**Notes On The Theory Of Perfect Numbers**
N. A. Carella, February, 2011.

***Abstract:*** A perfect number is a number whose divisors add up to twice the number itself. The existence of odd perfect numbers is a millennia-old unsolved problem. This note proposes a proof of the nonexistence of odd perfect numbers. More generally, the same analysis seems to generalize to a proof of the nonexistence of odd multiperfect numbers.

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## 1. INTRODUCTION

An integer $N \in \mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$ is said to be a *perfect number* if the sum of its aliquot divisors (proper divisors) is equal to the number itself. For the first perfect number $N = 6$, this description translates to $1 + 2 + 3 = 6$. Usually, this is written in terms of the sum of divisors function $\sigma(N) = \sum_{p \mid N} d$ as $\sigma(6) = 1 + 2 + 3 + 6 = 2 \cdot 6$. Elementary introductions to the theory of perfect number appear in [WK], [MW], [SP], [VT], [PK], and other sources. The history of these numbers is given in [DN, Vol. I], and other.

Perfect numbers have been around since ancient times. The basic algorithm for constructing even perfect numbers is recorded in Euclid's Elements. The complete existence result for even perfect numbers was established by Euler about two millenniums later, see Theorem 25. In contrast, the odd perfect number problem is one of the oldest unsolved problems in Mathematics.

The theory of odd perfect numbers spans a wide web of interesting partial results. A few new results on the current state of the theory of odd perfect numbers state the followings.

***Theorem* 1.** ([LP]) The radical $rad(N) = \prod_{p \mid N} p$ of a perfect number $N \geq 1$ satisfies $rad(N) < 2N^{17/26}$.

***Theorem* 2.** ([NP]) (i) An odd perfect number $N = 36a + 9$ has at least 9 distinct prime divisors.
(ii) An odd perfect number $N = 36a + 1$ has at least 12 distinct prime divisors.

The proofs of these results require extensive and delicate works on the prime factorizations of the sum of divisors function or the associated cyclotomic polynomials, and intricate integers divisibility criteria, see [HP], [CO], [CS], [IA], [JN], [GT], [HR], and other references for recent results. This note proposes a proof of the nonexistence of odd perfect numbers.

***Theorem* 3.** There are no odd perfect numbers. Specifically, $\sigma(N) \neq 2N$ for any odd integer $N \geq 1$.

The proof of Theorem 3 appears in Section 3.2. This analysis seems to be the first application of the Ramanujan series of the sum of divisors function to a divisibility problem. The analytic tools required for this work are developed in Section 2. Section 4 attempts a generalization of the previous analysis to odd multiperfect numbers, see Theorem 16. The last two sections A and B in the Appendix serve as references. Section A provides some elementary information on arithmetic functions, and related concepts. Section B is concerned with the theory of even perfect numbers, it is included for completeness.

## 2 POWER SERIES EXPANSIONS OF ARITHMETIC FUNCTIONS

The Ramanujan series $f(n)=\sum_{k\geq 1}c_k(n)k^{-s}$ of an arithmetic function $f:\mathbb{N}\to\mathbb{C}$ is quite similar to the Fourier series $\hat{f}(n)=\sum_{k\geq 1}f(k)e^{i2\pi kn}$ of an arithmetic function $f(n)$. One difference is that the basis { $c_k(n)$ : $k \geq 1$ } of the Ramanujan series satisfies several number theoretical properties. While the basis { $e^{i2\pi kn}:k\geq 1$} of the Fourier series satisfies fewer number theoretical properties.

### 2.1 Properties Of The Ramanujan Sum $c_k(N)$

The Ramanujan sum is defined by the exponential sum $c_k(N)=\sum_{\gcd(k,x)=1}e^{i2\pi Nx/k}$. This is an integer-valued function on the set of nonnegative numbers. The Mobius function $\mu(n)$ and the totient function $\varphi(n)$, see Section A in the Appendix, are intrinsic part of the properties of the Ramanujan sum $c_k(n)$, see [AP, p. 160] , [MH] for details.

***Lemma* 4.** Let $N \geq 1$ be an integer. Then
(1) $c_{km}(N)=c_k(N)c_m(N)$ for any pair of relatively prime integers $k$, $m \geq 1$.
(2) $c_k(N)=\mu(k/d)\varphi(k)/\varphi(k/d)$ for any integer $k \geq 1$, and $d = \gcd(k, N)$.
(3) $c_k(N)=\sum_{d\mid\gcd(k,N)}\mu(k/d)d$ for any integer $k \geq 1$.
(4) If $k \mid N$ then $c_k(N)=\varphi(k)$.
(5) $c_k(N)=\sum_{\gcd(k,x)}\cos(2\pi Nx/k)$ for any integer $k \geq 1$.

Proof (i): For a pair of relatively prime integers $k$ and $m$, the subset of integers { $mx + ky$ : $\gcd(k, x) = \gcd(k, x) = 1$ }, where $0 < x < k$, and $0 < y < m$, is the same as the subset { $x$ : $\gcd(km, x) = 1$ and $0 < x < km$ }. So

$$c_k(N)c_m(N)=\sum_{\gcd(k,x)=1}e^{i2\pi Nx/k}\sum_{\gcd(m,y)=1}e^{i2\pi Ny/m}=\sum_{\gcd(k,x)=\gcd(m,y)=1}e^{i2\pi N(ky+mx)/km}=c_{km}(N). \tag{1.}$$

The other verifications are quite similar. ■

These properties are very useful in the evaluation of the values $c_k(N)$, and the analysis of the power series $\sum_{k\geq 1}c_k(n)k^{-s}$.

***Example* 5.** Compute the value of $c_k(N)$ at $k = p^v$ , $p$ prime and $v \geq 0$. This is done using the formula $c_{p^v}(N)=\mu(p^v/d)\varphi(p^v)/\varphi(p^v/d)$, Lemma 4-3. The calculation is as follows:

$$c_{p^v}(N)=\begin{cases}\varphi(p^v) & \text{if } p^v \mid N,\\ \mu(p)\varphi(p^{v-1})=-\varphi(p^{v-1}) & \text{if } p^{v-1}\parallel N,\\ \mu(p^2)=0 & \text{if } p^{v-2}\parallel N,\\ \mu(p^v) & \text{if } p^0\parallel N,\end{cases} \tag{2.}$$

where $d = \gcd(p^v, N)$. Note that $p^u \parallel N$ is equivalent to $\gcd(p^u, N) = p^u$.

