ON THE SEMIGROUP OF ORDER-DECREASING PARTIAL ISOMETRIES OF A FINITE CHAIN

R. Kehinde, S. O. Makanjuola and A. Umar

Abstract

Let $\mathcal{I}_n$ be the symmetric inverse semigroup on $X_n = \{1, 2, \cdots, n\}$ and let $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ be its subsemigroups of order-decreasing partial isometries and of order-preserving order-decreasing partial isometries of X_n, respectively. In this paper we investigate the cycle structure of order-decreasing partial isometry and characterize the Green's relations on $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$. We show that $\mathcal{ODDP}_n$ is a $0-E-unitary$ ample semigroup. We also investigate the cardinalities of some equivalences on $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ which lead naturally to obtaining the order of the semigroups.[1] [2]

MSC2010: 20M18, 20M20, 05A10, 05A15.

1 Introduction and Preliminaries

Let $X_n = \{1, 2, \cdots, n\}$ and $\mathcal{I}_n$ be the partial one-to-one transformation semigroup on X_n under composition of mappings. Then $\mathcal{I}_n$ is an *inverse* semigroup (that is, for all $\alpha \in \mathcal{I}_n$ there exists a unique $\alpha' \in \mathcal{I}_n$ such that $\alpha = \alpha\alpha'\alpha$ and $\alpha' = \alpha'\alpha\alpha'$). The importance of $\mathcal{I}_n$ (more commonly known as the symmetric inverse semigroup or monoid) to inverse semigroup theory may be likened to that of the symmetric group $\mathcal{S}_n$ to group theory. Every finite inverse semigroup S is embeddable in $\mathcal{I}_n$, the analogue of Cayley's theorem for finite groups, and to the regular representation of finite semigroups. Thus, just as the study of symmetric, alternating and dihedral groups has made a significant contribution to group theory, so has the study of various subsemigroups of $\mathcal{I}_n$, see for example [1, 4, 5, 8, 9, 19].
A transformation $\alpha \in \mathcal{I}_n$ is said to be a *partial isometry* if (for all $x, y \in Dom\,\alpha$) $\mid x - y \mid = \mid x\alpha - y\alpha \mid$; *order-preserving (order-reversing)* if (for all $x, y \in\ Dom\,\alpha$) $x \leq y \implies x\alpha \leq y\alpha$ ($x\alpha \geq y\alpha$); and, is said to be *order-decreasing* if (for all $x \in Dom\,\alpha$) $x\alpha \leq x$. Semigroups of partial isometries on more restrictive but richer mathematical structures have been studied [2, 21]. Recently, the authors in [12] studied the semigroup of partial isometries of a finite chain, $\mathcal{DP}_n$ and its subsemigroup of order-preserving partial isometries $\mathcal{ODP}_n$. Ealier, one of the authors studied the semigroup of partial one-to-one order-decreasing(order-increasing) transformations of

[1] *Key Words*: partial one-one transformation, partial isometries, height, right (left) waist, right (left) shoulder and fix of a transformation, idempotents and nilpotents.

[2] This work was carried out when the first named author was visiting Sultan Qaboos University for a 3-month research visit in Fall 2010.

a finite chain, $\mathcal{I}_n^-$ [19]. This paper investigates the algebraic and combinatorial properties of $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$, the semigroups of order-decreasing partial isometries and of order-preserving order-decreasing partial isometries of an $n-$chain, respectively.

In this section we introduce basic terminologies and some preliminary results concerning the cycle structure of a partial order-decreasing isometry of X_n. In the next section, (Section 2) we characterize the classical Green's relations and their starred analogues, where we show that $\mathcal{ODDP}_n$ is a (nonregular) 0-E-unitary ample semigroup. We also show that certain Rees factor semigroups of $\mathcal{ODDP}_n$ are 0-E-unitary and categorical ample semigroups. In Section 3 we obtain the cardinalities of two equivalences defined on $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$. These equivalences lead to formulae for the order of $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ as well as new triangles of numbers not yet recorded in [17].

