{n+1,k}$ is the number of the normalized Dumont permutations $\sigma\in S_{2n+2}$ of the second kind such that $\sigma(1)=2k$. The following is an immediate corollary from the explicit bijections above.

Corollary 3.5. *The number of the Dellac configurations $D\in DC_n$ such that* $\min\{i:\ (i,n+1)\in D\}=k$ *is equal to $h_{n,k}$. The number of tuples $I^1,\dots,I^{n-1}$ subject to the condition $I^{l-1}\setminus\{l\}\subset I^l$ with an extra condition* $\min\{j:\ 1\in I^j\}=k$ *is equal to $h_{n,k}$.*

3.2. **The Poincaré polynomials.** For a tuple $\mathbf{I}=(I^1,\dots,I^{n-1})$ subject to the relation $I^{l-1}\setminus\{l\}\subset I^l$ we denote by $D_{\mathbf{I}}$ the corresponding Dellac configuration. For a Dellac configuration $D\in DC_n$ we define the length $l(D)$ of D as the number of pairs (l_1,j_1), (l_2,j_2) such that the boxes (l_1,j_1) and (l_2,j_2) are both in D and $l_1<l_2$, $j_1>j_2$. We call such a pair of boxes (l_1,j_1), (l_2,j_2) a disorder. This definition resembles the definition of the length of a permutation. We note that in the classical case the dimension of a cell attached to a permutation σ in a flag variety is equal to the number of pairs $j_1<j_2$ such that $\sigma(j_1)>\sigma(j_2)$ (which equals to the length of σ).

Theorem 3.6. *The dimension of a cell $C_{\mathbf{I}}$ is equal to $l(D_{\mathbf{I}})$.*

Proof. We prove the dimension formula by constructing explicitly the coordinates on the cell $C_{\mathbf{I}}$. Let

$$\mathbf{I}=(I^1,\dots,I^{n-1}),\ I^d=(i_1^d<\dots<i_d^d).$$

Recall the description of the cells $C_{I^d}\subset Gr(d,n)$ from Proposition 2.1. Using this description we construct the coordinates on $C_{\mathbf{I}}$ inductively on d. Let $(V_1,\dots,V_{n-1})\in C_{\mathbf{I}}$. For a number k we set $[k]_+=k$ if $k>0$ and $[k]_+=k+n$ if $k\le 0$.

We start with $d=1$. An element $V_1\in C_{I^1}$ is of the form $\mathbb{C}e_1^1$ with

$$e_1^1=v_{i_1^1}+a_1^1v_{[i_1^1-1]_+}+\dots+a^1_{[i_1^1-1]_+-1}v_2$$

(see Remark 2.2). We state that $[i_1^1-1]_+-1$ (which is exactly the number of the degrees of freedom we have so far) is exactly the number of boxes $(l,j)\in D_{\mathbf{I}}$ such that $l>1$ and $j<i_1^1$ (note that the box $(1,1)$ is necessarily in $D_{\mathbf{I}}$, but it does not add anything to the length of $D_{\mathbf{I}}$, since for any $(l.j)\in D_{\mathbf{I}}$ with $l>1$ we have $j>1$). In fact, the first column of $D_{\mathbf{I}}$ contains boxes in the first row and in the $([i_1^1-1]_++1)$-st row (see the proof of Proposition 3.1). Since any row of $D_{\mathbf{I}}$ contains exactly one box, the rows number $2,\dots,[i_1^1-1]_+$ are occupied by boxes in the columns from 2 to n. Therefore, the box $(1,[i_1^1-1]_++1)$ produces exactly $[i_1^1-1]_+-1$ disorders.

The second step is to construct the coordinates on those subspaces from C_{I^2} which contain pr_2V_1. There are two possibilities: either $i_1^1=2$ or $i_1^1\neq 2$. In the first case the condition $pr_2V_1\hookrightarrow V_2$ is empty. Therefore, we have to choose two basis vectors e_1^2,e_2^2 of $V_2\in C_{I^2}$, with the coordinates

$$e_1^2=v_{i_1^2}+a_1^1v_{[i_1^2-1]_+}+\dots+a^1_{[i_1^2-2]_+-1}v_3,$$
$$e_2^2=v_{i_2^2}+a_1^2v_{[i_2^2-1]_+}+\dots+a^2_{[i_2^2-2]_+-2}v_3.$$

We note that the number of coefficients of e_2^2 is $[i_2^2-2]_+-2$, because $i_1^2<i_2^2$ and hence adding appropriately normalized vector e_1^2 one can vanish the coefficient of e_2^2 in front of $v_{i_1^2}$. We note that since $i_1^1=2$, the second column of $D_{\mathbf{I}}$ contains boxes in the rows $([i_1^2-2]_++2)$ and $([i_1^2-2]_++2)$ (see the proof of Proposition 3.1). We state that $[i_1^2-2]_+-1+[i_2^2-2]_+-2$ (the number of degrees of freedom we have fixing the vectors e_1^2 and e_2^2) is exactly the number of boxes in the columns $3,4,\dots,n$, having disorders with boxes in the second column. In fact, each row from 3 to $[i_1^2-2]_+-1$ contains one box in the columns 3 and greater (recall $i_1^1=2$). This produces $[i_1^2-2]_+-1$ disorders with the box $(2,[i_1^2-2]_+-1)$. Similarly, we obtain $[i_2^2-2]_+-2$ disorders with the second box in the second column.