***Lemma* 6.** If $m$, and $N \geq 1$ are integers, then the Dirichlet sum $\sum_{d\mid m}c_d(N)=\begin{cases}m & \text{if } m\mid N,\\ 0 & \text{otherwise.}\end{cases}$

Proof: Let $\omega = e^{i2\pi/m}$ be a primitive the $m$th root of unity. Because $\sum_{d\mid n}\varphi(d)=n$, see Lemma 24-3 in the Appendix, summing $c_d(N)$ over the divisors of $m$, id est,

$$\sum_{d\,|\,n} c_d(N) = \sum_{d\,|\,n} \sum_{\gcd(x,m)=1} e^{i2\pi Nx/m} = \sum_{0\le x<n} \omega^{Nx} = \begin{cases} n & \text{if } n\,|\,N, \\ 0 & \text{otherwise}, \end{cases} \tag{3.}$$

is the same as summing over all the $m$th roots. ■

**2.2 Power Series Expansions Of Arithmetic Functions**

The fine details on the derivations, convergence, and other analytic aspects of the power series expansions of the functions σ(N) and φ(N) appear in [HD], [MH], and similar sources. An effort was made to include as much formal technical details and explanations as possible to make the materials accessible to a wider readerships.

***Theorem* 7.** Let $N \ge 1$ be an integer, and let $\Re e(s) \ge 2$ be a complex number. Then

$$\frac{\sigma_{s-1}(N)}{N^{s-1}} = \zeta(s)\sum_{k=1}^{\infty}\frac{c_k(N)}{k^s}. \tag{4.}$$

Proof: The Dirichlet convolution of the two power series is

$$\begin{aligned}\sum_{n\ge1}\frac{1}{n^s}\cdot\sum_{k\ge1}\frac{c_k(N)}{k^s} &= \sum_{m\ge1}\Big(\sum\nolimits_{d|m} c_{m/d}(N)\Big)m^{-s} \\ &= \sum_{m\,|N}\frac{1}{m^{s-1}} = \frac{\sigma_{s-1}(N)}{N^{s-1}},\end{aligned} \tag{5.}$$

where the second line follows from Lemma 6, see also [HW, p. 328] for a different proof. ■

The spectrum of the Ramanujan series of the sum of divisors function

$$\frac{\sigma(N)}{N} = \zeta(2)\sum_{k=1}^{\infty}\frac{c_k(N)}{k^2} = \frac{\pi^2}{6}\left(1+\frac{c_2(N)}{2^2}+\frac{c_3(N)}{3^2}+\frac{c_4(N)}{4^2}+\cdots\right) \tag{6.}$$

has large spikes of magnitudes $6c_k(N)/\pi^2k^2 = O(1/k)$ at the divisors $k \mid N$, and the rest of the terms $6c_k(N)/\pi^2k^2 = O(1/k^2)$ at the nondivisors $k \nmid N$ are essentially noise components. This is analogous to the spectrum of the Fourier series of a function.

Because $\sigma_{s-1}(N) = \zeta(s)\cdot\sum_{k\ge1} c_k(N)k^{-s}$ is an entire function of the complex number $s \in \mathbb{C}$, the power series $\sum_{k\ge1} c_k(N)k^{-s}$ has (simple?) poles in the same locations as the zeros of the zeta function $\zeta(s) = \sum_{n\ge1} n^{-s}$. Interesting enough, the zeros of the analytic continuation of the power series $\left(\sum_{k\ge1} c_k(N)k^{-s}\right)^{-1}$ contains the zeroes of the zeta function on the critical strip.

***Lemma* 8.** Let $N \ge 1$ be an integer, and let $\Re e(s) > 0$ be a complex number, then

(i) $\left(\sum_{k\ge1}\frac{c_k(N)}{k^s}\right)^{-1} = \frac{N^{s-1}}{(1-2^{1-s})\sigma_{s-1}(N)}\sum_{k\ge1}\frac{(-1)^{k+1}}{k^s}$ is analytic in the half plane $\{\ \Re e(s) > 0 : s \in \mathbb{C}\ \}$.

(ii) If $N \ge 1$ a perfect number, then $\left(\sum_{k\ge1}\frac{c_k(N)}{k^2}\right)^{-1} = \sum_{k\ge1}\frac{(-1)^{n+1}}{k^2}$.

***Proof***: For (i) start with the analytic continuation of ζ(s) and the power series of the sum of divisors function, that is, power series

$$\zeta(s) = (1-2^{1-s})^{-1}\sum_{k\geq 1}\frac{(-1)^{n+1}}{n^s}, \quad \text{and} \quad \frac{\sigma_{s-1}(N)}{\zeta(s)N^{s-1}} = \sum_{k\geq 1}\frac{c_k(N)}{k^s}, \tag{7.}$$

where $\Re\mathrm{e}(s) > 0$. Taking the Dirichlet product returns

$$\begin{aligned}\sum_{k\geq 1}\frac{(-1)^{k+1}}{k^s}\cdot\sum_{m\geq 1}\frac{c_m(N)}{m^s} &= \sum_{n\geq 1}\left(\sum_{d\mid n}(-1)^{d+1}c_{n/d}(N)\right)n^{-s} \\ &= \frac{\sigma_{s-1}(N)}{\zeta(s)N^{s-1}}\cdot(1-2^{1-s})\zeta(s) = (1-2^{1-s})\frac{\sigma_{s-1}(N)}{N^{s-1}}.\end{aligned} \tag{8.}$$

For (ii), use the hypothesis $\sigma(N)/N = 2$, and evaluate each power series at $s = 2$. ■

***Lemma* 9.** Let $s \in \mathbb{C}$ be a complex number, and let $N \geq 1$ be an integer. Then

$$\sum_{k=1}^{\infty}\frac{c_k(N)}{k^s} = \prod_p\left(1+\sum_{v=1}^{\infty}\frac{c_{p^v}(N)}{p^{vs}}\right). \tag{9.}$$

***Corollary* 10.** If $N \geq 1$ is an odd perfect number, then the following hold:

(i) $\displaystyle\sum_{k=1}^{\infty}\frac{c_k(N)}{k^2} = \frac{12}{\pi^2}$. (ii) $\displaystyle\sum_{k=\text{odd}}\frac{c_k(N)}{k^2} = \frac{16}{\pi^2}$. (iii) $\displaystyle\sum_{k=\text{even}}\frac{c_k(N)}{k^2} = \frac{-4}{\pi^2}$.

