Preprint, arXiv:1101.1946

ON SUMS OF APÉRY POLYNOMIALS AND RELATED CONGRUENCES

Zhi-Wei Sun

Department of Mathematics, Nanjing University
Nanjing 210093, People's Republic of China
zwsun@nju.edu.cn
http://math.nju.edu.cn/∼zwsun

Abstract. The Apéry polynomials are given by

$$A_n(x)=\sum_{k=0}^n\binom nk^2\binom{n+k}k^2x^k\ \ (n=0,1,2,\ldots).$$

(Those $A_n=A_n(1)$ are Apéry numbers.) Let p be an odd prime. We show that

$$\sum_{k=0}^{p-1}(-1)^kA_k(x)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}k^3}{16^k}x^k\pmod{p^2},$$

and that

$$\sum_{k=0}^{p-1}A_k(x)\equiv\left(\frac xp\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(256x)^k}\pmod p$$

for any p-adic integer $x\not\equiv0\pmod p$. This enables us to determine explicitly $\sum_{k=0}^{p-1}(\pm1)^kA_k$ mod p, and $\sum_{k=0}^{p-1}(-1)^kA_k$ mod p^2 in the case $p\equiv2\pmod 3$. Another consequence states that

$$\sum_{k=0}^{p-1}(-1)^kA_k(-2)\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p=x^2+4y^2\ (x,y\in\mathbb Z),\\0\pmod{p^2}&\text{if }p\equiv3\pmod 4.\end{cases}$$

We also prove that for any prime $p>3$ we have

$$\sum_{k=0}^{p-1}(2k+1)A_k\equiv p+\frac76p^4B_{p-3}\pmod{p^5}$$

where $B_0,B_1,B_2,\ldots$ are Bernoulli numbers.

2010 *Mathematics Subject Classification.* Primary 11A07, 11B65; Secondary 05A10, 11B68, 11E25.

Keywords. Apéry numbers and Apéry polynomials, Bernoulli numbers, binomial coefficients, congruences.

Supported by the National Natural Science Foundation (grant 10871087) and the Overseas Cooperation Fund (grant 10928101) of China.

1. Introduction

The well-known Apéry numbers given by

$$A_n=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2=\sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2\ (n\in\mathbb{N}=\{0,1,2,\ldots\}),$$

play a central role in Apéry's proof of the irrationality of $\zeta(3)=\sum_{n=1}^{\infty}1/n^3$ (see Apéry [Ap] and van der Poorten [Po]). They also have close connections to modular forms (cf. Ono [O, pp.198–203]). The Dedekind eta function in the theory of modular forms is defined by

$$\eta(\tau)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n)\quad \text{with } q=e^{2\pi i\tau},$$

where $\tau\in\mathbb{H}=\{z\in\mathbb{C}:\ \mathrm{Im}(z)>0\}$ and hence $|q|<1$. In 1987 Beukers [B] conjectured that

$$A_{(p-1)/2}\equiv a(p)\ (\mathrm{mod}\ p^2)\quad \text{for any prime } p>3,$$

where $a(n)$ $(n=1,2,3,\ldots)$ are given by

$$\eta^4(2\tau)\eta^4(4\tau)=q\prod_{n=1}^{\infty}(1-q^{2n})^4(1-q^{4n})^4=\sum_{n=1}^{\infty}a(n)q^n.$$

This was finally confirmed by Ahlgren and Ono [AO] in 2000.

We define Apéry polynomials by

$$A_n(x)=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n+k}{k}^2x^k=\sum_{k=0}^{n}\binom{n+k}{2k}^2\binom{2k}{k}^2x^k\quad (n\in\mathbb{N}).\quad (1.1)$$

Clearly $A_n(1)=A_n$. Motivated by the Apéry polynomials, we also introduce a new kind of polynomials:

$$W_n(x):=\sum_{k=0}^{n}\binom{n}{k}^2\binom{n-k}{k}^2x^k=\sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{2k}^2\binom{2k}{k}^2x^k\quad (n\in\mathbb{N}).\quad (1.2)$$

Recall that Bernoulli numbers $B_0,B_1,B_2,\ldots$ are rational numbers given by

$$B_0=1\ \text{ and }\ \sum_{k=0}^{n}\binom{n+1}{k}B_k=0\quad \text{for } n\in\mathbb{Z}^+=\{1,2,3,\ldots\}.$$

It is well known that $B_{2n+1}=0$ for all $n\in\mathbb{Z}^+$ and

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty}B_n\frac{x^n}{n!}\quad(|x|<2\pi).$$

Also, Euler numbers $E_0,E_1,E_2,\ldots$ are integers defined by

$$E_0=1\ \text{ and }\ \sum_{\substack{k=0\\ 2\mid k}}^{n}\binom{n}{k}E_{n-k}=0\quad\text{for } n\in\mathbb{Z}^+.$$

It is well known that $E_{2n+1}=0$ for all $n\in\mathbb{N}$ and

$$\sec x=\sum_{n=0}^{\infty}(-1)^nE_{2n}\frac{x^{2n}}{(2n)!}\quad\left(|x|<\frac{\pi}{2}\right).$$

Now we state our first theorem.

Theorem 1.1. (i) *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}(-1)^kA_k(x)\equiv\sum_{k=0}^{p-1}(-1)^kW_k(-x)\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}x^k\pmod{p^2}.\tag{1.3}$$

Also, for any p-adic integer $x\not\equiv 0\pmod p$, we have

$$\begin{aligned}\sum_{k=0}^{p-1}A_k(x)&\equiv\sum_{k=0}^{p-1}W_k(x)\pmod{p^2}\\&\equiv\left(\frac{x}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(256x)^k}\pmod{p},\end{aligned}\tag{1.4}$$

where $(-)$ denotes the Legendre symbol.

(ii) *For any positive integer n we have*

$$\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k(x)=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}\binom{n+k}{2k+1}\binom{2k}{k}x^k.\tag{1.5}$$

If $p>3$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)A_k\equiv p+\frac{7}{6}p^4B_{p-3}\pmod{p^5}\tag{1.6}$$

and

$$\sum_{k=0}^{p-1}(2k+1)A_k(-1)\equiv\left(\frac{-1}{p}\right)p-p^3E_{p-3}\pmod{p^4}. \tag{1.7}$$

(iii) *Given $\varepsilon\in\{\pm1\}$ and $m\in\mathbb{Z}^+$, for any prime p we have*

$$\sum_{k=0}^{p-1}(2k+1)\varepsilon^kA_k^m\equiv 0 \pmod{p}.$$

Remark 1.1. (i) Let p be an odd prime. The author [Su1, Su2] had conjectures on $\sum_{k=0}^{p-1}\binom{2k}{k}^3/m^k$ mod p^2 with $m=1,-8,16,-64,256,-512,4096$. Motivated by the author's conjectures on $\sum_{k=0}^{p-1}A_k(x)$ mod p^2 with $x=1,-4,9$ in an initial version of this paper, Guo and Zeng [GZ, Theorem 5.1] recently showed that

$$\sum_{k=0}^{p-1}A_k(x)\equiv\sum_{k=0}^{(p-1)/2}\binom{p+2k}{4k+1}\binom{2k}{k}^2x^k \pmod{p^2}.$$

(ii) The values of

$$s_n=\frac{1}{n}\sum_{k=0}^{n-1}(2k+1)A_k\in\mathbb{Z}$$

with $n=1,\ldots,8$ are

$$1,\ 8,\ 127,\ 2624,\ 61501,\ 1552760,\ 41186755,\ 1131614720$$

respectively. On June 6, 2011 Richard Penner informed the author an interesting application of (1.5): (1.5) with $x=1$ implies that s_n is the trace of the inverse of nH_n where H_n refers to the Hilbert matrix $(\frac{1}{i+j-1})_{1\leqslant i,j\leqslant n}$.

Can we find integers $a_0,a_1,a_2,\ldots$ such that $\sum_{k=0}^{p-1}a_k\equiv 4x^2-2p \pmod{p^2}$ if $p=x^2+y^2$ is a prime with x odd and y even? The following corollary provides an affirmative answer!

