10 conjectures in additive number theory

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January 25, 2011

Abstract

Following an idea of Rowland [Row] we give a conjectural way to generate increasing sequences of primes using "gcd-algorithms". These algorithms seem not so useless for searching primes since it appears we found sometime primes much more greater than the number of required iterations. In an other hand we propose new formulations of famous conjectures from the additive theory of numbers (the weak twin prime conjecture, the Polignac conjecture, the Goldbach conjecture or the very general Schinzel's hypothesis H). For the moment these are experimental results obtained using pari-gp [PAR].

Introduction

In [Row] the author solved a question related to the sequence defined recursively by $R(1) = 7$ and for $n \geq 2$ by:

- $R(n) = R(n-1) + \gcd(R(n-1), n)$

Namely he showed that $R(n) - R(n-1)$ is always a prime number or 1. At first glance this recursion is not very useful for finding primes since these differences generate primes quite randomly and the prime values are less than the number of required iterations. However, looking closer to Rowland recursion we found a way to exhibit the records values in the sequence of differences $R(n) - R(n-1)$ (these differences are given by $A137613$ in [Slo]). Namely $n+1$ when $R(n) = 2n+2$ are exaclty those records values. So this gives a method to exhibit primes in increasing order (and the growth is exponential). Those $n+1$ such that $R(n) = 2n+2$ begin:

$5, 11, 23, 47, 101, 233, 467, 941, 1889, 3779, 7559, 15131, 30323, 60647, \ldots$

Although this seems provable using Rowland reasoning, it looks like it is unknown since it is not mentioned before in [Row] and this sequence of records values is not in [Slo]. This is clearly a nice fact and we try to generalise this observation. In order to facilitate the computations we use the absolute value and run the algorithm "backwards" as we will see. So we succeeded to extend somewhat this result with the conjectures 1 and 2 and propose a method for building primes (section 1). In a conjecture 3 (section 2) we claim there are sequences of lesser of twin primes growing very fast. Then we provide a conjecture 3bis (cf. 2.4) related to the Polignac conjecture and a conjecture 3ter (cf. 2.5) dealing with the prime triplet conjecture. In a conjecture 4 (section 3) we propose a way to find very big primes compared to the number of iterations which could improve the conjecture 2 and be an efficient tool for searching primes. In a conjecture 5 (section 4) we propose a first constructive way to prove the Goldbach conjecture but with some restiction due to an exceptional set of non working cases. In a conjecture 6 (section 5) we propose to summarize all our observations with a reformulation of the Schinzel's hypothesis H. The conjecture 7 (cf. 5.2) gives another approach of the conjecture 6.

We also discuss the Shevelev conjecture who extends Rowland idea for twin primes. Indeed, V. Shevelev [She] introduced the related sequence ($A166944$ in [Slo]) :

- $S(1) = 2$
- $S(n) = S(n-1) + \gcd(S(n-1), n-1+(-1)^n)$

And noticed that the records values of differences ($A166945$ in [Slo]) :

$7, 13, 43, 139, 313, 661, 1321, ...$

gives for terms > 7 the greater prime of some twin primes pairs . As for Rowland sequence it can be seen that the indices where these records occur is given by the sequence of $n+1$ such that $S(n) = 2n+1$ (except for 7) and $(n, S(n)-S(n-1))$ is then a twin pair of primes. Using this algorithm backwards the conjecture 8 (section 6) propose a sligthly different way than the conjecture 3 or 6 or Shevelev conjectures to prove the weak twin prime conjecture.

Although Shevelev managed to do it, it is not easy to generalise this observation using the "forward" original Rowland recursion since we need to find where the records occur. However using the absolute value we shall see it is easy to obtain many increasing sequences of twin primes or of triplet of primes since we just have to check indices where zeros occur. Thus in a conjecture 9 (section 7) we propose a method for generating primes m-uplet of any type using periodic sequences in the gcd algorithm. Then we extend the idea to a family of polynomials giving another version of the Schinzel's hypothesis H. This allows us to perform easier computations than using the conjecture 6 for searching m-uplet.

We finally merge our "backward" recursion and Shevelev idea for approaching the Goldbach conjecture in the conjecture 10 (section 8) in a nicer way than the conjecture 5. There is apparently no more exception for n large enough. This right way to deal with this old conjecture using gcd-recursion was very hard to find. We also discuss a less known but hard conjecture of Legendre (section 9).

There is a curious fact in this study. It appears our methods work often better for large integer since once we found a good starting value we generate only primes, despite the probability to pick up big primes among large integers goes to zero.

1 Variation on Rowland recursion

After some attempts we arrive to this recursion seeming generating primes in a general way starting always with the same initial value 1. For a given integer value $m \geq 1$ we define the sequence $(a(n))_{n\geq 1}$ by $a(1) = 1$ and for $n \geq 2$ by the recursion:

- $a(n) = |a(n-1) - \gcd(a(n-1), mn-1)|$.

Now let us consider the values of n such that we get:

- $a(n) = 0 \Leftrightarrow a(n+1) = mn + m - 1$.

We claim that $\forall m \geq 1$ this sequence of indices n gives rise to an infinite sequence $(b_m(k))_{k\geq 1}$.

Example

For $m = 1$ the sequence$(a(n))_{n\geq 1}$ begins:

$1, 0, 2, 1, 0, 5, 4, 3, 2, 1, 0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 23, 22, 21, 20, 19, \ldots$

And the sequence $(b_1(k))_{k\geq 1}$ of indices where zeros appear in $(a(n))_{n\geq 1}$ begins:

$2, 5, 11, 23, 47, 79, 157, 313, 619, 1237, 2473, 4909, 9817, 19603, 39199, 78193, \ldots$

These listed numbers are prime numbers. This led us to a first conjecture.

1.1 Conjecture 1

We claim:

- $b_1(k)$ is prime for $k \geq 1$ and $b_1(k) \sim c2^k$ $(k \to \infty)$ with $c = 1.186\ldots$

See the APPENDIX 1 for a table supporting this conjecture. Although this conjecture is similar to the record sequence mentioned in the introduction, it is more interesting to us. Indeed we were able to generalise the result and much more seems true. So we make a stronger conjecture.

1.2 Conjecture 2

In general we claim that for any $m \geq 1$:

- $mb_m(k) + m - 1$ is prime for k large enough (usually $k \geq 2$ is working for small m).
- $mb_m(k) + m - 1 \sim c_m(m+1)^k$ $(k \to \infty)$ with $c_m > 0$.

Additional examples supporting the conjecture 2

For $m = 3$ the sequence $(a(n))_{n\geq 1}$ begins:

$1, 0, 8, 7, 0, 17, 16, 15, 14, 13, 12, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 71, \ldots$

And the sequence $(b_3(k))_{k\geq 1}$ begins:

$2, 5, 23, 89, 337, 1335, 5307, \ldots$

Next the values $3b_3(k) + 2$ appear to be prime values for $k \geq 2$:

$8, 17, 71, 269, 1013, 4007, 15923, 63521, 253949, 1014317, \ldots$

And $3b_3(n) + 2 \sim c_3 4^n$ $(n \to \infty)$ with $c_3 = 0.96\ldots$

For $m = 4$ the sequence $(a(n))_{n\geq 1}$ begins:

$1, 0, 11, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0, 59, 58, 57, \ldots$

And the sequence $(b_4(k))_{k\geq 1}$ begins:

$2, 14, 62, 314, 1574, 7846, 38020, \ldots$

Next the values $4b_4(k) + 3$ appear to be prime values only for $k \geq 1$:

$11, 59, 251, 1259, 6299, 31387, 152083, 758971, 3790651, 18953251, \ldots$

And $4b_4(n) + 3 \sim c_4 5^n$ $(n \to \infty)$ with $c_4 = 1.9408\ldots$

1.3 An efficient algorithm for finding primes?

Note we find sometime primes greater than n after making n iterations. So it could be an efficient method for finding primes since the gcd algorithm is well known and "fast" computation can be performed. For instance if $m = 28$ we compute 1000000 terms of the sequence $(a(n))_{n\geq 1}$. In this range $a(n)$ vanishes 5 times. This allows us to compute $(b_{28}(n))_{1\leq n\leq 5}$ wich gives 5 values of $28b_{28}(n) + 27$:

$83, 1147, 31891, 924811, 26819491.$

All these values are primes and the last one 26819491 gives a prime number larger than the 1000000 iterations (see APPENDIX 2 for more exemples). Moreover it appears this kind of algorithm can be adapted for finding bigger primes as shown thereafter.

1.3.1 A simple rule of construction

Observe from our definition we have for $n \geq 2$ (letting $b = b_m$):

- $a(b(n)+1) = (m+1)a(b(n-1)+1)+m+m\sum_{j=b(n-1)+1}^{b(n)-1}(a(j+1)-a(j)+1)$

Hence we have an "almost" recurrence relation between two consecutive records values which are conjectured to be prime for n large enough. This formula explains also why these record values are growing like $(m+1)^n$ since it appears we usually don't need to compute all terms in the sum. Indeed experiments show that $a(j+1)-a(j) = -1$ for $b(n-1) < n_0 \leq j \leq b(n)-1$ and this n_0 depends on $b(n-1)$ and stays "near" from this value. i.e. we claim that $n_0 - b(n-1) \ll \sqrt{b(n-1)}$ and is often much more smaller. So we can launch a computation and when the computer returns $a(j+1)-a(j) = -1$ sufficiently "often" we may stop the computation and suspect we are beyond this n_0. Therefore we built perhaps a prime greater than a starting prime. It could then be interesting to know rules in order to choose "good" values of m making n_0 the closest of $b(n-1)$ as possible.

1.3.2 Exemples

For instance consider $m = 10$ and the sequence of $a_n - a_{n-1}$ when $|a_n - a_{n-1}| > 1$ wich yields somewhere for $a_n - a_{n-1}$:

- $43213789, -3, -13, -15241, -43, -1889, -3, -433, -113, -3, -5827, -247$

The positive terms are our record values $a(b(k)+1)$ and are primes values. Here the computation returns nothing more using 10^5 iterations. Hence we are perhaps beyond our n_0 and the next record would be given by $11 \times 43213789 + 10 - 10 \times (2+12+15240+42+1888+2+432+112+2+5826+246) = 475113649$ which is a prime number and, as expected, our next record value. So we have found a prime number 11 times greater than the given prime 43213789 with few iterations.

But let us see this with a more striking exemple and take $m = 100000$. We compute $|a_n - a_{n-1}| > 1$ for $n \leq 10^6$ wich yields for these $a_n - a_{n-1}$:

- $299999, -59, 29994499999, -3, -7, -53, -3$

So we suspect we have nothing more after -3 until the next record value. Thus $100001 \times 29994499999 + 100000 - 100000 \times (2+6+52+2) = 2999479988299999$ should be the next record and it is indeed a prime value. So with few effort we found a prime number 100001 times greater than another one. A simple routine under pari-gp found easily big primes with this method (see the end of the APPENDIX 2).