Proof: To verify the first claim, assume $N$ is a perfect number and compute the value of the power series in Theorem 7 at $s = 2$. To verify the second and third claims, proceed to compute the even component. Specifically

$$\begin{aligned}\sum_{k=\text{even}}\frac{c_k(N)}{k^s} &= \sum_{m=\text{odd}}\frac{c_{2m}(N)}{(2m)^s} + \sum_{m=\text{odd}}\frac{c_{4m}(N)}{(4m)^s} + \sum_{m=\text{odd}}\frac{c_{8m}(N)}{(8m)^s} + \cdots \\ &= \sum_{m=\text{odd}}\frac{c_{2m}(N)}{(2m)^s} \\ &= \frac{-1}{2^s}\sum_{m=\text{odd}}\frac{c_m(N)}{m^s}\end{aligned}. \tag{10.}$$

Here all the power series vanish but one power series. This phenomenon occurs because $c_{2m}(N) = -c_m(N)$, and $c_{2^v m}(N) = \mu(2^v)c_m(N) = 0$ for odd $N$, $v \geq 2$, and odd $m \in \mathbb{N}$, see Lemma 4-1, and Example 5. Now take $s = 2$ in (10), and substitute it into the following:

$$\begin{aligned}\frac{12}{\pi^2} &= \sum_{k=1}^{\infty}\frac{c_k(N)}{k^2} \\ &= \sum_{k=\text{odd}}^{\infty}\frac{c_k(N)}{k^2} + \sum_{k=\text{even}}^{\infty}\frac{c_k(N)}{k^2} \\ &= \sum_{k=\text{odd}}^{\infty}\frac{c_k(N)}{k^2} - \frac{1}{2^2}\sum_{k=\text{odd}}^{\infty}\frac{c_k(N)}{k^2} \\ &= \frac{3}{4}\sum_{k=\text{odd}}^{\infty}\frac{c_k(N)}{k^2},\end{aligned} \tag{11.}$$

and solve for the respective values. ■

It should be explicated that the first statement (i), which is well known, is valid for both even and odd perfect integers, but statements (ii) and (iii) are exclusively properties of odd perfect numbers. For even perfect numbers, equation (10) has a different form, and it is nonnegative.

Let $rad(N)=\prod_{p\mid N} p$ be the radical of the integer $N$, also called squarefree kernel, and put $N_0 = \mathrm{rad}(N)N$.

***Theorem* 11.** Let $N \geq 1$ be an odd perfect number. Then

$$\frac{8}{3}\prod_{p\mid N}(1-1/p^2)=\sum_{d\mid N_0}\frac{c_d(N)}{d^2} \ . \tag{12.}$$

Proof: The result of Corollary 10-ii is rearranged as

$$\frac{16}{\pi^2}=\sum_{k=\text{odd}}\frac{c_k(N)}{k^2}=\sum_{\gcd(k,N)=1}\frac{c_k(N)}{k^2}+\sum_{\gcd(k,N)\neq 1}\frac{c_k(N)}{k^2}. \tag{13.}$$

Since the coefficient $c_k(N)=\mu(k)$ for $\gcd(k, N) = 1$, see Lemma 4-2, the first power series on the right side of (13) reduces to the simpler expression

$$\begin{aligned}\sum_{\gcd(k,N)=1}\frac{c_k(N)}{k^2}&=\sum_{\gcd(k,N)=1}\frac{\mu(k)}{k^2}=\prod_{\gcd(p,N)=1}(1-1/p^2)=\prod_{\gcd(p,N)\neq 1}(1-1/p^2)^{-1}\prod_{p\geq 2}(1-1/p^2)\\&=\frac{6}{\pi^2}\prod_{p\mid N}(1-1/p^2)^{-1},\end{aligned} \tag{14.}$$

since $6\pi^{-2}=\prod_{p\geq 2}(1-1/p^2)$. The second power series on the right side of (13) reduces to the simpler expression

$$\sum_{\gcd(k,N)\neq 1}\frac{c_k(N)}{k^2}=\sum_{1<d\mid N_0}\sum_{\gcd(m,N)=1}\frac{c_{dm}(N)}{(dm)^2}, \tag{15.}$$

where $\gcd(k, N) \neq 1$ was replaced with $k = dm$, where $d \mid N_0$, and the integers $m \in \mathbb{N}$ such that $\gcd(m, N) = 1$. The property $c_{dm}(N) = c_d(N)c_m(N)$, see Lemma 4-1, is now used to split the power series into two independent components.

$$\begin{aligned}\sum_{\gcd(k,N)\neq 1}\frac{c_k(N)}{k^2}&=\sum_{1<d\mid N_0}\sum_{\gcd(m,N)=1}\frac{c_{dm}(N)}{(dm)^2}\\&=\sum_{1<d\mid N_0}\sum_{\gcd(m,N)=1}\frac{c_d(N)c_m(N)}{(dm)^2}\\&=\sum_{1<d\mid N_0}\frac{c_d(N)}{d^2}\sum_{\gcd(m,N)=1}\frac{c_m(N)}{m^2}\\&=\left(\sum_{1<d\mid N_0}\frac{c_d(N)}{d^2}\right)\left(\frac{6}{\pi^2}\prod_{p\mid N}(1-1/p^2)^{-1}\right),\end{aligned} \tag{16.}$$

where the last line follows from (14). Combining these expressions returns

$$\begin{aligned}\frac{16}{\pi^2} &= \sum_{k=\text{odd}} \frac{c_k(N)}{k^2} \\ &= \sum_{\gcd(k,N)=1} \frac{c_k(N)}{k^2} + \sum_{\gcd(k,N)\neq 1} \frac{c_k(N)}{k^2} \\ &= \left(1 + \sum_{1<d\,|\,N_0} \frac{c_d(N)}{d^2}\right)\left(\frac{6}{\pi^2}\prod_{p\,|\,N}(1-1/p^2)^{-1}\right).\end{aligned} \tag{17.}$$

Rearranging the last equality proves the assertion. ■

***Theorem* 12.** Let $N \geq 1$ be an integer, let $rad(N) = \prod_{p\,|\,N} p$ be the radical of the integer $N$, and let $N_0 = \text{rad}(N)N$. Then

$$\frac{\sigma(N)}{N} = \left(\sum_{d\,|\,N_0} \frac{c_d(N)}{d^2}\right)\prod_{p\,|\,N}\left(1-1/p^2\right)^{-1}. \tag{18.}$$

