Pattern Matching in the Cycle Structures of Permutations.

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Abstract

In this paper, we study the occurrence of patterns in the cycle structures of permutations.

1 Introduction

The notion of patterns in permutations and words has proved to be a useful language in a variety of seemingly unrelated problems including the theory of Kazhdan-Lusztig polynomials, singularities of Schubert varieties, Chebyshev polynomials, rook polynomials for Ferrers boards, and various sorting algorithm including sorting stacks and sortable permutations. The study of occurrences of patterns in words and permutations is a new, but rapidly growing, branch of combinatorics which has its roots in the works by Rotem, Rogers, and Knuth in the 1970s and early 1980s. The first systematic study of permutation patterns was not undertaken until the paper by Simion and Schmidt [23] which appeared in 1985. The field has experienced explosive growth since 1992.

The goal of this paper is to initiate the study pattern matching conditions in the cycle structure of a permutation. First we recall the basic definitions for pattern matching in permutations. Given a sequence $\sigma = \sigma_1 \dots \sigma_n$ of distinct integers, let $\mathrm{red}(\sigma)$ be the permutation found by replacing the i^{th} largest integer that appears in σ by i. For example, if $\sigma = 2\ 7\ 5\ 4$, then $\mathrm{red}(\sigma) = 1\ 4\ 3\ 2$. Given a permutation $\tau = \tau_1 \dots \tau_j$ in the symmetric group S_j, we say a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ has a *τ-match starting at position i* provided $\mathrm{red}(\sigma_i \dots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-mch}(\sigma)$ be the number of τ-matches in the permutation σ. Similarly, we say that τ *occurs* in σ if there exist $1 \leq i_1 < \dots < i_j \leq n$ such that $\mathrm{red}(\sigma_{i_1} \cdots \sigma_{i_j}) = \tau$. We say that σ *avoids* τ if there are no occurrences of τ in σ.

These definitions can naturally be extended to sets of permutations. That is, given a set of permutations Υ in the symmetric group S_j, define a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$ to have a *Υ-match starting at position i* provided $\mathrm{red}(\sigma_i \dots \sigma_{i+j-1}) \in \Upsilon$. Let $\Upsilon\text{-mch}(\sigma)$ be the number of Υ-matches in the permutation σ. Similarly, we say that Υ *occurs* in σ if there exist

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$1 \leq i_1 < \cdots < i_j \leq n$ such that $\text{red}(\sigma_{i_1} \cdots \sigma_{i_j}) \in \Upsilon$. We say that σ *avoids* Υ if there are no occurrences of Υ in σ.

In this paper, we want to study matching conditions within the cycle structure of a permutation. Suppose that $\tau = \tau_1 \dots \tau_j$ is a permutation in S_j and σ is a permutation in S_n with k cycles $C_1 \dots C_k$. We shall always write cycles in the form $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $c_{0,i}$ is the smallest element in C_i and p_i is the length of C_i and we arrange the cycles by increasing smallest elements. That is, we arrange the cycles of σ so that $c_{0,1} < \cdots < c_{0,k}$. Then we say that σ has a *cycle τ-match* (c-τ-match) if there is an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \geq j$ and an r such that $\text{red}(c_{r,i}c_{r+1,i} \dots c_{r+j-1,i}) = \tau$ where we take indices of the form $r + s$ modulo p_i. Let c-τ-mch(σ) be the number of cycle τ-matches in the permutation σ. For example, if $\tau = 2\ 1\ 3$ and $\sigma = (1, 10, 9)(2, 3)(4, 7, 5, 8, 6)$, then 9 1 10 is a cycle τ-match in the first cycle and 7 5 8 and 6 4 7 are cycle τ-matches in the third cycle so that c-τ-mch$(\sigma) = 3$. Similarly, we say that τ *cycle occurs* in σ if there exists an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \geq j$ and there is an r with $0 \leq r \leq p_i - 1$ and indices $0 \leq i_1 < \cdots < i_{j-1} \leq p_i - 1$ such that $\text{red}(c_{r,i}c_{r+i_1,i} \dots c_{r+i_{j-1},i}) = \tau$ where the indices $r + i_s$ are taken mod p_i. We say that σ *cycle avoids* τ if there are no cycle occurrences of τ in σ. For example, if $\tau = 1\ 2\ 3$ and $\sigma = (1, 10, 9)(2, 3)(4, 8, 5, 7, 6)$, then 4 5 7, 4 5 6, and 5 6 8 are cycle occurrences of τ in the third cycle.

We can extend of the notion of cycle matches and cycle occurrences to sets of permutations in the obvious fashion. That is, suppose that Υ is a set of permutations in S_j and σ is a permutation in S_n with k cycles $C_1 \dots C_k$. Then we say that σ has a *cycle Υ-match* (c-Υ-match) if there is an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \geq j$ and an r such that $\text{red}(c_{r,i} \dots c_{r+j-1,i}) \in \Upsilon$ where we take indices of the form $r + s$ modulo p_i. Let c-Υ-mch(σ) be the number of cycle Υ-matches in the permutation σ. Similarly, we say that Υ *cycle occurs* in σ if there exists an i such that $C_i = (c_{0,i}, \dots, c_{p_i-1,i})$ where $p_i \geq j$ and there is an r with $0 \leq r \leq p_i - 1$ and indices $0 \leq i_1 < \cdots < i_{j-1} \leq p_i - 1$ such that $\text{red}(c_{r,i}c_{r+i_1,i} \dots c_{r+i_{j-1},i}) \in \Upsilon$ where the indices $r + i_s$ are taken mod p_i. We say that σ *cycle avoids* Υ if there are no cycle occurrences of Υ in σ.

Given $\Upsilon \subseteq S_j$, we let $\mathcal{AS}_n(\Upsilon)$ ($\mathcal{CAS}_n(\Upsilon)$) denote the set of permutations of S_n which avoid (cycle avoid) Υ and $aS_n(\Upsilon) = |\mathcal{AS}_n(\Upsilon)|$ ($caS_n(\Upsilon) = |\mathcal{CAS}_n(\Upsilon)|$). Similarly, we let $\mathcal{NMS}_n(\Upsilon)$ ($\mathcal{NCMS}_n(\Upsilon)$) denote the set of permutations of S_n which have no Υ-matches (no cycle Υ-matches) Υ and $nmS_n(\Upsilon) = |\mathcal{NMS}_n(\Upsilon)|$ ($ncmS_n(\Upsilon) = |\mathcal{NCMS}_n(\Upsilon)|$). Throughout this paper, when $\Upsilon = \{\tau\}$ is a singleton, we shall just write the τ rather than $\{\tau\}$. Thus for example, we shall write $\mathcal{AS}_n(\tau)$ for $\mathcal{AS}_n(\Upsilon)$ when $\Upsilon = \{\tau\}$.

Given α and β in S_j, we say that α and β are *Wilf equivalent* if $aS_n(\alpha) = aS_n(\beta)$ for all n. We say that α and β are *matching Wilf equivalent* (m-Wilf equivalent) if $nmS_n(\alpha) = nmS_n(\beta)$ for all n. For any permutation $\sigma = \sigma_1 \dots \sigma_n$, we let σ^r be the reverse of σ and σ^c be the complement of σ. That is, $\sigma^r = \sigma_n \dots \sigma_1$ and $\sigma^c = (n + 1 - \sigma_1) \dots (n + 1 - \sigma_n)$. It is well known that Wilf equivalence classes and m-Wilf equivalence classes are closed under reverse and complementation. We say that α and β are *cycle avoidance Wilf equivalent* (ca-Wilf equivalent) if $caS_n(\alpha) = caS_n(\beta)$ for all n and we say that α and β are *cycle matching Wilf equivalent* (cm-Wilf equivalent) if $ncmS_n(\alpha) = ncmS_n(\beta)$ for all n. If α and β are cycle avoidance Wilf equivalent, we shall write $\alpha \sim_{ca} \beta$. If α and β are cycle matching Wilf equivalent, we shall write $\alpha \sim_{cm} \beta$. Similarly, for sets of permutations Γ and Δ in S_j, we say that Γ and Δ are cycle avoidance Wilf equivalent (ca-Wilf equivalent) if $caS_n(\Gamma) = caS_n(\Delta)$ for all n and we say that Γ and Δ are cycle matching Wilf equivalent (cm-Wilf equivalent) if $ncmS_n(\Gamma) = ncmS_n(\Delta)$ for all n.