Corollary 1.1. *Let p be any odd prime. Then*

$$\begin{aligned}\sum_{k=0}^{p-1}(-1)^kA_k(-2)&\equiv\sum_{k=0}^{p-1}(-1)^kA_k\left(\frac14\right)\\ &\equiv\begin{cases}4x^2-2p \pmod{p^2} & \text{if } p\equiv1\pmod 4\ \&\ p=x^2+y^2\ (2\nmid x),\\ 0\pmod{p^2} & \text{if } p\equiv 3\pmod 4.\end{cases}\end{aligned} \tag{1.8}$$

Proof. It is known (cf. Ishikawa [I]) that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{64^k}\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1\pmod 4\ \&\ p=x^2+y^2\ (2\nmid x),\\0\pmod{p^2}&\text{if }p\equiv3\pmod 4.\end{cases}$$

The author conjectured that we can replace 64^k by $(-8)^k$ in the congruence, and this was recently confirmed by Z. H. Sun [S3]. So, applying (1.3) with $x=-2,1/4$ we obtain (1.8). □

Corollary 1.2. *Let p be an odd prime. Then*

$$\sum_{k=0}^{p-1}A_k\equiv c(p)\pmod p \tag{1.9}$$

where

$$c(p):=\begin{cases}4x^2-2p&\text{if }p\equiv1,3\pmod 8\ \&\ p=x^2+2y^2\ (x,y\in\mathbb{Z}),\\0&\text{if }(\frac{-2}{p})=-1,\ i.e.,\ p\equiv5,7\pmod 8.\end{cases}$$

Also,

$$\sum_{k=0}^{p-1}(-1)^kA_k\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}(-1)^kA_k\left(\frac{1}{16}\right)$$
$$\equiv\begin{cases}4x^2-2p\pmod p&\text{if }p\equiv1\pmod 3\text{ and }p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }p\equiv2\pmod 3.\end{cases} \tag{1.10}$$

Proof. By [M95] and [Su3], we have

$$\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{256^k}\equiv c(p)\pmod{p^2}$$

as conjectured in [RV]. (Here we only need the mod p version which was proved in [M95].) So (1.9) follows from (1.4). The author [Su2] conjectured that

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k}\equiv\left(\frac{-1}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{256^k}$$
$$\equiv\begin{cases}4x^2-2p\pmod{p^2}&\text{if }p\equiv1\pmod 3\text{ and }p=x^2+3y^2\ (x,y\in\mathbb{Z}),\\0\pmod{p^2}&\text{if }p\equiv2\pmod 3.\end{cases}$$

This was confirmed by Z. H. Sun [S3] in the case $p\equiv2\pmod 3$, and the mod p version in the case $p\equiv1\pmod 3$ follows from (4)-(5) in Ahlgren [A, Theorem 5]. So we get (1.10) by applying (1.3) with $x=1,1/16$. □

Remark 1.2. The author conjectured that (1.9) also holds modulo p^2, and that (1.10) is also valid modulo p^2 in the case $p\equiv1\pmod 3$.

Corollary 1.3. *For any odd prime p and integer x, we have*

$$\sum_{k=0}^{p-1}(2k+1)A_k(x) \equiv p\left(\frac{x}{p}\right) \pmod{p^2}. \tag{1.11}$$

Proof. This follows from (1.5) in the case $n = p$, for, $p \mid \binom{p+k}{2k+1}$ for every $k = 0, \ldots, (p-3)/2$, and $p \mid \binom{2k}{k}$ for all $k = (p+1)/2, \ldots, p-1$. □

We deduce Theorem 1.1(i) from our following result which has its own interest.

Theorem 1.2. *Let p be an odd prime and let x be any p-adic integer.*

(i) *If $x \equiv 2k \pmod{p}$ with $k \in \{0, \ldots, (p-1)/2\}$, then we have*

$$\sum_{r=0}^{p-1}(-1)^r\binom{x}{r}^2 \equiv (-1)^k\binom{x}{k} \pmod{p^2}. \tag{1.12}$$

(ii) *If $x \equiv k \pmod{p}$ with $k \in \{0, \ldots, p-1\}$, then*

$$\sum_{r=0}^{p-1}\binom{x}{r}^2 \equiv \binom{2x}{k} \pmod{p^2}. \tag{1.13}$$

Remark 1.3. In contrast with (1.12) and (1.13), we recall the following identities (cf. [G, (3.32) and (3.66)]):

$$\sum_{k=0}^{2n}(-1)^k\binom{2n}{k}^2 = (-1)^n\binom{2n}{n} \quad \text{and} \quad \sum_{k=0}^{n}\binom{n}{k}^2 = \binom{2n}{n}.$$

Corollary 1.4. *Let p be an odd prime.*

(i) (Conjectured in [RV] and proved in [M03]) *We have*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{16^k} \equiv \left(\frac{-1}{p}\right) \pmod{p^2}.$$

(ii) (Conjectured by the author [Su1] and confirmed in [S2]) *If $p \equiv 1 \pmod{4}$ and $p = x^2 + y^2$ with $x \equiv 1 \pmod{4}$ and $y \equiv 0 \pmod{2}$, then*

$$\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^2}{(-16)^k} \equiv (-1)^{(p-1)/4}\left(2x - \frac{p}{2x}\right) \pmod{p^2}. \tag{1.14}$$

Proof. Since $\binom{-1/2}{r}=\binom{2r}{r}/(-4)^r$ for all $r=0,1,\ldots$, applying (1.13) with $x=-1/2$ and $k=(p-1)/2$ we immediately get the congruence in part (i).

When $p=x^2+y^2$ with $x\equiv 1 \pmod 4$ and $y\equiv 0 \pmod 2$, by (1.12) with $x=-1/2$ and $k=(p-1)/2$ we have

$$\begin{aligned}\sum_{r=0}^{p-1}\frac{\binom{2r}{r}^2}{(-16)^r}\equiv&(-1)^{(p-1)/4}\binom{-1/2}{(p-1)/4}=\frac{\binom{(p-1)/2}{(p-1)/4}}{4^{(p-1)/4}}\\ \equiv&(-1)^{(p-1)/4}\left(2x-\frac{p}{2x}\right) \pmod{p^2} \quad \text{(by [CDE] or [BEW]).}\end{aligned}$$

This proves (1.14). □

Corollary 1.5. *Let $a_n:=\sum_{k=0}^n\binom{n}{k}^2C_k$ for $n=0,1,2,\ldots$, where C_k denotes the Catalan number $\binom{2k}{k}/(k+1)=\binom{2k}{k}-\binom{2k}{k+1}$. Then, for any odd prime p we have*

$$a_1+\cdots+a_{p-1}\equiv 0 \pmod{p^2}. \tag{1.15}$$

Remark 1.4. We find no prime $p\leqslant 5{,}000$ with $\sum_{k=1}^{p-1}a_k\equiv 0 \pmod{p^3}$ and no composite number $n\leqslant 70{,}000$ satisfying $\sum_{k=1}^{n-1}a_k\equiv 0 \pmod{n^2}$. We conjecture that (1.15) holds for no composite $p>1$.

The author [Su1, Remark 1.2] conjectured that for any prime $p>5$ with $p\equiv 1 \pmod 4$ we have

$$\sum_{k=0}^{p^a-1}\frac{k^3\binom{2k}{k}^3}{64^k}\equiv 0 \pmod{p^{2a}} \quad \text{for } a=1,2,3,\ldots.$$

This was recently confirmed by Z. H. Sun [S3] in the case $a=1$. Note that

$$\frac{k^3\binom{2k}{k}^3}{64^k}=(-1)^kk^3\binom{-1/2}{k}^3=\frac{(-1)^{k-1}}{8}\binom{-3/2}{k-1}^3 \quad \text{for all } k=1,2,3,\ldots.$$

So, for any prime $p>5$ with $p\equiv 1 \pmod 4$ we have

$$\sum_{r=0}^{p-1}(-1)^r\binom{-3/2}{r}^3\equiv 0 \pmod{p^2}.$$

Since $-3/2\equiv -2(p+3)/4 \pmod p$, the result just corresponds to the case $x=-3/2$ of our following general theorem.