For standard concepts in semigroup and symmetric inverse semigroup theory, see for example [11, 16, 14]. In particular E(S) denotes the set of idempotents of S. Let

$$\mathcal{DDP}_n = \{\alpha \in \mathcal{DP}_n : (\forall\ x \in Dom\ \alpha)\ x\alpha \leq x\}. \tag{1}$$

be the subsemigroup of $\mathcal{I}_n$ consisting of all order-decreasing partial isometries of X_n. Also let

$$\mathcal{ODDP}_n = \{\alpha \in \mathcal{DDP}_n : (\forall\ x, y \in Dom\ \alpha)\ x \leq y \Longrightarrow x\alpha \leq y\alpha\} \tag{2}$$

be the subsemigroup of $\mathcal{DDP}_n$ consisting of all order-preserving order-decreasing partial isometries of X_n. Then we have the following result.

Lemma 1.1 *$\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ are subsemigroups of $\mathcal{I}_n$.*

Remark 1.2 *$\mathcal{DDP}_n = \mathcal{DP}_n \cap \mathcal{I}_n^-$ and $\mathcal{ODDP}_n = \mathcal{ODP}_n \cap \mathcal{I}_n^-$, where $\mathcal{I}_n^-$ is a semigroup of partial one-to-one order-decreasing transformations of X_n.*

As in [12], we prove a sequence of lemmas that help us understand the cycle structure of order-decreasing partial isometries. These lemmas also seem to be useful in investigating the combinatorial questions in Section 3. First, let α be in $\mathcal{I}_n$. Then the *height* of α is $h(\alpha) = \mid Im\, \alpha \mid$, the *right [left] waist* of α is $w^+(\alpha) = max(Im\, \alpha)\, [w^-(\alpha) = min(Im\, \alpha)]$, the *right [left] shoulder* of α is $\varpi^+(\alpha) = max(Dom\, \alpha)\ \ [\varpi^-(\alpha) = min(Dom\, \alpha)]$, and *fix* of α is denoted by $f(\alpha)$, and defined by $f(\alpha) = |F(\alpha)|$, where

$$F(\alpha) = \{x \in X_n : x\alpha = x\}.$$

Lemma 1.3 *[12, Lemma 1.2] Let $\alpha \in \mathcal{DP}_n$ be such that $h(\alpha) = p$. Then $f(\alpha) = 0$ or 1 or p.*

Corollary 1.4 *[12, Corollary 1.3] Let* $\alpha \in \mathcal{DP}_n$. *If* $f(\alpha) = p > 1$ *then* $f(\alpha) = h(\alpha)$. *Equivalently, if* $f(\alpha) > 1$ *then* α *is an idempotent.*

Lemma 1.5 *Let* $\alpha \in \mathcal{DDP}_n$. *If* $i \in F(\alpha)$ $(1 \leq i \leq n)$ *then for all* $x \in Dom\ \alpha$, *such that* $x < i$ *we have* $x\alpha = x$.

Proof. Note that for all $x \in Dom\,\alpha$ we have $x\alpha \leq x < i$ and so $i - x = \mid i\alpha - x\alpha \mid = \mid i - x\alpha \mid = i - x\alpha \Longrightarrow x = x\alpha$. □

Corollary 1.6 *Let* $\alpha \in \mathcal{DDP}_n$. *If* $F(\alpha) = \{i\}$ *then* $Dom\,\alpha \subseteq \{i, i+1, \cdots, n\}$.

Lemma 1.7 *[12, Lemma 1.4] Let* $\alpha \in \mathcal{DP}_n$. *If* $1 \in F(\alpha)$ *or* $n \in F(\alpha)$ *then for all* $x \in Dom\alpha$, *we have* $x\alpha = x$. *Equivalently, if* $1 \in F(\alpha)$ *or* $n \in F(\alpha)$ *then* α *is a partial identity.*

Lemma 1.8 *[12, Lemma 1.5] Let* $\alpha \in \mathcal{ODP}_n$ *and* $n \in Dom\,\alpha \cap Im\,\alpha$. *Then* $n\alpha = n$.

Lemma 1.9 *[12, Lemma 1.6] Let* $\alpha \in \mathcal{ODP}_n$ *and* $f(\alpha) \geq 1$. *Then* α *is an idempotent.*

Lemma 1.10 *Let* $\alpha \in \mathcal{ODDP}_n$. *Then* $x - x\alpha = y - y\alpha$ *for all* $x, y \in Dom\,\alpha$.

