Unbounding Ext

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Abstract. We produce examples in the cohomology of algebraic groups which answer two questions of Parshall and Scott. Specifically, if $G = SL_2$, then we show: (a) $\dim \operatorname{Ext}^2_G(L, L)$ can be arbitrarily large for a simple module L; and (b) the sequence $\max_{L-\text{irred}} \dim H^k(G, L)$ grows exponentially fast with k.

Introduction

Let G be a simply connected, semisimple algebraic group with associated root system Φ defined over an algebraically closed field k of characteristic $p > 0$. Let B be a Borel subgroup of G with maximal torus T defining a set of dominant weights $X^+(T)$, a subset of the weight lattice $X(T)$ of T . Recall that the simple G-modules are denoted by $L(\lambda)$ for $\lambda \in X^+(T)$. In the case $G = SL_2$ we identify $X^+(T) \cong \mathbb{Z}$. Also denote by $X_{e,p}$ the subset of $X^+(T)$ consisting of weights whose p-adic expansion is no longer than e.

In [PS11] the authors find a constant $c := c(\Phi, n, e)$ such that

$$\dim \operatorname{Ext}^n_G(L(\lambda), L(\mu)) \leq c$$

for all simply connected, semisimple algebraic groups with root system Φ (thus independent of the characteristic p of k) and all $\lambda \in X_{e,p}$.

In the case $n = 1$, the authors are able to drop the dependence on e to yield a constant $c := c(\Phi)$ such that

$$\dim \operatorname{Ext}^1_G(L(\lambda), L(\mu)) \leq c$$

for all simply connected, semisimple algebraic groups with root system Φ. In [loc. cit., Remark 7.4(b)] the authors ask if the dependence on the length e of the p-adic expansion of λ can be dropped for $n > 1$.

Let $p > 2$ and let $G = SL_2$. In Theorem 1 we give a sequence of weights λ_r, μ_r for G such that $\dim \operatorname{Ext}^2_G(L(\lambda_r), L(\mu_r)) = r$, answering this question in the negative. This is the subject of §1.

In a further paper, [PSar], the same authors make the following definitions: For an algebraic group G and rational G-module V, put

$$\gamma_m(V) = \max_{L-\text{irred}} \dim \operatorname{Ext}^m_G(V, L)$$
$$\gamma_m(\Phi, e, p) = \max_{\lambda \in X_{e,p}} \gamma_m(L(\lambda))$$
$$\gamma_m(\Phi, e) = \max_p \gamma_m(\Phi, e, p)$$

which are finite by [PS11, 7.1]. They prove

Theorem 0.1 ([PSar, 6.1]). *(i) The sequence $\{\log \gamma_m(\Phi, e)\}$ has polynomial rate of growth at most* 4.

(ii) For any fixed prime p, the sequence $\{\log \gamma_m(\Phi, e, p)\}$ has polynomial rate of growth at most 3.

They then ask if these bounds can be improved to polynomial rates of growth in the case of cohomology. To wit, the following is Question 6.2 in [loc. cit]:

Question 0.2. *Let Φ be a finite root system. Do there exist constants $C = C(\Phi)$ and $f = f(\Phi)$ such that*

$$\dim H^m(G, L) \leq Cm^f$$

for all semisimple, simply connected groups G over an algebraically closed field k (of arbitrary characteristic) having root system Φ and all irreducible rational G-modules L?

Let again $G = SL_2$ and let p be arbitrary. We use the algorithm in [Par07] to show that the sequence $\max_{L-\text{irred}} \dim H^m(G, L)$ is exponential, answering this second question in the negative. For simplicity we prove this first in the case $p = 2$ showing that the sequence $H^m(G, L_m) = H_{m-1}$ where L_m is the mth Frobenius twist of the natural module for G and H_m is the number of partitions of unity into mth powers of $1/2$. This is our Theorem 2. We prove this in §2 and offer a number of extensions to this result, including to the case $p > 2$.