If σ is a permutation in S_n with k cycles $C_1 \dots C_k$, then we let the *cycle reverse of* σ, denoted by σ^{cr}, be the permutation which arises from σ by replacing each cycle $C_i = (c_{0,i}, c_{1,i}, \dots, c_{p_i-1,i})$ by its reverse cycle $C_i^{cr} = (c_{0,i}, c_{p_i-1,i}, \dots c_{1,i})$. For example, if $\sigma = (1,10,9)(2,3)(4,7,5,8,6)$, then $\sigma^{cr} = (1,9,10)(2,3)(4,6,8,5,7)$. We let the cycle complement of σ, denoted by σ^{cc}, be the permutation that results from σ by replacing each element i in the cycle structure of σ by $n+1-i$. For example, if $\sigma = (1,10,9)(2,3)(4,7,5,8,6)$, then $\sigma^{cr} = (10,1,2)(9,8)(7,4,6,3,5) = (1,2,10)(3,5,7,4,6)(8,9)$. Note that in general σ^r, σ^c, σ^{cr} and σ^{cc} are all distinct. For example, $\sigma = 2\ 3\ 1\ 5\ 4$ so that it cycle structure is $(1,2,3)(4,5)$, then

$$\begin{aligned}
\sigma^r &= 4\ 5\ 1\ 3\ 2, \\
\sigma^c &= 4\ 3\ 5\ 1\ 2, \\
\sigma^{cr} &= (1,3,2)(4,5) = 3\ 1\ 2\ 5\ 4, \text{ and} \\
\sigma^{cc} &= (5,4,3)(2,1) = 2\ 1\ 5\ 3\ 4.
\end{aligned}$$

It is easy to see that for any permutation $\sigma \in S_n$,

1. σ has a cycle τ-match if and only if σ^{cr} has a cycle τ^r-match,
2. σ has a cycle τ-match if and only if σ^{cc} has a cycle τ^c-match,
3. σ has a cycle τ occurrence if and only if σ^{cr} has a cycle τ^r occurrence, and
4. σ has a cycle τ occurrence if and only if σ^{cc} has a cycle τ^c occurrence.

It then easily follows that for all permutations τ, $ncmS_n(\tau) = ncmS_n(\tau^r) = ncmS_n(\tau^c)$ so that τ, τ^r, and τ^c are all cycle matching Wilf equivalent. Similarly, $caS_n(\tau) = caS_n(\tau^r) = caS_n(\tau^c)$ so that τ, τ^r, and τ^c are all cycle avoidance Wilf equivalent. Finally we observe that our definitions also ensure that for any $\tau = \tau_1 \dots \tau_j \in S_j$, any cyclic rearrangement of τ, $\tau^{(i)} = \tau_i \dots \tau_j \tau_1 \dots \tau_{i-1}$ also has the property that for any $\sigma \in S_n$, τ cycle occurs in σ if and only if $\tau^{(i)}$ cycle occurs in σ. Thus for all $1 \leq j$, $caS_n(\tau) = caS_n(\tau^{(i)})$ so that τ and $\tau^{(i)}$ are cycle avoidance Wilf equivalent.

Given a permutation $\sigma = \sigma_1 \dots \sigma_n \in S_n$, we let $\mathrm{des}(\sigma) = |\{i : \sigma_i > \sigma_{i+1}\}|$. We say that σ_j is a *left-to-right minima* of σ if $\sigma_j < \sigma_i$ for all $i < j$. We let $\mathrm{LtRMin}(\sigma)$ denote the number of left-to-right minma of σ. Given a cycle $C = (c_0, \dots, c_{p-1})$ where c_0 is the smallest element in the cycle, we let $\mathrm{cdes}(C) = 1 + \mathrm{des}(c_0 \dots c_{p-1})$. Thus $\mathrm{cdes}(C)$ counts the number of descent pairs as we traverse once around the cycle because the extra factor of 1 counts the descent pair $c_{p-1} > c_0$. For example if $C = (1,5,3,7,2)$, then $\mathrm{cdes}(C) = 3$ which counts the descent pairs 53, 72, and 21 as we traverse once around C. By convention, if $C = (c_0)$ is one-cycle, we let $\mathrm{cdes}(C) = 1$. If σ is a permutation in S_n with k cycles $C_1 \dots C_k$, then we define $\mathrm{cdes}(\sigma) = \sum_{i=1}^k \mathrm{cdes}(C_i)$. We let $\mathrm{cyc}(\sigma)$ denote the number of cycles of σ.

The main goal of this paper is to study the generating functions

$$CA_\Upsilon(t) = 1 + \sum_{n \geq 1} caS_n(\Upsilon) \frac{t^n}{n!}, \tag{1}$$

and

$$NCM_\Upsilon(t) = 1 + \sum_{n \geq 1} ncmS_n(\Upsilon) \frac{t^n}{n!} \tag{2}$$

for $\Upsilon \subseteq S_j$ as well as refinements of such generating functions such as

$$\begin{aligned}
CA_{\Upsilon}(t,x) &= 1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{CAS}_n(\Upsilon)} x^{\mathrm{cyc}(\sigma)},\\
CA_{\Upsilon}(t,x,y) &= 1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{CAS}_n(\Upsilon)} x^{\mathrm{cyc}(\sigma)}y^{\mathrm{cdes}(\sigma)},\\
NCM_{\Upsilon}(t,x) &= 1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NCMS}_n(\Upsilon)} x^{\mathrm{cyc}(\sigma)}, \text{ and}\\
NCM_{\Upsilon}(t,x,y) &= 1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{NCMS}_n(\Upsilon)} x^{\mathrm{cyc}(\sigma)}y^{\mathrm{cdes}(\sigma)}.
\end{aligned}$$

We know of several ways to approach the this problem. The most direct way is to use the theory of exponential structures to reduce the problem down to studying pattern matching in n-cycles. That is, let $\mathcal{L}_m$ denote the set m-cycles in S_m. Suppose that R is a ring and for each $m \geq 1$, we have a weight function $W_m : \mathcal{L}_m \to R$. We let $W(L_m) = \sum_{C\in\mathcal{L}_m} W_m(C)$. Now suppose that $\sigma \in S_n$ and the cycles of σ are $C_1,\ldots,C_k$. If C_i is of size m, then we let $W(C_i) = W_m(red(C_i))$ where $red(C_i)$ is the m-cycle in S_m that results by replacing j-th smallest element in C_i by j for $j = 1,\ldots,m$. For example, if $C_i = (1,5,7,10,4)$, then $red(C_i) = (1,3,4,5,2)$. Then we define the weight of σ, $W(\sigma)$, by

$$W(\sigma) = \prod_{i=1}^{k} W(C_i).$$

We let $\mathcal{C}_{n,k}$ denote the set of all permutations of S_n with k cycles. This given, the following theorem easily follows from the theory of exponential structures, see [24].

Theorem 1.

$$1+\sum_{n=1}^{\infty}\frac{t^n}{n!}\sum_{k=1}^{n}x^k\sum_{\sigma\in\mathcal{C}_{n,k}}W(\sigma) = e^{x\sum_{m\geq 1}\frac{W(L_m)t^m}{m!}}. \tag{3}$$

Let $\Upsilon \subseteq S_j$. Then we will be most interested in the special case of weight functions W_m where $W_m(C) = 1$ if C cycle avoids a set of permutations and $W_m(C) = 0$ otherwise or where $W_m(C) = 1$ if C has no cycle Υ-matches and $W_m(C) = 0$ otherwise. We let $\mathcal{CAS}_{n,k}(\Upsilon)$ denote the set of permutations σ of S_n with k cycles such that σ cycle avoids Υ and we let $caS_{n,k}(\Upsilon) = |\mathcal{CAS}_{n,k}(\Upsilon)|$. We let $\mathcal{NCMS}_{n,k}(\Upsilon)$ denote the set of permutations σ of S_n with k cycles such that σ has no cycle Υ-matches and $ncmS_{n,k}(\Upsilon) = |\mathcal{NCMS}_{n,k}(\Upsilon)|$. Similarly, we let $\mathcal{L}^{ca}_m(\Upsilon)$ be the set of m cycles γ in S_m such γ cycle avoids Υ, $L^{ca}_m(\Upsilon) = |\mathcal{L}^{ca}_m(\Upsilon)|$, $\mathcal{L}^{ncm}_m(\Upsilon)$ denote the set of m cycles γ in S_m such γ has no cycle Υ-matches, and $L^{ncm}_m(\Upsilon) = |\mathcal{L}^{ncm}_m(\Upsilon)|$. Then a special case of Theorem 1 is the following theorem.

