THE MÉNAGE PROBLEM WITH A KNOWN MATHEMATICIAN

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ABSTRACT. We give a solution of the following combinatorial problem: "A known mathematician N found himself with his wife among the guests, which were $n(\geq 3)$ married couples. After seating the ladies on every other chair at a circular table, N was the first offered to choose an arbitrary chair but not side by side with his wife. To find the number of ways of seating of other men after N chose a chair (under the condition that no husband is beside his wife)." We discuss also the problem: "For which values of n the number of ways of seating of other men does not depend on a choice by N his chair?"

1. INTRODUCTION

In 1891, Lucas [2] formulated the following "ménage problem":

Problem 1. *To find the number M_n of ways of seating n married couples at a circular table, men and women in alternate positions, so that no husband is next to his wife.*

After seating the ladies by $2n!$ ways we have

$$M_n = 2n!U_n, \tag{1.1}$$

where U_n is the number of ways of seating men.

Earlier Muir [4] solved a problem posed by Tait (cf. [4]): to find the number H_n of permutations π of $\{1, ..., n\}$ for which $\pi(i) \neq i$ and $\pi(i) \neq i+1$ (mod n), $i = 1, ..., n$. By a modern language, $H_n = per(J_n - I - P)$, where J_n is $n \times n$ matrix composed by 1's only, $I = I_n$ is the identity matrix and $P = P_n$ is the incidence matrix corresponding to the cycle $(1, 2, ..., n)$ (cf. [3]). Simplifying Muir's solution, Cayley [1] found a very simple recursion for $H_n : H_2 = 0, H_3 = 1$, and for $n \geq 4$,

$$(n-2)H_n = n(n-2)H_{n-1} + nH_{n-2} + 4(-1)^{n+1}. \tag{1.2}$$

Only in 1934 due to celebrated research by Touchard [9], it became clear that

$$H_n = U_n \tag{1.3}$$

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and thus formulas (1.1)-(1.2) give a recursion solution of the ménage problem. Moreover, Touchard gave a remarkable explicit formula

$$U_n = \sum_{k=0}^{n}(-1)^k \frac{2n}{2n-k}\binom{2n-k}{k}(n-k)! \tag{1.4}$$

A beautiful proof of (1.4 with help of the rook technique one can find in [5].

The first terms of the sequence $\{U_n\}$, for $n \geq 2$, are (cf. [8])

$$0, 1, 2, 13, 80, 579, 4738, 43387, 439792, 4890741, 59216642, ... \tag{1.5}$$

Note that formulas for U_n in other forms are given by Wayman and Moser [10] and Shevelev [6].

In the present paper we study the following problem.

Problem 2. *A known mathematician N found himself with his wife among the guests, which were $n(>= 3)$ married couples. After seating the ladies on every other chair at a circular table, N was the first offered to choose an arbitrary chair but not side by side with his wife. To find the number of ways of seating of other men, after N chose a chair, under the condition that no husband is beside his wife.*

We also discuss a close problem:

Problem 3. *For which values of n the number of ways of seating of other men in Problem 2 does not depend on a choice by mathematician N his chair?*

2. A comment to representation of solution of Problem 1 by $per(J_n - I - P)$ in connection with Problem 2

Denote $2n$ chairs at a circular table by the symbols

$$1, \overline{1}, 2, \overline{2}, ..., n, \overline{n} \tag{2.1}$$

over clockwise. Ladies occupy either chairs $\{1, ..., n\}$ or chairs $\{\overline{1}, ..., \overline{n}\}$. Let they occupy, say, chairs $\{\overline{1}, ..., \overline{n}\}$. Then to every man we give a number i, if his wife occupies the chair $\overline{i}$. Now the i-th man, for $i = 1, ..., n-1$, can occupy every chair except of chairs $i, i+1$, while the n-th man cannot occupy chairs n and 1. Denoting in the corresponding $n \times n$ incidence matrix the prohibited positions by 0's and other positions by 1's, we obtain the matrix $J_n - I - P$. Now, evidently, to every seating the men corresponds a diagonal of 1's in this matrix. This means that

$$U_n = per(J_n - I - P). \tag{2.2}$$

Let in Problem 2 the wife of mathematician N occupy, say, chair $\overline{1}$.