In D. Hemmer's review of [Par07], he admits to being unsure how difficult the recursions would be to use for actual computation. We therefore hope our theorem can be seen as a vindication of Parker's algorithm as useful for producing interesting general results about Ext-groups arising from SL_2-modules.

At the end of the paper, we make a number of remarks indicating, as far as we can, various possible extensions to this work, and relevance to questions of [GKKL07].

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1 Unbounding Ext

Let $G = SL_2$ defined over an algebraically closed field k of characteristic $p > 3$. The following result is the main result from [Ste10].

Lemma 1.1. *Let $V = L(r)^{[d]}$ be any Frobenius twist (possibly trivial) of the irreducible G-module $L(r)$ with highest weight r where r is one of*

$$\begin{aligned}&2p\\&2p^2-2p-2\\&2p-2+(2p-2)p^e \ (e>1)\end{aligned}$$

Then $H^2(G,V) \cong k$. For all other irreducible G-modules V, $H^2(G,V) = 0$.

Now we can prove

Theorem 1. *Let $V_n = L(1) \otimes L(1)^{[1]} \otimes \cdots \otimes L(1)^{[n]}$.*

Then $\mathrm{Ext}^2_G(V_n, V_n) = n$.

Proof. By Steinbergs tensor product theorem, V_n is simple; thus it is self-dual and we have

$$\begin{aligned}\mathrm{Ext}^2_G(V_n,V_n) &\cong \mathrm{Ext}^2_G(k, V_n\otimes V_n^*)\\&\cong H^2(G,(L(1)\otimes L(1))\otimes(L(1)\otimes L(1))^{[1]}\otimes\cdots\otimes(L(1)\otimes L(1))^{[n]})\\&\cong H^2(G,(L(2)+k)\otimes(L(2)+k)^{[1]}\otimes\cdots\otimes(L(2)+k)^{[n]})\\&\cong H^2(G,k)+H^2(G,L(2))+H^2(G,L(2)^{[1]})+\cdots+H^2(G,L(2)^{[n]})\\&\quad+H^2(G,L(2)\otimes L(2)^{[1]})+\dots\end{aligned}$$

where the third isomorphism follows since when $p > 2$, $L(1) \otimes L(1)$ has composition factors $L(2)$ and k which do not extend each other.

Now, by the Lemma, the only terms in this expression which are non-zero are $H^2(G, L(2)^{[r]})$ with $r > 0$. Thus $\dim \mathrm{Ext}^2_G(V_n, V_n) = n$ as required. □

Remark 1.2. In fact one knows from [McN02] that if $p \geq h$, then for any $r > 0$, we have $H^2(G, \mathfrak{g}^{[r]}) \cong k$ for any simply connected, simple algebraic group G, where $\mathfrak{g}$ denotes the Lie algebra of G.

Then one can construct a similar example to the above for any G. One takes any simple module $L = L(\lambda)$ such that L is a faithful representation of G, with p big enough so that $L \otimes L^*$ is completely reducible. Then it will contain $\mathfrak{g}$ and k as direct summands. If the weights of $L \otimes L^*$ are less than p^r then one can take $V_n = L \otimes L^{[r]} \otimes L^{[2r]} \dots L^{[nr]}$ with the property that $\dim \operatorname{Ext}^2(V_n, V_n) \geq n$.

We now know that Parshall and Scott's restriction on the length of the p-adic expansion of L is necessary to have a finite bound $\max \dim \operatorname{Ext}^n_G(L, L') \leq c(\Phi, e)$ with the maximum taken over all irreducible modules L, L' with $e_p(L) < e$. In which case, It might be interesting to see how the sequence

$$\{f_e\} := \{\max \dim \operatorname{Ext}^n_G(L, L')\}$$

grows with e for fixed values of n and Φ, where the maximum is taken over all p and irreducible G-modules L, L' with $e_p(L) < e$. In the case $n = 2$ our examples show that f_e is at least linear.

2 Exponential growth of H^m

Let $G = SL_2$ defined over an algebraically closed field k whose characteristic will be $p = 2$ until further notice.