Theorem 2.

$$CA_{\Upsilon}(t,x) = 1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{k=1}^{n}caS_{n,k}(\Upsilon)x^k = e^{x\sum_{m\geq 1}L^{ca}_m(\Upsilon)\frac{t^m}{m!}}, \tag{4}$$

$$NCM_{\Upsilon}(t,x) = 1+\sum_{n\geq 1}\frac{t^n}{n!}\sum_{k=1}^{n}ncmS_{n,k}(\Upsilon)x^k = e^{x\sum_{m\geq 1}L^{ncm}_m(\Upsilon)\frac{t^m}{m!}}, \tag{5}$$

$$CA_{\Upsilon}(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{CAS}_{n,k}(\Upsilon)} y^{\mathrm{cdes}(\sigma)} = e^{x \sum_{m\geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ca}(\Upsilon)} y^{\mathrm{cdes}(C)}}, \quad (6)$$

and

$$NCM_{\Upsilon}(t,x,y) = 1 + \sum_{n\geq 1} \frac{t^n}{n!} \sum_{k=1}^{n} x^k \sum_{\sigma \in \mathcal{NCMS}_{n,k}(\Upsilon)} y^{\mathrm{cdes}(\sigma)} = e^{x \sum_{m\geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ncm}(\Upsilon)} y^{\mathrm{cdes}(C)}}. \quad (7)$$

For example, suppose that $\tau = 1\ 2$. It is clear that any cycle of length k where $k \geq 2$ has both a cycle occurrence of τ and a cycle τ-match so that $L_m^{ca}(12) = L_m^{ncm}(12) = 0$ if $m \geq 2$. Since 1-cycles can not have any cycle occurrences of τ or any cycle τ-matches by definition, it follows that

$$y = \sum_{C \in \mathcal{L}_1^{ca}(12)} y^{\mathrm{cdes}(C)} = \sum_{C \in \mathcal{L}_1^{ncm}(12)} y^{\mathrm{cdes}(C)}.$$

Thus

$$CA_{12}(t,x,y) = NCM_{12}(t,x,y) = e^{xyt}.$$

Next consider $\tau = 1\ 2\ 3$. It is easy to see that for $k \geq 3$, the only k-cycle which cycle avoids τ is the cycle $(1, k, k-1, \ldots, 2)$. Let

$$A_m(y) = \sum_{C \in \mathcal{L}_m^{ca}(123)} y^{\mathrm{cdes}(C)},$$

then clearly $A_1(y) = y$ since $\mathrm{cdes}((1)) = 1$, $A_2(y) = y$ since $\mathrm{cdes}((1,2)) = 1$, and for $k \geq 3$, $A_k(y) = y^{k-1}$ since $\mathrm{cdes}((1,k,\ldots,2)) = k-1$. Thus

$$CA_{123}(t,x,y) = e^{x\left(yt + \sum_{m\geq 2} \frac{y^{m-1}t^m}{m!}\right)} = e^{x\left(yt + \frac{1}{y}(e^{yt}-1-yt)\right)}.$$

It turns out that if $\tau \in S_j$ is a permutation that starts with 1, then we can reduce the problem of finding $NCM_\tau(t,x)$ and $NCM_\tau(t,x,y)$ to the usual problem of finding the generating function of permutations that have no τ-matches. That is, suppose we are given a permutation $\sigma \in S_n$ with k-cycles $C_1 \cdots C_k$. Assume we have arranged the cycles so that the smallest element in each cycle is on the left and we arrange the cycles by decreasing smallest elements. Then we let $\bar{\sigma}$ be the permutation that arise from $C_1 \cdots C_k$ by erasing all the parenthesis and commas. For example, if $\sigma = (7,10,9,11)\ (4,8,6)\ (1,5,3,2)$, then $\bar{\sigma} = 7\ 10\ 9\ 11\ 4\ 8\ 6\ 1\ 5\ 3\ 2$. It is easy to see that the minimal elements of the cycles correspond to left-to-right minima in $\bar{\sigma}$. It is also easy to see that under our bijection $\sigma \to \bar{\sigma}$, that $cdes(\sigma) = \mathrm{des}(\bar{\sigma}) + 1$ since every left-to-right minima is part of a descent pair in $\bar{\sigma}$. For example, if $\sigma = (7,10,9,11)\ (4,8,6)\ (1,5,3,2)$ so that $\bar{\sigma} = 7\ 10\ 9\ 11\ 4\ 8\ 6\ 1\ 5\ 3\ 2$, $\mathrm{cdes}((7,10,9,11)) = 2$, $\mathrm{cdes}((4,8,6)) = 2$, and $\mathrm{cdes}((1,5,3,2)) = 4$ so that $\mathrm{cdes}(\sigma) = 2+2+4 = 8$ while $\mathrm{des}(\bar{\sigma}) = 7$. This given, we have the following lemma.

Lemma 3. *If $\tau \in S_j$ and τ starts with 1, then for any $\sigma \in S_n$,*

1. *σ has k cycles if and only if $\bar{\sigma}$ has k left-to-right minima,*
2. $\mathrm{cdes}(\sigma) = 1 + \mathrm{des}(\bar{\sigma})$, *and*
3. *σ has no cycle-τ-matches if and only if $\bar{\sigma}$ has no τ-matches.*

Proof. For (3), suppose that $\bar{\sigma} = \bar{\sigma}_1 \ldots \bar{\sigma}_n$ and $\bar{\sigma}_i = 1$. Since τ starts with 1, it is easy to see that any τ-match in $\bar{\sigma}$ must either occur weakly to the right of $\bar{\sigma}_i$ or strictly to left of $\bar{\sigma}_i$. That is, 1 can be part of τ-match in $\bar{\sigma}$ only if the τ-match starts at position i. If a τ-match occurred weakly to the right of $\bar{\sigma}_i$, then that τ-match would correspond to a cycle-τ-match in C_k in σ.

Next suppose that the τ-match occurred strictly to the left of $\bar{\sigma}_i = 1$. Then we claim that we can make a similar argument with respect to the cycles $C_1 \cdots C_{k-1}$. That is, suppose that C_{k-1} starts with m. Then m must be the smallest element among $\bar{\sigma}_1 \ldots \bar{\sigma}_{j-1}$. Suppose that $\bar{\sigma}_s = m$ where $1 \leq s < j$. Then again we can argue that any τ-match in $\bar{\sigma}_1 \ldots \bar{\sigma}_{j-1}$ must occur either weakly to the right of $\bar{\sigma}_s$ or strictly to left of $\bar{\sigma}_s$. If the τ-match in $\bar{\sigma}_1 \ldots \bar{\sigma}_{j-1}$ occurs weakly to the right of $\bar{\sigma}_s$, then it would correspond to a cycle-τ-match in C_{k-1}. Continuing on in this way, we see that any τ-match in $\bar{\sigma}$ must correspond to a cycle τ-match in C_i for some i.

Vice versa, it is easy to see that since τ starts with 1, the only way that a cycle-τ-match in C_i can involve the smallest element $c_{0,i}$ in the cycle C_i is if $c_{0,i}$ corresponds to the 1 in τ in cycle match. But this easily implies that any τ-cycle match in C_i must also correspond to a τ-match in the elements of $\bar{\sigma}$ corresponding to C_i.