1.3.3 Probability to got a prime value from the first record

We also observe something which could have practical use for searching primes if people work together like for the GIMPS where they share computer power.

We keep previous notations and our definition of a_n (depending on m) so that the first record is simply $3m-1$. Then we tried to estimate the chance to got a prime value with the second record value for various m after making $\lfloor m^{\alpha} \rfloor$ iterations and where $0 < \alpha \leq 1$. i.e. for a given m we define:

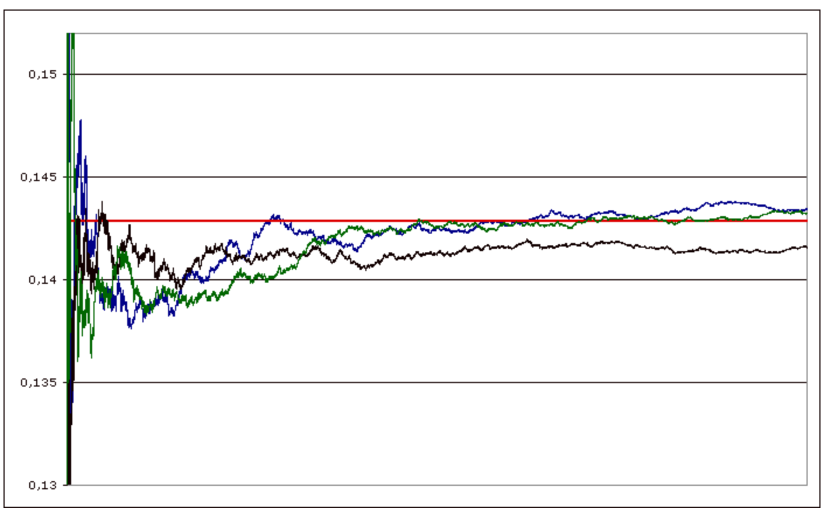
- $f_{\alpha}(m) = 1$ if the second record value which equals $(m+1)(3m-1)+m+m\sum_{j=4}^{\lfloor m^{\alpha} \rfloor}(a(j+1)-a(j)+1)$ is a prime value and otherwise $f_{\alpha}(m) = 0$.
- $L(\alpha) = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f_{\alpha}(k)$

Then we conjecture that:

- $L(\alpha)$ exists with simply $L(\alpha) = \alpha$ and so we know the probability to got a prime with this method.

To see this last point with experiments suppose E is a set of 30000 random values m satisfying $10^{14} < m < 10^{15}$. We take $\alpha = \frac{1}{7}$ so that we have to make around 100 iterations only to got the value $(m+1)(3m-1)+m+m\sum_{j=4}^{\lfloor m^{\alpha} \rfloor}(a(j+1)-a(j)+1)$. We then launch the computation 3 times (so that E changes each time) and we plot the different graph of $\frac{1}{n}\sum_{i=1}^{n} f_{\alpha}(m_i)$ (graphs are black, blue and green) where $m_1 < m_2 < ... \in E$ compared to the graph of $y = \frac{1}{7}$ (red).

(fig.1)



This supports the claim $L(\alpha) = \alpha$. In other word this means that if we take $\lfloor \alpha^{-1} \rfloor$ random values of m we are fairly certain the formula $(m+1)(3m-1)+m+m\sum_{j=4}^{\lfloor m^{\alpha} \rfloor}(a(j+1)-a(j)+1)$ will produce at least a prime number among them. Note we may start form the second record value (or the third...) instead of the first but we aim to work with big values of m so it seems inappropriate to compute the second record.

Remark

Although this method for finding primes seems satisfying it is not evident to see whether it is very efficient since we need to combine two computations (one for finding the candidates and one for testing primalty) and so far we find no rule in order to force n_0 to stay very close from $b(n-1)$. Perhaps the conjecture 4 would be better for that purpose (see 3.4.). By the way this kind of algorithms is interesting in its own since variations of the Rowland-Shevelev algorithm led us to formulate in a new way old conjectures from the additive theory of numbers. Hereafter we come across conjectures like the Polignac conjecture, the Schintzel hypothesis H or the Goldbach conjecture and our approach suggests analytic or probabilistic study.

2 An increasing sequence of twin primes

This was somewhat surprising to find such a sequence since it is not known whether there are infinitely many twin primes. Here we consider the recursion $a(1) = 1$ and for $n \geq 2$:

- $a(n) = |a(n-1) - \gcd(a(n-1), n^p - 1)|$

2.1 Conjecture 3

Suppose $p \geq 2$ is a prime number then we claim:

- there are infinitely many values of n such that $a(n) = 0$.
- for n large enough (usually $n > 2$ is working for small p) we have $a(n) = 0 \Rightarrow n$ is prime and $\frac{(n+1)^p-1}{n}$ is prime.

2.2 Corollary

There are infinitely many twin primes since the conjecture yields for $p = 2$:

- for n large enough we have $a(n) = 0 \Rightarrow n$ is prime and $n + 2$ is prime

and this happens infinitely many times.

2.3 Tables (in the sequel $\delta(n) = 1$ if n is prime and 0 otherwise.)

$p = 2$

Values of n such that $a(n) = 0$	$n+2$	$\delta(n)$	$\delta(n+2)$
2	4	1	0
11	13	1	1
137	139	1	1
19181	19183	1	1
367953497	367953499	1	1

$p = 3$

Values of n such that $a(n) = 0$	$\frac{(n+1)^3-1}{n}$	$\delta(n)$	$\delta(\frac{(n+1)^3-1}{n})$
2	23	1	1
23	601	1	1
119333	142432291	1	1

We can also start from another initial value. For instance let $p = 2$ and choose $a(1) = 2$ this gives

Values of n such that $a(n) = 0$	$n + 2$	$\delta(n)$	$\delta(n + 2)$
3	5	1	1
17	19	1	1
281	283	1	1
79559	79561	1	1
6329815697	6329815699	1	1

See the APPENDIX 3 for more experiments supporting the conjecture for $p = 2$ and various starting values.

2.4 Conjecture 3bis

In the same vein we find an algorithm seeming generating primes pairs of type $(p, p + 2m)$. Let $a(1) = 4m^2$ and define:

- $a(n) = |a(n-1) - \gcd(a(n-1), n(n+2m))|$

Then we claim that for n large enough:

- $a(n) = 0 \Rightarrow n + 1$ and $n + 2m + 1$ are primes and this happens infinitely many times and there are infinitely many pairs $(n + 1, n + 2m + 1)$ of consecutive primes.

Thus the Polignac conjecture would be true. Here a table for $2m = 4$

Values of n when $a(n) = 0$	$n + 5$	$\delta(n + 1)$	$\delta(n + 5)$
12	17	1	1
192	197	1	1
38196	38201	1	1
1459118862	1459118867	1	1

See APPENDIX 4 for experiments with other small values of m. The conjecture 9 will give an easier way for computation of such pairs of primes since we avoid the square in the gcd.

Remark It is also possible to use Rowland-Shevelev recursion for generating increasing sequences of twin primes or things like that. For instance let:

- $a_1 = 2$ and $a_n = a_{n-1} + \gcd(a_{n-1}, n(n-2))$

Then the sequence of records for the differences $a_n - a_{n-1}$ yields an increasing sequence of lower of twin primes. See the end of the APPENDIX 3.

2.5 Conjecture 3ter

We generate primes triplet of type $(p, p+2, p+6)$. Let $a(1) = 4$ and define:

- $a(n) = |a(n-1) - \gcd(a(n-1), n(n+2)(n+6))|$

Then we claim

- $a(n) = 0 \Rightarrow (n+1, n+3, n+7)$ is a prime triplet.

Here a table

Values of n such that $a(n) = 0$	$\delta(n+1)$	$\delta(n+3)$	$\delta(n+7)$
4	1	1	1
40	1	1	1
82006	1	1	1

Although it is hard to perform convincing experiments, it is coherent with our previous conjectures and some very general rule should exist. This conjecture can be extended to any sort of prime triplet and to m-uplet. The conjecture 9 is better fitted for practical computation of m-uplet.

3 Finding big primes

Here we merge the conjectures 2 and 3 to obtain another conjectural way to unearth increasing sequences of primes. The rate of growth is multiple exponential and this could produce very big primes compared to the number of required iterations. We then discuss probabilities issues and propose a method to got very big primes similarly as what is described in 1.3.

3.1 Conjecture 4

Suppose $p \geq 2$ is prime and let $a(1) = p$ and:

- $a(n) = \left|a(n-1) - \gcd(a(n-1), pn^2 - 1)\right|$

Then we claim:

- there are infinitely many values of n such that $a(n) = 0$.
- for n large enough (usually $n > 2$ is working for small p) we have $a(n) = 0 \Rightarrow p(n+1)^2 - 1$ is prime.

See APPENDIX 5 for the begining of tables for some values of p. Could we by chance get so many prime numbers?

3.2 Why stopping here?

In fact it seems one can go further. Suppose $a(1) = 2$ and:

- $a(n) = \big|a(n-1) - \gcd(a(n-1), 2n^3 - 1)\big|$

Then we claim:

- there are infinitely many values of n such that $a(n) = 0$.
- $a(n) = 0 \Rightarrow 2(n+1)^3 - 1$ is prime for n large enough.