Proof: Put $s = 2$ in Theorem 7. Then

$$\frac{\sigma(N)}{N} = \frac{\pi^2}{6}\sum_{k=1}^{\infty}\frac{c_k(N)}{k^2} = \frac{\pi^2}{6}\left(\sum_{\gcd(k,N)=1}\frac{c_k(N)}{k^2} + \sum_{\gcd(k,N)\neq 1}\frac{c_k(N)}{k^2}\right). \tag{19.}$$

Since $\gcd(k, N) \neq 1$ implies that $k = dm$, where $d \mid N_0$, and the integers $m \in \mathbb{N}$, such that $\gcd(m, N) = 1$, the power series is rewritten as

$$\frac{\sigma(N)}{N} = \frac{\pi^2}{6}\sum_{k=1}^{\infty}\frac{c_k(N)}{k^2} = \frac{\pi^2}{6}\left(\sum_{\gcd(k,N)=1}\frac{c_k(N)}{k^2} + \sum_{1<d\,|\,N_0}\sum_{\gcd(m,N)=1}\frac{c_{dm}(N)}{(dm)^2}\right). \tag{20.}$$

Here the finite sum index $d$ ranges over the divisors of $N_0 = \text{rad}(N)N$, which are the nonvanishing terms. Continuing

$$\begin{aligned}\frac{\sigma(N)}{N} &= \frac{\pi^2}{6}\left(\sum_{\gcd(k,N)=1}\frac{c_k(N)}{k^2} + \sum_{1<d\,|\,N_0}\sum_{\gcd(m,N)=1}\frac{c_d(N)c_m(N)}{d^2m^2}\right) \\ &= \frac{\pi^2}{6}\left(\sum_{\gcd(k,N)=1}\frac{c_k(N)}{k^2} + \sum_{1<d\,|\,N_0}\frac{c_d(N)}{d^2}\sum_{\gcd(m,N)=1}\frac{c_m(N)}{m^2}\right).\end{aligned} \tag{21.}$$

That follows from property $c_{dm}(N) = c_d(N)c_m(N)$, see Lemma 4-1, which is used to split the power series into two independent components. Next simplify it:

$$\begin{aligned}\frac{\sigma(N)}{N} &= \frac{\pi^2}{6}\left(1 + \sum_{1<d\,|\,N_0}\frac{c_d(N)}{d^2}\right)\sum_{\gcd(k,N)=1}\frac{c_k(N)}{k^2} \\ &= \frac{\pi^2}{6}\left(\sum_{d\,|\,N_0}\frac{c_d(N)}{d^2}\right)\left(\sum_{\gcd(k,N)=1}\frac{c_k(N)}{k^2}\right).\end{aligned} \tag{22.}$$

Substitute the identity

$$\sum_{\gcd(k,N)=1} \frac{c_k(N)}{k^2} = \frac{6}{\pi^2} \prod_{p \mid N} (1 - 1/p^2)^{-1} \tag{23.}$$

to obtain the asserted formula. ■

In the above formula for computing the sum of divisors function σ(*N*)/*N*, the divisor *d* runs over the divisors of the integer $N_0 = \text{rad}(N)N$. Why is the index *d* extended from *N* to $N_0$? The reason is that $c_d(N) \neq 0$ for all $d \mid N_0$. However, $c_d(N) = 0$ for any $d > 1$ such that $\gcd(d, N) \neq 1$, but $d \nmid N_0$, use Lemma 4-3 to verify this.

Formula (18) is valid for any integer $N \in \mathbb{N}$, but it is not an efficient method for computing the sum of divisors function σ(*N*)/*N*. However, it has some nice theoretical properties, which will be used to abridge the proof of Theorem 3. The example below is intended to illustrate the mechanic of the calculations.

***Example* 13.** Show that $N = 6$ is a perfect number. This is done as follows: Put $N_0 = \text{rad}(N)N = (2\cdot3)6 = 2^2\cdot3^2$. For $d \mid N_0$, let $\gcd(d, N) = f$, and use $c_d(N) = \mu(d/f)\varphi(d)/\varphi(d/f)$ to evaluate the coefficients. In particular, for $d = 12$, this is:

$$c_{12}(6) = \frac{\mu(12/f)\varphi(12)}{\varphi(12/f)} = \frac{\mu(12/6)\varphi(12)}{\varphi(12/6)} = -\varphi(12) = -4, \tag{24.}$$

since $f = \gcd(d, N) = \gcd(12, 6) = 6$, and so on. This procedure, formula (18), returns

$$\begin{aligned} \frac{\sigma(6)}{6} &= \left( \sum_{d \mid 2^2\cdot3^2} \frac{c_d(6)}{d^2} \right) \prod_{p \mid 6} \left(1 - 1/p^2\right)^{-1} \\ &= \left( \frac{c_1(6)}{1^2} + \frac{c_2(6)}{2^2} + \frac{c_3(6)}{3^2} + \frac{c_4(6)}{4^2} + \frac{c_6(6)}{6^2} + \frac{c_9(6)}{9^2} + \frac{c_{12}(6)}{12^2} + \frac{c_{18}(6)}{18^2} + \frac{c_{36}(6)}{36^2} \right) \left(1 - 1/2^2\right)^{-1} \left(1 - 1/3^2\right)^{-1} \\ &= \left( \frac{1}{1^2} + \frac{1}{2^2} + \frac{2}{3^2} + \frac{-2}{4^2} + \frac{2}{6^2} + \frac{-3}{9^2} + \frac{-4}{12^2} + \frac{-3}{18^2} + \frac{6}{36^2} \right) \left(1 - 1/2^2\right)^{-1} \left(1 - 1/3^2\right)^{-1} = 2\,. \end{aligned} \tag{25.}$$

In comparison, the standard method is simply $\dfrac{\sigma(6)}{6} = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{6} = 2$.