Thus we have proved that for any σ, σ has cycle-τ-match if only if $\bar{\sigma}$ has a τ-match. □

We should note that if a permutation τ does not start with 1, then it may be that case that $ncmS_n(\tau) \neq nmS_n(\tau)$. For example, $\tau =$ 3 1 4 2 is the smallest permutation such that neither τ, τ^r, τ^c, nor $(\tau^r)^c$ starts with one. For example, even though we do not know how to compute closed forms for $NCM(t)$ and $NM(t)$, we have computed the following table.

n	$L_n^{ncm}(3142)$	$NCM_n(3142)$	$NM_n(3142)$
1	1	1	1
2	1	2	2
3	2	6	6
4	5	23	23
5	20	110	110
6	92	632	632
7	532	4236	4237
8	3565	32448	32465

One consequence of Lemma 3 is that we can automatically obtain refinements of generating functions for the number of permutations with no τ-matches when τ starts with 1. That is, let

$$\begin{aligned} NM_\tau(t,x) &= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NMS}_n(\tau)} x^{\mathrm{LtRMin}(\sigma)} \text{ and} \\ NM_\tau(t,x,y) &= \sum_{n \geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NMS}_n(\tau)} x^{\mathrm{LtRMin}(\sigma)} y^{1+\mathrm{des}(\sigma)}. \end{aligned}$$

Then we have the following corollary of Lemma 3.

Corollary 4. *If $\tau \in S_j$ and τ starts with 1, then*

$$NCM_\tau(t,x) = NM_\tau(t,x) \text{ and} \tag{8}$$
$$NCM_\tau(t,x,y) = NM_\tau(t,x,y). \tag{9}$$

Then by Theorem 2 and Lemma 3, if $\tau \in S_j$ and τ starts with 1, we have that

$$\begin{aligned} NM_\tau(t,1) &= \sum_{n\geq 0} NM_n(\tau)\frac{t^n}{n!} \\ &= NCM(t,1) \\ &= e^{\sum_{m\geq 1} L_m^{ncm}(\tau)\frac{t^m}{m!}} \end{aligned}$$

so that

$$ln(NM_\tau(t,1)) = \sum_{m\geq 1} L_m^{ncm}(\tau)\frac{t^m}{m!}. \tag{10}$$

But then

$$\begin{aligned} NM(t,x) &= NCM(t,x) && (11) \\ &= e^{x\sum_{m\geq 1} L_m^{ncm}(\tau)\frac{t^m}{m!}} \\ &= e^{x\ln(NM_\tau(t,1))} = (NM_\tau(t,1))^x && (12) \end{aligned}$$

Thus if we can compute $NM_\tau(t,1)$ for a permutation $\tau \in S_j$ that starts with 1, we automatically can compute $NM_\tau(t,x)$. For example, Goulden and Jackson [8] proved that when $\tau = 1\ 2 \ldots k$, then

$$NM_\tau(t) = \frac{1}{\sum_{i\geq 0} \frac{t^{ki}}{(ki)!} - \frac{t^{ki+1}}{(ki+1)!}}. \tag{13}$$

Hence, we automatically have the following refinement of Goulden and Jackson's result.

Theorem 5. *If $\tau = 12\ldots k$ where $k \geq 2$, then*

$$NM_{j\ldots 2\ 1}(t,x) = \left(\frac{1}{\sum_{i\geq 0} \frac{t^{ki}}{(ki)!} - \frac{t^{ki+1}}{(ki+1)!}}\right)^x. \tag{14}$$

An example, where one can use the full power of Theorem 1 is the following. In section 2, we shall show that

$$\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(132)} y^{\mathrm{cdes}(C)} = ln\left(\frac{1}{1-y\int_0^t e^{(1-y)s-y\frac{s^2}{2}}ds}\right). \tag{15}$$

Then it follows that

$$\begin{aligned} NCM(t,x,y) &= \sum_{n\geq 0}\frac{t^n}{n!}\sum_{\sigma\in S_n} x^{cyc(\sigma)}y^{\mathrm{cdes}(\sigma)} && (16) \\ &= \sum_{n\geq 0}\frac{t^n}{n!}\sum_{k=1}^{n} x^k \sum_{\sigma\in\mathcal{NCMS}_{n,k}(\tau)} y^{\mathrm{cdes}(\sigma)} \\ &= e^{x\ ln\left(\frac{1}{1-y\int_0^t e^{(1-y)s-y\frac{s^2}{2}}ds}\right)} \\ &= \left(\frac{1}{1-y\int_0^t e^{(1-y)s-y\frac{s^2}{2}}ds}\right)^x. \end{aligned}$$

The outline of this paper is as follows. In Section 2, we determine the generating function $CA_\tau(t,x,y)$ and $NCM_\tau(t,x,y)$ for all $\tau \in S_3$ as well as compute $CA_\Upsilon(t,x,y)$ and $NCM_\Upsilon(t,x,y)$ for certain subsets $\Upsilon \subseteq S_3$. In section 3, we shall compute $NCM_\tau(t,x,y)$ for all $\tau = \tau_1 \ldots \tau_j \in S_j$ where $\tau_1 =$ and $\tau_j = 2$ and for all $\tau = \tau_1 \ldots \tau_{j+p} \in S_{j+p}$ of the form $\tau = 1\ 2 \ldots J-1\ \gamma\ j$ where $j \geq 3$ and γ is a permutation of $j+1, \ldots, j+p$. Finally, in Section 4, we shall briefly describe two other approaches to computing the generating function $NCM_\tau(t,x,y)$.

2 Patterns of length 3

In this section, we study $CA_\tau(t,x,y)$ and $NCM_\tau(t,x,y)$ for $\tau \in S_3$.

First we consider $CA_\tau(t,x)$ for $\tau \in S_3$. It follows from our remarks in the introduction that both cycle avoidance Wilf equivalence and cycle matching Wilf equivalence are closed under the operation of reverse and complement. Thus

1. $1\ 2\ 3 \sim_{ca} 3\ 2\ 1$ and $1\ 2\ 3 \sim_{cm} 3\ 2\ 1$ and
2. $1\ 3\ 2 \sim_{ca} 2\ 3\ 1 \sim_{ca} 2\ 1\ 3 \sim_{ca} 3\ 1\ 2$ and $1\ 3\ 2 \sim_{cm} 2\ 3\ 1 \sim_{cm} 2\ 1\ 3 \sim_{cm} 3\ 1\ 2$.

Now since cycle avoidance Wilf equivalence is closed under cycle rearrangements, it follows that $1\ 2\ 3 \sim_{ca} 2\ 3\ 1$ which means that all permutations of length three are cycle avoidance Wilf equivalent. Thus for all permutations τ of length three, we have

$$CA_\tau(t) = CA_{123}(t) = e^{e^t-1}.$$

But since

$$CA_\tau(t) = e^{\sum_{m\geq 1} L_m^{ca}(\tau)\frac{t^m}{m!}}$$

for all $\tau \in S_3$, it must be the case that

$$\sum_{m\geq 1} L_m^{ca}(\tau)\frac{t^m}{m!} = e^t - 1$$

for all $\tau \in S_3$ and, hence,

$$CA_\tau(t,x) = e^{x\sum_{m\geq 1} L_m^{ca}(\tau)\frac{t^m}{m!}} = e^{x(e^t-1)}$$

for all $\tau \in S_3$. However it is not the case that the generating functions $CA_\tau(t,x,y)$ are equal for all $\tau \in S_3$. That is, suppose that α is a cyclic rearrangement of β. Then it is easy to see that $\mathcal{L}_m^{ca}(\alpha) = \mathcal{L}_m^{ca}(\beta)$ for all $m \geq 1$ so that

$$\sum_{C\in\mathcal{L}_m^{ca}(\alpha)} y^{\text{cdes}(C)} = \sum_{C\in\mathcal{L}_m^{ca}(\beta)} y^{\text{cdes}(C)}. \tag{17}$$

But then it follows from Theorem 2 that we must have $CA_\alpha(t,x,y) = CA_\beta(t,x,y)$. It thus follows that from our results in the introduction that

$$CA_{123}(t,x,y) = CA_{312}(t,x,y) = CA_{231}(t,x,y) = e^{x\left(yt+\frac{1}{y}(e^{yt}-1-yt)\right)}.$$