Theorem 1.3. *Let $p>3$ be a prime and let x be a p-adic integer with $x\equiv -2k \pmod p$ for some $k\in\{1,\ldots,\lfloor (p-1)/3\rfloor\}$. Then we have*

$$\sum_{r=0}^{p-1}(-1)^r\binom{x}{r}^3\equiv 0 \pmod{p^2}. \tag{1.16}$$

Similar to Apéry numbers, the central Delannoy numbers (see [CHV]) are defined by

$$D_n=\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k}=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}\ (n\in\mathbb{N}).$$

Such numbers arise naturally in many enumeration problems in combinatorics (cf. Sloane [S]); for example, D_n is the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0),(0,1)$ and $(1,1)$.

Now we give our result on central Delannoy numbers.

Theorem 1.4. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{p-1}D_k\equiv\left(\frac{-1}{p}\right)-p^2E_{p-3} \pmod{p^3}, \tag{1.17}$$

We also have

$$\sum_{k=0}^{p-1}(2k+1)(-1)^kD_k\equiv p-\frac{7}{12}p^4B_{p-3} \pmod{p^5} \tag{1.18}$$

and

$$\sum_{k=0}^{p-1}(2k+1)D_k\equiv p+2p^2q_p(2)-p^3q_p(2)^2 \pmod{p^4}, \tag{1.19}$$

where $q_p(2)$ denotes the Fermat quotient $(2^{p-1}-1)/p$.

In the next section we will show Theorems 1.1-1.2 and Corollary 1.5. Section 3 is devoted to our proofs of Theorems 1.3 and 1.4. In Section 4 we are going to raise some related conjectures for further research.

2. Proofs of Theorems 1.1-1.2 and Corollary 1.5

We first prove Theorem 1.2.

Proof of Theorem 1.2. (i) We now consider the first part of Theorem 1.2. Set

$$f_k(y):=\sum_{r=0}^{p-1}(-1)^r\binom{2k+py}{r}^2\quad \text{for } k\in\mathbb{N}. \tag{2.1}$$

We want to prove that

$$f_k(y) \equiv (-1)^k \binom{2k+py}{k} \pmod{p^2} \tag{2.2}$$

for any p-adic integer y and $k \in \{0, 1, \ldots, (p-1)/2\}$.

Applying the Zeilberger algorithm via `Mathematica 7`, we find that

$$\begin{aligned}(py+2k+2)&f_{k+1}(y) + 4(py+2k+1)f_k(y)\\ =&\frac{(p(y-1)+2k+3)^2 F_k(y)}{(py+2k+1)(py+2k+2)^2}\binom{py+2k+2}{p-1}^2,\end{aligned} \tag{2.3}$$

where

$$F_k(y) = 14+34k+20k^2-10p-12kp+2p^2+17py+20kpy-6p^2y+5p^2y^2.$$

Now fix a p-adic integer y. Observe that

$$\begin{aligned} f_{(p-1)/2}(y) =& \sum_{r=0}^{p-1}(-1)^r \binom{p-1+py}{r}^2 = \sum_{r=0}^{p-1}(-1)^r \prod_{0<s\leqslant r}\left(1-\frac{p(y+1)}{s}\right)^2 \\ \equiv& \sum_{r=0}^{p-1}(-1)^r\left(1-\sum_{0<s\leqslant r}\frac{2p(y+1)}{s}\right) = 1-\sum_{r=1}^{p-1}(-1)^r\sum_{s=1}^{r}\frac{2p(y+1)}{s} \\ =&1-2p(y+1)\sum_{s=1}^{p-1}\frac{1}{s}\sum_{r=s}^{p-1}(-1)^r = 1-p(y+1)\sum_{j=1}^{(p-1)/2}\frac{1}{j} \\ \equiv&(-1)^{(p-1)/2}\binom{p-1+py}{(p-1)/2} \pmod{p^2}. \end{aligned}$$

For each $k \in \{0, \ldots, (p-3)/2\}$, clearly $py+2k+1, py+2k+2 \not\equiv 0 \pmod{p}$, and also

$$(p(y-1)+2k+3)^2\binom{py+2k+2}{p-1}^2 \equiv 0 \pmod{p^2}$$

since $\binom{py+2k+2}{p-1} = \frac{p}{py+2k+3}\binom{py+2k+3}{p} \equiv 0 \pmod{p}$ if $0 \leqslant k < (p-3)/2$. Thus, by (2.3) we have

$$f_k(y) \equiv -\frac{py+2k+2}{4(py+2k+1)}f_{k+1}(y) \pmod{p^2} \quad \text{for } k=0,\ldots,\frac{p-3}{2}.$$

If $0 \leqslant k < (p-1)/2$ and

$$f_{k+1}(y) \equiv (-1)^{k+1}\binom{2(k+1)+py}{k} \pmod{p^2},$$

then

$$\begin{aligned}f_k(y)\equiv&-\frac{py+2k+2}{4(py+2k+1)}(-1)^{k+1}\binom{2(k+1)+py}{k+1}\\ =&\frac{(-10^k(py+2k+2)^2}{4(k+1)(py+k+1)}\binom{2k+py}{k}\equiv(-1)^k\binom{2k+py}{k}\pmod{p^2}.\end{aligned}$$

Therefore (2.2) holds for all $k=0,1,\ldots,(p-1)/2$. This proves Theorem 1.2(i).

(ii) The second part of Theorem 1.2 can be proved in a similar way. Here we mention that if we define

$$g_k(y):=\sum_{r=0}^{p-1}\binom{k+py}{r}^2\quad\text{for } k\in\mathbb{N}\tag{2.4}$$

then by the Zeilberger algorithm we have the recursion

$$\begin{aligned}&(py+k+1)g_{k+1}(y)-2(2py+2k+1)g_k(y)\\ =&-\frac{(p(y-1)+k+2)^2(3py-2p+3k+3)}{(py+k+1)^2}\binom{py+k+1}{p-1}^2.\end{aligned}$$

It follows that if $k\in\{0,\ldots,p-2\}$ and y is a p-adic integer then

$$\begin{aligned}&g_{k+1}(y)\equiv\binom{2(k+1)+2py}{k+1}\pmod{p^2}\\ \Longrightarrow\ &g_k(y)\equiv\binom{2k+2py}{k}\pmod{p^2}.\end{aligned}\tag{2.5}$$

In view of this, we have the second part of Theorem 1.2 by induction.

The proof of Theorem 1.2 is now complete. □

Proof of Corollary 1.5. Observe that

$$\sum_{n=0}^{p-1}a_n=\sum_{k=0}^{p-1}C_k\sum_{n=k}^{p-1}\binom{n}{k}^2=\sum_{k=0}^{p-1}C_k\sum_{j=0}^{p-1-k}\binom{k+j}{k}^2.$$

If $0\leqslant k\leqslant p-1$ and $p-k\leqslant j\leqslant p-1$, then

$$\binom{k+j}{k}=\frac{(k+j)!}{k!j!}\equiv 0\pmod{p}.$$

Therefore

$$\sum_{n=0}^{p-1}a_n\equiv\sum_{k=0}^{p-1}C_k\sum_{j=0}^{p-1}\binom{k+j}{k}^2=\sum_{k=0}^{p-1}\sum_{j=0}^{p-1}\binom{x_k}{j}^2,$$

where $x_k = -k-1 \equiv p-1-k \pmod p$. Applying Theorem 1.2(ii) we get

$$\sum_{n=0}^{p-1} a_n \equiv \sum_{k=0}^{p-1} C_k \binom{2x_k}{p-1-k} = \sum_{k=0}^{p-1} (-1)^k \binom{p+k}{2k+1} C_k \pmod{p^2}.$$

So it suffices to show that for any $n \in \mathbb{Z}^+$ we have

$$\sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} C_k = 1. \tag{2.6}$$

We prove (2.6) by induction. Clearly, (2.6) holds for $n=1$. Let n be any positive integer. By the Chu-Vandermonde identity

$$\sum_{k=0}^{n} \binom{x}{k}\binom{y}{n-k} = \binom{x+y}{n}$$

(see, e.g., [GKP, p. 169]), we have

$$\sum_{k=0}^{n-1} \binom{n+1}{k+1}\binom{n+k}{k}(-1)^k = \sum_{k=0}^{n} \binom{n+1}{n-k}\binom{-n-1}{k} = -\binom{-n-1}{n}.$$

Thus

$$\begin{aligned}
&\sum_{k=0}^{n} (-1)^k \binom{n+1+k}{2k+1} C_k - \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k+1} C_k \\
=&(-1)^n C_n + \sum_{k=0}^{n-1} (-1)^k \binom{n+k}{2k} C_k \\
=&(-1)^n C_n + \frac{1}{n+1} \sum_{k=0}^{n-1} \binom{n+1}{k+1}\binom{n+k}{k}(-1)^k \\
=&(-1)^n C_n - \frac{1}{n+1}\binom{-n-1}{n} = 0.
\end{aligned}$$

This concludes the induction step. We are done. $\square$

Now we can apply Theorem 1.2 to deduce the first part of Theorem 1.1.