Let us measure the distance between N and his wife via the number of spaces between the separating them chairs over clockwise. Now, if N occupies the r-th chair, then the distance equals to $r-1$. In the incidence matrix, the r-th chair of the first man corresponds to position $(1, r)$. Denote the matrix obtained by the removing the first row and the r-th column of the matrix $J_n - I - P$ by $(J_n - I - P)[1|\ r]$. Then, we obtain the following lemma.

Lemma 1. *If N chose a chair at the distance $r-1$ from his wife, then the number of seating of other men equals to $per((J_n - I - P)[1|\ r])$.*

Note that, if to consider numeration 2.1 over *counterclockwise,* then we obtain a quite symmetric result in which r corresponds to $n-r+3$, $r = 3, ..., n$, such that as a corollary of Lemma 1 we have

(2.3) $per((J_n - I - P)[1|\ r]) = per((J_n - I - P)[1|\ n-r+3]),\ \ r = 3, ..., n.$

3. Rook lemmas

Here we place several results of the classic Kaplansky-Riordan rook theory (cf. [5], Ch. 7-8).

Let M be a rectangle (quadratic) $(0,1)$-matrix M.

Definition 1. *The polynomial*

$$R_M(x) = \sum_{j=0}^{n} \nu_j(M) x^j \tag{3.1}$$

where $\nu_0 = 1$ and ν_j is the number of ways of putting j non-taking rooks on positions 1's of M, is called rook polynomial.

Note that n is the maximal number for which there exists at least one possibility to put n non-taking rooks on positions 1's of M.

Lemma 2. *If M is a quadratic matrix with the rook polynomial (3.1), then*

$$per(J_n - M) = \sum_{j=0}^{n} (-1)^j \nu_j(M)(n-j)! \tag{3.2}$$

Definition 2. *Two submatrices M_1 and M_2 of $(0,1)$-matrix M are called* disjunct *if no 1's of M_1 in the same row or column as those of M_2.*

From Definition 1 the following lemma evidently follows.

Lemma 3. *If $(0,1)$-matrix M consists of two disjunct submatrices M_1 and M_2, then*

$$R_M(x) = R_{M_1}(x) R_{M_2}(x). \tag{3.3}$$

Consider a position (i, j) of 1 in matrix M. Denote $M^{(0(i,j))}$ the matrix obtained from M after replacing 1 in position (i, j) by 0. Denote $M^{(i,j)}$ the matrix obtained from M by removing the i-th row and j-column.

Lemma 4. *We have*

$$R_M(x) = xR_{M^{(i,j)}} + R_{M^{(0(i,j))}}. \tag{3.4}$$

Consider so-called simplest connected staircase $(0, 1)$-matrices. Such matrix is called k-staircase, if the number of its 1's equals to k. For example, the following several matrices are 5-staircase:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

and the following matrices are 6-staircase:

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Lemma 5. *For every $k \geq 1$, all k-staircase matrices M have the same rook polynomial*

$$R_M(x) = \sum_{i=0}^{\lfloor \frac{k+1}{2} \rfloor} \binom{k-i+1}{i} x^i. \tag{3.5}$$

4. Solution of Problem 2

According to Lemma 2, in order to calculate permanent of matrix $(J_n - I - P)[1|\ r]$, we can find rook polynomial of matrix $J_{n-1} - (J_n - I - P)[1|\ r]$. We use an evident equation

$$J_{n-1} - (J_n - I - P)[1|\ r] = (I_n + P)[1|\ r]. \tag{4.1}$$

Pass from matrix $(I_n + P)$ to matrix $(I_n + P)[1|\ r]$. We have (here $n = 10,\ \ r = 5$)

$$\begin{pmatrix} 1&1&0&0&0&0&0&0&0&0 \\ 0&1&1&0&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0&0 \\ 0&0&0&1&1&0&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0&0 \\ 0&0&0&0&0&1&1&0&0&0 \\ 0&0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&0&1&1 \\ 1&0&0&0&0&0&0&0&0&1 \end{pmatrix} \to \begin{pmatrix} 0&1&1&0&0&0&0&0&0 \\ 0&0&1&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0&0 \\ 0&0&0&0&1&1&0&0&0 \\ 0&0&0&0&0&1&1&0&0 \\ 0&0&0&0&0&0&1&1&0 \\ 0&0&0&0&0&0&0&1&1 \\ 1&0&0&0&0&0&0&0&1 \end{pmatrix} \tag{4.2}$$