## 3 ODD PERFECT NUMBERS

The theory of odd perfect numbers spans a wide web of congruence relations, divisibility conditions, inequalities, numerical algorithms, and so on. The research on odd perfect numbers consists of an amalgamation of theory and computational analysis. Some of the current research efforts on odd perfect number are focused on the sizes of the large prime factors, the number of prime factors counting multiplicities $\Omega(N) \geq m$, the least number of prime factors $\omega(N) \geq k$, and other related parameters. These works are centered on the Euler normal form $N = p_0^{4v_0+1} \cdot p_1^{2v_1} \cdot p_2^{2v_2} \cdots p_k^{2v_k}$, $v_i \geq 1$, and $p_0 \equiv 1 \bmod 4$, the formula

$$\sigma(N) = \prod_{i=1}^{k} \sigma(p_i^{\alpha_i}) = \prod_{i=1}^{k} \frac{p_i^{\alpha_i+1} - 1}{p_i - 1}, \tag{26.}$$

and its cyclotomic polynomials representation

$$\sigma(N)=\prod_{i=1}^{k}\prod_{1<d\ \mid\ \alpha_i+1}\Phi_d(p_i^{\alpha_i})\,, \tag{27.}$$

where $\Phi_n(x)=x^{\varphi(n)}+a_{\varphi(n)-1}x^{\varphi(n)-1}+\cdots+a_1x\pm 1$ is the $n$th *cyclotomic polynomial*, and

$x^n-1=\prod_{d\mid n}\Phi_d(x)$. The previous expansion of σ(*N*) is obtained by taking the product of $\sigma(p^\alpha)=\prod_{1<d\ \mid\ \alpha+1}\Phi_d(p^\alpha)$ over all the prime power $p^\alpha$ divisors of *N*. Any result derive from these identities requires extensive and delicate works involving divisibility results and numerical calculations of the factors of the cyclotomic polynomials, see [HG], [GT], [IA] and earlier works.

### 3.1 Review of the Literature

A cursory sample of the papers in the literature is described below. The reader should confer the literature for a complete listing. A short survey appears in [RN, p. 98].

*Odd Perfect Number Congruences:* A labyrinth of congruence relations has been developed to investigate odd perfect numbers. These congruence relations constraint the prime divisors and their exponents to certain specific cases. One of the earliest is the Euler normal form $N=p_0^{4v_0+1}\cdot p_1^{2v_1}\cdot p_2^{2v_2}\cdots p_k^{2v_k}$, $v_i \geq 1$, and $p_0 \equiv 1 \bmod 4$. Touchard congruence claims that an odd perfect number satisfies $N\equiv 1, 9 \bmod 36$. The original proof is based on congruences of the power divisor functions, not very easy, see [TD]. But a recent proof in [HJ] simplifies it. New congruences are developed in [GR]

*Smallest, The Largest, And Total Numbers of Prime Factors:* Estimates on the smallest and the largest prime factors of an odd perfect numbers if a topic of much interest. Elementary determinations of the upper estimates of the smallest prime divisors is given in [BJ], see also [RN, p. 98]. On the other end, there are the estimates of the largest prime divisors, see [IA], [NI], [HK], and previous works.

*Smallest Odd Perfect Numbers*: Several authors have found estimates of the smallest odd perfect number. In [BT], it was determined that $N > 10^{300}$, and there is a new effort claiming to have pushed it to $N > 10^{1500}$, see [PR].

*Density Odd Perfect Numbers*: For a fixed ω(*N*) = *k*, there is finitely many potential odd perfect numbers, the analysis has been sharpened by several authors, see [PK, p. 247] for elaboration.

*Number of Primes Factors In Odd Perfect Numbers*:
B. Pierce, 1842?, proved that an odd perfect number has at least four prime factors, specifically, ω(*N*) ≥ 4.
J. Sylvester, 1888, proved that ω(*N*) ≥ 5.
D. Gradstein, 1925, proved that ω(*N*) ≥ 6.
N. Robbins, 1972, proved that ω(*N*) ≥ 7.
C. Pomerance, 1974, proved that ω(*N*) ≥ 7.
K. Chien, 1979, proved that ω(*N*) ≥ 8.
P. Hagis, 1979, proved that ω(*N*) ≥ 8.
P. Nielsen, 2008, proved that ω(*N*) ≥ 9.

### 3.2 Odd Perfect Numbers Theorem: Nullus Equus Numerus Perfectus Theorema

A different approach to the proof of the nonexistence of odd perfect numbers will be considered here. The proof is based on power series representation of the sum of divisors function and other related formulas. This analysis seems to be a new direction in the analysis of the divisibility problem *N* | σ(*N*). Surprisingly, only two of the simplest properties of odd perfect numbers will be utilized:

(i) *N* is an odd number.

(ii) σ(*N*)/*N* = 2.

***Theorem* 3.** There are no odd perfect numbers. Specifically, $\sigma(N) \neq 2N$ for any odd integers $N \geq 1$.

Proof: Assume $N \geq 1$ is an odd perfect number. Let $rad(N) = \prod_{p|N} p$ be the radical of the integer $N$, and put $N_0$ = rad($N$)$N$. By Theorem 12, the inequality

$$\sum_{d \mid N_0} \frac{c_d(N)}{d^2} < \sum_{d \mid N} \frac{1}{d}, \tag{28.}$$

holds for any integer $N \in \mathbb{N}$. By hypothesis $N$ is an odd perfect number $\sigma(N)/N = \sum_{d \mid N} 1/d = 2$, so applying the previous estimate to Theorem 11 lead to

$$\frac{8}{3}\prod_{p \mid N}(1-1/p^2) = \sum_{d \mid N_0} \frac{c_d(N)}{d^2} < \sum_{d \mid N} \frac{1}{d} = 2. \tag{29.}$$

Furthermore, since 2 is not a divisor of $N$, the product satisfies the inequality

$$2 < \frac{8}{3}\left(\frac{6}{\pi^2}(1-1/2^2)^{-1}\right) = \frac{8}{3}\prod_{p \geq 3}(1-1/p^2) \leq \frac{8}{3}\prod_{p \mid N}(1-1/p^2). \tag{30.}$$

Combining the last two expressions leads to

$$2 < \frac{8}{3}\prod_{p \mid N}(1-1/p^2) = \sum_{d \mid N_0} \frac{c_d(N)}{d^2} < 2. \tag{31.}$$

Quod Erat Demonstrandum Deo Volente.

## 4. GENERALIZATION

Let $t \geq 2$ be a fixed integer. An integer $N \geq 1$ is called *t-multiperfect* if $\sigma(N) = tN$. The same analysis applied to the case $t$ = 2 seems to extend to a proof of the nonexistence of odd multiperfect numbers. The required generalized results are duplicated here. From another perspective, the power series analysis proposed here applies equally well to the divisibility problem $N \mid \sigma(N)$. Confer [BN], [LV] for new results, and [MP] and [SL] for numerical data.