Proof of Theorem 1.1(i). Let $\varepsilon\in\{\pm1\}$. Then

$$\begin{aligned}\sum_{m=0}^{p-1}\varepsilon^m A_m(x)=&\sum_{m=0}^{p-1}\varepsilon^m\sum_{k=0}^{m}\binom{m+k}{2k}^2\binom{2k}{k}^2x^k\\ =&\sum_{k=0}^{p-1}\binom{2k}{k}^2x^k\sum_{m=k}^{p-1}\varepsilon^m\binom{m+k}{2k}^2\\ =&\sum_{k=0}^{p-1}\binom{2k}{k}^2x^k\sum_{r=0}^{p-1-k}\varepsilon^{k+r}\binom{2k+r}{r}^2\\ =&\sum_{k=0}^{p-1}\binom{2k}{k}^2\varepsilon^kx^k\sum_{r=0}^{p-1-k}\varepsilon^r\binom{p-1-2k-p}{r}^2\end{aligned}$$

Set $n=(p-1)/2$. Clearly $\binom{2k}{k}\equiv0\pmod p$ for $k=n+1,\ldots,p-1$, and

$$\binom{p-1-2k-p}{r}\equiv\binom{p-1-2k}{r}=0\pmod p$$

if $0\leqslant k\leqslant n$ and $p-1-2k<r\leqslant p-1$. Therefore

$$\sum_{m=0}^{p-1}\varepsilon^m A_m(x)\equiv\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^k\sum_{r=0}^{p-1}\varepsilon^r\binom{2(n-k)-p}{r}^2\pmod{p^2}.$$

Similarly,

$$\begin{aligned}\sum_{m=0}^{p-1}\varepsilon^m W_m(\varepsilon x)=&\sum_{m=0}^{p-1}\varepsilon^m\sum_{k=0}^{\lfloor m/2\rfloor}\binom{m}{2k}^2\binom{2k}{k}^2(\varepsilon x)^k\\ =&\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^k\sum_{m=2k}^{p-1}\varepsilon^m\binom{m}{2k}^2\\ =&\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^k\sum_{r=0}^{p-1-2k}\varepsilon^{2k+r}\binom{2k+r}{r}^2\\ \equiv&\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^k\sum_{r=0}^{p-1}\varepsilon^r\binom{2(n-k)-p}{r}^2\pmod{p^2}.\end{aligned}$$

So we have

$$\sum_{m=0}^{p-1}\varepsilon^m A_m(x)\equiv\sum_{m=0}^{p-1}\varepsilon^m W_m(\varepsilon x)\equiv\sum_{k=0}^{n}\binom{2k}{k}^2\varepsilon^kx^kS_k(\varepsilon)\pmod{p^2},\tag{2.7}$$

where

$$S_k(\varepsilon):=\sum_{r=0}^{p-1}\varepsilon^r\binom{2(n-k)-p}{r}^2.$$

Applying Theorem 1.2(i) we get

$$\begin{aligned}S_k(-1)\equiv&(-1)^{n-k}\binom{2(n-k)-p}{n-k}=(-1)^{n-k}\binom{-2k-1}{n-k}\\=&\binom{n+k}{n-k}=\binom{n+k}{2k}\equiv\frac{\binom{2k}{k}}{(-16)^k}\pmod{p^2}.\end{aligned}$$

(The last congruence can be easily deduced, see. e.g., [S2, Lemma 2.2].) Combining this with (2.7) in the case $\varepsilon=-1$ we immediately obtain (1.3).

In view of Theorem 1.2(ii),

$$S_k(1)\equiv\binom{4(n-k)-2p}{2(n-k)}\pmod{p^2}.$$

Recall that $\binom{n+k}{n-k}(-16)^k\equiv\binom{2k}{k}\pmod{p^2}$. So, in view of (2.7) with $\varepsilon=1$, we have

$$\begin{aligned}\sum_{m=0}^{p-1}A_m(x)\equiv\sum_{m=0}^{p-1}W_m(x)\equiv&\sum_{k=0}^{n}\binom{n+k}{n-k}^2(-16)^{2k}x^k\binom{4(n-k)-2p}{2(n-k)}\\=&\sum_{j=0}^{n}\binom{n+(n-j)}{j}^2256^{n-j}x^{n-j}\binom{4j-2p}{2j}\\=&16^{p-1}\sum_{k=0}^{n}\frac{\binom{4k-2p}{2k}\binom{2k-p}{k}^2}{256^k}x^{n-k}\pmod{p^2}\end{aligned}$$

If x is a p-adic integer with $x\not\equiv 0\pmod p$, then

$$\begin{aligned}&16^{p-1}\sum_{k=0}^{n}\frac{\binom{4k-2p}{2k}\binom{2k-p}{k}^2}{256^k}x^{n-k}\\\equiv&\left(\frac{x}{p}\right)\sum_{k=0}^{n}\frac{\binom{4k}{2k}\binom{2k}{k}^2}{(256x)^k}\equiv\left(\frac{x}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(256x)^k}\pmod p,\end{aligned}$$

and therefore (1.4) holds. □

Lemma 2.1. *Let $k\in\mathbb{N}$. Then, for any $n\in\mathbb{Z}^+$ we have the identity*

$$\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}^2=\frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2. \tag{2.8}$$

Proof. Obviously (2.8) holds when $n=1$.

Now assume that $n>1$ and (2.8) holds. Then

$$\begin{aligned}&\sum_{m=0}^{n}(2m+1)\binom{m+k}{2k}^2\\=&\frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2+(2n+1)\binom{n+k}{2k}^2\\=&\frac{(n+k+1)^2}{2k+1}\binom{n+k}{2k}^2=\frac{(n+1-k)^2}{2k+1}\binom{(n+1)+k}{2k}^2.\end{aligned}$$

Combining the above, we have proved the desired result by induction. □

Lemma 2.2. *Let $p>3$ be a prime. Then*

$$\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{2k+1}\equiv -pE_{p-3}\pmod{p^2}. \tag{2.9}$$

Proof. Observe that

$$\begin{aligned}\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{2k+1}=&\frac{1}{2}\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\left(\frac{(-1)^k}{2k+1}+\frac{(-1)^{p-1-k}}{(2(p-1-k)+1)}\right)\\=&-p\sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{(-1)^k}{(2k+1)(2k+1-2p)}\\\equiv&-\frac{p}{4}\sum_{k=0}^{p-1}(-1)^k\left(k+\frac{1}{2}\right)^{p-3}\pmod{p^2}.\end{aligned}$$

So we have reduced (2.9) to the following congruence

$$\sum_{k=0}^{p-1}(-1)^k\left(k+\frac{1}{2}\right)^{p-3}\equiv 4E_{p-3}\pmod{p}. \tag{2.10}$$

Recall that the Euler polynomial of degree n is defined by

$$E_n(x)=\sum_{k=0}^{n}\binom{n}{k}\frac{E_k}{2^k}\left(x-\frac{1}{2}\right)^{n-k}.$$

It is well known that

$$E_n(x)+E_n(x+1)=2x^n.$$

Thus

$$\begin{aligned}&2\sum_{k=0}^{p-1}(-1)^k\left(k+\frac12\right)^{p-3}\\ =&\sum_{k=0}^{p-1}\left((-1)^kE_{p-3}\left(k+\frac12\right)-(-1)^{k+1}E_{p-3}\left(k+1+\frac12\right)\right)\\ =&E_{p-3}\left(\frac12\right)-(-1)^pE_{p-3}\left(p+\frac12\right)\\ \equiv&2E_{p-3}\left(\frac12\right)=2\frac{E_{p-3}}{2^{p-3}}\equiv 8E_{p-3}\pmod p\end{aligned}$$

and hence (2.10) follows. We are done. □

For each $m=1,2,3,\ldots$ those rational numbers

$$H_n^{(m)}:=\sum_{0<k\leqslant n}\frac{1}{k^m}\quad (n=0,1,2,\ldots)$$

are called harmonic numbers of order m. We simply write H_n for $H_n^{(1)}$. A well-known theorem of Wolstenholme asserts that $H_{p-1}\equiv 0\pmod{p^2}$ and $H_{p-1}^{(2)}\equiv 0\pmod p$ for any prime $p>3$.