Here the begining of the table

Values of n such that $a(n) = 0$	$2(n+1)^3 - 1$	$\delta(2(n+1)^3 - 1)$
3	127	1
125	4000751	1
4000877	128084306502569672303	1

However it is not easy to generalise this one. We can also find good initial values and/or a good factor before n^3. For instance let $w(1) = 3$ and:

- $a(n) = \big|a(n-1) - \gcd(a(n-1), 10n^3 - 1)\big|$

Then we get

Values of n such that $a(n) = 0$	$10(n+1)^3 - 1$	$\delta(10(n+1)^3 - 1)$
4	1249	1
1240	19112405209	1

And we can provide this other impressive example with exponent 7. Suppose $a(1) = 3$ and:

- $a(n) = \big|a(n-1) - \gcd(a(n-1), 2n^7 - 1)\big|$

Values of n such that $a(n) = 0$	$2(n+1)^7 - 1$	$\delta(2(n+1)^7 - 1)$
2	4373	1
4352	59231218330987879606185473	1

3.3 How many chances we have to catch a prime?

Continuing this way it seems possible to find very big primes using the (k, b, c) recursion:

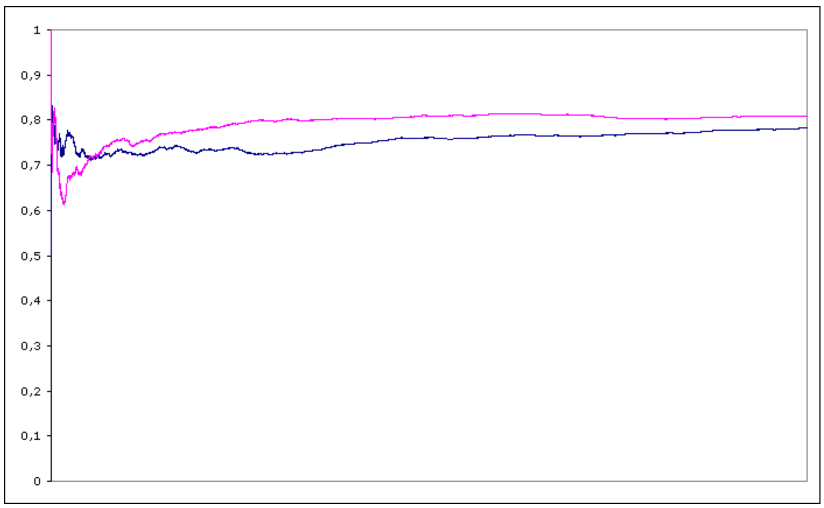
- $a(1) = k$ and $a(n) = |a(n-1) - \gcd(a(n-1), bn^c - 1)|$

with suitable choices of (k, b, c). A natural question is then: what is the chance to get a prime when the algorithm reachs the first zero starting with any value N? To evaluate this chance let $r_k = \min\{i \geq 1 \mid a(i) = 0\}$ and:

- $\Upsilon(N) = \frac{1}{N} \# \{k \mid 1 \leq k \leq N \,\&\, \delta(b(r_k+1)^c - 1) = 1\}$

wich represents the chance to get a prime reached by the algorithm starting with N. As $N \to \infty$ it appears this chance is not zero. For instance if $(b,c) = (2,2)$ we have $\Upsilon(N) \simeq 0.8$ as $N \to \infty$. Which gives an efficient method to got big primes since the first zero is reached after a number of iterations of order N and thus we have more than 80% of chance to got a prime of size $2N^2$. Here is a graph supporting this claim. We plot $\Upsilon(N)$ for $N = 1, 2, 3, \ldots, 20000$ and for $(b,c) = (3,2)$ (pink) $(b,c) = (5,2)$ (blue)

(fig.2)



And we believe $\lim_{N\to\infty} \Upsilon(N) = L(c)$ exists and depends only on c and it is clear that $c < c' \Rightarrow L(c) > L(c')$.

3.4 A rule of construction

As seen before in 1.3. there is here also a general relation between consecutive records. Namely we still consider for any initial value:

- $a(n) = |a(n-1) - \gcd(a(n-1), bn^c - 1)|$

and we define $(w(k))_{k\geq 1}$ as the sequence of values taken by $a_n - a_{n-1}$ when $|a_n - a_{n-1}| > 1$. Then the records values of $(w(k))_{k\geq 1}$ are given by k such that $w(k) > 0$. Let us now write this increasing sequence of k using a sequence $(\alpha_j)_{j\geq 1}$. Then we have the following simple relationship between 2 consecutive records values $w(\alpha_j)$ and $w(\alpha_{j+1})$ (details ommited):

- $w(\alpha_{j+1}) = b\left(w(\alpha_j) + 1 + \left(\frac{w(\alpha_j)+1}{b}\right)^{1/c} + \sum_{i=\alpha_j+1}^{\alpha_{j+1}-1} (a_{i+1} - a_i + 1)\right)^c - 1$

Thus as in 1.3.1. experiments show that it isn't necessary to compute all terms in the sum to got the next record value. Indeed there is again a value n_0 conjectured to be closed of α_j such that $\alpha_j < n_0 \leq i < \alpha_{j+1} \Rightarrow a_{i+1} - a_i = -1$. Therefore this gives sometime an efficient method for building a bigger prime from a prime record value or a non prime record value. The quadratic case seems well working. For instance let us consider this quadratic case:

- $a(1) = 1$ and $a(n) = \big|a(n-1) - \gcd(a(n-1), 32n^2 - 1)\big|$

Then we get the sequence of values $a_n - a_{n-1}$ for those $n \leq 10^6$ such that $|a_n - a_{n-1}| > 1$:

$127, -7, -17, -7, -7, 294911, -1289, 2760686028799, -113, -103, -7, -7, -113.$

The 3 first records $(127, 294911, 2760686028799)$are prime values and supposing there is no more value until the next record (say X) the formula above yields:

$$X = 32\left(2760686028800 + \left(\tfrac{2760686028800}{32}\right)^{1/2} - 112 - 102 - 6 - 6 - 112\right)^2 - 1$$

Giving $X = 243884447023448880167715967$ which is still a prime value. We provide also an exemple for the cubic case:

- $a(1) = 2$ and $a(n) = \big|a(n-1) - \gcd(a(n-1), 2n^3 - 1)\big|$

Then we get the sequence of values $a_n - a_{n-1}$ for those $n \leq 10^5$ such that $|a_n - a_{n-1}| > 1$:

$-3, 53, -5, -3, 265301, -109, -31, -17, -3, -5, -3.$

So we suspect there is no more value until the next record value (say X) and the formula above yields:

$$X = 2\left(265301 + 1 + \left(\tfrac{265301+1}{2}\right)^{1/3} - 108 - 16 - 2 - 4 - 2\right)^3 - 1$$

giving $X = 37299785868725741$ which indeed is the next record value and is a prime value.

For higher exponent it seems less easy to find many working exemples but we think it would be worth to explore this method further in order to check its possible efficiency. The main question would be: are there any conditions forcing records values to be prime values and making n_0 very close from the working record value?

3.5 Generating big twin primes

We can also do the same kind of task (cf. 3.3.) for twin primes borrowing from Shevelev the idea for a quadratic case. Let:

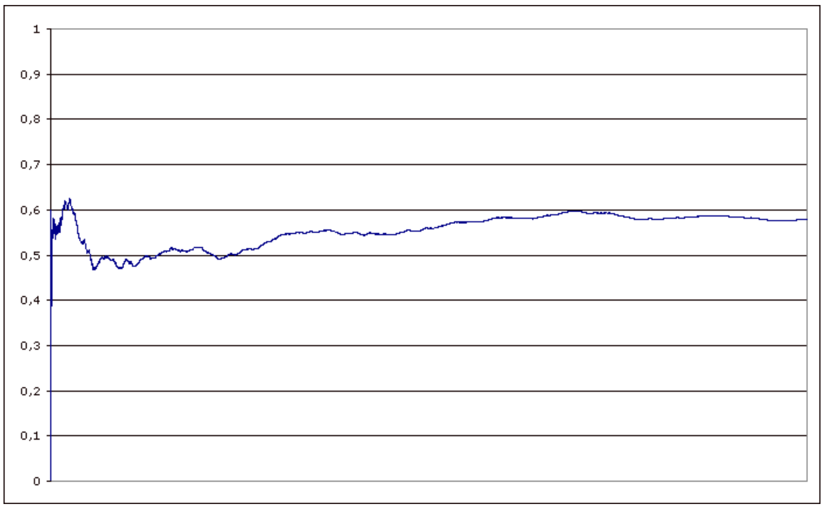
- $a(0) = k$ and $a(n) = a(n-1) - \gcd(a(n-1), 2n^2 + (-1)^n)$.

Define:

- $\Upsilon_{twin}(N) = \frac{1}{N} \# \left\{ k \mid 1 \leq k \leq N \,\&\, \delta(2(r_k+1)^2-1)\delta(2(r_k+1)^2+1) = 1 \right\}$

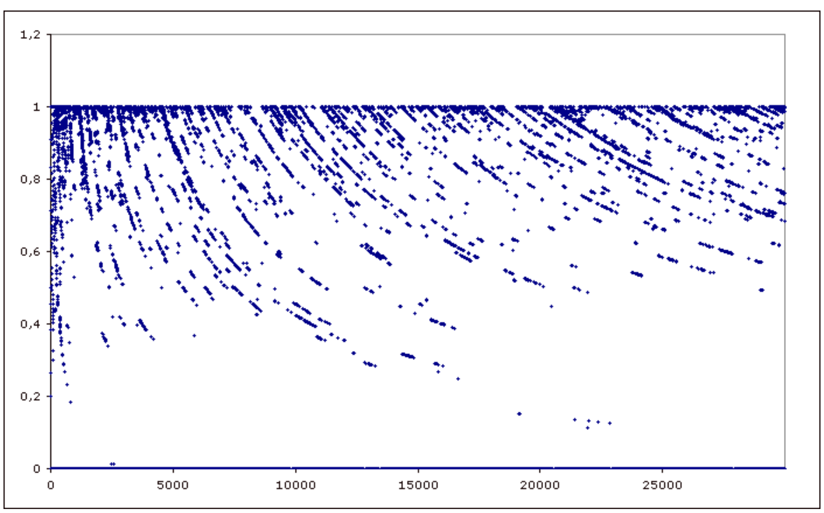
wich represents the chance to get a pair of twin primes reached by the algorithm starting with N. We plot below $\Upsilon_{twin}(N)$ for $N = 1, 2, 3, \ldots, 20000$

(fig.3)



And the graph in the range $2 \leq N \leq 30000$ of $\frac{r_N}{N}\delta(2(r_N+1)^2-1)\delta(2(r_N+1)^2+1)$

(fig.4)



The fact the x axis is blue means there are many zero values.