Now we use Lemma 4 to the latter matrix in case $i = n, \;\; j = 1$. Denote

$$A = ((I_n + P)[1| \;\; r])^{(n-1,1)}, \;\; B = ((I_n + P)[1| \;\; r])^{(0(n-1,1))}. \tag{4.3}$$

According to (3.4), we have

$$R_{(I_n+P)[1| \;\; r]}(x) = xR_A(x) + R_B(x). \tag{4.4}$$

Note that matrix A has the form (here $n = 10, \;\; r = 5$)

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{4.5}$$

which is $(n-2) \times (n-2)$ matrix with $2n - 6$ 1's. This matrix consists of two disjunct matrices: $(r-2) \times (r-2)$ matrix A_1 of the form (here $r = 5$)

$$A_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.6}$$

which is $2r - 5$-staircase matrix, and $(n-r) \times (n-r)$ matrix (here $n = 10, \;\; r = 5$)

$$A_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix} \tag{4.7}$$

which is $2(n-r) - 1$-staircase matrix.

Thus, by Lemmas 3 and 5, we have

$$R_A(x) = \sum_{i=0}^{r-2} \binom{2r-i-4}{i} x^i \sum_{i=0}^{n-r} \binom{2(n-r)-i}{i} x^i$$

$$= \sum_{i=0}^{r-2} \binom{2r-i-4}{i} x^i \sum_{j=0}^{n-r+1} \binom{2(n-r)-j+1}{j-1} x^{j-1}. \tag{4.8}$$

Note that, since $\binom{n}{-1} = 0$, then we write formally the lower limit in interior sum $j = 0$. Furthermore, matrix B has the form (here $n = 10, \;\; r = 5$)

$$B=\begin{pmatrix}0&1&1&0&0&0&0&0&0\\0&0&1&1&0&0&0&0&0\\0&0&0&1&0&0&0&0&0\\0&0&0&0&1&0&0&0&0\\0&0&0&0&1&1&0&0&0\\0&0&0&0&0&1&1&0&0\\0&0&0&0&0&0&1&1&0\\0&0&0&0&0&0&0&1&1\\0&0&0&0&0&0&0&0&1\end{pmatrix} \tag{4.9}$$

which is $(n-1)\times(n-1)$ matrix with $2n-5$ 1's. This matrix consists of two disjunct matrices: $(r-2)\times(r-1)$ matrix B_1 of the form (here $r=5$)

$$B_1=\begin{pmatrix}0&1&1&0\\0&0&1&1\\0&0&0&1\end{pmatrix} \tag{4.10}$$

which is $2r-5$-staircase matrix, and $(n-r+1)\times(n-r)$ matrix (here $n=10,\ \ r=5$)

$$B_2=\begin{pmatrix}1&0&0&0&0\\1&1&0&0&0\\0&1&1&0&0\\0&0&1&1&0\\0&0&0&1&1\\0&0&0&0&1\end{pmatrix} \tag{4.11}$$

which is $2(n-r)$-staircase matrix.

Thus, by Lemmas 3 and 5, we have

$$R_B(x)=\sum_{i=0}^{r-2}\binom{2r-i-4}{i}x^i\sum_{j=0}^{n-r+1}\binom{2(n-r)-j+1}{j}x^j. \tag{4.12}$$

Note that, since $\binom{n-r}{n-r+1}=0$, then we write formally the upper limit in interior sum $j=n-r+1$. Now, using Lemma 4 for $M=(I_n+P)[1|\ r]$, from (4.7) and (4.11) we find

$$R_{(I_n+P)[1|\ r]}(x)=\sum_{i=0}^{r-2}\binom{2r-i-4}{i}x^i\sum_{j=0}^{n-r+1}\binom{2(n-r)-j+2}{j}x^j$$

$$=\sum_{k=0}^{n-1}x^k\sum_{i=0}^{\min(k,\ r-2)}\binom{2r-i-4}{i}\binom{2(n-r)-k+i+2}{k-i}. \tag{4.13}$$

Note that in the interior sum in (4.13) it is sufficient to take summation over interval $[\max(r+k-n-1,0),\min(k,r-2)]$. Thus, by Lemma 2 and