Proof. let $x, y \in Dom\,\alpha$ be such that $x > y$. Then by the order-preserving and isometry properties we see that $\mid x - y \mid = \mid x\alpha - y\alpha \mid \Longrightarrow x - y = x\alpha - y\alpha \Longrightarrow x - x\alpha = y - y\alpha$. The case $x < y$ is similar. □

2 Green's relations and their starred analogues

For the definitions of Green's relations we refer the reader to Howie [**?**, Chapter 2]. First we have

Theorem 2.1 *Let* $\mathcal{DDP}_n$ *and* $\mathcal{ODDP}_n$ *be as defined in (1) and (2) respectively. Then* $\mathcal{DDP}_n$ *and* $\mathcal{ODDP}_n$ *are* $\mathcal{J}$*-trivial.*

Proof. It follows from [19, Lemma 2.2] and Remark 1.2. □

Now since $\mathcal{ODDP}_n$ contains some nonidempotent elements:

$$\begin{pmatrix} x \\ y \end{pmatrix} (x > y)$$

it follows immediately that

Corollary 2.2 *For $n > 1$, $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ are non-regular semigroups.*

On the semigroup S the relation $\mathcal{L}^*(\mathcal{R}^*)$ is defined by the rule that $(a,b) \in \mathcal{L}^*(\mathcal{R}^*)$ if and only if the elements a, b are related by the Green's relation $\mathcal{L}(\mathcal{R})$ in some oversemigroup of S. The join of the equivalences $\mathcal{L}^*$ and $\mathcal{R}^*$ is denoted by $\mathcal{D}^*$ and their intersection by $\mathcal{H}^*$. For the definition of the starred analogue of the Green's relation $\mathcal{J}$, see [7] or [19].

A semigroup S in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent is called *abundant* [7].

By [3, Lemma1.6] and [**?**, Proposition 2.4.2 & Ex. 5.11.2] we deduce the following lemma.

Lemma 2.3 *Let $\alpha, \beta \in \mathcal{DDP}_n$. Then*

(1) $\alpha \leq_{\mathcal{R}^*} \beta$ if and only if $Dom\,\alpha \subseteq Dom\,\beta$;

(2) $\alpha \leq_{\mathcal{L}^*} \beta$ if and only if $Im\,\alpha \subseteq Im\,\beta$;

(3) $\alpha \leq_{\mathcal{H}^*} \beta$ if and only if $Dom\,\alpha \subseteq Dom\,\beta$ and $Im\,\alpha \subseteq Im\,\beta$.

Proof. It is enough to observe that $\mathcal{ODDP}_n$ and $\mathcal{DDP}_n$ are full subsemigroups of $\mathcal{I}_n$ in the sense that $E(\mathcal{ODDP}_n)$=$E(\mathcal{DDP}_n)$=$E(\mathcal{I}_n)$. □

An abundant semigroup S in which $E(S)$ is a semilattice is called *adequate* [6]. Of course inverse semigroups are adequate since in this case $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$.

As in [6], for an element a of an adequate semigroup S, the (unique) idempotent in the $\mathcal{L}^*$-class($\mathcal{R}^*$-class) containing a will be denoted by $a^*(a^+)$. An adequate semigroup S is said to be *ample* if ea=$a(ea)^*$ and ae=$(ae)^+a$ for all elements a in S and all idempotents e in S. Ample semigroups were known as *typeA* semigroups.

Theorem 2.4 *Let $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ be as defined in (1) and (2) respectively. Then $\mathcal{DDP}_n$ and $\mathcal{ODDP}_n$ are non-regular ample semigroups.*

Proof. The proofs are similar to that of [19, theorem 2.6]. □

Theorem 2.5 *Let $S = \mathcal{ODDP}_n$ be as defined in (2). Then $\alpha \leq_{\mathcal{D}^*} \beta$ if and only if there exists an order-preserving isometry $\theta : Dom\,\alpha \to Im\,\beta$.*

Let $E' = E \setminus 0$. A semigroup S is said to be $0-E-$*unitary* if $(\forall e \in E')(\forall s \in S)$ $es \in E' \Longrightarrow s \in E'$. The structure theorem for 0-E-unitary inverse semigroup was given by Lawson [15], see also Szendrei [18] and Gomes and Howie [10].