In this section we show that the sequence of simple modules $\{L_n = L(2^n)\}$ has the property that the sequence $\dim H^n(G, L_n)$ has exponential growth with n.

We need the following two formulae from [Par07], valid when $p = 2$.

Theorem 2.1. *Let M be in* $\operatorname{mod}(G)$ *and $b, q \in \mathbb{N}$ with $q > 0$. Then*

$$\operatorname{Ext}^q_G(\Delta(2b), M^{[1]}) \cong \bigoplus_{n=0}^{n=q} \operatorname{Ext}^{q-n}_G(\Delta(n+b), M), \tag{1}$$

$$\operatorname{Ext}^q_G(\Delta(2b+1), M^{[1]} \otimes L(1)) \cong \operatorname{Ext}^q_G(\Delta(b), M), \tag{2}$$

where $\Delta(r)$ denotes the Weyl module for G of high weight r.

Note that the above formulae are also clearly valid when $q = 0$; however, our analysis of the algorithm is slightly more transparent if we do not use these formulae in the case $q = 0$.

Using (1) and (2) it is possible to calculate $H^q(G, L)$ inductively for any simple G-module L. We give such a recipe now.

Firstly, by Steinberg's Tensor Product Theorem, $L \cong L(a_0) \otimes L(a_1)^{[1]} \otimes L(a_2)^{[2]} \otimes \cdots \otimes L(a_n)^{[n]}$ for some $n \in \mathbb{N}$, with (as $p = 2$) each $a_i \in \{0, 1\}$, i.e. L is the trivial module k, or a tensor products of different Frobenius twists of the natural module $L(1)$ for G. By the linkage principle, if $H^q(G, L) \neq 0$, then $L = M^{[1]}$ for some simple module M.

Thus, taking $b = 0$, we apply (1) to express $H^q(G, M^{[1]}) \cong \mathrm{Ext}^q_G(k, M^{[1]})$ in terms of Exts of equal or lower degree between Δ-modules and another simple module M of lower weight.

We may then ignore about half of these Ext terms since, if the parities of the highest weights of M and a given $\Delta(r)$ module are different then this Ext term vanishes by linkage. For the remainder, apply equation (2) if M is a simple module of odd high weight; and then continue to expand each surviving Ext term using equation (1). Eventually this process terminates with a sum of terms $\mathrm{Ext}^q_G(\Delta(r), k)$ with $q > 0$, which are 0 by [Jan03, II.4.13] and terms $\mathrm{Ext}^0_G(\Delta(r_i), N_i) \cong \mathrm{Hom}_G(\Delta(r_i), N_i)$ for some known collection of simple modules N_i. We call these Ext^0 terms 'leaves'.

As each N_i is simple and $\Delta(r_i)$ has a simple head, each of these leaves is then visibly either isomorphic to k or 0 (according to whether or not the highest weight of N_i is the integer r_i) and so the desired value of $\dim H^q(G, L)$ has been calculated.

Given a simple module L and a degree m of cohomology, we wish to enumerate these Ext^0 leaves. To this end we make the following (possibly somewhat vague) definition, which we hope will be elucidated by the following examples.

Definition 2.2. For a given degree m of cohomology, and simple module L, an a-string is a list of non-negative integers $(a_1, \ldots, a_n)$ specifying an Ext^0-leaf, arising from recursive expansion of $\mathrm{Ext}^m_G(\Delta(0), L)$ by expanding the $\mathrm{Ext}^{m - \sum_{i=1}^{r}(a_i)}$ term at stage r.

Observe that since we do not expand Ext^0 terms, the last value, a_n, in an a-string is non-zero. As we end up at at a leaf, we must have $\sum_{i=1}^{n} a_i = m$.