The perfect or multiperfect number problem is an integer divisibility problem. It asks for the integers $N \in \mathbb{N}$ which divide the arithmetic function $\sigma(N)$. Since $\sigma(N) < c_0N$logloglog $N$ for almost every integer $N \geq 1$, it is quite possible that $N \mid \sigma(N)$ can occur infinitely often for small even integers $2 \leq t < c_1$logloglog $N$, where $c_0$, $c_1$, $c_2 > 0$ are constants. On the other hand, $\sigma(N) > c_1N$loglog $N$ does occur infinitely often, but on a subset of integers of zero density. Therefore, it is less likely to have $N \mid \sigma(N)$ infinitely often for small even integers $t \geq 2$ in the range $c_1$logloglog $N < t < c_2$loglog $N$.

As an example, the Nicomachus conjecture claims that the sequence of integers $N = 2^p - 1$, $p$ prime, satisfies $N \mid \sigma(N)$ for infinitely many integers, see (46) and [PM].

Since for $t > e^{\gamma}$loglog $N$, the equation $\sigma(N) = tN$ is trivially false, it is assumed that the even integer $t \geq 2$ is in the range $2 \leq t < c_0$loglog $N$ for all integers $N \geq 1$, where $c_0 > 0$ is a constant.

***Corollary* 14.** Let $t \geq 2$ be a fixed integer. If $N \geq 1$ is an odd $t$-multiperfect number, then the following hold:

(i) $\displaystyle\sum_{k=1}^{\infty} \frac{c_k(N)}{k^2} = \frac{6t}{\pi^2}$. (ii) $\displaystyle\sum_{k=\text{odd}} \frac{c_k(N)}{k^2} = \frac{8t}{\pi^2}$. (iii) $\displaystyle\sum_{k=\text{even}} \frac{c_k(N)}{k^2} = \frac{-2t}{\pi^2}$.

Proof: Same as the proof of Corollary 10, mutatis mutandis. ■

***Theorem* 15.** Let $t \geq 2$ be a fixed integer, and let $N \geq 1$ be an odd $t$-multiperfect number. Then

$$\frac{4t}{3}\prod_{p \mid N}(1-1/p^2) = \sum_{d \mid N_0}\frac{c_d(N)}{d^2} . \tag{32.}$$

Proof: Start with Corollary 14-ii, and implement the needed changes in the proof of Theorem 11. ■

***Theorem* 16.** There are no odd $t$-multiperfect numbers. Specifically, $\sigma(N) \neq tN$ for any odd integers $N \geq 1$, and any fixed integer $t \geq 2$.

Proof: On the contrary, assume $N \geq 1$ is an odd $t$-multiperfect number. Let $rad(N) = \prod_{p|N} p$ be the radical of the integer $N$, and put $N_0 = \text{rad}(N)N$. By Theorem 12, the inequality

$$\sum_{d \mid N_0}\frac{c_d(N)}{d^2} < \sum_{d \mid N}\frac{1}{d} , \tag{33.}$$

holds for any integer $N$. By hypothesis $N$ is a $t$-multiperfect number $\sigma(N)/N = \sum_{d \mid N} 1/d = t$ , so applying the previous estimate to Theorem 15 leads to

$$\frac{4t}{3}\prod_{p \mid N}(1-1/p^2) = \sum_{d \mid N_0}\frac{c_d(N)}{d^2} < \sum_{d \mid N}\frac{1}{d} = t . \tag{34.}$$

Furthermore, since 2 is not a divisor of $N$, the product satisfies the inequality

$$t < \frac{4t}{3}\left(\frac{6}{\pi^2}(1-1/2^2)^{-1}\right) = \frac{4t}{3}\prod_{p \geq 3}(1-1/p^2) \leq \frac{4t}{3}\prod_{p \mid N}(1-1/p^2) . \tag{35.}$$

Combining the last two expressions leads to

$$t < \frac{8}{3}\prod_{p \mid N}(1-1/p^2) = \sum_{d \mid N_0}\frac{c_d(N)}{d^2} < t . \tag{36.}$$

Is est a reductio ad absurdum, ergo $\sigma(N) \neq tN$. ■

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# APPENDIX

Elementary technical materials on a few arithmetic functions are collected as reference for the convenience of the reader in Section A. Basic knowledge of these results will be required to establish the analysis of perfect number based on these arithmetic functions and related concepts. Section B has information on the theory of even perfect numbers. It is included for completeness.

## A. SOME ARITHMETIC FUNCTIONS

The literature on the theory of each of arithmetic functions is a world in itself. For example, an extensive survey of the literature on the sum of divisors function appears in [SD, Chapter 3]. And a cornucopia of details on various aspects of the divisor function, totient function, and other related open problems are given in [GY, p. 45].

**A.1 Sum of Divisors Function.** The sum of divisors function is defined by $\sigma(N)=\sum_{d\mid N} d$ . Its first few values are

| | | |
|---|---|---|
| $\sigma(1) = 1$, | $\sigma(4) = 1 + 2 + 4 = 7$, | $\sigma(7) = 1 + 7 = 8$, |
| $\sigma(2) = 1 + 2 = 3$, | $\sigma(5) = 1 + 5 = 6$, | $\sigma(8) = 1 + 2 + 4 + 8 = 15$, |
| $\sigma(3) = 1 + 3 = 4$, | $\sigma(6) = 1 + 2 + 3 + 6 = 12$, | $\sigma(9) = 1 + 3 + 9 = 13$, … . |

***Lemma* 17.** Let $N \geq 1$ be an integer, and let the symbol $p^{\alpha} \parallel N$ denotes the maximum prime power divisor of $N$. Then

(i) $\sigma(MN)=\sigma(M)\sigma(N)$ if $\gcd(M, N) = 1$.

(ii) $\sigma(N)=\prod_{p^{\alpha}\parallel N}\frac{p^{\alpha+1}-1}{p-1}$.

(iii) $\frac{\sigma(M)}{M}\leq\frac{\sigma(N)}{N}$, if $M \mid N$.

(iv) $\frac{\sigma(N)}{N}<\prod_{p\mid N}\frac{p}{p-1}$.

Typical applications of these properties of $\sigma(N)$ to the theory of odd perfect numbers are as follow.

***Lemma* 18.** An odd perfect number $N \geq 1$ has at least 3 odd prime divisors.

Proof: Assume that an odd perfect number $N = p^a \cdot q^b$ has two odd prime divisors $p$ and $q$, with $a, b \geq 1$. By Lemma 17-iv, the inequality

$$\frac{\sigma(p^a\cdot q^b)}{p^a\cdot q^b}<\frac{p}{p-1}\frac{q}{q-1}\leq\frac{3}{3-1}\frac{5}{5-1}<2 \qquad (37.)$$

holds. This contradict the hypothesis $\sigma(N) = 2N$. Therefore, the integer $N = p^a \cdot q^b$ cannot be an odd perfect number. ■

Following this pattern, the proofs for at least 4 odd prime divisors, et cetera, become exponentially more complicated.