Roughly speaking it shows we have 50% chance to come across a twin prime of size N^2 after N iterations starting from a large random value of N. Certainly one should search for simple conditions on N in order to increase this chance. For instance let us start from $2N^2$ when $2N^2-1$ and $2N^2+1$ are already primes. This sequence of N begins:

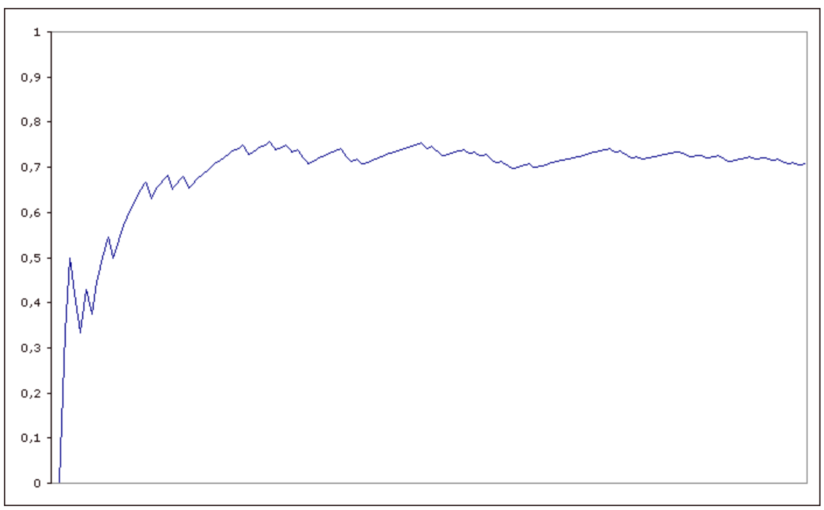
- $3, 6, 21, 24, 36, 42, 45, 87, 102, 132, 153, 186, 204, 228, 237, 273, 297, 300, 321, ...$

Let now $v(n)$ denotes the n-th term of this sequence and let:

- $\Upsilon_v(N) = \frac{1}{N}\#\left\{k \mid 1 \leq k \leq N \,\&\, \delta(2(r_{v(k)}+1)^2-1)\delta(2(r_{v(k)}+1)^2+1) = 1\right\}$

We plot below $\Upsilon_v(N)$ for $N = 1, 2, 3, \ldots, 140$

(fig.5)



So it seems we have slightly more chances to got a twin prime pair starting with $2v(n)^2$ ($\sim 70\%$ of chance to got a twin prime of order n^4) than starting with n ($\sim 50\%$ of chance producing primes of order n^2). Although some computation is needed to got the sequence $v(n)$ it could be an efficient method for generating big primes. More importantly, this observation is a striking one regarding the conjectures like 2 or 3. Indeed, it appears that when we catch a prime with a given property and run the gcd-algortihm another time (starting around this value) we have more chance to get a new prime with this property. Heuristically this should explain why there are apparently infinite "chains" of primes generated by the algorithm in conjectures 2, 3 or similar ones when we have a good starting value. Repeating the process seems to force the algorithm to reach primes every time once we are on a right track.

4 Conjecture 5 : on the Goldbach conjecture

We propose a first constructive way to prove this famous conjecture. This is somewhat unsatisfactory since there are very few non working starting values in the range where we performed the computation. Consider $N \geq 2$ and let $a(1) = N - 2$ and define for $n \geq 2$:

- $a(n) = a(n-1) - \gcd(a(n-1), (n-1)(2N - n + 1))$

Then we claim there is always a unique $g_N \in \{2, 3, ..., N - 2\}$ such that :

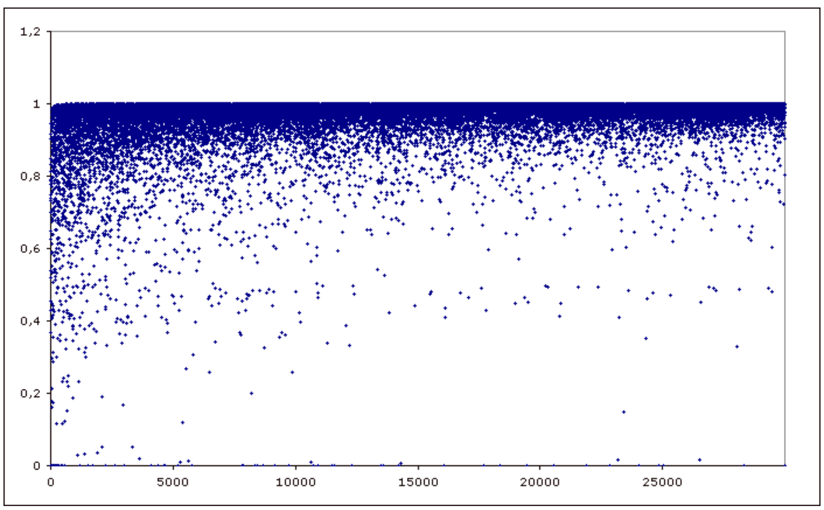
- $a(g_N) = 0$

and we have:

- g_N and $2N - g_N$ are simultanuously primes except for very few N (a set conjectured to be of measure zero).

Hereafter we plot $\frac{g_N}{N}\delta(g_N)\delta(2N - g_N)$ for $2 \leq N \leq 30000$.

(fig.6)



We can see on the x axis some zero values which become very sparse when N increases (perhaps there is no more zero value for N sufficiently large). In the conjecture 10 we provide a variation of this conjecture where clearly there is no exceptional set of non working values for N large enough.

5 The Schinzel's hypothesis H

Finally this kind of method should have considerable application. One of them, regarding what we discuss before, could be a new version of the Schinzel's hypothesis H [Sch]. Indeed, suppose $P(x)$ is a polynomial with integer coefficients and define the sequence S as follows.

- $S(1) \in \mathbb{N}$
- $S(n) = |S(n-1) - \gcd(S(n-1), P(n))|$

Under some assumptions the values of n such that $S(n) = 0$ imply $P(n+1)$ has nice arithmetical properties. This is what we have seen previously. But let us consider three additional examples.

$P(x) = x^2 + 1$

We claim that for a suitable starting value $S(1)$ such as $S(1) = 2$ we have:

- $S(n) = 0$ for infinitely many values of n.

$S(n) = 0 \Rightarrow P(n+1)$ is a prime number.

Here a table with the first values

Values of n such that $S(n) = 0$	$P(n+1)$	$\delta(P(n+1))$
3	17	1
13	197	1
203	41617	1
41813	1748410597	1

$P(x) = x^3 + 1 = (x+1)(x^2 - x + 1)$

We claim that for a suitable starting value $S(1)$ such as $S(1) = 2$ we have:

- $S(n) = 0$ for infinitely many values of n.
- $S(n) = 0 \Rightarrow (n+2)$ and $(n^2 + n + 1)$ are prime numbers.

Here a table with the first values

Values of n such that $S(n) = 0$	$n+2$	n^2+n+1	$\delta(n+2)\delta(n^2+n+1)$
3	5	13	1
69	71	4831	1
299391	299393	89635270273	1

$P(x) = (2x - 3)(x^2 - x + 1)$

We claim that for a suitable starting value $S(1)$ we have:

- $S(n) = 0$ for infinitely many values of n.
- $S(n) = 0 \Rightarrow (2n - 1)$ and $(n^2 + n + 1)$ are prime numbers.

Here a table with the first values for $S(1) = 1, 2, 5$

$S(1)$	n such that $S(n)=0$	$2n-1$	n^2+n+1
1	2	3	7
	24	47	601
	24186	48371	584986783
2	3	5	13
	69	137	4831
	658657	1317713	434093205307
5	6	11	43
	414	827	171811
	141629682	283259363	20058966965050807

And all the values for $2n-1$ and n^2+n+1 in the table are primes.

Above exemples allow us to unify all our previous observations (except for the conjecture 5 which is a special case needing more thought to be generalized). This is the next conjecture.

5.1 Conjecture 6

Clearly something very general is working and one can imagine to state a deep conjecture. Suppose $P(x)=\prod_{j=1}^{m}Q_j(x)$ where P is polynomial with integer coefficients (not all zeros). Suppose each Q_j is irreducible and $Q_1(k),Q_2(k),\ldots,Q_m(k)$ can simultanously be primes for large k. Then we claim there are infinitely many values of $S(1)$ such that:

- $S(n)=0$ for infinitely many values of n.
- $S(n)=0 \Rightarrow Q_j(n+1)$ is simultaneously prime for $1\le j\le m$.

5.1.1 Remark

This could have direct application such as generating not only big primes (see conjecture 4) but also big primes with given property like twin primes. For instance let:

- $P(x+1)=(x^2+1)(x^2+3)$
- $S(1)=4$

Then we compute big twin primes compared to the number of iterations:

Values of n such that $S(n)=0$	n^2+1	n^2+3	$\delta(n^2+1)\delta(n^2+3)$
2	5	7	1
14	197	199	1
32374	1048075877	1048075879	1

5.2 Conjecture 7

Instead of considering an infinite sequence we consider a starting value and see what happens. This gives a modified version of the conjecture 6 with more details suggesting some analytic study.

Suppose that $P(x)$ is a polynomial with integer coefficients satisfying $P(n) \geq 0$ and $P(+\infty) = +\infty$. Let us define the sequence a as follows.

- $a(1) = N \in \mathbb{N}$
- $a(n) = a(n-1) - \gcd(a(n-1), P(n))$

Then we conjecture without any other assumption on P:

- $\exists\, f(N) \in \{1, 2, \ldots, N\}$ such that $a(f(N)) = 0$.
- $\liminf_{N\to\infty} \frac{f(N)}{N} \geq 0$ and $\limsup_{N\to\infty} \frac{f(N)}{N} = 1$.

Now suppose as above $P(x) = \prod_{j=1}^{m} Q_j(x)$ where P is polynomial with integer coefficients (not all zeros). Suppose each Q_j is irreducible and $Q_1(k), Q_2(k), \ldots, Q_m(k)$ can be simultanously prime for large k. Then there are infinitely many N such that we get:

- $Q_j(f(N)+1)$ is prime for $j \in \{1, 2, \ldots, m\}$

And more precisely there is a positive proportion of N such that $Q_j(f(N)+1)$ is prime for $j \in \{1, 2, \ldots, m\}$. Let us define this proportion as follows:

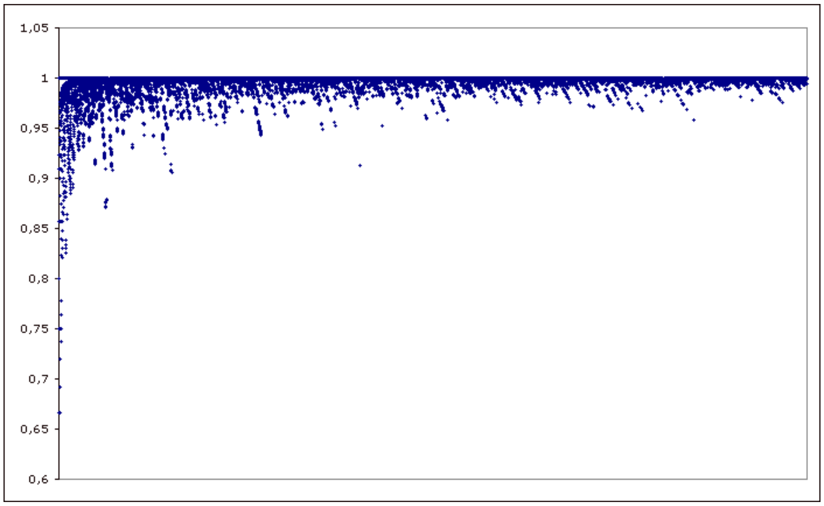
- $L(P) = \lim_{n\to\infty} \frac{1}{n} \# \Big\{ k \mid 1 \leq k \leq n \,\&\, \prod_{j=1}^{m} \delta(Q_j(f(k)+1)) = 1 \Big\}$.