Theorem 2.6 *$\mathcal{ODDP}_n$ is a $0-E-unitary$ ample subsemigroup of $\mathcal{I}_n$.*

Proof. It follows from [12, Theorem 2.4]. □

Remark 2.7 *Note that $\mathcal{DDP}_n$ is not 0-E-unitary:*

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \in E(\mathcal{DDP}_n) \text{ but } \begin{pmatrix} 2 & 3 \\ 2 & 1 \end{pmatrix} \notin E(\mathcal{DDP}_n).$$

For natural numbers n, p with $n \geq p \geq 0$, let

$$L(n,p) = \{\alpha \in \mathcal{ODDP}_n : h(\alpha) \leq p\} \tag{3}$$

be a two-sided ideal of $\mathcal{ODDP}_n$, and for $p > 0$, let

$$Q(n,p) = L(n,p)/L(n,p-1) \tag{4}$$

be its Rees quotient semigroup. Then $Q(n,p)$ is a 0-E-unitary semigroup whose nonzero elements may be thought of as the elements of $\mathcal{ODDP}_n$ of height p. The product of two elements of $Q(n,p)$ is 0 whenever their product in $\mathcal{ODDP}_n$ is of height less than p.
A semigroup S is said to be *categorical* [10] if

$$(\forall a, b, c \in S),\ abc = 0 \Longrightarrow ab = 0 \,\text{or}\, bc = 0$$

.

Theorem 2.8 *Let $Q(n,p)$ be as defined in (4). Then $Q(n,p)$ is a $0-E-$ unitary categorical semigroup.*

Proof. It follows from [12, thrm2.6]. □

Remark 2.9 *Note that $\mathcal{ODDP}_n$ is not categorical:*

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = 0$$

but

$$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \neq 0 \text{ and } \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \end{pmatrix} \neq 0.$$

3 Combinatorial results

For a nice survey article concerning combinatorial problems in the symmetric inverse semigroup and some of its subsemigroups we refer the reader to Umar [20].

Now recall the definitions of *height* and *fix* of $\alpha \in \mathcal{I}_n$ from the paragraph after Lemma 1.1. As in Umar [20], for natural numbers $n \geq p \geq m \geq 0$ we define

$$F(n;p) = \mid \{\alpha \in S : h(\alpha) = \mid Im\,\alpha \mid = p\} \mid, \tag{5}$$

$$F(n;m) = \mid \{\alpha \in S : f(\alpha) = m\} \mid \tag{6}$$

where S is any subsemigroup of $\mathcal{I}_n$. Also, let $i = a_i = a$, for all $a \in \{p, m\}$, and $0 \leq i \leq n$.

Lemma 3.1 *Let $S = \mathcal{ODDP}_n$. Then $F(n;p_1) = F(n;1) = \binom{n+1}{2}$ and $F(n;p_n) = F(n;n) = 1$, for all $n \geq 1$.*

Proof. Consider $\alpha = \begin{pmatrix} x \\ x\alpha \end{pmatrix}$, where $x \geq x\alpha$. If $x\alpha = i$ then $x \in \{i, i+1, \cdots, n\}$ and so x has $n-i+1$ degrees of freedom. Hence there are $\sum_{i=1}^{n} (n-i+1) = \frac{n(n+1)}{2} = \binom{n+1}{2}$, order-decreasing partial isometries of height 1. For the second statement, it is not difficult to see that there is exactly one order-decreasing partial isometry of height n: $\begin{pmatrix} 1 & 2 & \ldots & n \\ 1 & 2 & \ldots & n \end{pmatrix}$ (the identity). □

Lemma 3.2 *Let $S = \mathcal{ODDP}_n$. Then $F(n;p) = F(n-1;p-1) + F(n-1;p)$, for all $n \geq p \geq 2$.*

Proof. Let $\alpha \in \mathcal{ODDP}_n$ and $h(\alpha) = p$. Then it is clear that $F(n;p) = \mid A \mid + \mid B \mid$, where $A = \{\alpha \in \mathcal{ODDP}_n : h(\alpha) = p \,\text{and}\, n \notin Dom\,\alpha \cup Im\,\alpha\}$ and $B = \{\alpha \in \mathcal{ODDP}_n : h(\alpha) = p \,\text{and}\, n \in Dom\,\alpha \cup Im\,\alpha\}$. Define a map $\theta : \{\alpha \in \mathcal{ODDP}_{n-1} : h(\alpha) = p\} \to A$ by $(\alpha)\theta = \alpha'$ where $x\alpha' = x\alpha\,(x \in Dom\,\alpha$. This is clearly a bijection since $n \notin Dom\,\alpha \cup Im\,\alpha$. Next, recall the definitions of $\varpi^+(\alpha)$ and $w^+(\alpha)$ from the paragraph after Lemma 1.1. Now, define a map $\Phi : \{\alpha \in \mathcal{ODDP}_{n-1} : h(\alpha) = p-1\} \to B$ by $(\alpha)\Phi = \alpha'$ where