***Lemma* 19.** An odd perfect number $N \geq 1$ is not divisible by 3·5·7.

Proof: Assume that $3\cdot5\cdot7 \mid N$. The Euler form of an odd perfect number calls for $N=p_0^{4v_0+1}\cdot p_1^{2v_1}\cdot p_2^{2v_2}\cdots p_k^{2v_k}$ , $v_i \geq 1$, and $p_0 \equiv 1 \bmod 4$. This implies that $N$ is divisible by $3^2\cdot5\cdot7^2$. Moreover, by Lemma 17-iii, the inequality

$$\frac{\sigma(N)}{N}\geq\frac{\sigma(3^2\cdot5\cdot7^2)}{3^2\cdot5\cdot7^2}=\frac{13}{9}\cdot\frac{6}{5}\cdot\frac{57}{49}>2 \qquad (38.)$$

holds. Therefore, an odd perfect integer $N$ cannot be divisible by 3·5·7. ■

The integers $n \in \mathbb{N} = \{ 0, 1, 2, 3, \ldots \}$ are classified as *deficient* if $\sigma(N) < 2N$, or *abundant* if $\sigma(N) > 2N$. The density function $A(x) = \#\{ n : \sigma(n)/n \geq x \}/x$ is a topic of current interest, see [DE] for an estimate of the value $.2474 < A(2) < .2480$. Roughly speaking, this asserts that 1 in 4 integers is an abundant integer.

**A.2 Euler Function.** The Euler function is defined by $\varphi(N) = N\prod_{p\mid N}(1-1/p)$. Its first few values are

$\varphi(1) = 1$, $\varphi(4) = 4(1 - 1/2) = 2$, $\varphi(7) = 7(1 - 1/7) = 6$,
$\varphi(2) = 2(1 - 1/2) = 1$, $\varphi(5) = 5(1 - 1/5) = 4$, $\varphi(8) = 8(1 - 1/2) = 4$,
$\varphi(3) = 3(1 - 1/3) = 2$, $\varphi(6) = 6(1 - 1/2)(1 - 1/3) = 2$, $\varphi(9) = 9(1 - 1/3) = 6, \ldots$ .

***Lemma* 20.** Let $N \geq 1$ be an integer. Then
(i) $\varphi(MN) = \varphi(M)\varphi(N)$ if $\gcd(M, N) = 1$. (ii) $\varphi(N) = N\prod_{p\mid N}(1-1/p)$.
(iii) $\varphi(N) < 2N/\log\log N$ .

**A.3. Mobius Function.** The Mobius function is defined by

$$\mu(d) = \begin{cases} (-1)^v & \text{if } n = p_1p_2\cdots p_v, \\ 0 & \text{if } n \neq \text{squarefree}. \end{cases} \tag{39.}$$

A sample of calculations for small integers is shown here:

$\mu(80) = \mu(2^4\cdot 5) = 0$, $\mu(81) = \mu(3^4) = 0$, $\mu(82) = \mu(2\cdot 41) = (-1)^2 = 1$,
$\mu(83) = (-1)^1 = -1$, $\mu(84) = \mu(2^2\cdot 3\cdot 7) = 0$, $\mu(85) = \mu(5\cdot 17) = (-1)^2 = 1$.

This is a multiplicative but not completely multiplicative function. More precisely,

(i) $\mu(mn) = \mu(m)\mu(n)$, if $\gcd(m, n) =1$,
(ii) $\mu(p^2) \neq \mu(p)\mu(p)$, if $p$ is a prime.

There is no obvious pattern on the occurrences of 0, 1 or −1. The next two Lemmas provide the mean value of the Mobius function and its second (moment) variance.

***Lemma* 21.** Let $n \geq 1$ be an integer. Then, the Dirichlet sum $\sum_{d\mid n}\mu(d) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases}$

Proof: Since $\mu(d) = 0$ if $d \geq 1$ is not a squarefree integer, it is sufficient to consider only the squarefree divisors $d$ of the squarefree kernel $\text{rad}(n) = p_1p_2\cdots p_v$, $v \geq 1$, of the integer $n \geq 1$. So, summing over all the squarefree divisors of $n > 1$ yields

$$\sum_{d\mid n}\mu(d) = \sum_{k\geq 0}\binom{v}{k}(-1)^k = (1-1)^v = 0, \tag{40.}$$

because there are $(v \mid k)$ ways of selecting a subset $\{ q_1, q_2, \ldots, q_k \}$ of $k \geq 0$ primes from the set of primes $\{ p_1, \ldots, p_v \}$, and $\mu(q_1q_2\cdots q_k) = (-1)^k$. ■

***Lemma* 22.** Let $n \geq 1$ be an integer. Then, the Dirichlet sum $\sum_{d\mid n}\mu^2(d) = 2^{\omega(n)}$, where $\omega(n) \geq 1$ is the number of prime divisors of the integer $n \geq 1$.

Proof: Since $\mu(d) = 0$ if $d \geq 1$ is not a squarefree integer, it is sufficient to consider only the squarefree divisors $d$ of the squarefree kernel $\text{rad}(n) = p_1p_2\cdots p_v$ , $v \geq 1$, of the integer $n \geq 1$. So, summing over all the divisors of $d > 1$ yields

$$\sum_{d \mid n} \mu^2(d) = \sum_{k \geq 0} \binom{v}{k} (-1)^{2k} = (1+1)^v = 2^v, \tag{41.}$$

because there are $(v \mid k)$ ways of selecting a subset $\{ q_1, \ldots, q_k \}$ of $k \geq 0$ primes from the set of primes $\{ p_1, \ldots, p_v \}$, and $\mu^2(q_1 q_2 \cdots q_k) = (-1)^{2k}$. ■

The large second moment (variance) $\sum_{d \mid n} \mu^2(d)$ of the sum $\sum_{d \mid n} \mu(d)$ can be used to show that the values of $\sum_{d \leq x} \mu(d)$ have wide variations about the mean value 0.

There are many complicated problems involving sums of the Mobius function. Some of the following finite sums are of current research interest:

(i) $\sum_{d \mid n,\, d \leq x} \mu(d)$, (ii) $\sum_{n \leq x} \mu(n)$, (iii) $\sum_{n \leq x} \mu(n)((x/n))$,

where $x \geq 1$ is a fixed real number, and $((\,x\,))$ is the fractional part function.