Then we claim:

- Q_j is of degree 1 for any $j \in \{1, 2, \ldots, m\} \Rightarrow L(P) = 1$.
- One Q_j is irreducible of degree 2 $\Rightarrow L(P) < 1$ (see 3.3. with a sample giving $L(2x^2 - 1) \sim 0.8$).

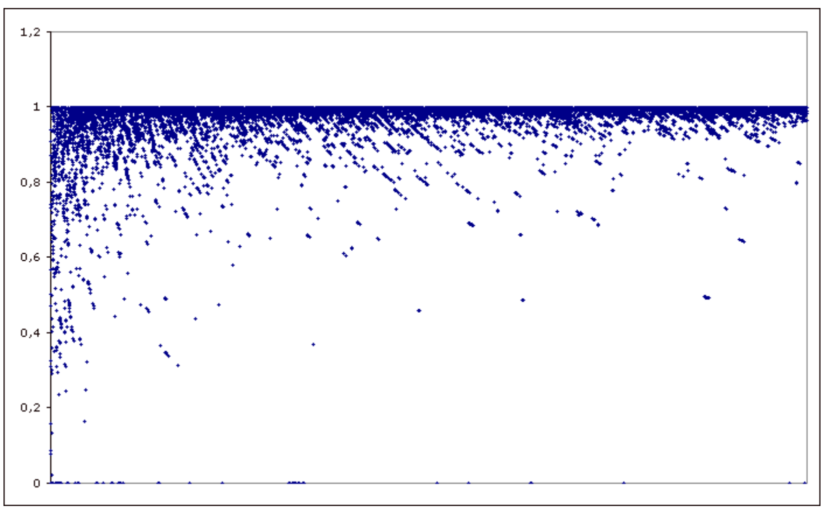
Here the graph of $\frac{f(N)}{N}\delta(f(N)+1)$ when $P(x) = x$ for $1 \leq N \leq 20000$

(fig.7)



Here the graph of $\frac{f(N)}{N}\delta(2f(N)+3)\delta(f(N)+3)\delta(f(N)+7)$ when $P(x) = (2x+1)(x+2)(x+6)$ for $N = 100k$ and $1 \leq k \leq 20000$

(fig.8)



One can see the values of N such that $\delta(2f(N)+3)\delta(f(N)+3)\delta(f(\alpha N)+7) = 0$ become sparse when N increases. We believe that for N large enough this set is of measure zero since Q_j are of degree 1.

6 Conjecture 8

We propose a way to prove again there are infinitely many twin primes but in a slightly different way than before. The recursion is the backwards version of the

Shevelev recursion. First we give a conjectural way to prove there are infinitely many prime numbers (this is very similar to what we said in conjecture 5 and should be easy to prove). Although it is not the simplest way to prove there are infinitely many primes, it seems important to start with the usual primes in order to see better how it is also working for twin primes. The two cases are indeed very similar.

6.1 There are infinitely many primes

Let:

- $a(1) = N - 2 \geq 0$
- $n \geq 2 \Rightarrow a(n) = a(n-1) - \gcd(a(n-1), n-1)$

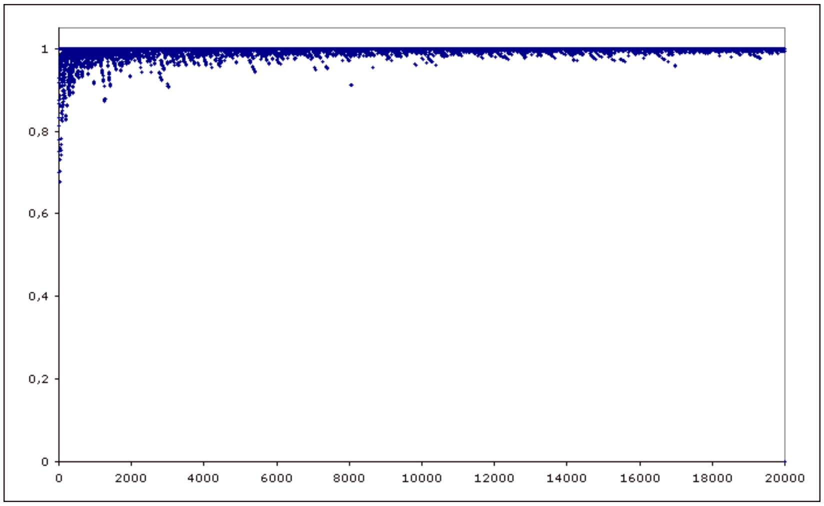
Then we claim we have:

1. $\forall N \geq 4$, $\exists f(N) \in \{2, \ldots, N-1\}$ such that $a(f(N)) = 0$.
2. $f(N) \sim N$ $(N \to \infty)$ and more precisely we claim $f(N) = N + o(N^{1/2} \log N)$
3. $f(N)$ is prime.
4. $f(N) = N - 1 \Leftrightarrow N - 1$ is a prime number.
5. $f(N) = N - 2 \Leftrightarrow N - 2$ is an odd prime number.
6. $f(N) = N - 3$ or $N - 4 \Leftrightarrow N - 3$ (resp $N - 4$) is a prime number of form $6k + 1$.
7. $f(N) = N - 5$ or $N - 6 \Leftrightarrow N - 5$ (resp $N - 6$) is a prime number of form $30k + 1$.

Since $f(N) \to \infty$ as $N \to \infty$ (from 2.) there are infinitely many primes (from 3.).

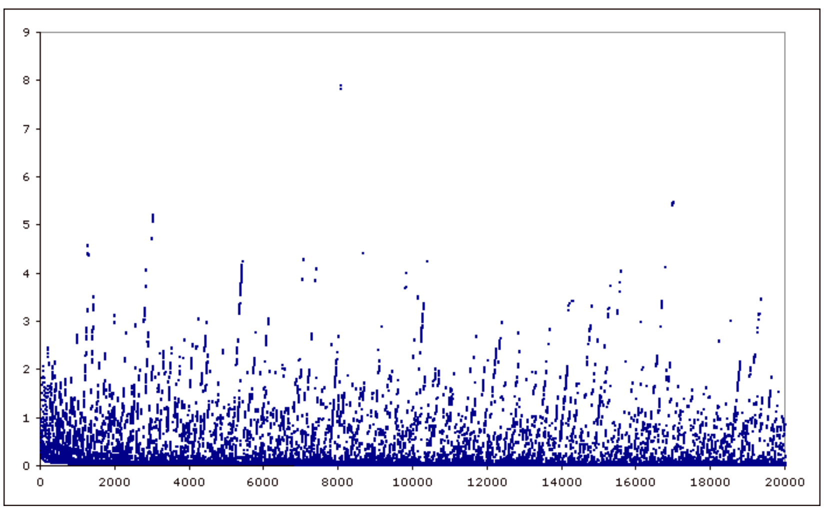
Here the graph of $\frac{f(N)}{N-1}\delta(f(N))$ for $2 \leq N \leq 20000$.

(fig.9)



To see the more precise behaviour claimed in 2. we plot $\frac{N-f(N)}{n^{1/2}}$ for $3 \leq N \leq 20000$.

(fig.10)

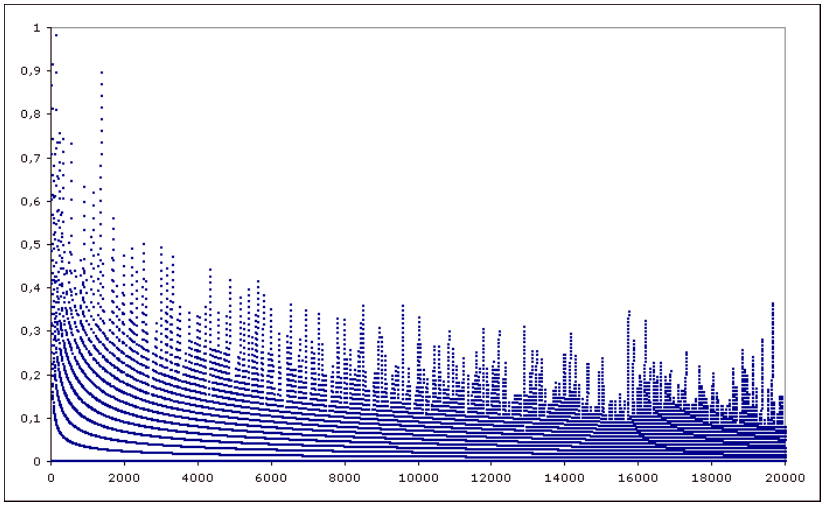


This is quite erratic and divided by $\log n$ it should converge very slowly to zero but more experiments are needed to confirm the tendancy.

Remark

It is worth to compare this sequence $f(N)$ to $p(\pi(N))$ where $p(n)$ denotes the n-th prime and $\pi(x)$ is the prime couting function. So that $p(\pi(N))$ is the largest prime $\leq N$. Here we still plot $\frac{N-p(\pi(N))}{n^{1/2}}$ for $3 \leq N \leq 20000$.

(fig.11)



Things are less erratic and here the graph goes more certainly to zero without dividing by $\log n$. We will discuss about an important consequence of this conjectured behaviour in section 9.

6.2 There are infinitely many twin primes

Let:

- $a(1) = N - 2 \geq 0$
- $a(n) = a(n-1) - \gcd(a(n-1), n + (-1)^n)$

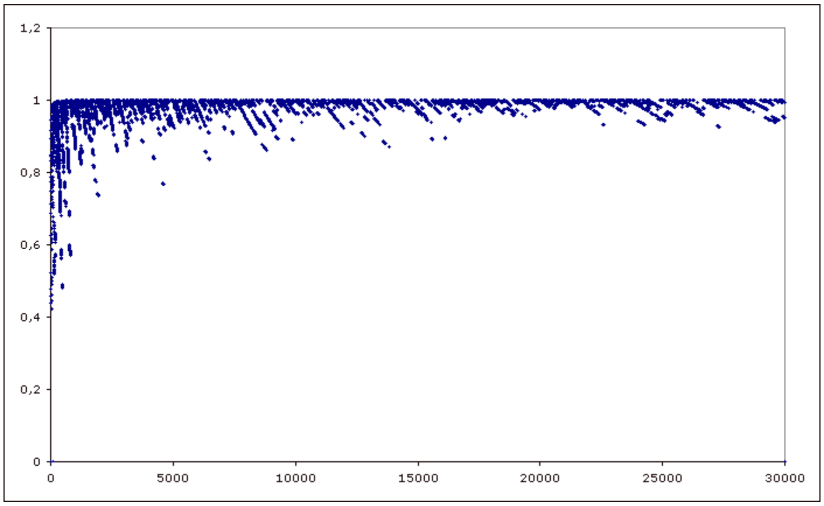
Then we claim we have:

1. $\forall N \geq 2$, $\exists h(N) \in \{1, 2, \ldots, N\}$ such that $a(h(N)) = 0$.
2. $h(N) \sim N$ $(N \to \infty)$.
3. $\forall N \geq 98$, $h(N)$ and $h(N) + 2$ are necessarily simultanously primes.
4. For $N \geq 4$ we have $h(N) = N - 1 \Leftrightarrow (N - 1, N + 1)$ is a pair of twin primes.
5. For $N \geq 13$ we have $h(N) = N - 2 \Leftrightarrow N$ is the greater of a pair of twin primes.
6. For $N \geq 14$ we have $h(N) = N - 3 \Leftrightarrow N - 3$ is the lesser of a pair of twin primes.