(4.1), we have

$$per((J_n - I - P)[1|\ r]) =$$

(4.14)

$$\sum_{k=0}^{n-1}(-1)^k(n-k-1)! \sum_{i=\max(r+k-n-1,\ 0)}^{\min(k,\ r-2)} \binom{2r-i-4}{i}\binom{2(n-r)-k+i+2}{k-i}.$$

By Lemma 1, formula (4.14) solves Problem 2. ■

Remark 1. *It is well known* ([5],*Ch.*8)*, that if in Problem 1 to replace a circular table by a straight one, than the incidence matrix of the problem is obtained from* $J_n - I - P$ *by removing 1 in position* $(n,\ 1)$*. Therefore, a solution of the corresponding problem to Problem 2, for a fixed* $r \geq 3$*, is given by* $per(J_{n-1} - B)$*, where* B *is the matrix* (*4.9*)*. Thus, by Lemma 2 and* (*4.12*)*, we analogously have*

$$per(J_{n-1} - B) =$$

(4.15)

$$= \sum_{k=0}^{n-2}(-1)^k(n-k-1)! \sum_{i=\max(r+k-n,\ 0)}^{\min(k,\ r-2)} \binom{2r-i-4}{i}\binom{2(n-r)-k+i+1}{k-i}.$$

5. Discussion of Problem 3

Expanding $U_n = per(J_n - I - P)$ over the first row, we have

$$U_n = \sum_{r=3}^{n} per((J_n - I - P)[1|\ r]). \tag{5.1}$$

According to Lemma 1, in conditions of Problem 3, a value of $per((J_n - I - P)[1|\ r])$ does not depend on r. This means that, by (5.1), we have

$$per((J_n - I - P)[1|\ r]) = \frac{U_n}{n-2},\ \ r = 3, ..., n. \tag{5.2}$$

Note that, in view of (2.3), it is sufficient to consider in (5.2) $r = 3, ..., \lfloor \frac{n+3}{2} \rfloor$.

For the first time, Problem 3 was announced by the author in [7] with conjecture that the solution supplies the set of those n for which $n - 2 | U_n$. Such solutions were verified for $n = 3,\ 4,\ 6$. Let us show that this conjecture is not true. Reducing (4.14) for $r = 3$, let us find a necessary condition for the suitable n.

Lemma 6. *If, for a given n, Problem 3 is solved in affirmative, then we have*

$$\sum_{k=0}^{n-3}(-1)^k\binom{2n-k-4}{k}(n-k-2)!(n-k-2)=\frac{U_n}{n-2}. \tag{5.3}$$

Proof. By (5.2) and (4.14) for $r=3$, we have

$$\frac{U_n}{n-2}=per((J_n-I-P)[1|\ 3])=$$

$$\sum_{k=0}^{n-1}(-1)^k(n-k-1)!B_{n,\ k}, \tag{5.4}$$

where

$$B_{n,\ k}=\sum_{i=\max(k-n+2,\ 0)}^{\min(k,\ 1)}\binom{2-i}{i}\binom{2n-4-k+i}{k-i},\ \ k=0,...,n-1. \tag{5.5}$$

It is easy to see that

$$B_{n,\ 0}=B_{n,\ n-1}=1;$$

$$B_{n,\ k}=\binom{2n-4-k}{k}+\binom{2n-3-k}{k-1},\ \ k=1,...,n-2.$$

Therefore, by (5.4), we have

$$\frac{U_n}{n-2}=(n-1)!+(-1)^{n-1}+$$

$$\sum_{k=1}^{n-2}(-1)^k(\binom{2n-4-k}{k}+\binom{2n-3-k}{k-1})(n-k-1)!=$$

$$(n-1)!+(-1)^{n-1}+$$

$$\sum_{k=1}^{n-2}(-1)^k\binom{2n-4-k}{k}(n-k-1)!-\sum_{k=0}^{n-3}(-1)^k\binom{2n-4-k}{k}(n-k-2)!=$$

$$(n-1)!-(n-2)!+$$

$$\sum_{k=1}^{n-3}(-1)^k\binom{2n-4-k}{k}((n-k-1)!-(n-k-2)!)$$

and (5.3) follows. ■

However, for $n = 10$, $\frac{U_n}{n-2} = 54974$, but the left hand side of (5.3) equals to 54888.

Conjecture 1. *Set* $\{3,\ 4,\ 6\}$ *contains only solutions of Problem 3.*

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