Next consider $\tau = 1\ 3\ 2$. It is easy to see that for $k \geq 3$, the only k-cycle which cycle avoids τ is the cycle $(1, 2, \ldots, k)$. Thus

$$\sum_{C \in \mathcal{L}^{ca}_m(132)} y^{\text{cdes}(C)} = y,$$

for all $k \geq 1$. Hence

$$CA_{132}(t, x, y) = CA_{213}(t, x, y) = CA_{321}(t, x, y) = e^{x\left(\sum_{m\geq 1} \frac{yt^m}{m!}\right)} = e^{xy(e^t - 1)}.$$

Next we shall consider the generating functions $NCM_\tau(t, x, y)$ for $\tau \in S_3$. We claim that is enough to compute $NCM_{123}(t, x, y)$ and $NCM_{132}(t, x, y)$. That is, for any $j \geq 2$ and $\tau \in S_j$, we can compute $NCM_{\tau^r}(t, x, y)$ and $NCM_{\tau^c}(t, x, y)$ from $NCM_\tau(t, x, y)$. Note that it follows from Theorem 2 that

$$\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C \in \mathcal{L}^{ncm}_n(tau)} y^{\text{cdes}(C)} = ln\left(NCM_\tau(t, 1, y)\right). \tag{18}$$

Since $\sum_{C \in \mathcal{L}^{ncm}_1(123)} y^{\text{cdes}(C)} = y$, it follows that

$$\sum_{n\geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}^{ncm}_n(\tau)} y^{\text{cdes}(C)} = ln\left(NCM_\tau(t, 1, y)\right) - yt. \tag{19}$$

Given any n-cycle C in S_n, let C^{cr} denotes its cycle-reverse and C^{cc} denotes its cycle-complement. Then $C \in \mathcal{L}^{ncm}_n(\tau)$ if and only if $C^{cr} \in \mathcal{L}^{ncm}_n(\tau^r)$ and $C \in \mathcal{L}^{ncm}_n(\tau)$ if and only if $C^{cc} \in \mathcal{L}^{ncm}_n(\tau^c)$ Now if $n \geq 2$, then it is easy to see that $n - \text{cdes}(C) = \text{cdes}(C^{cr}) = \text{cdes}(C^{cc})$. That is, each descent as we read once around the cycle C becomes a rise as we read around the cycles of C^{cr} and C^{cc} and each rise as we read once around the cycle C becomes a descent as we read around the cycles of C^{cr} and C^{cc}. Note, however, that if C is a one-cycle, then $C^{cr} = C^{cc} = C$ and $\text{cdes}(C) = \text{cdes}(C^{cr}) = \text{cdes}(C^{cc}) = 1$ so that it is not the case that $\text{cdes}(C^{cr}) = \text{cdes}(C^{cr}) = 1 - \text{cdes}(C)$. Thus we have to treat the one-cycles separately. Thus we have that

$$\begin{aligned}
\sum_{n\geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}^{ncm}_n(\tau)} y^{n-\text{cdes}(C)} &= \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}^{ncm}_n(\tau^r)} y^{\text{cdes}(C)} \\
&= \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}^{ncm}_n(\tau^c)} y^{\text{cdes}(C)}.
\end{aligned}$$

It follows that if $\tau \in S_j$ where $j \geq 2$ and

$$G(t, y) = \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}^{ncm}_n(\tau)} y^{\text{cdes}(C)}, \tag{20}$$

then

$$G(ty, y^{-1}) = \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}^{ncm}_n(\tau^r)} y^{\text{cdes}(C)} = \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C \in \mathcal{L}^{ncm}_n(\tau^c)} y^{\text{cdes}(C)}. \tag{21}$$

Thus by (19), we have that

$$\begin{aligned} ln\left(NCM_\tau(ty,1,y^{-1})\right) - t &= \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau^r)} y^{\mathrm{cdes}(C)} \\ &= \sum_{n\geq 2} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau^c)} y^{\mathrm{cdes}(C)} \end{aligned}$$

so that

$$\begin{aligned} ty - t + ln\left(NCM_\tau(ty,1,y^{-1})\right) &= \sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau^r)} y^{\mathrm{cdes}(C)} \\ &= \sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau^c)} y^{\mathrm{cdes}(C)} \end{aligned}$$

Then we can apply Theorem 2 to obtain the following result.

Theorem 6. *Let $\tau \in S_j$ where $j \geq 2$. Then*

$$NCM_{\tau^r}(t,x,y) = NCM_{\tau^c}(t,x,y) = e^{x(yt-t+ln(NCM_\tau(ty,1,y^{-1})))}. \tag{22}$$

Next we shall show that we can find an explicit expression $NCM_{123}(t,x,y)$ using some results of Mendes and Remmel [16]. Suppose that we want to compute the generating function

$$\begin{aligned} NCM_\tau(t,x,y) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{NCMS}_n(\tau)} x^{cyc(\sigma)} y^{\mathrm{cdes}(\sigma)} \\ &= e^{x\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)}} \end{aligned} \tag{23}$$

in the case where τ starts with 1. Then by Corollary 4, we know that

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{NMS}_n(\tau)} x^{LtRMin(\sigma)} y^{1+des(\sigma)}. \tag{24}$$

Now suppose that we can compute

$$NM_\tau(t,1,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma\in\mathcal{NMS}_n(\tau)} y^{1+des(\sigma)}. \tag{25}$$

Then we know that

$$NM_\tau(t,1,y) = e^{\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)}}$$

so that

$$\sum_{n\geq 1} \frac{t^n}{n!} \sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(\sigma)} = ln\left(NM_\tau(t,1,y)\right).$$

But then it follows that

$$NCM_\tau(t,x,y) = NM_\tau(t,x,y) = e^{x\ ln(NM_\tau(t,1,y))}. \tag{26}$$

Thus we need only compute (25). However, Mendes and Remmel [16] proved the following theorem.

Theorem 7. *If $\tau = j \dots 2\ 1$ where $j \geq 2$, then*

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NMS}_n(\tau)} y^{\mathrm{des}(\sigma)} = \left(\sum_{n\geq 0} \frac{t^n}{n!} \sum_{i\geq 0} (-1)^i \mathcal{R}_{n-1,i,j-1} y^i \right)^{-1} \tag{27}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-i$ ones such that j zeroes never appear consecutively.

Replacing y by $1/y$ and then replacing t by yt in (27) yields

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NMS}_n(\tau)} y^{n-\mathrm{des}(\sigma)} = \left(\sum_{n\geq 0} \frac{t^n}{n!} \sum_{i\geq 0} (-1)^i \mathcal{R}_{n-1,i,j-1} y^{n-i} \right)^{-1}. \tag{28}$$

It is easy to see that if $\sigma \in S_n$ has no $j \dots 2\ 1$-matches, then the reverse of σ, σ^r has no $1\ 2 \dots j$-matches and that $n - \mathrm{des}(\sigma)$ equals $1 + \mathrm{des}(\sigma^r)$. Thus it follows that if $\alpha = 1\ 2 \dots j$, then

$$\sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NMS}_n(\alpha)} y^{1+\mathrm{des}(\sigma)} = \left(\sum_{n\geq 0} \frac{t^n}{n!} \sum_{i\geq 0} (-1)^i \mathcal{R}_{n-1,i,j-1} y^{n-i} \right)^{-1}. \tag{29}$$

Thus we have the following theorem.

Theorem 8. *For $j \geq 2$ and $\tau = 12 \dots j$,*

$$\begin{aligned} NCM_\tau(t,x,y) &= \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCMS}_n(\tau)} x^{cyc(\sigma)} y^{\mathrm{cdes}(\sigma)} \\ &= e^{x\ ln\left(\frac{1}{\sum_{n\geq 0} \frac{t^n}{n!} \sum_{i\geq 0} (-1)^i \mathcal{R}_{n-1,i,j-1} y^{n-i}}\right)} \\ &= \left(\frac{1}{\sum_{n\geq 0} \frac{t^n}{n!} \sum_{i\geq 0} (-1)^i \mathcal{R}_{n-1,i,j-1} y^{n-i}} \right)^x. \end{aligned} \tag{30}$$

where $\mathcal{R}_{n,i,j}$ is the number of rearrangements of i zeroes and $n-i$ ones such that j zeroes never appear consecutively.