Since $h(N) \to \infty$ as $N \to \infty$ (from 2) there are infinitely many twin primes (from 3).

Here the graph of $\frac{h(N)}{N}\delta(h(N))\delta(h(N) + 2)$ for $1 \leq N \leq 30000$

(fig.12)



We see there is no more zero value for $N \geq 98$.

7 Conjecture 9

Let $m \in \mathbb{N}$ and define the sequence a as follows:

- $a(1) \in \mathbb{N}$
- $a(n) = |a(n-1) - \gcd(a(n-1), mn + b(n))|$

Where $(b_n)_{n \geq 1}$ is a periodic sequence of period length β such that:

- $mn + b_1, mn + b_2, \ldots, mn + b_\beta$ can be simultanously prime for n large.

Then we claim we have:

- $a(n) = 0$ for infinitely many values of n.
- for n large enough we have $a(n) = 0 \Rightarrow m(n+1) + b_1, m(n+1) + b_2, \ldots, m(n+1) + b_\beta$ are simultanously prime.
- the rate of growth of the sequence of n_i such that $a(n_i) = 0$ is like $(m+1)^i$.

See APPENDIX 6 for experiments where we provide examples for β-periodic sequences and $\beta \in \{2, 3, 4, 5, 6\}$.

Generalisation

Although we check few cases of this generalisation it is worth to mention this result which is an interesting variation of the conjectures 6 and 9. Suppose Q_j is irreducible and $Q_1(k), Q_2(k), \ldots, Q_\beta(k)$ can be simultanously prime for large k. Suppose $(b_n)_{n\geq 1}$ is a periodic sequence of period length β such that $\{b(i)\}_{1\leq i\leq \beta}$ is a permutation of $\{i\}_{1\leq i\leq \beta}$ and define the sequence a as follows:

- $a(1) \in \mathbb{N}$
- $a(n) = \left|a(n-1) - \gcd(a(n-1), Q_{b(n)}(n))\right|$

Then we claim we have:

- $a(n) = 0$ for infinitely many values of n.
- for n large enough we have $a(n) = 0 \Rightarrow Q_1(n+1), Q_2(n+1), \ldots, Q_\beta(n+1)$ are simultanously prime.

Extension of the conjecture 6

We can extend what we done in conjecture 6 (and also in conjecture 7) with twin primes to prime triplet or any sort of m-uplet. For instance let us see how this is also working with prime triplet of type $(p, p+2, p+6)$. Let $(b(n))_{n\geq 1}$ be the 3-periodic sequence $\{2, 6, 0\}$ and define the sequence:

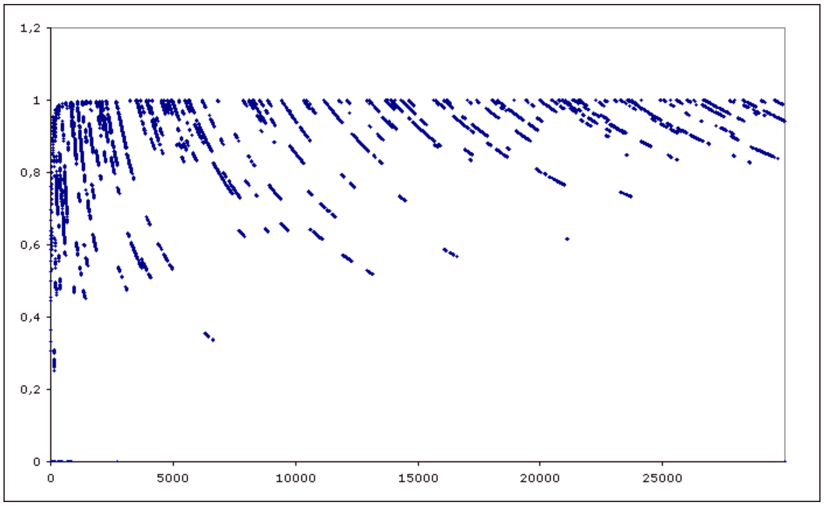
- $a(n) = N - 6 \geq 0$
- $a(n) = a(n-1) - \gcd(a(n-1), n + b(n))$

Then we claim:

1. $\forall N \geq 6\ \exists f(N) \in \{1, 2, \ldots, N\}$ such that $a(f(N)) = 0$.
2. $f(N) \sim N\ (N \to \infty)$.
3. $\forall N \geq 2735$, $(f(N)+1, f(N)+3, f(N)+7)$ is a prime triplet.
4. For $N \geq 5$ we have $h(N) = N - 1 \Leftrightarrow (N, N+2, N+6)$ is a prime triplet of type $(p, p+2, p+6)$ (sequence $A022004$ in [Slo])

Here the graph of $\frac{f(N)}{N}\delta(f(N)+1)\delta(f(N)+3)\delta(f(N)+7)$ for $6 \leq N \leq 30000$

(fig.13)



We see that for $N \geq 2735$ there is no more zero value.

8 Conjecture 10: on the Goldbach conjecture

Finally we found a way to adapt the method to the Goldbach conjecture. This was not easy since this recursion is very sensitive (to the initial value and to what we put in the gcd). This formulation of the conjecture is nicer than the conjecture 5 and is very similar to the conjecture 6 and our formulation of the weak twin prime conjecture. Indeed the method works in both cases for N large enough. Namely define the sequence a for $N \geq 2$ by:

- $a_1 = N - 2$ and for $n \geq 2$ by $a_n = a_{n-1} - \gcd(a_{n-1}, N - (-1)^n(N - n))$

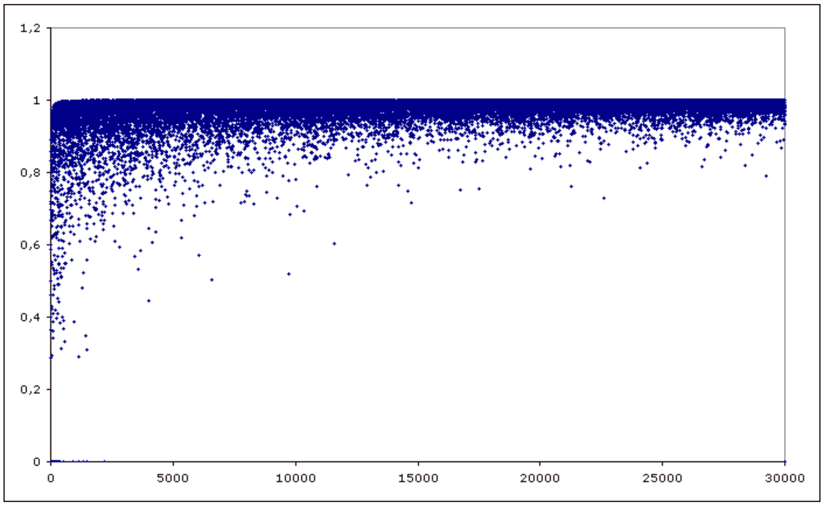
Then we claim:

- there is a least $g_N \in \{2, 3, ..., N - 1\}$ such that $a_{g_N} = 0$.
- $g_N \sim N$ $(N \to \infty)$
- for $N \geq 2208$ we have $g_N + 1$ and $2N - g_N - 1$ which are simultanously primes.

Thus the Goldbach conjecture would be true.

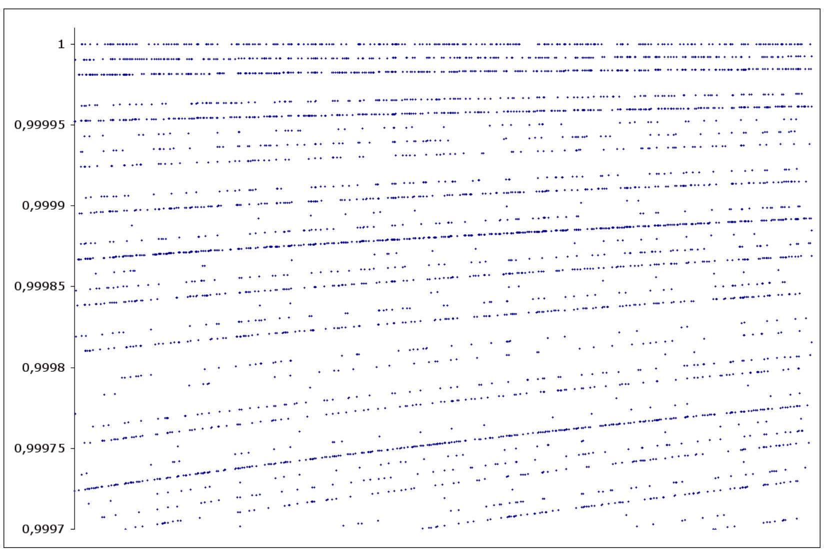
Hereafter a graph of $\frac{g_N}{N}\delta(g_N + 1)\delta(2N - g_N - 1)$ for $2 \leq N \leq 30000$.

(fig.14)



We see there is no more zero value for $N \geq 2208$.
Next we provide a zoom of this picture in the range $100000 \leq N \leq 130000$.

(fig.15)



The lines appearing regularly are due to the following observation.

Relation with the weak twin prime conjecture

Moreover we claim that $\forall m \geq 2 \in \mathbb{N}$

- $g_N = N - m$ for infinitely many values of N.

And we add withouth any other condition on N:

- $g_N = N - m \Leftrightarrow N - m + 1$ and $N + m - 1$ are both primes.

Thus for $m = 2$ this means there are infinitely many twin primes.