Example 2.3. Let $m = 6$ and $L = L(24)$. Then there is an a-string 402:

$$\begin{aligned}
\mathrm{Ext}^6(\Delta(0), L(24)) &\cong \mathrm{Ext}^6(\Delta(0), L(12)) + \mathrm{Ext}^5(\Delta(1), L(12)) + \mathrm{Ext}^4(\Delta(2), L(12)) \\
&\quad + \mathrm{Ext}^3(\Delta(3), L(12)) + \underline{\mathrm{Ext}^2(\Delta(4), L(12))} + \mathrm{Ext}^1(\Delta(4), L(12)) \\
&\quad + \mathrm{Ext}^0(\Delta(6), L(12)) \\
\mathrm{Ext}^2(\Delta(4), L(12)) &\cong \underline{\mathrm{Ext}^2(\Delta(2), L(6))} + \mathrm{Ext}^1(\Delta(3), L(6)) + \mathrm{Ext}^0(\Delta(4), L(6)) \\
\mathrm{Ext}^2(\Delta(2), L(6)) &= \mathrm{Ext}^2(\Delta(1), L(3)) + \mathrm{Ext}^1(\Delta(2), L(3)) + \underline{\mathrm{Ext}^0(\Delta(3), L(3))} \\
\mathrm{Ext}^0(\Delta(3), L(3)) &= k.
\end{aligned}$$

Note that not all strings of non-negative integers are valid a-strings even if they add up to m. For instance, in this setting, strings such as 33 or 321 are meaningless as a-strings since they gives rise to a chain

$$\mathrm{Ext}^6(\Delta(0), L(24)) \geq \mathrm{Ext}^3(\Delta(3), L(12)) = 0$$

as the parity of 3 and 12 is different, and so one cannot consider the next term in the sequence.

Also there are a-strings which result in Ext^0 leaves which are zero. For instance the string 42 is valid as an a-string:

$$\mathrm{Ext}^6(\Delta(0), L(24)) \geq \mathrm{Ext}^2(\Delta(4), L(12)) \geq \mathrm{Ext}^0(\Delta(4), L(6))$$

but zero. We call an a-string which results in a non-zero leaf, a non-trivial a-string. Thus we have $\dim H^m(G, L) = \sharp\{\text{non-trivial } a\text{-strings}\}$. We wish to give a lower bound on the number of non-trivial a-strings.

Firstly though, observe that any a-string can be made into a longer (a, n)-string of length n say by adding 0s to the end. We can of course, recover the original a-string from an (a, n)-string by removing all 0s from the end. If the highest weight of L is no more than 2^n then the length of any valid a-string can be no longer than n and so we have a bijection between a-strings and (a, n)-strings.

Let $L = L(2^n)$. So all a-strings can still be made into (a, n)-strings. For this L, we make the

Claim. An (a, n)-string $(a_1, \dots, a_n)$ is non-trivial provided there exists a string of positive integers $(b_1, b_2, \dots, b_n)$ with

(i) $2b_1 = a_1$,

(ii) $a_i + b_{i-1} = 2b_i$ for $2 \leq i \leq n-1$,

(iii) $b_n = a_n$. and

(iv) $b_{n-1} + b_n = 1$

As each a_i is positive, the resulting string has the property (ii'): $2b_i \geq b_{i-1}$ for $2 \leq i \leq n-1$. Note also that if such a string exists, it has property (iv) $m = \sum a_i = (\sum_{i=1}^{n} b_i) + b_{n-1}$; so $\sum_{i=1}^{n-1} b_i = m - 1$. We call a string satisfying properties (i), (ii') (iii), and (iv) a b-string, and observe that if a b-string exists for a given (a, n)-string, one can recover the original a-string.

To prove the claim, we trace the highest weight of the Δ-module on the left. One finds that at step $r < n$ one is considering a term

$$\mathrm{Ext}^{m-\sum_{i=1}^{r} a_i}\left(\Delta\left(\frac{\frac{\frac{\frac{a_1}{2}+a_2}{2}+\dots}{\ddots} + a_{r-1}}{2} + a_r\right), L(2^{n-r})\right),$$

and one can expand this term precisely if the parity of the left hand side is even. By induction one sees that the big fraction inside the Δ term is equal to b_{r-1} and so the highest weight of the Δ-module is $a_r + b_{r-1} = 2b_r$ as required. Finally, taking $r = n-1$ and expanding one last time we find we are considering $\mathrm{Ext}^0(\Delta(b_{n-1}+b_n), L(1))$ which is non-zero (and one-dimensional) precisely if $b_{n-1} + b_n = 1$ as required.