Now assume $i_1^1\neq 2$. Then the space pr_2V_1 is nontrivial and spanned by a single vector $e_1^2=pr_2e_1^1$. Therefore in order to specify V_2 we need to fix one more vector e_2^2 such that $\mathrm{span}(e_1^2,e_2^2)\in C_{I^2}$. Recall that since $i_1^1\neq 2$ we have $I^2\setminus I^1=\{j\}$. Also, the second column of $D_{\mathbf{I}}$ contains boxes in the second row and in the row number $[j-2]_++2$ (see the proof of Proposition 3.1). The box $(2,2)$ does not produce any disorder with boxes in the columns greater than 2. As for the box $(2,[j-2]_++2)$, the number of disorders it produces is equal to the number of degrees of freedom of choosing the vector e_2^2 (the argument is very similar to the ones above in the case $i_1^1=2$).

Now let us consider the general induction step. Assume that we have already computed the number of degrees of freedom while fixing the subspaces $V_1,\dots,V_{d-1}$. Our goal is to show that the number of degrees of freedom of V_d is equal to the number of disorders produced by the boxes in the d'th column with the boxes in columns l with $l>d$. As in the previous case, one has to consider two cases: $d\in I^{d-1}$ and $d\notin I^{d-1}$. The proof is very similar to the one in the case $d=2$ and we omit it. □

Corollary 3.7. *The Poincaré polynomial* $P_n(t) = P_{\mathcal{F}^a}(t)$ *is given by*

$$P_n(t) = \sum_{D \in DC_n} t^{2l(D)}.$$

Let $q = t^2$. Then P_n are polynomials in q with $P_n(1) = h_n$. Thus the Poincaré polynomials of the degenerate flag varieties provide a natural q-version of the normalized median Genocchi numbers (it would be interesting to compare our q-version with the one in [HZ]).

Example 3.8. The first four polynomials $P_n(q)$ are as follows:

$$P_1(q) = 1, \qquad P_2(q) = 1 + q,$$
$$P_3(q) = 1 + 2q + 3q^2 + q^3,$$
$$P_4(q) = 1 + 3q + 7q^2 + 10q^3 + 10q^4 + 6q^5 + q^6.$$

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References

[A] I. Arzhantsev, *Flag varieties as equivariant compactifications of* $\mathbb{G}_a^n$, arXiv:1003.2358.
[AS] I. Arzhantsev, E. Sharoiko, *Hassett-Tschinkel correspondence: modality and projective hypersurfaces*, arXiv:0912.1474.
[Ba] D. Barsky, *Congruences pour les nombres de Genocchi de 2e espèce*, Groupe d'étude d'Analyse ultramétrique, 8e année, no. 34, 1980/81, 13 pp.
[De] H. Dellac, *Problem 1735*, L'Intermédiaire des Mathématiciens, 7 (1900), 9–10.
[Du] D. Dumont, *Interprétations combinatoires des nombres de Genocchi*, Duke Math. J., 41 (1974) , 305–318.
[DR] D. Dumont and A. Randrianarivony, *Dérangements et nombres de Genocchi*, Discrete Math. 132 (1994), 37–49.
[DZ] D. Dumont and J. Zeng, *Further results on Euler and Genocchi numbers*, Aequationes Mathemicae 47 (1994), 31–42.
[Fe1] E. Feigin, *The PBW filtration*, Represent. Theory 13 (2009), 165-181.
[Fe2] E. Feigin, *The PBW filtration, Demazure modules and toroidal current algebras*, SIGMA, 4 (2008), 070, 21 p.
[Fe3] E. Feigin, $\mathbb{G}_a^M$ *degeneration of flag varieties*, arXiv:1007.0646,
[Fu] W. Fulton, *Young tableaux, with applications to representation theory and geometry.* Cambridge University Press, 1997.
[FFL] B. Feigin, E. Feigin, P. Littelmann, *Zhu's algebras, C_2-algebras and abelian radicals*, arXiv:0907.3962.
[FFoL1] E. Feigin, G. Fourier, P. Littelmann, *PBW filtration and bases for irreducible modules in type A_n*, arXiv:1002.0674.
[FFoL2] E. Feigin, G. Fourier, P. Littelmann, *PBW filtration and bases for symplectic Lie algebras*, arXiv:1010.2321.
[FL] E. Feigin, P. Littelmann, *Zhu's algebra and the C_2-algebra in the symplectic and the orthogonal cases*,2010 J. Phys. A: Math. Theor. 43 135206.
[G] I.Gessel *Applications of the classical umbral calculus*, Algebra Universalis **49** (2003), 397–434.

[HZ] G.-N. Han and J. Zeng, *On a q-sequence that generalizes the median Genocchi numbers*, Ann. Sci. Math. Québec 23 (1999), 63–72.

[HT] B. Hassett, Yu. Tschinkel, *Geometry of equivariant compactifications of* $\mathbb{G}_a^n$, Int. Math. Res. Notices 20 (1999), 1211-1230.

[Kr] G. Kreweras, *Sur les permutations comptées par les nombres de Genocchi de 1-ière et 2-ième espèce*, Europ. J. Combinatorics 18 (1997), 49–58.

[K1] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory,* Progress in Mathematics vol. 204, Birkhauser, Boston, 606 (2002).

[K2] S. Kumar, *The nil Hecke ring and singularity of Schubert varieties*, Inventiones Math. 123, 471-506 (1996).

[S] N. J. A. Sloane, *Sequence A000366*, The On-Line Encyclopedia of Integer Sequences, http://oeis.org.

[Vien] G. Viennot, *Interprétations combinatoires des nombres d'Euler et de Genocchi*, Seminar on Number Theory, 1981/1982, No. 11, 94 pp., Univ. Bordeaux I, Talence, 1982.

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