Now if $\tau = 123$, then we can obtain a more explicit formula for $NCM_\tau(t,x,y)$ using the following observations of Mendes and Remmel [16]. That is, suppose that we start with a word $w = w_1 \dots w_n$ which is a sequence in $\{0,1\}^*$ with no two consecutive zeros. Then we can uniquely factor w by cutting the word before each 0. For example, if $w = 11110110111010101110$ then we would factor w as

$$1111|011|0111|01|01|0111|0.$$

It is easy to see that each such word w is of the form

$$\{1\}^*\{01^i : i \geq 1\}^*(\epsilon + 0)$$

where ϵ is the empty word. Thus if U is the set of a words in $\{0,1\}^*$ with no two consecutive zeros and we weight each word in $w \in U$ by $WT(w) = y^{1(w)} z^{0(w)} t^{|w|}$ where $1(w)$ is the number of 1's in w, $0(w)$ is the number of 0's in w, and $|w|$ is the length of w, then it follows that

$$\begin{aligned} U(t,y,z) &= \sum_{w \in U} WT(w) \\ &= \frac{1}{1-yt} \frac{1}{1-\sum_{n\geq 2} y^{n-1} z t^n}(1+zt) \\ &= \frac{1+zt}{(1-yt-yzt^2)}. \end{aligned} \tag{31}$$

But then it is easy to see that

$$\sum_{i \geq 0} (-1)^i \mathcal{R}_{n-1,i,j-1} y^{n-i} = yU(t,y,-1)|_{t^{n-1}}. \tag{32}$$

Thus we have the following corollary of Theorem 8.

Corollary 9.

$$NCM_{123}(t,x,y) = \sum_{n\geq 0} \frac{t^n}{n!} \sum_{\sigma \in \mathcal{NCMS}_n(123)} x^{cyc(\sigma)} y^{\mathrm{cdes}(\sigma)} \tag{33}$$

$$\begin{aligned} &= e^{x \; ln\left(\frac{1}{\sum_{n\geq 0} \frac{t^n}{n!}\left(\frac{y(1-t)}{1-yt+yt^2}|_{t^{n-1}}\right)}\right)} \\ &= \left(\frac{1}{\sum_{n\geq 0} \frac{t^n}{n!}\left(\frac{y(1-t)}{1-yt+yt^2}|_{t^{n-1}}\right)}\right)^x \end{aligned} \tag{34}$$

One can use our generating functions for $NCM_{123}(t,x,y)$ to compute the initial values of $L_n^{ncm}(123)$ and $NCM_n(123)$.

n	$L_n^{ncm}(123)$	$NCM_n(123)$
1	1	1
2	1	2
3	1	5
4	3	17
5	9	70
6	39	349
7	189	2017
8	1107	13358
9	7281	99377
10	54351	822041

If one looks in the OEIS, one will see that both sequences occur. That is, the sequence of $L_n^{ncm}(123)$ is sequence A080635 and counts the number of permutations on n letters without double falls and without an initial fall. The sequence for $NCM_n(123)$ counts the number of permutations in S_n which have no 123-matches as expected.

Next we will compute $NCM_{132}(t,x,y)$. In this case, we will directly compute

$$L_{132}(t,y) = \sum_{m\geq 1} \frac{t^m}{m!} \sum_{C \in \mathcal{L}_m^{ncm}} y^{\mathrm{cdes}(C)}. \tag{35}$$

We start with a general observation. Suppose $\tau = \tau_1 \ldots \tau_j \in S_j$ where $\tau_1 = 1$. We can write any n-cycle C in the form $C = (\alpha_1, \ldots, \alpha_n)$ where $\alpha_1 = 1$. It is easy to see that the only cycle τ-match in C that can involve $\alpha_1 = 1$ is $\alpha_1\ \alpha_2 \ldots \alpha_j$. This means that the only possible cycle τ-matches in C must be of the form $\alpha_i\ \alpha_{i+1} \ldots \alpha_{i+j-1}$ where $i \leq n-j+1$. Thus the problem of finding n-cycles with no cycle τ-matches is equivalent to the problem of finding permutations $\sigma = \sigma_1 \ldots \sigma_n$ where $\sigma_1 = 1$ and σ has no τ-matches. Let S_n^1 denote the set of all permutations $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ such that $\sigma_1 = 1$ and let $S_{n,\tau}^1 = S_n^1 \cap \mathcal{NMS}_n(\tau)$ be the set of permutations of S_n^1 with no τ-matches. Then

$$A_{n,\tau}(y) = \sum_{\sigma \in S_{n,\tau}^1} y^{1+\mathrm{des}(\sigma)} = \sum_{C \in \mathcal{L}_n^{ncm}} y^{\mathrm{cdes}(C)}. \tag{36}$$

It turns out that in many cases we can find recurrences for $A_{n,\tau}(y)$ by classifying the permutations $\sigma = \sigma_1 \ldots \sigma_n \in S_n$ such that $\sigma_1 = 1$ according the position of 2 in σ. Let $\mathcal{E}_{n,k,\tau}$ denote the set of permutations $\sigma = \sigma_1 \ldots \sigma_n \in S_n^1(\tau)$ such that $\sigma_k = 2$.

Now fix $\tau = 1\ 3\ 2$ and let $A_m(y) = A_{m,\tau}(y)$ and $\mathcal{E}_{n,k} = \mathcal{E}_{n,k,\tau}$. Our goal is compute $A(t,y) = \sum_{m \geq 1} \frac{A_m(y) t^m}{m!}$. Now $A_1(y) = A_2(y) = y$ since the permutation 1 has no τ-matches and $1+\mathrm{des}(1) = 1$ and the permutation 1 2 has no τ-matches and $1+\mathrm{des}(12) = 1$. There are two permutations in S_3 that start with 1, namely, 1 2 3 and 1 3 2 and only 1 2 3 has no τ-matches so that $A_3(y) = y$ since $1+\mathrm{des}(123) = 1$. Now suppose that $n \geq 4$. Every permutation in $\mathcal{E}_{n,2}$ is of the form $1\ 2\ \sigma_3 \ldots \sigma_n$. Clearly, the only τ-matches must be of the form $\sigma_i\ \sigma_{i+1}\ \sigma_{i+2}$ where $i \geq 2$ so that $\mathcal{E}_{n,2}$ contributes $A_{n-1}(y)$ to $A_n(y)$. Every permutation in $\mathcal{E}_{n,3}$ is of the form $1\ \sigma_2\ 2 \ldots \sigma_n$ where $\sigma_2 \geq 3$. Thus all such permutations have a τ-match so that $\mathcal{E}_{n,3}$ contributes nothing to $A_n(y)$. For $4 \leq k \leq n$, the elements of the $\mathcal{E}_{n,k}$ are of the form

$$1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n.$$

In such a case, the only way that 2 can be part of τ-match is if the τ-match is $2\ \sigma_{k+1}\ \sigma_{k+2}$. It follows that an element of $\mathcal{E}_{n,k}$ contributes to $A_n(y)$ only if there is no τ-match in $\sigma_1 \ldots \sigma_{k-1}$ and there is no τ-match in $2\ \sigma_{k+1} \ldots \sigma_n$. Note that since $\sigma_{k-1}2$ is adescent pair,

$$1 + \mathrm{des}(1\ \sigma_2 \ldots \sigma_{k-1}\ 2\ \sigma_{k+1} \ldots \sigma_n) = 1 + \mathrm{des}(1\ \sigma_2 \ldots \sigma_{k-1}) + 1 + \mathrm{des}(2\ \sigma_{k+1} \ldots \sigma_n).$$

Hence the contribution of $\mathcal{E}_{n,k}$ to $A_n(y)$ is just $\binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y)$ since there are $\binom{n-2}{k-2}$ to choose the elements which make up $\sigma_2, \ldots, \sigma_{k-1}$. Thus for $n \geq 4$,

$$A_n(y) = A_{n-1}(y) + \sum_{k=4}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y). \tag{37}$$

Dividing both sides of (37) by $(n-2)!$, we obtain that for all $n \geq 4$,

$$\frac{A_n(y)}{(n-2)!} = \frac{A_{n-1}(y)}{(n-2)!} + \sum_{k=2}^{n-2} \frac{A_{k+1}(y)}{k!} \frac{A_{n-k-1}(y)}{(n-2-k)!}. \tag{38}$$