(i) $x\alpha' = x\alpha\,(x \in Dom\,\alpha)$ and $n\alpha' = n$ (if $\varpi^+(\alpha) = w^+(\alpha)$);

(ii) $x\alpha' = x\alpha\,(x \in Dom\,\alpha)$ and $n\alpha' = n - \varpi^+(\alpha) + w^+(\alpha) < n$ (if $\varpi^+(\alpha) > w^+(\alpha)$).

In all cases $h(\alpha') = p$, and case (i) coincides with $n \in Dom\,\alpha' \cap Im\,\alpha'$; and case (ii) coincides with $n \in Dom\,\alpha' \setminus Im\,\alpha'$. Note that $\varpi^+(\alpha) \geq w^+(\alpha)$, by the order-decreasing property. Thus Φ is onto. Moreover, it is not difficult to see that Φ is one-to-one. Hence Φ is a bijection, as required. This establishes the statement of the lemma. □

Proposition 3.3 *Let $S = \mathcal{ODDP}_n$ and $F(n;p)$ be as defined in (2) and (5), respectively. Then $F(n;p) = \binom{n+1}{p+1}$, where $n \geq p \geq 1$.*

Proof. (By Induction).

Basis Step: $F(n;1) = \binom{n+1}{1+1} = \binom{n+1}{2}$ and $F(n;n) = 1$ are true by Lemma 3.1

Inductive Step: Suppose $F(n;p)$ is true for all $n \geq p \geq 1$.

Consider $F(n+1;p) = F(n;p-1) + F(n;p) = \binom{n+1}{p} + \binom{n+1}{p+1}$ $= \binom{n+2}{p+1} = \binom{(n+1)+1}{p+1}$, which is the formula for $F(n+1;p)$. Hence the statement is true for all $n \geq p \geq 1$. □

Theorem 3.4 *Let $\mathcal{ODDP}_n$ be as defined in (2). Then*

$$\mid \mathcal{ODDP}_n \mid = 2^{n+1} - (n+1).$$

Proof. It is enough to observe that $\mid \mathcal{ODDP}_n \mid = \sum_{p=0}^{n} F(n;p)$.

Lemma 3.5 *Let $S = \mathcal{ODDP}_n$. Then $F(n;m) = \binom{n}{m}$, for all $n \geq m \geq 1$.*

Proof. It follows directly from [12, Lemma 3.7] and the fact that all idempotents are necessarily order-decreasing. □

Proposition 3.6 *Let U_n be a subsemigroup of $\mathcal{I}_n^-$ and $F(n;m)$ be as defined in (6). Then $F(n;0) = \mid U_{n-1} \mid$.*

Proof. First, we define a map $\theta : U_{n-1} \longrightarrow \{\alpha \in U_n : f(\alpha) = 0\}$ by $\theta(\alpha) = \alpha'$ where for all $i\,(>1)$ in $Dom\,\alpha$,

$$i\alpha' = (i-1)\alpha.$$

Since $n \notin Dom\,\alpha$ and $i\alpha' = (i-1)\alpha < i$ for all $i > 1$, it follows that $i\alpha'$ has the same degrees of freedom as $(i-1)\alpha$, for all $i > 1$. It is also clear that $f(\alpha') = 0$. Thus θ is a bijection onto $\{\alpha \in U_n : f(\alpha) = 0\}$. □

Remark 3.7 *The triangles of numbers $F(n;p)$ and $F(n;m)$, are as at the time of submitting this paper not in Sloane[17]. However, the sequence $F(n+1;m_0) = \mid \mathcal{ODDP}_n \mid$ is [17, A000325]. For some computed values of $F(n;p)$ and $F(n;m)$ in $\mathcal{ODDP}_n$, see Tables 3.1 and 3.2.*

$n\backslash p$	0	1	2	3	4	5	6	7	$\sum F(n;p) = \mid \mathcal{ODDP}_n \mid$
0	1								1
1	1	1							2
2	1	3	1						5
3	1	6	4	1					12
4	1	10	10	5	1				27
5	1	15	20	15	6	1			58
6	1	21	35	35	21	7	1		121
7	1	28	56	70	56	28	8	1	248