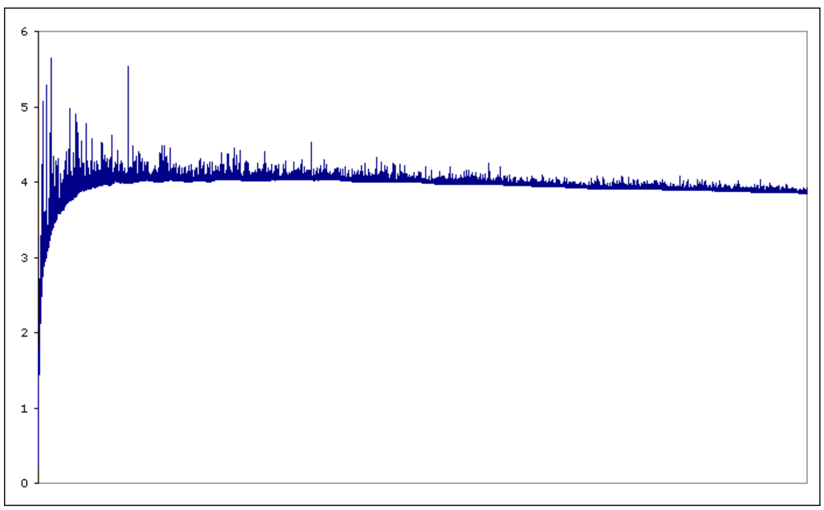
The Goldbach constant

It is worth to consider analytic aspects of this formulation of the Goldbach conjecture. For instance we claim:

- $\sum_{k=3}^{n} \left(1 - \frac{g_k}{k-1}\right) \sim C\sqrt{n}$ $(n \to \infty)$ where $C \leq 4$.

Hereafter the graph of $n^{-1/2} \sum_{k=3}^{n} \left(1 - \frac{g_k}{k-1}\right)$ for $1 \leq n \leq 2000000$

(fig.16)



9 On a conjecture of Legendre

In this study we give new formulation using gcd-algorithms for 3 problems among a list of 4 problems considered by Landau in 1912 as "unattackable at the present state of science"[Lan] [Pin]:

- are they infinitely many primes numbers of form $n^2 + 1$?
- the twin prime conjecture.
- the Goldbach conjecture.

So a Landau problem is missing but our gcd-algorithm formulation still works for this fourth probem attributed to Legendre. This remaining conjecture says:

- there is always a prime between 2 consecutive squares.

Our belief is that for N large enough the algorithm given in 6.1 starting with $a(1) = (N+1)^2 - 2$ produces a prime value $f((N+1)^2)$ greater than N^2. But this comes not from our asymptotic claim in 6.1.

Remark

Of course the situation is clearer if we use the sequence $p(\pi(N))$ introduced in 6.1. and the conjectured asmptotic behaviour which works for proving the Legendre conjecture (the behaviour toward zero of $\frac{N-p(\pi(N))}{\sqrt{N}}$ shown in 6.1.). But for this one one must consider RH or things like that. Our hope is that the sequence $f(N)$ despite its random nature allow us to avoid such considerations.

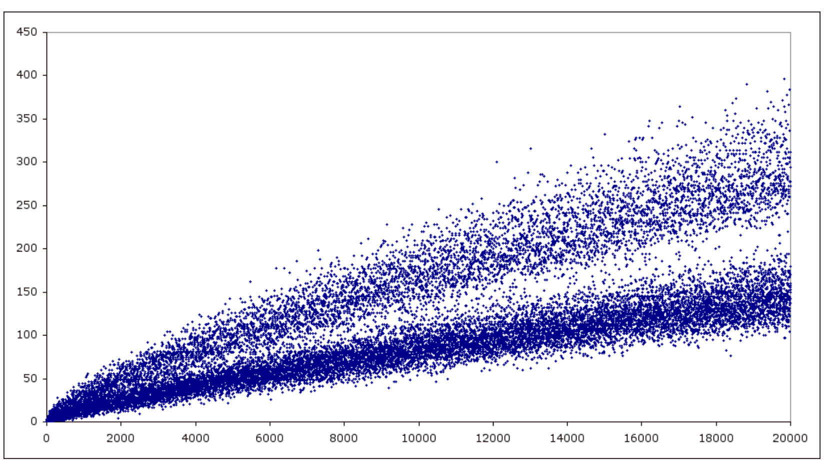
A Goldbach variation

We came also across a stronger hypothesis than the original Legendre conjecture and it is worth to present this "Goldbach variation". We don't know if this last conjecture could be helpful for finding a "gcd-formulation" of Legendre conjecture like our conjecture 10 for the Goldbach conjecture (we found nothing clear even for N large enough). Namely we claim:

- $\forall N \geq 175$ there is at least one k satisfying $2 \leq k \leq 2N$ such that N^2+k+1 and $(N+1)^2 - k$ are simultanously primes.

Here the graph of the function of N for $2 \leq N \leq 20000$ representing the number of k verifying $2 \leq k \leq 2N$ and such that $N^2 + k + 1$ and $(N+1)^2 - k$ are simultanously primes.

(fig.17)



It looks like the classical Goldbach comet.

Concluding remark

The Schinzel hypothesis doesn't cover the Goldbach conjecture but in view of our conjecture 10 we think there is a possible extension of the conjecture 6. We mention that some people didn't think Rowland simple theorem and related ideas of V. Shevelev could lead to interesting results regarding the theory of the prime numbers. We hope we show this advice is wrong and that some very important rules need to be better understood and that our results could inspired some analytic or probabilistic studies. Regarding the probabilistic approach we think we have other striking arguments. In our study we consider only natural objects in the gcd such as polynomials or periodic sequences. It appears we can play with less predictive functions and for instance we observe we can generate twin primes with any function r taking values in $\{0, 2\}$ sufficiently "randomly".[1] To see this let us consider the differences of the Beatty sequence for π i.e. $v_n = \lfloor \pi n \rfloor - \lfloor \pi(n-1) \rfloor$ which takes values in $\{3, 4\}$ and is not a periodic sequence (this is the sequence A063438 in [Slo]).

Then let $r_n = 2(v_n - 3)$ wich takes values in $\{0, 2\}$ and define the sequence:

- $a(1) = N - 2 \geq 0$
- $a(n) = a(n-1) - \gcd(a(n-1), n + r_n)$

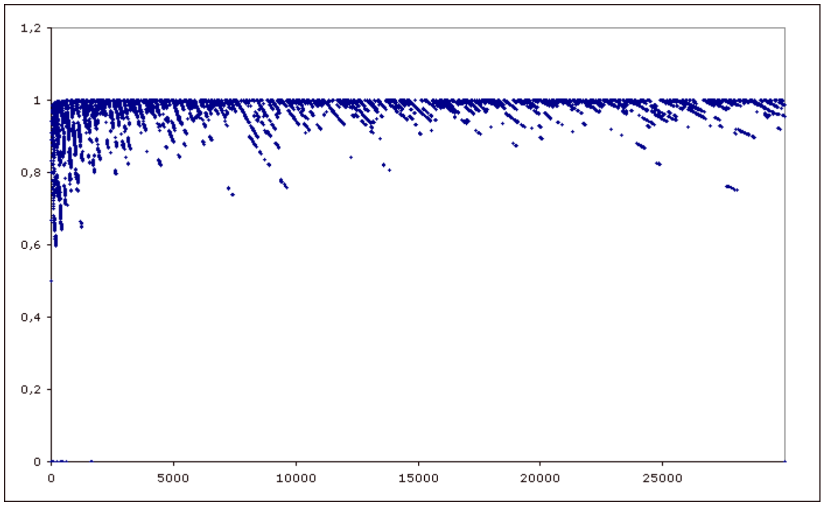
Then we claim:

1. $\forall N \geq 2$, $\exists f(N) \in \{1, 2, \ldots, N\}$ such that $a(f(N)) = 0$.
2. $f(N) \sim N$ $(N \to \infty)$.
3. $\forall N \geq 1649$, $f(N) + 1$ and $f(N) + 3$ are necessarily simultanously primes.

Hereafter we plot $\frac{f(N)}{N}\delta(f(N)+1)\delta(f(N)+3)$ for $2 \leq N \leq 30000$.

(fig.18)

[1] This needs to be clarified but we think $\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} r_j \notin \{0, 2\}$ is a sufficient condition and we believe the random process comes mainly from the gcd.



We can see there is no more zero value for $N \geq 1649$.

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APPENDIX 1

Table related to the conjecture 1

k	$b_1(k)$	$\delta(b_1(k))$	$b_1(k)2^{-k}$
1	2	1	1.000000000
2	5	1	1.250000000
3	11	1	1.375000000
4	23	1	1.437500000
5	47	1	1.468750000
6	79	1	1.234375000
7	157	1	1.226562500
8	313	1	1.222656250
9	619	1	1.208984375
10	1237	1	1.208007812
11	2473	1	1.207519531
12	4909	1	1.198486328
13	9817	1	1.198364257
14	19603	1	1.196472167
15	39199	1	1.196258544
16	78193	1	1.193130493
17	156019	1	1.190330505
18	311347	1	1.187694549
19	622669	1	1.187646865
20	1244149	1	1.186512947
21	2487739	1	1.186246395
22	4975111	1	1.186158895
23	9950221	1	1.186158776
24	19900399	1	1.186156213
25	39800797	1	1.186156183
26	79601461	1	1.186154201
27	159202369	1	1.186150081
28	318404629	1	1.186149675
29	636788881	1	1.186111720

APPENDIX 2

In the following table one can see the number of iterations n giving $a(n) = 0$ (left column) and the corresponding value $mn + m - 1$ in the middle column. The right colum gives the value of the function "isprime" using pari-gp.