Indeed, this argument shows that any b-string gives rise to a non-trivial a-string. So it suffices to count b-strings. We do this now in the case $n = m-1$.

Take $n = m$. If $b_{n-1} = 0$ then $b_1 = \cdots = b_{n-2} = 0$ by property (ii'); thus $b_n = m = 1$ and $n = 0$ which is nonsense. So $b_{n-1} = 1$. Then we wish to find all sequences $b_1, \dots, b_{n-2}$ with $\sum b_i = n-2$ and $b_i \geq 2b_{i-1}$. Reversing the order; call a string of integers a c-string if $c_1 = 1$ and $c_i \leq 2c_{i-1}$ with $\sum_{i=1}^{n-1} c_i = n-1$. For each n, the number of c-strings is then precisely the sequence H_{n-1} from [FP87, p150]. That is, the dimension of $H^m(G, L(2^m))$ is the integer H_{m-1}: the number of 'level number sequences' associated to binary trees, or the number of partitions of 1 into nth powers of 1/2.[1] We have from [FP87] the inequality

$$F_n \leq H_n \leq 2^{n-1}.$$

As $F_n \sim \left(\frac{1+\sqrt{(5)}}{2}\right)^n$, it follows immediately that H_n grows exponentially, but we give a quick proof here that $H_{2n+1} \geq 2^n$:

[1]See [OEI11, http://oeis.org/A002572] for more on this sequence

Observe

$$1, \underbrace{2, 2, \ldots, 2}_{n} \underbrace{0, 0, \ldots, 0}_{n}$$

is a c-string. For any choice of subset of the 2s in the first underbrace, we may replace each 2 by the string $1, 1$ and remove a 0 from the right to have another c-string. Running through the different choices of the 2^n subsets we see that they are all distinct; and thus

Theorem 2. *For $m > 2$,*

$$\dim H^{2m}(G, L(2^{2m})) \geq 2^m$$

and so $H^m(G, L(2^m))$ grows exponentially with m.

Remark 2.4. The longest b-string without 0s at the front is clearly

$$\underbrace{1, 1, \ldots, 1}_{m-1}, 0.$$

It follows from this that $\dim H^m(G, L(1)^{[r]}) < \dim H^m(G, L(1)^{[m]})$ if and only if $r < m$, with equality otherwise. So for $p = 2$ rational stability occurs for the module $L(1)$ at the Frobenius twist m, in other words the value of ϵ in [CPSvdK77, Corollary 6.8] can be as large as m.

Remark 2.5. We note that the rate of growth of H^m is not too severely under-estimated by a sequence $\{C.2^{m/2}\}$. The following are the precise numbers up to $n = 31$:

```
H^4(G,L(2^4))=2
H^5(G,L(2^5))=3
H^6(G,L(2^6))=5
H^7(G,L(2^7))=9
H^8(G,L(2^8))=16
H^9(G,L(2^9))=28
H^10(G,L(2^10))=50
H^11(G,L(2^11))=89
H^12(G,L(2^12))=159
H^13(G,L(2^13))=285
H^14(G,L(2^14))=510
H^15(G,L(2^15))=914
H^16(G,L(2^16))=1639
H^17(G,L(2^17))=2938
H^18(G,L(2^18))=5269
H^19(G,L(2^19))=9451
H^20(G,L(2^20))=16952
H^21(G,L(2^21))=30410
H^22(G,L(2^22))=54555
H^23(G,L(2^23))=97871
H^24(G,L(2^24))=175586
H^25(G,L(2^25))=315016
H^26(G,L(2^26))=565168
H^27(G,L(2^27))=1013976
H^28(G,L(2^28))=1819198
H^29(G,L(2^29))=3263875
H^30(G,L(2^30))=5855833
H^31(G,L(2^31))=10506175
```

which appears to grow at about the same rate as $1.8^m \sim 3.2^{m/2}$.