Table 3.1

$n\backslash m$	0	1	2	3	4	5	6	7	$\sum F(n;m) = \mid \mathcal{ODDP}_n \mid$
0	1								1
1	1	1							2
2	2	2	1						5
3	5	3	3	1					12
4	12	4	6	4	1				27
5	27	5	10	10	5	1			58
6	58	6	15	20	15	6	1		121
7	121	7	21	35	35	21	7	1	248

Table 3.2

Lemma 3.8 *[12, Lemma 3.11] Let $\alpha \in \mathcal{DP}_n$. Then α is either order-preserving or order-reversing.*

Next, we prove similar results for $\mathcal{DDP}_n$

Lemma 3.9 *Let $\alpha \in \mathcal{DDP}_n$. For $1 < i < n$, if $F(\alpha) = \{i\}$ then for all $x \in Dom\,\alpha$ we have that $x + x\alpha = 2i$.*

Proof. Let $F(\alpha) = \{i\}$ and suppose $x \in Dom\,\alpha$. Obviously, $i + i\alpha = i + i = 2i$. If $x < i$ then $x\alpha > i$, for otherwise we would have $i - x = \mid i\alpha - x\alpha \mid = \mid i - x\alpha \mid = i - x\alpha \Longrightarrow x = x\alpha$, which is a contradiction. Thus, $i - x = \mid i\alpha - x\alpha \mid = \mid i - x\alpha \mid = \mid x\alpha - i \mid = x\alpha - i \Longrightarrow x + x\alpha = 2i$. The case $x > i$ is similar. □

Lemma 3.10 *Let $S = \mathcal{DDP}_n$. Then $F(n;m) = \binom{n}{m}$, for all $n \geq m \geq 2$.*

Proof. It follows directly from [12, Lemma 3.18] and the fact that all idempotents are necessarily order-decreasing. □

Proposition 3.11 *Let $S = \mathcal{DDP}_n$. Then $F(2n;m_1) = F(2n;1) = 2^{n+1} - 2$ and $F(2n-1;m_1) = F(2n-1;1) = 3.2^{n-1} - 2$, for all $n \geq 1$.*

Proof. Let $F(\alpha) = \{i\}$. Then by Lemma 3.9, for any $x \in Dom\,\alpha$ we have $x + x\alpha = 2i$. Thus, by corollary 1.6, there $2i - 2$ possible elements for $Dom\,\alpha : (x, x\alpha) \in \{(i,i), (i+1, i-1), (i+2, i-2), \cdots (2i-1, 1)\}$. However, (excluding (i,i)) we see that there are $\sum_{j=0} \binom{i-1}{j} = 2^{i-1}$, possible partial isometries with $F(\alpha) = \{i\}$, where $2i-1 \leq n \iff i \leq (n+1)/2$. Moreover, by symmetry we see that $F(\alpha) = \{i\}$ and $F(\alpha) = \{n-i+1\}$ give rise to equal number of decreasing partial isometries. Note that if n is odd the equation $i = n - i + 1$ has one solution. Hence, if $n = 2a - 1$ we have

$$2\sum_{i=1}^{a-1} 2^{i-1} + 2^{a-1} = 2(2^{a-1} - 1) + 2^{a-1} = 3.2^{n-1} - 2$$

decreasing partial isometries with exactly one fixed point; if $n = 2a$ we have

$$2\sum_{i=1}^{a} 2^{i-1} = 2(2^a - 1) = 2^{a+1} - 2$$

decreasing partial isometries with exactly one fixed point. □

Theorem 3.12 *Let $\mathcal{DDP}_n$ be as defined in (1). Then*

$$\mid \mathcal{DDP}_n \mid = 3a_{n-1} - 2a_{n-2} - 2^{\lfloor \frac{n}{2} \rfloor} + n + 1.$$

Proof. It follows from Proposition 3.6, Lemma 3.10, Proposition 3.11 and the fact that $\mid \mathcal{DDP}_n \mid = \sum_{m=0}^{n} F(n;m)$. □

Remark 3.13 *The triangles of numbers $F(n;m)$ and the sequences $\mid \mathcal{DDP}_n \mid = F(n+1;m_0)$, are as at the time of submitting this paper not in Sloane [17]. For some computed values of $F(n;m)$ in $\mathcal{DDP}_n$, see Table 3.3.*

$n \backslash m$	0	1	2	3	4	5	6	7	$\sum F(n;m) = \mid \mathcal{DDP}_n \mid$
0	1								1
1	1	1							2
2	2	2	1						5
3	5	4	3	1					13
4	13	6	6	4	1				30
5	30	10	10	10	5	1			66
6	66	14	15	20	15	6	1		137
7	137	22	21	35	35	21	7	1	279

Table 3.3

Acknowledgements. The first named author would like to thank Bowen University, Iwo and Sultan Qaboos University for their financial support and hospitality, respectively.