$m = 5$

n such that $a(n) = 0$	$5n + 4$	$\delta(5n + 4)$
2	14	0
17	89	1
95	479	1
575	2879	1
3419	17099	1
19967	99839	1
119801	599009	1
718571	3592859	1
4311419	21557099	1
25867229	129336149	1

$m = 6$

n such that $a(n) = 0$	$6n + 5$	$\delta(6n + 5)$
2	17	1
16	101	1
76	461	1
466	2801	1
3258	19553	1
22774	136649	1
159306	955841	1
1114124	6684749	1
77796204	46777229	1
54573434	327440609	1

$m = 7$

n such that $a(n) = 0$	$7n + 6$	$\delta(7n + 6)$
2	21	0
23	167	1
113	797	1
899	6299	1
6973	48817	1
55633	389437	1
444901	3114313	1
3558575	24910031	1
28468585	199280101	1

$m = 8$

n such that $a(n) = 0$	$8n + 7$	$\delta(8n + 7)$
2	23	1
20	167	1
188	1511	1
1682	13463	1
15020	120167	1
134504	1076039	1
1210544	9684359	1
10894874	87158999	1
98053784	784430279	1

$m = 9$

n such that $a(n) = 0$	$9n + 8$	$\delta(9n + 8)$
2	26	0
29	269	1
299	2699	1
2935	26423	1
28869	259829	1
288385	2595473	1
2883809	25954289	1
28832339	259491059	1

$m = 100$

n such that $a(n) = 0$	$100n + 99$	$\delta(100n + 99)$
2	299	0
226	22699	1
22810	2281099	1
2303908	230390899	1

To see further how this is working nicely let us take $m = 1000$.

n such that $a(n) = 0$	$1000n + 999$	$\delta(1000n + 999)$
2	2999	1
2986	2986999	1
2917174	2917174999	1

Finding big primes (from 1.3.1) We take $m = 2^k$ for $100 \leq k \leq 200$ so that $b(1) = 3$ and $a(b(1)+1) = 3.2^k - 1$ is the first record value (not necessarily prime of course and it would be interesting to find conditions on the initial value in order to have very few terms to compute). Next we stop the algorithm after $\sim 10^5$ iterations (few seconds are needed each time). In the following table we keep the values of k such that $(m+1)a(b(1)+1)+m+m\sum_{j=4}^{100000}(a(j+1) - a(j) + 1)$ is a prime value of size $\sim 3.4^k$and should be our second record value $a(b(2)+1)$. We notice we came across 11 primes in that range.

k	$a(b(2)+1)$
100	4820814132776970826625886270541990288599672051495735016816639
107	78984218751417890023438520762741628349070105828337814558023352319
118	331283824645947061796868281389238893531663193824530694464237235871940607
127	86844066927987146567678238756515930889442064948849015334398126094787194912767
131	22232081133564709521325629121668078276785918415807297904592970941163697569005567
132	88928324534258838085302516486672313230587227346404196132053363353462480809492479
141	23312026706708748851033542881882226879708773352492831418233933930240272261700160323583
149	1527776982250864564701334266307033620788600839243814368422923338893418735708806455350001663
158	400497569235170640449066569906791021488007090321237481912409760223102559592991367888359129087999
164	1640438043587258943279376670338216024014877072467663636588378426222267002018688816605037919597494271
172	107507747624534602106757229467285325349838983867263171688888770338883656854354650572386871166337445527551

pari gp code

```
for(k=100,200,m=2^k;a=1;S=0;M=0;

for(n=2,10^5,t=a;a=abs(a-gcd(a,m*n-1));M=M+if(a-t>0,a-t,0);S=S+if(a-t<0,a-t+1,0);

if(abs(t-a)>1,if(isprime((m+1)*M+m+m*S)==1,print(k,'' '',(m+1)*M+m+m*S,"")))))
```

APPENDIX 3

Let $a(1) = m$ and

- $a(n) = \left| a(n-1) - \gcd(a(n-1), n^2 - 1) \right|$.

Here a table for various starting values m allowing us to exhibit distinct pairs of twin primes (the other values $m = 10k$ for $1 \le k \le 14$ produce also pairs of twin primes but they are all in this list keeping the distinct pairs only).

m	n such that $a(n) = 0$	$\delta(n)$	$\delta(n+2)$
10	11	1	1
	137	1	1
	19181	1	1
20	17	1	1
	281	1	1
	79559	1	1
30	29	1	1
	881	1	1
	777011	1	1
40	41	1	1
	1787	1	1
	3198731	1	1
60	59	1	1
	3527	1	1
	12448001	1	1
70	71	1	1
	5099	1	1
	26010041	1	1
100	101	1	1
	10499	1	1
	110258891	1	1
110	107	1	1
	11699	1	1
	136890881	1	1
140	137	1	1
	19181	1	1
	367953497	1	1

And 2 more starting values

m	n such that $a(n) = 0$	$\delta(n)$	$\delta(n+2)$
200	197	1	1
	39161	1	1
	1533646397	1	1
300	281	1	1
	79559	1	1
	6329815697	1	1

We also provide this computation but using Rowland-Shevelev recursion:

- $a_1 = 3$ then $a_n = a_{n-1} + \gcd(a_{n-1}, n(n-2))$

Here the table of records of the differences $a_n - a_{n-1}$ which are the lower terms of some twin prime pairs.

records of $a_n - a_{n-1}$
3
5
11
41
101
239
521
1049
2111
4229
10331
20747
41519
83219
166847
333791
669479
1341017
2682539
5365229
10732751
21466259
42932567
85865321
171730679
343461647
686929511
1373891861
2747784329
5495586839

APPENDIX 4

Here $a(1) = 4m^2$ and $a(n) = |a(n-1) - \gcd(a(n-1), n(n+2m))|$ and the 3 first values of n such that $a_n = 0$.

m	$n+1$ when $a(n) = 0$	$\delta(n+1)$	$\delta(n+2m+1)$
2	13	1	1
	193	1	1
	38197	1	1
3	31	1	1
	1091	1	1
	1197193	1	1
4	59	1	1
	4013	1	1
	16138511	1	1
5	97	1	1
	10477	1	1
	109880317	1	1
6	139	1	1
	20521	1	1
	421370778	1	1

APPENDIX 5

$p = 2$

Values of n such that $a(n) = 0$	$2(n+1)^2 - 1$	$\delta(2(n+1)^2 - 1)$
3	31	1
35	2591	1
2627	13812767	1
11993333	287680120871111	1

$p = 3$

Values of n such that $a(n) = 0$	$3(n+1)^2 - 1$	$\delta(3(n+1)^2 - 1)$
3	47	1
51	8111	1
7665	176302667	1
176310323	93255991046954927	1

$p = 5$

Values of n such that $a(n) = 0$	$5(n+1)^2 - 1$	$\delta(5(n+1)^2 - 1)$
3	79	1
83	35279	1
34647	6002419519	1

$p = 7$

Values of n such that $a(n) = 0$	$7(n+1)^2 - 1$	$\delta(7(n+1)^2 - 1)$
5	251	1
257	465947	1
461009	1487711540699	1

$p = 11$

Values of n such that $a(n) = 0$	$11(n+1)^2 - 1$	$\delta(11(n+1)^2 - 1)$
11	1583	1
1419	22180399	1
22181509	5412213244681099	1

$p = 17$

Values of n such that $a(n) = 0$	$17(n+1)^2 - 1$	$\delta(17(n+1)^2 - 1)$
11	2447	1
2417	99394307	1
87543523	130285766103755791	1

APPENDIX 6

We define the sequence a as follows:

- $a(1) = k$
- $a(n) = |a(n-1) - \gcd(a(n-1), mn + b(n))|$

We then give tables supporting the conjecture 9 for various k, m and b periodic sequence which is given by its period $\{b_1, b_2, ..., b_\beta\}$.

2 periodic sequence

$$b = \{0, 2\}, m = 1, k = 100$$

n such that $a(n) = 0$	$\delta(n+1)\delta(n+3)$
100	1
196	1
310	1
616	1
1228	1
2380	1
4648	1
8860	1
17026	1
33808	1
67408	1
134680	1
267718	1
535348	1
1069216	1
2138398	1
4275640	1
8545696	1
17091376	1
34182748	1
68365468	1
136730638	1
273461158	1
546917140	1
1093813726	1
2187610990	1

$$b = \{-1, 1\}, m = 10, k = 100$$

n such that $a(n) = 0$	$\delta(10n+9)\delta(10n+11)$
101	1
1115	1
12203	1
130013	1
1427183	1
15692309	1
172614683	1

$$b = \{-2, 2\}, m = 3, k = 100$$

n such that $a(n) = 0$	$\delta(3n+1)\delta(3n+5)$
34	1
116	1
434	1
1576	1
6102	1
21154	1
84606	1
338386	1
1351382	1
5405526	1
21622094	1

3 periodic sequence

$$b = \{2, 6, 0\}, m = 1, k = 3000$$

n such that $a(n) = 0$	$\delta(n+1)\delta(n+3)\delta(n+7)$
2686	1
5230	1
10456	1
19420	1
29566	1
54496	1
105526	1
211060	1
408430	1
802126	1
1600216	1
3200200	1
6393910	1
12783496	1
25566676	1
51095410	1
102190390	1
204347176	1

$b = \{0, 4, 6\}, m = 1, k = 3000$

n such that $a(n) = 0$	$\delta(n+1)\delta(n+5)\delta(n+7)$
2082	1
3462	1
6546	1
12372	1
23052	1
44262	1
85086	1
167016	1
313986	1
622476	1
1237206	1
2452752	1
4882326	1
9753276	1
19504866	1

$b = \{-2, 2, -4\}, m = 5, k = 2000$

n such that $a(n) = 0$	$\delta(5n+1)\delta(5n+3)\delta(5n+7)$
1772	1
9806	1
58274	1
343772	1
2057378	1
12342518	1
73895180	1

4 periodic sequence

$$b = \{1, 7, 11, -1\}, m = 1, k = 20000$$

n such that $a(n) = 0$	$\delta(n)\delta(n+2)\delta(n+8)\delta(n+12)$
19421	1
36779	1
70841	1
138239	1
236771	1
443159	1
882239	1
1758389	1
3376979	1
6631901	1
13236539	1
26425379	1
52658999	1
104785649	1
209560319	1
418973999	1

$$b = \{2, 6, 18, 26\}, m = 1, k = 100000$$

n such that $a(n) = 0$	$\prod_{j=1}^{4} \delta(n + b(j) + 1)$
83200	1
150190	1
294754	1
573844	1
1107784	1
2208064	1
4171774	1
8332840	1
16461094	1
32756680	1
65166814	1
130175344	1
260331034	1
520484380	1
1040389234	1
2080515244	1
4161006904	1
8321226490	1

5 periodic sequence

$$b = \{2, 8, 12, 14, 18\}, m = 1, k = 1500000$$

n such that $a(n) = 0$	$\prod_{j=1}^{5} \delta(n + 1 + b(j))$
1212424	1
2270674	1
4271158	1
8358658	1
15875398	1
31562608	1
62555878	1
125087098	1
249509788	1
477331018	1
954642034	1
1905881278	1
3809937208	1

$$b = \{2, 8, 12, 14, 18\}, m = 1, k = 2000000$$

n such that $a(n) = 0$	$\prod_{j=1}^{5} \delta(n + 1 + b(j))$
1460728	1
2839924	1
4218154	1
8068438	1
16130884	1
32240278	1
64123234	1
127725328	1
254416288	1
507764278	1

6 periodic sequence

$$b = \{0, 2, 6, 14, 30, 62\}, m = 1, k = 2000000$$

n such that $a(n) = 0$	$\prod_{j=1}^{6} \delta(n+1+b(j))$
1460728	1
2839924	1
4218154	1
8068438	1
16130884	1
32240278	1
64123234	1
127725328	1
254416288	1
507764278	1