This would suggest that the best likely result in the spirit of Theorem 0.1 given in the introduction would be that the sequence $\{\log \gamma_m(\Phi, e)\}$ has polynomial growth at most 1 for any Φ (in other words, is linear with m). In any case, Theorem 2 shows that Parshall and Scott's estimate is certainly in the right ball-park.

Remark 2.6. One can replace the weight 2^m with any other weight $r.2^m$ with the result that the sequence $\{\dim H^m(SL_2, L(r.2^m))\}$ grows exponentially fast. We have written a computer program using Parker's algorithm to calculate the dimensions of cohomology groups. The output from the program giving dimensions for $H^m(SL_2, L(r.2^{m-2}))$ is given below.

```
H^3(G,L(3.2))=1
H^4(G,L(3.2^2))=1
H^5(G,L(3.2^3))=2
H^6(G,L(3.2^4))=4
H^7(G,L(3.2^5))=6
H^8(G,L(3.2^6))=11
H^9(G,L(3.2^7))=20
H^10(G,L(3.2^8))=35
H^11(G,L(3.2^9))=63
H^12(G,L(3.2^10))=113
H^13(G,L(3.2^11))=201
H^14(G,L(3.2^12))=361
H^15(G,L(3.2^13))=647
H^16(G,L(3.2^14))=1159
H^17(G,L(3.2^15))=2080
H^18(G,L(3.2^16))=3730
H^19(G,L(3.2^17))=6689
H^20(G,L(3.2^18))=12001
H^21(G,L(3.2^19))=21528
H^22(G,L(3.2^20))=38619
H^23(G,L(3.2^21))=69287
H^24(G,L(3.2^22))=124304
H^25(G,L(3.2^23))=223010
H^26(G,L(3.2^24))=400108
H^27(G,L(3.2^25))=717838
H^28(G,L(3.2^26))=1287890
H^29(G,L(3.2^27))=2310651
H^30(G,L(3.2^28))=4145619
H^31(G,L(3.2^29))=7437818
H^32(G,L(3.2^30))=13344508
```

The combinatorics become more complicated when one changes the value of r away from 1, though proofs of exponentiality using the above methods are available. One notices from the numbers, though, that the dimensions appear to grow at about the same rate as $1.8^m \sim 3.2^{m/2}$.

Remark 2.7. For $p > 2$ one can use essentially the same method to show that the sequence $\{\dim H^m(SL_2, L_m)\}$ also has exponential growth, where $L_m = L(2.p^m)$.

We outline the changes necessary to show this:

The relevant recursions are

$$\mathrm{Ext}^q_G(\Delta(pb+i), M^{[1]} \otimes L(i)) \cong \bigoplus_{n \text{ even},\ 0\le n\le q} \mathrm{Ext}^{q-n}(\Delta(b+n), M) \tag{3}$$

$$\mathrm{Ext}^q_G(\Delta(pb+i), M^{[1]} \otimes L(\bar{i})) \cong \bigoplus_{n \text{ odd},\ 0\le n\le q} \mathrm{Ext}^{q-n}(\Delta(b+n)), M) \tag{4}$$

$$\mathrm{Ext}^q_G(\Delta(pb+p-1), M^{[1]} \otimes L(p-1)) \cong \mathrm{Ext}^q_G(\Delta(b), M) \tag{5}$$

where $0 \le i \le p-2$ and $\bar{i} = p-2-i$.

We use just equation (3) above, starting with $b = i = 0$. Then one continues to expand terms of the form $\mathrm{Ext}^q(\Delta(s), L_m)$ provided $p|s$ and q is even and counts Ext^0-leaves as before.