Lemma 2.3. *Let $p>3$ be a prime. Then*

$$\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1}\equiv -\frac74 B_{p-3}\pmod p. \tag{2.11}$$

Proof. Clearly,

$$\sum_{k=1}^{p-1}\frac{1}{k^3}=\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^3}+\frac{1}{(p-k)^3}\right)\equiv 0\pmod p.$$

By [ST, (5.4)], $\sum_{k=1}^{p-1}H_k/k^2\equiv B_{p-3}\pmod p$. Therefore

$$\begin{aligned}\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k}=&\sum_{k=1}^{p-1}\frac1k\sum_{j=1}^{k}\frac{1}{j^2}=\sum_{j=1}^{p-1}\frac{H_{p-1}-H_{j-1}}{j^2}\\ \equiv&-\sum_{k=1}^{p-1}\frac{H_k}{k^2}+\sum_{k=1}^{p-1}\frac{1}{k^3}\equiv -B_{p-3}\pmod p.\end{aligned}$$

On the other hand,

$$\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k}=\sum_{k=1}^{(p-1)/2}\left(\frac{H_k^{(2)}}{k}+\frac{H_{p-k}^{(2)}}{p-k}\right)$$
$$\equiv\sum_{k=1}^{(p-1)/2}\left(\frac{H_k^{(2)}}{k}+\frac{H_{p-1}-H_{k-1}^{(2)}}{-k}\right)\equiv 2\sum_{k=1}^{(p-1)/2}\frac{H_k^{(2)}}{k}-H_{(p-1)/2}^{(3)}\pmod p.$$

It is known (see, e.g., [S1, Corollary 5.2]) that

$$H_{(p-1)/2}^{(3)}=\sum_{k=1}^{(p-1)/2}\frac{1}{k^3}\equiv -2B_{p-3}\pmod p.$$

So we have

$$\sum_{k=1}^{(p-1)/2}\frac{H_k^{(2)}}{k}\equiv\frac12\left(\sum_{k=1}^{p-1}\frac{H_k^{(2)}}{k}+H_{(p-1)/2}^{(3)}\right)\equiv\frac{-B_{p-3}-2B_{p-3}}{2}=-\frac32 B_{p-3}\pmod p.$$

Clearly

$$H_{(p-1)/2}^{(2)}=\frac12\sum_{k=1}^{(p-1)/2}\left(\frac{1}{k^2}+\frac{1}{(p-k)^2}\right)=\frac12 H_{p-1}^{(2)}\equiv 0\pmod p.$$

Observe that

$$\begin{aligned}\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1}\equiv&-\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{p-1-2k}=-\sum_{k=1}^{(p-1)/2}\frac{H_{(p-1)/2-k}^{(2)}}{2k}\\ \equiv&-\frac12\sum_{k=1}^{(p-1)/2}\frac1k\left(H_{(p-1)/2}^{(2)}-\sum_{j=0}^{k-1}\frac{1}{((p-1)/2-j)^2}\right)\\ \equiv&2\sum_{k=1}^{(p-1)/2}\frac1k\sum_{j=0}^{k-1}\frac{1}{(2j+1)^2}\equiv 2\sum_{k=1}^{(p-1)/2}\frac1k\left(H_{2k}^{(2)}-\sum_{j=1}^{k}\frac{1}{(2j)^2}\right)\\ =&4\sum_{k=1}^{(p-1)/2}\frac{H_{2k}^{(2)}}{2k}-\frac12\sum_{k=1}^{(p-1)/2}\frac{H_k^{(2)}}{k}\pmod p\end{aligned}$$

and

$$\begin{aligned}\sum_{k=1}^{(p-1)/2}\frac{H_{2k}^{(2)}}{2k}=&\sum_{\substack{k=1\\ 2\mid k}}^{p-1}\frac1k\sum_{j=1}^{k}\frac{1}{j^2}=\sum_{\substack{k=1\\ 2\mid k}}^{p-1}\frac{1}{k^3}+\sum_{\substack{1\leqslant j<k\leqslant p-1\\ 2\mid k}}\frac{1}{j^2k}\\ \equiv&\frac18 H_{(p-1)/2}^{(3)}-\frac38 B_{p-3}\quad\text{(by Pan [P, (2.4)])}\\ \equiv&\frac18(-2B_{p-3})-\frac38 B_{p-3}=-\frac58 B_{p-3}\pmod p.\end{aligned}$$

So we finally get

$$\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1}\equiv 4\left(-\frac{5}{8}B_{p-3}\right)-\frac{1}{2}\left(-\frac{3}{2}B_{p-3}\right)=-\frac{7}{4}B_{p-3}\pmod p.$$

This concludes the proof of (2.11). □

Proof of Theorem 1.1(ii). (i) Let n be any positive integer. Then

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)A_m(x)&=\sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{m+k}{2k}^2\binom{2k}{k}^2x^k\\&=\sum_{k=0}^{n-1}\binom{2k}{k}^2x^k\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}^2\\&=\sum_{k=0}^{n-1}\binom{2k}{k}^2x^k\frac{(n-k)^2}{2k+1}\binom{n+k}{2k}^2\quad\text{(by (2.8))}\\&=\sum_{k=0}^{n-1}\frac{(n-k)^2}{2k+1}\binom{n}{k}^2\binom{n+k}{k}^2x^k.\end{aligned}$$

Since

$$(n-k)\binom{n}{k}=n\binom{n-1}{k}\qquad\text{for all}\ \ k=0,\dots,n-1,$$

we have

$$\begin{aligned}\frac{1}{n}\sum_{m=0}^{n-1}(2m+1)A_m(x)&=\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{n-k}{2k+1}\binom{n}{k}\binom{n+k}{k}^2x^k\\&=\sum_{k=0}^{n-1}\binom{n-1}{k}\frac{n-k}{2k+1}\binom{n+k}{2k}\binom{2k}{k}\binom{n+k}{k}x^k\\&=\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}\binom{n+k}{2k+1}\binom{2k}{k}x^k.\end{aligned}$$

This proves (1.5).

Now we fix a prime $p>3$. By the above,

$$\sum_{m=0}^{p-1}(2m+1)A_m(x)=\sum_{k=0}^{p-1}\frac{p^2}{2k+1}\binom{p-1}{k}^2\binom{p+k}{k}^2x^k.\tag{2.12}$$

For $k\in\{0,\dots,p-1\}$, clearly

$$\begin{aligned}\binom{p-1}{k}^2\binom{p+k}{k}^2&=\prod_{0<j\leqslant k}\left(\frac{p-j}{j}\cdot\frac{p+j}{j}\right)^2=\prod_{0<j\leqslant k}\left(1-\frac{p^2}{j^2}\right)^2\\&\equiv\prod_{0<j\leqslant k}\left(1-\frac{2p^2}{j^2}\right)\equiv 1-2p^2H_k^{(2)}\pmod{p^4}.\end{aligned}$$

Thus (2.12) implies that

$$\sum_{m=0}^{p-1}(2m+1)A_m(x)=\sum_{k=0}^{p-1}\frac{p^2}{2k+1}\left(1-2p^2H_k^{(2)}\right)x^k\pmod{p^5}.\qquad(2.13)$$

Since $H_{(p-1)/2}^{(2)}\equiv 0\pmod p$, taking $x=-1$ in (2.13) and applying (2.10) we obtain

$$\sum_{m=0}^{p-1}(2m+1)A_m(-1)\equiv\sum_{k=0}^{p-1}\frac{p^2(-1)^k}{2k+1}\equiv\frac{p^2(-1)^{(p-1)/2}}{2(p-1)/2+1}-p^3E_{p-3}\pmod{p^4}$$

and hence (1.7) holds.