If we multiply both sides of (38) by t^{n-2} and sum, we obtain the differential equation

$$\frac{\partial^2 A(t,y)}{\partial t^2} - y - yt = \frac{\partial A(t,y)}{\partial t} - y - yt + \left(\frac{\partial A(t,y)}{\partial t} - y - yt \right) \frac{\partial A(t,y)}{\partial t}$$

so that $A(t,y)$ satisfies the second order partial differential equation

$$\frac{\partial^2 A(t,y)}{\partial t^2} = \frac{\partial A(t,y)}{\partial t}(1-y-yt) + \left(\frac{\partial A(t,y)}{\partial t}\right) \tag{39}$$

with initial conditions $A_0(y) = 0$ and $A_1(y) = y$. One can check that the solution to (39) is

$$A(t,y) = ln\left(\frac{1}{1 - y\int_0^t e^{(1-y)s - ys^2/2} ds}\right) \tag{40}$$

Hence

$$\begin{aligned} L_{132}(t,y) &= \sum_{m\geq 1} \frac{t^m}{m!} \sum_{C\in\mathcal{L}_m^{ncm}(132)} y^{\mathrm{cdes}(C)}. \\ &= ln\left(\frac{1}{1 - y\int_0^t e^{(1-y)s - ys^2/2} ds}\right) \end{aligned} \tag{41}$$

Thus we have the following theorem.

Theorem 10.

$$\begin{aligned} NCM_{132}(t,x,y) &= e^{x\ ln\left(\frac{1}{1-y\int_0^t e^{(1-y)s-ys^2/2}ds}\right)} \\ &= \frac{1}{\left(1 - y\int_0^t e^{(1-y)s-ys^2/2} ds\right)^x}. \end{aligned} \tag{42}$$

We note that specialization

$$NCM_{132}(t,1,1) = \frac{1}{1 - \int_0^t e^{-s^2/2} ds}$$

has been proved by Elizalde and Noy [6].

One can use our generating functions for $NCM_{132}(t,x,y)$ to compute the initial values of $L_n^{ncm}(132)$ and $NCM_n(132)$.

n	$L_n^{ncm}(132)$	$NCM_n(132)$
1	1	1
2	1	2
3	1	5
4	2	16
5	7	63
6	28	296
7	131	1623
8	720	10176
9	4513	71793
10	31824	562848

If one looks in the OEIS, then both the sequences for $L_n^{ncm}(132)$ and $NCM_n(132)$ occur. The sequence for $L_n^{ncm}(132)$ is sequence A052319 which counts the number of increasing rooted trimmed trees with n nodes. Here an increasing tree is a tree labeled with $1,\ldots,n$ where the numbers increase as you move away from the root. A tree with a forbidden limb of length k is a tree where the path from any leaf inward hits a branching node or another leaf within k steps. A trimmed tree is a tree with a forbidden limb of length 2. The sequence for $NCM_n(132)$ is the number of permutations that have no 132-matches as expected.

We end this section with some results on $CA_\Upsilon(t,x,y)$ and $NCM_\Upsilon(t,x,y)$ where $\Upsilon \subseteq S_3$. For certain Υ's, this problem is uninteresting. For example, if Υ contains both 1 2 3 and 1 3 2, then any k-cycle $C = (\sigma_1,\sigma_2,\ldots,\sigma_k)$ where $\sigma_1 = 1$ and $k \geq 3$ will have a cycle Υ-match since $\sigma_1\ \sigma_2\ \sigma_3$ must be either a cycle 1 2 3-match or a cycle 1 3 2-match. Thus in this case $\mathcal{L}_1^{ca}(\Upsilon) = \mathcal{L}_1^{ncm}(\Upsilon) = \{(1)\}$, $\mathcal{L}_2^{ca}(\Upsilon) = \mathcal{L}_2^{ncm}(\Upsilon) = \{(1,2)\}$, and $\mathcal{L}_k^{ca}(\Upsilon) = \mathcal{L}_k^{ncm}(\Upsilon) = \emptyset$ for $k \geq 3$. It then follows from Theorem 2 that

$$CA_\Upsilon(t,x,y) = NCM_\Upsilon(t,x,y) = e^{x\left(yt+\frac{yt^2}{2}\right)}$$

A more interesting case is when $\Upsilon = \{123, 321\}$. First observe that since any cycle contains a cycle occurrence of 1 3 2 if and only if it contains a cycle occurrence of 3 2 1, then it is the case that any k-cycle C where $k \geq 3$ must have a cycle occurrence of either 1 2 3 or 3 2 1. Thus

$$CA_\Upsilon(t,x,y) = e^{x\left(yt+\frac{yt^2}{2}\right)}$$

Let $C = (\sigma_1,\ldots,\sigma_n)$ be an n-cycle such that $\sigma_1 = 1$. If $n \geq 3$, then we must have $\sigma_2 > \sigma_3$ since otherwise there will be a cycle 1 2 3-match. But then we must have $\sigma_3 < \sigma_4$ since otherwise there would be cycle 3 2 1-match. Continuing on in this way, we see that $\sigma_2 \ldots \sigma_n$ must be an alternating permutation. That is, we must have

$$\sigma_2 > \sigma_3 < \sigma_4 > \sigma_5 < \sigma_6 > \sigma_7 \cdots .$$

However, this means if $n = 2k+1 \geq 3$, then there are no n cycles which have no cycle Υ-matches since since we are forced to have $\sigma_{2k} > \sigma_{2k+1} > \sigma_1$ which is a cycle 3 2 1-match. If $n = 2k$ and $\sigma_2 \ldots \sigma_n$ is alternating, then C will have no cycle Υ-matches. For such σ it is easy to see that $1 + \mathrm{des}(\sigma) = k$. Thus in this case, $L_{2k+1}^{ncm}(\Upsilon) = 0$ for $k \geq 1$ and $L_{2k}^{ncm}(\Upsilon)$ is just the number of odd alternating permuations of length $2k-1$ for $k \geq 1$.

If we let Alt_n denote the number of Alternating permutations of length n, then André [1, 2] proved that

$$\sum_{n\geq 0} Alt_{2n+1} \frac{t^{2n+1}}{(2n+1)!} = \frac{sin(t)}{cos(t)}. \tag{43}$$

Thus

$$\begin{aligned}\sum_{n\geq 1} L_{2n}^{ncm}(\Upsilon)\frac{t^{2n}}{(2n)!} &= \sum_{n\geq 1} Alt_{2n-1}\frac{t^{2n}}{(2n)!} \\ &= \int_0^t \frac{sin(z)}{cos(z)}dz = -ln(|cos(t)|.\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{n\geq 1}\frac{t^{2n}}{(2n)!}\sum_{C\in\mathcal{L}_{2n}^{ncm}(\Upsilon)} y^{\mathrm{cdes}(C)} &= \sum_{n\geq 1} y^n L_{2n}^{ncm}(\Upsilon)\frac{t^{2n}}{(2n)!} \\ &= -ln(|cos(t\sqrt{y})|.\end{aligned}$$

and

$$\sum_{n\geq 1}\frac{t^{n}}{(n)!}\sum_{C\in\mathcal{L}_{n}^{ncm}(\Upsilon)} y^{\mathrm{cdes}(C)} = ty - ln(|cos(t\sqrt{y})|. \tag{44}$$

It follows that

$$NCM_{\Upsilon}(t,x,y) = e^{x(ty-ln(|cos(t\sqrt{y})|)} = \frac{e^{xyt}}{cos(t\sqrt{y})^x} = e^{xyt}sec(t\sqrt{y})^x. \tag{45}$$

Thus we have proved the following theorem.

Theorem 11.

$$\sum_{n\geq 1}\frac{t^{n}}{(n)!}\sum_{C\in\mathcal{L}_{n}^{ncm}(\{123,321\})} y^{\mathrm{cdes}(C)} = ty - ln(|cos(t\sqrt{y})| \tag{46}$$

and

$$NCM_{\{123,321\}}(t,x,y) = e^{xyt}(sec(t\sqrt{y}))^x. \tag{47}$$

3 General results

In this section, we shall describe how we can compute $NCM_\tau(t,x,y)$ for certain general classes of permutations τ. We start by considering permutations $\tau = \tau_1 \dots \tau_j$ where $\tau_1 = 1$ and $\tau_j = 2$. In that case, we have the following theorem.