***Lemma* 23.** (Inversion Formulae) Let $f, g : \mathbb{N} \to \mathbb{C}$ be complex valued functions, and linearly independent over the complex numbers. Then the following inversion formulae hold.

(i) If $f(n) = \sum_{d \mid n} g(d)$ then $g(n) = \sum_{d \mid n} \mu(d) f(n/d)$, Additive Inversion.

(ii) If $f(n) = \prod_{d \mid n} g(d)$ then $g(n) = \prod_{d \mid n} f(n/d)^{\mu(d)}$, Multiplicative Inversion.

The linear independence condition is required since the inversion formula can fails for linearly dependent functions $f(n) = cg(n)$, $c \in \mathbb{C}$ constant.

***Lemma* 24.** For any integer $N \geq 1$, the followings are Mobius transform pairs.

(1a) $\sum_{d \mid n} d = \sigma(n)$, (1b) $\sum_{d \mid n} \mu(d) \sigma(n/d) = n$.

(2a) $\sum_{d \mid n} 1 = d(n)$, (2b) $\sum_{d \mid n} \mu(d) d(n/d) = 1$.

(3a) $\sum_{d \mid n} \varphi(d) = n$, (3b) $n \sum_{d \mid n} \mu(d)/d = \varphi(n)$.

where $d(n)$ is the number of divisor function, $\sigma(n)$ is the sum of divisor function, $\varphi(n)$ is the totient function.

Proof: For the first claim, simply take $f(n) = \sigma(n)$ and $g(n) = n$, and apply the inversion formula. ■

## B. EVEN PERFECT NUMBERS

The basic algorithm for constructing even perfect number is ancient and it is recorded in Euclid's Elements, Book IX, Proposition 36, see [EU]. The procedure is as follows:

Add consecutive powers of 2 until the sum is a prime. Then, the sum times the last power of 2 is an even perfect number. Symbolically, this given by

$$2^{p-1}(1 + 2 + 2^2 + \cdots + 2^{p-1}) = 2^{p-1}(2^p - 1). \tag{42.}$$

In ancient time about 3 or 4 perfect numbers were known:

$$2(2^2 - 1) = 6, \qquad 2^2(2^3 - 1) = 28, \qquad 2^4(2^5 - 1) = 496, \qquad 2^6(2^7 - 1) = 8128. \tag{43.}$$

The next few even perfect numbers were determined after the 1450's. The initial theoretical result of Euclid was completed by Euler about two millenniums later.

***Theorem* 25.** (Euclid-Euler) An even integer $N \geq 1$ is a perfect number if and only if $N = 2^{p-1}(2^p - 1)$, and $2^p - 1$ is a prime.

*Proof:* Suppose $N = 2^{p-1}(2^p - 1)$, with $2^p - 1$ prime. Then

$$\sigma(N) = \sigma(2^{p-1}(2^p - 1)) = \sigma(2^{p-1})\sigma(2^p - 1) = (2^p - 1)\sigma(2^p - 1) = (2^p - 1)2^p = 2N. \tag{44.}$$

This implies that $N$ is a perfect number. Conversely, assume $N = 2^{p-1}(2^p - 1)$ is a perfect number. Then

$$2N = \sigma(N) = \sigma(2^{p-1}(2^p - 1)) = \sigma(2^{p-1})\sigma(2^p - 1) = (2^p - 1)\sigma(2^p - 1). \tag{45.}$$

Ergo, $2N = 2^p(2^p - 1) = (2^p - 1)\sigma(2^p - 1)$. Idem quod, $2^p = \sigma(2^p - 1)$. Now assume $2^p - 1 = p_1 \cdots p_k$, $p_i$ prime. By Fermat Little Theorem, $p_i = 2pa_i + 1$, for $i = 1, 2, \ldots, k$. So, $2^p = \sigma(2^p - 1) = 2^k \cdot p^k a_1 \cdots a_k$, which is a contradiction since $p \neq 2^u$, $u \geq 1$. Therefore, $2^p - 1$ is prime and $2^p = \sigma(2^p - 1)$. ■

The assumption $2^p - 1 = p_1^{e_1} \cdots p_k^{e_k}$ with $e_i > 1$ leads to the same conclusion. Various versions of this converse second part of Theorem ?10 are widely available in the literature. In [VT], six different proofs of the converse second part of the above result are given, see also [HW, p. 312].

An integer $N \geq 1$ is called *superperfect* if $\sigma(\sigma(N)) = 2N$. For example, $N = 16$, and 64 are superperfect numbers.

***Corollary* 26.** An even integer $N \geq 1$ is superperfect number if and only if $N = 2^p$ and $2^p - 1$ is a prime.

The Lucas-Lehmer algorithm is a very efficient and effective algorithm used in the search for both Mersenne primes and for even perfect numbers. Extremely large Mersenne primes and perfect numbers have been determined in the last few decades. Currently, as of 2011, there are 47 Mersenne primes and the same number of even perfect numbers. The largest ones are $2^{43112609} - 1$, and $N = 2^{43112609}(2^{43112609} - 1)$ respectively. The prime $2^{43112609} - 1$ has 12978189 decimal digits.

***Theorem* 27.** (Lucas-Lehmer) Define the nonlinear sequence $S_n = S_{n-1}^2 - 2$, $S_0 = 4$. Let $p$ be a prime. Then the number $2^p - 1$ is a prime if and only if $S_{p-1} \equiv 0 \bmod 2^p - 1$.

An elementary proof of this result appears in [BR]. The main research problem on even perfect numbers is whether or not there are infinite many even perfect numbers. This is essentially the content of the following statements.

***Conjecture* 28.** (Gillies) Let $p$ be a prime and let $M_p$ be a Mersenne number. If $0 < A < B < \sqrt{M_p}$, and both B/A and $M_p \to \infty$, then the number of prime divisors of $M_p$ in the interval $[A, B]$ is Poisson distributed with mean $\log\log B / \log(\max\{A, 2p\})$.

Confer the literature for the recent refinements of this conjecture. This conjecture appears to be one of the most important unsolved problems in the theory of even perfect numbers. A heuristic argument suggests that the number of Mersenne primes up to $x$ is given by

$$N(x) = \#\left\{ M_p \leq x : M_p = 2^p - 1 \text{ is prime} \right\} = e^{\gamma}(\log 2)^{-1} \log\log x + o(\log\log x) . \tag{46.}$$