Take in fact $m = 2m'$; then an appropriate a-string $(a_1, \ldots, a_m)$ with $\sum a_i = m$

is one for which

$$\left(\frac{\dfrac{\frac{\frac{a_1}{p}+a_2}{p}+\dots \ \ddots + a_{r-1}}{\ddots}}{p}+a_r\right)$$

is an integer for each $r \leq m$, where every a_i is even and $2.p^m = \sum a_i p^i$. The continued fraction's integrality condition is equivalent to finding a b-string subject to $a_1 = 3b_1$ and $a_i + b_{i-1} = pb_i$ for each $i < m$; this also implies that each b_i with $i < m$ is even. Interpreting the other restraints, we see such a b-string also satisfies $pb_i \geq b_{i-1}$ for $2 \leq i \leq n-1$ and set $b_m = a_m$. We want that $m = \sum a_i = (p-1)\sum_{i=1}^{m-1} b_i + b_{n-1} + b_n$ with also $b_{n-1} + b_n = 2$. Any string of non-negative integers satisfying these properties will work to give an a-string. One can then cook up exponentially many b-strings in a similar way to that done for $p = 2$.

Remark 2.8. It can be shown using Parker's recursions that when $p > 2$,

$$H^m(G, L(2r+1)) = 0 \text{ for all } m, r \geq 0,$$

showing that the analogue of Remark 2.6 does not hold for $p > 2$.

Remark 2.9. We have used Parker's equations to show that there is a sequence of simple modules L_m with the value of $\dim \mathrm{Ext}_G^m(\Delta(0), L_m)$ growing exponentially. One can show similarly that there is a sequence M_m with $\dim \mathrm{Ext}_G^m(\Delta(r), M_m)$ growing exponentially. In fact, if $r < p^s$ then it is easy to see that $M_m = L_m^{[s]} \otimes L(r)$ will work. (One uses the fact that $\mathrm{Ext}_G^q(\Delta(pb+i), L(i) \otimes M^{[1]}) \geq \mathrm{Ext}_G^q(\Delta(b), M)$.)

Remark 2.10. Brian Parshall asked by private communication if one could get exponential sequences $\{\dim H^m(G, L_m)\}$ for other G. We believe the answer is probably 'yes' but as yet cannot give such a sequence. However we make some hopefully promising observations:

Firstly, let G be any simple algebraic group with torus T. If $\lambda,\ \mu \in X^+(T)$ with $\lambda - \mu = m\beta$ for some $m \in \mathbb{Z}$ and β a simple root, then, as observed in [Par07], we have by [CPS04, Corollary 10],

$$\mathrm{Ext}^q(\Delta(\lambda), L(\mu)) \cong \mathrm{Ext}^q_{SL_2}(\Delta(2m_\beta), L(2n_\beta)),$$

where $m_\beta = \langle \lambda, \beta \rangle$ and $n_\beta = \langle \mu, \beta \rangle$.

Now take $G = SL_3$ and $p = 2$. We choose $\lambda_m = (2^m, 0)$ on the α-wall of the dominant chamber, where $\alpha = (2, -1)$ and $\beta = (-1, 2)$ are the simple roots for SL_3 as elements of $X(T)$. Then take $\mu_m = \lambda_m + 2^m\beta = (0, 2^{m+1})$ and observe that in (2.10), with $\lambda = \lambda_m$, and $\mu = \mu_m$ we have $m_\beta = 0$ with $n_\beta = 2^{m+1}$.

Then we know from Remark 2.6 that the right hand side of 2.10 grows exponentially.

If one knew that the number of composition factors M_m of $\Delta(\lambda_m)$ admitting non-zero values of $\mathrm{Ext}^m(M_m, L(\mu_m))$ grew subexponentially, then one could find a sequence of such M_m with the dimension of this latter Ext group growing exponentially. Since $M_m^* \otimes L(\mu_m)$ is irreducible by Steinberg, we would then have $\dim H^m(G, M_m^* \otimes L(\mu_m))$ giving the desired result. Unfortunately, using [Par01, Theorem 4.12] one can show there are $2^{m-2} + 2$ composition factors in $\Delta(\lambda_m)$.