Now we prove (1.6). In view of (2.13) with $x=1$, we have

$$\begin{aligned}\sum_{m=0}^{p-1}(2m+1)A_m\equiv&\frac{p^2}{2(p-1)/2+1}\left(1-2p^2H_{(p-1)/2}^{(2)}\right)\\&+p^2\sum_{k=0}^{(p-3)/2}\left(\frac{1-2p^2H_k^{(2)}}{2k+1}+\frac{1-2p^2H_{p-1-k}^{(2)}}{2(p-1-k)+1}\right)\\=&p-2p^3H_{(p-1)/2}^{(2)}+2p^3\sum_{k=0}^{(p-3)/2}\frac{2p+2k+1}{(2k+1)(4p^2-(2k+1)^2)}\\&-2p^4\sum_{k=0}^{(p-3)/2}\left(\frac{H_k^{(2)}}{2k+1}+\frac{H_{p-1}^{(2)}-\sum_{0<j\leqslant k}(p-j)^{-2}}{2p-(2k+1)}\right)\\\equiv&p-2p^3H_{(p-1)/2}^{(2)}-4p^4\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^3}\\&-2p^3\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^2}-4p^4\sum_{k=0}^{(p-3)/2}\frac{H_k^{(2)}}{2k+1}\pmod{p^5}.\end{aligned}$$

By [S1, Corollaries 5.1 and 5.2],

$$H_{p-1}^{(2)}\equiv\frac{2}{3}pB_{p-3}\pmod{p^2},\quad H_{(p-1)/2}^{(2)}\equiv\frac{7}{3}pB_{p-3}\pmod{p^2},$$

$$\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^2}=H_{p-1}^{(2)}-\frac{H_{(p-1)/2}^{(2)}}{4}\equiv\frac{p}{12}B_{p-3}\pmod{p^2},$$

and

$$\sum_{k=0}^{(p-3)/2}\frac{1}{(2k+1)^3}=H_{p-1}^{(3)}-\frac{H_{(p-1)/2}^{(3)}}{8}\equiv 0-\frac{-2B_{p-3}}{8}=\frac{B_{p-3}}{4}\pmod p.$$

Combining these with Lemma 2.3, we finally obtain

$$\begin{aligned}\sum_{k=0}^{p-1}(2k+1)A_k \equiv& p-2p^3\frac{7}{3}pB_{p-3}-4p^4\frac{B_{p-3}}{4}-2p^3\frac{p}{12}B_{p-3}-4p^4\left(-\frac{7}{4}B_{p-3}\right)\\ =& p+\frac{7}{6}p^4B_{p-3} \pmod{p^5}\end{aligned}$$

So far we have proved the second part of Theorem 1.1. □

Part (iii) of Theorem 1.1 is easy.

Proof of Theorem 1.1(iii). As $A_0=1$ and $A_1=3$, the desired congruence with $p=2$ holds trivially.

Below we assume that $p>2$. If $k\in\{0,1\ldots,p-1\}$, then

$$\begin{aligned}A_{p-1-k}=&\sum_{j=0}^{p-1}\binom{(p-1-k)+j}{2j}^2\binom{2j}{j}^2\\ \equiv&\sum_{j=0}^{p-1}\binom{j-k-1}{2j}^2\binom{2j}{j}^2=\sum_{j=0}^{k}\binom{j+k}{2j}^2\binom{2j}{j}^2=A_k \pmod{p}\end{aligned}$$

Thus

$$\begin{aligned}\sum_{k=0}^{p-1}(2k+1)\varepsilon^kA_k^m=&\sum_{k=0}^{p-1}(2(p-1-k)+1)\varepsilon^{p-1-k}A_{p-1-k}^m\\ \equiv&-\sum_{k=0}^{p-1}(2k+1)\varepsilon^kA_k^m \pmod{p}\end{aligned}$$

and hence we have the desired congruence. □

3. Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. Define

$$w_k(y):=\sum_{r=0}^{p-1}(-1)^r\binom{py-2k}{r}^3 \quad \text{for } k\in\mathbb{N}. \tag{3.1}$$

We want to show that $w_k(y)\equiv 0 \pmod{p^2}$ for any p-adic integer y and $k\in\{1,\ldots,\lfloor(p-1)/3\rfloor\}$.

By the Zeilberger algorithm, for $k=0,1,2,\ldots$ we have

$$\begin{aligned}&(py-2k)^2w_k(y)+3(3py-2(3k+1))(3py-2(3k+2))w_{k+1}(y)\\ &\quad=\frac{P(k,p,y)(p(1-y)+2k-1)^3}{(py-2k)^3(py-2k-1)^3}\binom{py-2k}{p-1}^3\end{aligned} \tag{3.2}$$

where $P(k,p,y)$ is a suitable polynomial in k,p,y with integer coefficients such that $P(0,p,y)\equiv 0 \pmod{p^2}$. (Here we omit the explicit expression of $P(k,p,y)$ since it is complicated.) Note also that

$$w_1(0)=\sum_{r=0}^{p-1}(-1)^r\binom{-2}{r}^3=\sum_{r=0}^{p-1}(r+1)^3=\frac{p^2(p+1)^2}{4}\equiv 0 \pmod{p^2}.$$

Fix a p-adic integer y. If $y\neq 0$, then (3.2) with $k=0$ yields

$$\begin{aligned}&3(3py-2)(3py-4)w_1(y)\\ \equiv&\frac{P(0,p,y)(p(1-y)-1)^3}{(py)^3(py-1)^3}\left(\frac{py}{p-1}\binom{p(y-1)+p-1}{p-2}\right)^3\equiv 0 \pmod{p^2}\end{aligned}$$

and hence $w_1(y)\equiv 0 \pmod{p^2}$. If $1<k+1\leqslant\lfloor(p-1)/3\rfloor$, then by (3.2) we have

$$(py-2k)^2w_k(y)+3(3py-2(3k+1))(3py-2(3k+2))w_{k+1}(y)\equiv 0 \pmod{p^3}$$

since

$$\binom{py-2k}{p-1}=\frac{p}{py-2k+1}\binom{py-2k+1}{p}\equiv 0 \pmod{p}.$$

Thus, when $1<k+1\leqslant\lfloor(p-1)/3\rfloor$ we have

$$w_k(y)\equiv 0 \pmod{p^2} \implies w_{k+1}(y)\equiv 0 \pmod{p^2}.$$

So, by induction, $w_k(y)\equiv 0 \pmod{p^2}$ for all $k=1,\ldots,\lfloor(p-1)/3\rfloor$.