References

[1] Borwein, D., Rankin, S. and Renner, L. Enumeration of injective partial transformations. *Discrete Math.* **73** (1989), 291–296.

[2] Bracci, L., and Picasso, L. E. Representations of semigroups of partial isometries. *Bull. Lond. Math. Soc.* **39** (2007), 792–802.

[3] El-Qallali, A., and Fountain, J. B. Idempotent-connected abundant semigroups. *Proc. Roy. Soc. Edinburgh Sect. A*, **91** (1981), 79–90.

[4] Fernandes, V. H. The monoid of all injective orientation-preserving partial transformations on a finite chain. *Comm. Algebra* **32** (2000), 3401–3426.

[5] Fernandes, V. H., Gomes, G. M. S. and Jesus, M. M. The cardinal and idempotent number of various monoids of transformations on a finite chain. Algebra Colloquium. (*To appear*).

[6] Fountain, J. B. Adequate semigroups. *Proc. Edinburgh Math. Soc.* **22** (1979), 113–125.

[7] Fountain, J. B. Abundant semigroups. *Proc. London Math. Soc.* **(3) 44** (1982), 103–129.

[8] Ganyushkin, O. and Mazorchuk, V. *Classical Finite Transformation Semigroups: An Introduction*, Springer, London, 2009.

[9] Garba, G. U. Nilpotents in semigroups of partial one-to-one order-preserving mappings. *Semigroup Forum*, **48** (1994), 37–49.

[10] Gomes, G. M. S and Howie, J. M. A P-theorem for inverse semigroups with zero. *Portugaliae Math.* **53** (1996), 257–278.

[11] Howie, J. M. *Fundamentals of semigroup theory.* Oxford: Clarendon Press, 1995.

[12] Kehinde, R. and Umar, A. On the semigroup of partial isometries of a finite chain. (*Submitted*).

[13] Laradji, A. and Umar, A. On certain finite semigroups of order-decreasing transformations I. *Semigroup Forum*, **69** (2004), 184–200.

[14] Lawson, M. V. *Inverse semigroups: the theory of partial symmetries*, Singapore: World Scientific, 1998.

[15] Lawson, M. V. The structure of 0-E-unitary inverse semigroups I: the monoid case *Proc. Edinb. Math. Soc.*, **42** (1999), 497–520.

[16] Limpscomb, S. *Symmetric Inverse Semigroups*, Mathematical Surveys of The American mathematical Society, no. 46, Providence, R. I., 1996.

[17] Sloane, N. J. A. – *The On-Line Encyclopedia of Integer Sequences*, @http://www.research.att.com/∼njas/sequences/.

[18] Szendrei, M. B. A generalization of McAlister's P-theorem for E-unitary regular semigroups, *Acta Sci Math (Szeged)* **57** (1987), 229–249.

[19] Umar, A. On the semigroups of partial one-to-one order-decreasing finite transformations, *Proc. Roy. Soc. Edinburgh*, **123A** (1993), 355–363.

[20] Umar, A. Some combinatorial problems in the theory of symmetric inverse semigroups, *Algebra and Discrete Math. 9* (2010), 115–126.

[21] Wallen, Lawrence J. Semigroups of partial isometries. *Bull. Amer. Math. Soc.* **75** (1969), 763–764.

R. Kehinde
Department of Mathematics and Statistics
Bowen University
P. M. B. 284,Iwo, Osun State
Nigeria.
E-mail:kennyrot2000@yahoo.com

S.Ø. Makanjuola
Department of Mathematics
University of Ilorin
P. M. B. 1515,Ilorin, Kwara State
Nigeria.
E-mail:somakanjuola@unilorin.edu.ng

A. Umar
Department of Mathematics and Statistics
Sultan Qaboos University
Al-Khod, PC 123 – OMAN
E-mail:aumarh@squ.edu.om