Remark 2.11. It is remarkable that the dimension of the modules in our sequences $\{L_m\}$ can be so small: when $G = SL_2$, as little as $\dim L_m = 2$ for all m when $p = 2$ and $\dim L_m = 3$ when $p > 2$. In Remark 2.7 we used modules of dimension 3 but we expect that we could still use modules of dimension 2.

This brings to mind some of the questions raised in [GKKL07]. We list some apposite results from that paper:

Theorem. *(i) Let G be a finite simple group, F a field and M an FG module. Then* $\dim H^2(G, M) \leq 17.5 \dim M$.

(ii) Let G be a finite group, F a field and M an irreducible FG module. Then $\dim H^2(G, M) \leq 18.5 \dim M$.

(iii) Let F be an algebraically closed field of characteristic $p > 0$ *and k a positive integer. Then there exists a sequence of finite groups* G_i, $i \in \mathbb{N}$ *and irreducible faithful* FG_i*-modules* M_i *such that*

(a) $\lim_{i\to\infty} \dim M_i = \infty$,

(b) $\dim H^k(G_i, M_i) \geq e(\dim M_i)^{k-1}$ *for some constant* $e = e(k, p) > 0$, *and*

(c) if $k \geq 3$ *then* $\lim_{i\to\infty} \frac{\dim H^k(G_i, M_i)}{\dim M_i} = \infty$.

It is then pointed out that (iii) above precludes the possibility of generalising item (ii) above to higher degrees of cohomology. Nonetheless, following questions are raised.

Questions. *(i) For which k is it true that there is an absolute constant* C_k *such that* $\dim H^k(G, V) < C_k$ *for all absolutely irreducible FG-modules V and all finite simple groups G with F an algebraically closed field (of any characteristic)?*

(ii) For which positive integers k is it true that there is an absolute constant d_k *such that* $\dim H^k(G, V) < d_k. \dim V^{k-1}$ *for all absolutely irreducible faithful FG-modules V and all finite groups G with F an algebraically closed field (of any characteristic)?*

Note that there is no answer to question (i), for any $k > 0$, even in the possibly easier case where G is a simple algebraic group. The highest value of $\dim H^1(G, V)$ on record (see [Sco03]) is 3, where $G = SL_6$. Assuming Lusztig's character formula holds, we could take $p = 7$ and $V = L(45454)$ to achieve this value. If we did have a positive answer to Question (i), this would imply a positive answer to Question (ii) in the case G is taken to be a finite simple group.

In any case, our examples are relevant to Question (ii), when G is taken to be a simple group. Consider the case when G is algebraic. If G is SL_2 we suspect that $\max_{p,L-\text{irred}} \dim H^m(G, L) \leq H_{m-1}$ with equality occurring if and only if $p = 2$ and L is a sufficiently high twist of $L(1)$. Then for all G, it is conceivable, owing to the low dimensions of the module involved, that the largest value of $\dim H^k(G, V)/(\dim V)^{k-1}$ occurs in the case $G = SL_2$, for $p = 2$ and where $V = L(1)^{[r]}$ is a twist of the natural module for G, since then, $\dim V = 2$ and the lowest it could possibly be. But while the rate of growth of $\max_r \dim H^k(G, L(1)^{[r]})$ is exponential, our computed values appear to show that it grows at about the rate 1.8^k, so that $\dim H^k(G, V)/(\dim V)^{k-1} \sim 1.8^k/2^{k-1}$ would tend to zero. (Since $H_k \leq 2^{k-1}$ we can be sure that these numbers will tend to a value no more than 1/2.)

Thus it is conceivable that one could ask for a single constant $d \geq d_k$ that works for all k in Question (ii), when G is simple and algebraic. Ignoring the case where $k = 1$ (and Questions (i) and (ii) coincide), possibly even $d = 1$ may work. This is then relevant to the finite group situation by considering generic cohomology. One has from [CPSvdK77] that $H^m(G, V^{[e]}) \cong H^m(G(q), V)$ for high enough values of e and q. Our example provides some small evidence then, that for $k > 1$, one might replace d_k with a universal constant in Question (ii) if G is a finite simple group.

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