In view of the above, we have completed the proof of Theorem 1.3. □

Lemma 3.1. *Let $n\in\mathbb{N}$. Then we have*

$$\sum_{k=0}^{n}\binom{x+k-1}{k}=\binom{x+n}{n}. \tag{3.3}$$

Proof. By the Chu-Vandermonde identity (see, e.g., [GKP, p. 169]),

$$\sum_{k=0}^{n}\binom{-x}{k}\binom{-1}{n-k}=\binom{-x-1}{n}$$

which is equivalent to (3.3). Of course, it is easy to prove (3.3) by induction. □

Proof of Theorem 1.4. (i) Observe that

$$\begin{aligned}\sum_{n=0}^{p-1} D_n &= \sum_{n=0}^{p-1}\sum_{k=0}^{n}\binom{n+k}{2k}\binom{2k}{k} = \sum_{k=0}^{p-1}\binom{2k}{k}\sum_{n=k}^{p-1}\binom{n+k}{2k}\\ &= \sum_{k=0}^{p-1}\binom{2k}{k}\sum_{j=0}^{p-1-k}\binom{j+2k}{j}\\ &= \sum_{k=0}^{p-1}\binom{2k}{k}\binom{2k+1+p-1-k}{p-1-k} \text{ (by Lemma 3.1)}\\ &= \sum_{k=0}^{p-1}\binom{2k}{k}\binom{p+k}{2k+1} = \sum_{k=0}^{p-1}\frac{k+1}{2k+1}\binom{2k+1}{k}\binom{p+k}{2k+1}\end{aligned}$$

and thus

$$\sum_{n=0}^{p-1} D_n = \sum_{k=0}^{p-1}\frac{k+1}{2k+1}\binom{p+k}{k}\binom{p}{k+1} = p + \sum_{k=1}^{p-1}\frac{p}{2k+1}\binom{p-1}{k}\binom{p+k}{k}.$$

For $k = 1, \ldots, p-1$ we clearly have

$$\binom{p-1}{k}\binom{p+k}{k} = (-1)^k\prod_{j=1}^{k}\left(1-\frac{p^2}{j^2}\right) \equiv (-1)^k(1-p^2H_k^{(2)}) \pmod{p^4};$$

in particular,

$$\binom{p-1}{(p-1)/2}\binom{p+(p-1)/2}{(p-1)/2} \equiv (-1)^{(p-1)/2} = \left(\frac{-1}{p}\right) \pmod{p^3}$$

since $H_{(p-1)/2}^{(2)} \equiv 0 \pmod{p}$. Therefore

$$\begin{aligned}\sum_{n=0}^{p-1} D_n &\equiv \sum_{\substack{k=0\\ k\neq(p-1)/2}}^{p-1}\frac{p}{2k+1}(-1)^k + \left(\frac{-1}{p}\right)\\ &\equiv \left(\frac{-1}{p}\right) - p^2E_{p-3} \pmod{p^3} \quad \text{(by (2.8)).}\end{aligned}$$

This proves (1.17).

(ii) Now we prove (1.18) and (1.19).

Let n be any positive integer. Then

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)(-1)^m D_m &= \sum_{m=0}^{n-1}(2m+1)(-1)^m\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}\\ &= \sum_{k=0}^{n-1}\binom{2k}{k}\sum_{m=0}^{n-1}(2m+1)(-1)^m\binom{m+k}{2k}\end{aligned}$$

By induction, we have the identity

$$\sum_{m=0}^{n-1}(2m+1)(-1)^m\binom{m+k}{2k}=(-1)^n(k-n)\binom{n+k}{2k}. \tag{3.4}$$

Thus

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)(-1)^mD_m=&(-1)^{n-1}\sum_{k=0}^{n-1}\binom{2k}{k}(n-k)\binom{n+k}{2k}\\ =&(-1)^{n-1}\sum_{k=0}^{n-1}(n-k)\binom{n}{k}\binom{n+k}{k}\\ =&(-1)^{n-1}n\sum_{k=0}^{n-1}\binom{n-1}{k}\binom{n+k}{k}.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{m=0}^{n-1}(2m+1)D_m=&\sum_{m=0}^{n-1}(2m+1)\sum_{k=0}^{m}\binom{m+k}{2k}\binom{2k}{k}\\ =&\sum_{k=0}^{n-1}\binom{2k}{k}\sum_{m=0}^{n-1}(2m+1)\binom{m+k}{2k}\\ =&n\sum_{k=0}^{n-1}C_k(n-k)\binom{n+k}{2k}=\sum_{k=0}^{n-1}\frac{n^2}{k+1}\binom{n-1}{k}\binom{n+k}{k}.\end{aligned}$$

In view of the above,

$$\begin{aligned}&\frac{1}{p}\sum_{m=0}^{p-1}(2m+1)(-1)^mD_m=\sum_{k=0}^{p-1}\binom{p-1}{k}\binom{p+k}{k}\\ \equiv&\sum_{k=0}^{p-1}(-1)^k-p^2\sum_{k=1}^{p-1}\sum_{0<j\leqslant k}\frac{(-1)^k}{j^2}=1-p^2\sum_{j=1}^{p-1}\frac{1}{j^2}\sum_{k=j}^{p-1}(-1)^k\\ \equiv&1-p^2\sum_{i=1}^{(p-1)/2}\frac{1}{(2i)^2}=1-\frac{p^2}{4}H_{(p-1)/2}^{(2)}\equiv1-\frac{7}{12}p^3B_{p-3}\pmod{p^4}\end{aligned}$$

and hence (1.18) holds. Similarly,

$$\begin{aligned}&\frac{1}{p}\sum_{m=0}^{p-1}(2m+1)D_m=\sum_{k=0}^{p-1}\frac{p}{k+1}\binom{p-1}{k}\binom{p+k}{k}\\ \equiv&\binom{p+(p-1)}{p-1}+p\sum_{k=0}^{p-2}\frac{(-1)^k}{k+1}\left(1-p^2H_k^{(2)}\right)\pmod{p^5}\\ \equiv&\binom{2p-1}{p-1}-p\sum_{k=1}^{p-1}\frac{1+(-1)^k}{k}\equiv1-pH_{(p-1)/2}\pmod{p^3}.\end{aligned}$$

(We have employed Wolstenholme's congruences $\binom{2p-1}{p-1}\equiv 1 \pmod{p^3}$ and $H_{p-1}\equiv 0 \pmod{p^2}$.) To obtain (1.19) it suffices to apply Lehmer's congruence (cf. [L])

$$H_{(p-1)/2}\equiv -2q_p(2)+p\,q_p^2(2) \pmod{p^2}.$$

The proof of Theorem 1.4 is now complete. □

4. Some related conjectures

Our following conjecture was motivated by Theorem 1.1(i).

Conjecture 4.1. *Let $p>3$ be a prime.*
(i) *If $p\equiv 1 \pmod 3$, then*

$$\sum_{k=0}^{p-1}(-1)^kA_k\equiv\sum_{k=0}^{p-1}\frac{\binom{2k}{k}^3}{16^k} \pmod{p^3}. \tag{4.1}$$

If $p\equiv 1,3 \pmod 8$, then

$$\sum_{k=0}^{p-1}A_k\equiv\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{256^k} \pmod{p^3}. \tag{4.2}$$

(ii) *If x belongs to the set*

$$\{1,-4,9,-48,81,-324,2401,9801,-25920,-777924,96059601\}\bigcup\left\{\frac{81}{256},-\frac{9}{16},\frac{81}{32},-\frac{3969}{256}\right\}$$

and $x\not\equiv 0 \pmod p$, then we must have

$$\sum_{k=0}^{p-1}A_k(x)\equiv\left(\frac{x}{p}\right)\sum_{k=0}^{p-1}\frac{\binom{4k}{k,k,k,k}}{(256x)^k} \pmod{p^2}.$$

Remark 4.1. For those

$$x=-4,9,-48,81,-324,2401,9801,-25920,-777924,96059601,\frac{81}{256},$$

the author (cf. [Su2]) had conjectures on $\sum_{k=0}^{p-1}\binom{4k}{k,k,k,k}/(256x)^k \bmod p^2$. Motivated by this, Z. H. Sun [S2] guessed $\sum_{k=0}^{p-1}\binom{4k}{k,k,k,k}/(256x)^k \bmod p^2$ for $x=-9/16,\ 81/32,\ -3969/256$ in a similar way.

Inspired by parts (ii) and (iii) of Theorem 1.1, we raise the following conjecture.

Conjecture 4.2. *For any $\varepsilon \in \{\pm 1\}$, $m, n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$, we have*

$$\sum_{k=0}^{n-1}(2k+1)\varepsilon^k A_k(x)^m \equiv 0 \pmod{n}. \tag{4.3}$$

If $p > 3$ is a prime, then

$$\sum_{k=0}^{p-1}(2k+1)A_k \equiv p - \frac{7}{2}p^2 H_{p-1} \pmod{p^6} \tag{4.4}$$

and

$$\sum_{k=0}^{p-1}(2k+1)A_k(-3) \equiv p\left(\frac{p}{3}\right) \pmod{p^3}. \tag{4.5}$$

Remark 4.2. After reading an initial version of this paper, Guo and Zeng [GZ] proved the author's following conjectural results:

(a) For any $n \in \mathbb{Z}^+$ and $x \in \mathbb{Z}$ we have

$$\sum_{k=0}^{n-1}(2k+1)(-1)^k A_k(x) \equiv 0 \pmod{n}.$$

If p is an odd prime and x is an integer, then

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k(x) \equiv p\left(\frac{1-4x}{p}\right) \pmod{p^2}.$$

(b) For any prime $p > 3$ we have

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k \equiv p\left(\frac{p}{3}\right) \pmod{p^3}$$

and

$$\sum_{k=0}^{p-1}(2k+1)(-1)^k A_k(-2) \equiv p - \frac{4}{3}p^2 q_p(2) \pmod{p^3}.$$

Recall that for a prime p and a rational number x, the *p-adic valuation* of x is given by

$$\nu_p(x) = \sup\{a \in \mathbb{Z} : \text{ the denominator of } p^{-a}x \text{ is not divisible by } p\}.$$

Just like the Apéry polynomial $A_n(x) = \sum_{k=0}^n \binom{n}{k}^2\binom{n+k}{k}^2 x^k$ we define

$$D_n(x) = \sum_{k=0}^n \binom{n}{k}\binom{n+k}{k} x^k.$$

Actually $D_n((x-1)/2)$ coincides with the Legendre polynomial $P_n(x)$ of degree n.

Our following conjecture involves p-adic valuations.

Conjecture 4.3. (i) *For any* $n\in\mathbb{Z}$ *the numbers*

$$s(n)=\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k\left(\frac14\right)$$

and

$$t(n)=\frac{1}{n^2}\sum_{k=0}^{n-1}(2k+1)(-1)^kD_k\left(-\frac14\right)^3$$

are rational numbers with denominators $2^{2\nu_2(n!)}$ *and* $2^{3(n-1+\nu_2(n!))-\nu_2(n)}$ *respectively. Moreover, the numerators of* $s(1),s(3),s(5),\ldots$ *are congruent to* 1 *modulo* 12 *and the numerators of* $s(2),s(4),s(6),\ldots$ *are congruent to* 7 *modulo* 12*. If* p *is an odd prime and* $a\in\mathbb{Z}^+$*, then*

$$s(p^a)\equiv t(p^a)\equiv 1 \pmod p.$$

For $p=3$ *and* $a\in\mathbb{Z}^+$ *we have*

$$s(3^a)\equiv 4 \pmod{3^2}\quad and\quad t(3^a)\equiv -8 \pmod{3^5}.$$

(ii) *Let* p *be a prime. For any positive integer* n *and* p*-adic integer* x*, we have*

$$\nu_p\left(\frac1n\sum_{k=0}^{n-1}(2k+1)(-1)^kA_k(x)\right)\geqslant\min\{\nu_p(n),\,\nu_p(4x-1)\}\tag{4.6}$$

and

$$\nu_p\left(\frac1n\sum_{k=0}^{n-1}(2k+1)(-1)^kD_k(x)^3\right)\geqslant\min\{\nu_p(n),\,\nu_p(4x+1)\}.\tag{4.7}$$

Motivated by Theorem 1.3, we pose the following conjecture.

Conjecture 4.4. *Let* p *be an odd prime and let* $n\geqslant 2$ *be an integer. Suppose that* x *is a* p*-adic integer with* $x\equiv -2k \pmod p$ *for some* $k\in\{1,\ldots,\lfloor (p+1)/(2n+1)\rfloor\}$*. Then we have*

$$\sum_{r=0}^{p-1}(-1)^r\binom{x}{r}^{2n+1}\equiv 0\pmod{p^2}.\tag{4.8}$$

References

[A] S. Ahlgren, *Gaussian hypergeometric series and combinatorial congruences*, in: Symbolic computation, number theory, special functions, physics and combinatorics (Gainesville, FI, 1999), pp. 1-12, Dev. Math., Vol. 4, Kluwer, Dordrecht, 2001.

[AO] S. Ahlgren and K. Ono, *A Gaussian hypergeometric series evaluation and Apéry number congruences*, J. Reine Angew. Math. **518** (2000), 187–212.

[Ap] R. Apéry, *Irrationalité de $\zeta(2)$ et $\zeta(3)$. Journees arithmétiques de Luminy*, Astérisque **61** (1979), 11–13.

[B] F. Beukers, *Another congruence for the Apéry numbers*, J. Number Theory **25** (1987), 201–210.

[BEW] B. C. Berndt, R. J. Evans and K. S. Williams, *Gauss and Jacobi Sums*, John Wiley & Sons, 1998.

[CHV] J.S. Caughman, C.R. Haithcock and J.J.P. Veerman, *A note on lattice chains and Delannoy numbers*, Discrete Math. **308** (2008), 2623–2628.

[CDE] S. Chowla, B. Dwork and R. J. Evans, *On the mod p^2 determination of $\binom{(p-1)/2}{(p-1)/4}$*, J. Number Theory **24** (1986), 188–196.

[G] H. W. Gould, *Combinatorial Identities*, Morgantown Printing and Binding Co., 1972.

[GZ] V. J. W. Guo and J. Zeng, *Proof of some conjectures of Z.-W. Sun on congruences for Apéry polynomials*, preprint, `http://arxiv.org/abs/1101.0983`.

[I] T. Ishikawa, *Super congruence for the Apéry numbers*, Nagoya Math. J. **118** (1990), 195–202.

[L] E. Lehmer, *On congruences involving Bernoulli numbers and the quotients of Fermat and Wilson*, Ann. of Math. **39** (1938), 350–360.

[M03] E. Mortenson, *A supercongruence conjecture of Rodriguez-Villegas for a certain truncated hypergeometric function*, J. Number Theory **99** (2003), 139–147.

[M05] E. Mortenson, *Supercongruences for truncated ${}_{n+1}F_n$ hypergeometric series with applications to certain weight three newforms*, Proc. Amer. Math. Soc. **133** (2005), 321–330.

[O] K. Ono, *Web of Modularity: Arithmetic of the Coefficients of Modular Forms and q-series*, Amer. Math. Soc., Providence, R.I., 2003.

[P] H. Pan, *On a generalization of Carlitz's congruence*, Int. J. Mod. Math. **4** (2009), 87–93.

[Po] A. van der Poorten, *A proof that Euler missed... Apéry's proof of the irrationality of $\zeta(3)$*, Math. Intelligencer **1** (1978/79), 195–203.

[RV] F. Rodriguez-Villegas, *Hypergeometric families of Calabi-Yau manifolds*, in: Calabi-Yau Varieties and Mirror Symmetry (Toronto, ON, 2001), pp. 223-231, Fields Inst. Commun., **38**, Amer. Math. Soc., Providence, RI, 2003.

[S] N. J. A. Sloane, Sequence A001850 in OEIS (On-Line Encyclopedia of Integer Sequences), `http://oeis.org/A001850`.

[S1] Z. H. Sun, *Congruences concerning Bernoulli numbers and Bernoulli polynomials*, Discrete Appl. Math. **105** (2000), 193–223.

[S2] Z. H. Sun, *Congruences concerning Legendre polynomials*, Proc. Amer. Math. Soc. **139** (2011), 1915–1929.

[S3] Z. H. Sun, *Congruences concerning Legendre polynomials II*, preprint, 2010. `http://arxiv.org/abs/1012.3898`.

[Su1] Z. W. Sun, *On congruences related to central binomial coefficients*, J. Number Theory **131** (1011), in press.

[Su2] Z. W. Sun, *Super congruences and Euler numbers*, Sci. China Math., to appear. `http://arxiv.org/abs/1001.4453`.

[Su3] Z. W. Sun, *On sums involving products of three binomial coefficients*, preprint, arXiv:1012.3141. `http://arxiv.org/abs/1012.3141`.

[ST] Z. W. Sun and R. Tauraso, *New congruences for central binomial coefficients*, Adv. in Appl. Math. **45** (2010), 125–148.