Theorem 12. *Let $\tau = \tau_1 \dots \tau_j \in S_j$ where $j \geq 3$ and $\tau_1 = 1$ and $\tau_j = 2$. Then*

$$NCM_\tau(t,x,y) = \frac{1}{(1-\int_0^t e^{(y-1)s-\frac{y^{\mathrm{des}(\tau)}s^{j-1}}{(j-1)!}}ds)^x} \tag{48}$$

Proof. Note that in the special case where $j = 3$, the only permutation satisfying the hypothesis of the theorem is $\tau = 1\ 3\ 2$. Thus in this special case, the result follows from Theorem 10. Thus assume that we fix a $\tau = \tau_1 \dots \tau_j \in S_j$ where $\tau_1 = 1$ and $\tau_j = 2$ and $j \geq 4$.

Our first goal is to compute

$$A(t,y) = \sum_{n\geq 1} A_n(y)\frac{t^n}{n!} \tag{49}$$

where $A_n(y) = \sum_{\sigma\in S_{n,\tau}^1} y^{\mathrm{des}(\sigma)+1}$. Now it is easy to see that $A_n(y) = \sum_{\sigma\in S_n^1} y^{\mathrm{des}(\sigma)+1}$ for $1 \leq n \leq j-1$. Thus

$$\begin{aligned} A(t,y) &= yt + y\frac{t^2}{2} + (y+y^2)\frac{t^3}{3!} + \cdots \\ \frac{\partial A(t,y)}{\partial t} &= y + yt + (y+y^2)\frac{t^2}{2!} + \cdots \text{ and} \\ \frac{\partial^2 A(t,y)}{\partial t^2} &= y + (y+y^2)t + \cdots .\end{aligned}$$

For $n \geq j$, we shall prove a recursive formula for $A_n(y)$. We consider three cases for $\sigma = \sigma_1 \dots \sigma_n \in S^1_{n,\tau}$ depending on the position of 2 in σ.

Case 1. $\sigma_2 = 2$.
In this case because $j \geq 4$, the only possible τ-matches in σ must occur in $\sigma_2 \dots \sigma_n$. Since $\mathrm{des}(\sigma) + 1 = \mathrm{des}(\sigma_2 \dots \sigma_n) + 1$, it follows that the contribution of the permutations in this case to $A_n(y)$ is just $A_{n-1}(y)$.

Case 2. $\sigma_k = 2$ where $k \notin \{2, j\}$.
In this case, we have $\binom{n-2}{k-2}$ ways to choose the elements D_k that will constitute $\sigma_2 \dots \sigma_{k-1}$. Once we have chosen D_k, we have to consider the ways in which we can arange the elements of D_k to form $\sigma_2 \dots \sigma_{k_1}$ and the ways that we can arrange $[n] - (D_k \cup \{1, 2\})$ to form $\sigma_{k+1} \dots \sigma_n$ so that

$$\sigma = 1\ \sigma_2 \dots \sigma_{k-1}\ 2\ \sigma_{k+1} \dots \sigma_n \tag{50}$$

has no τ-matches. However it is easy to see that since $k \notin \{2, j\}$ that the only τ-matches for σ of the form (50) can occur in either entirely in $1\ \sigma_2 \dots \sigma_{k-1}$ or entirely in $2\ \sigma_{k+1} \dots \sigma_n$. Moreover it is the case that

$$\mathrm{des}(\sigma) + 1 = \mathrm{des}(1\ \sigma_2 \dots \sigma_{k-1}) + 1 + \mathrm{des}(2\ \sigma_{k+1} \dots \sigma_n) + 1$$

since $\sigma_{k-1} > 2$. Thus the contribution to $A_n(y)$ of the permutations in this case is

$$\binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y).$$

Case 3. $\sigma_j = 2$.
In this case, we have $\binom{n-2}{j-2}$ ways to choose the elements D_j that will constitute $\sigma_2 \dots \sigma_{j-1}$. Once we have chosen D_j, we have to consider the ways in which we can arange the elements of D_j to form $\sigma_2 \dots \sigma_{j_1}$ and we can arrange $[n] - (D_j \cup \{1, 2\})$ to form $\sigma_{j+1} \dots \sigma_n \sigma_{k+1} \dots \sigma_n$ so that

$$\sigma = 1\ \sigma_2 \dots \sigma_{j-1}\ 2\ \sigma_{j+1} \dots \sigma_n \tag{51}$$

has no τ-matches. Unlike Case 2, it is not enough just to ensure that $1\ \sigma_2 \dots \sigma_{j-1}$ and $2\ \sigma_{j+1} \dots \sigma_n$ have no τ-matches. That is, we must also ensure that $\mathrm{red}(\sigma_2 \dots \sigma_{j-1}) \neq \mathrm{red}(\tau_2 \dots \tau_{j-1})$ since otherwise $1\ \sigma_2 \dots \sigma_{j-1}\ 2$ would be τ-match. Note that in such a situation $\mathrm{des}(1\ \sigma_2 \dots \sigma_{j-1}) + 1 = \mathrm{des}(\tau)$. Thus the contributions to $A_n(y)$ of the permutations in this case is

$$\binom{n-2}{j-2} (A_{j-1}(y) - y^{\mathrm{des}(\tau)}) A_{n-j+1}(y).$$

It follows that for $n \geq j$,

$$A_n(y) = A_{n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y) - \binom{n-2}{j-2} y^{\mathrm{des}(\tau)} A_{n-j+1}(y) \tag{52}$$

or, equivalently,

$$\frac{A_n(y)}{(n-2)!} = \frac{A_{n-1}(y)}{(n-2)!} + \left(\sum_{k=3}^{n} \frac{A_{k-1}(y)}{(k-2)!} \frac{A_{n-k+1}(y)}{(n-k)!} \right) - \frac{y^{\mathrm{des}(\tau)}}{(j-2)!} \frac{A_{n-j+1}(y)}{(n-j)!}. \tag{53}$$

Now for any formal power series $f(t) = \sum_{n\geq 1} f_n t^n$, we let $f(t)|_{t\leq j}$ denote $f_0+f_1t+\cdots+f_jt^j$. We then then multiple both sides of (53) by t^{n-2} and sum and we will get the differential equation

$$\begin{aligned}
&\frac{\partial^2 A(t,y)}{\partial t^2} - \left(\frac{\partial^2 A(t,y)}{\partial t^2}|_{t\leq j-3}\right) \\
&= \frac{\partial A(t,y)}{\partial t} - \left(\frac{\partial A(t,y)}{\partial t}|_{t\leq j-3}\right) + \\
&\left(\frac{\partial A(t,y)}{\partial t} - y\right)\frac{\partial A(t,y)}{\partial t} - \left(\left(\frac{\partial A(t,y)}{\partial t} - y\right)\frac{\partial A(t,y)}{\partial t}|_{t\leq j-3}\right) - \\
&\frac{y^{\mathrm{des}(\tau)}}{(j-2)!}\frac{\partial A(t,y)}{\partial t}.
\end{aligned}$$

Thus

$$\begin{aligned}
\frac{\partial^2 A(t,y)}{\partial t^2} &= (1-y-y^{\mathrm{des}(\tau)})\frac{\partial A(t,y)}{\partial t} + \left(\frac{\partial A(t,y)}{\partial t}\right)^2 + \\
&\left(\frac{\partial^2 A(t,y)}{\partial t^2}|_{t\leq j-3}\right) - \left(\frac{\partial A(t,y)}{\partial t}|_{t\leq j-3}\right) - \left(\left(\frac{\partial A(t,y)}{\partial t} - y\right)\frac{\partial A(t,y)}{\partial t}|_{t\leq j-3}\right).
\end{aligned}$$

We claim that

$$0 = \left(\frac{\partial^2 A(t,y)}{\partial t^2}|_{t\leq j-3}\right) - \left(\frac{\partial A(t,y)}{\partial t}|_{t\leq j-3}\right) - \left(\left(\frac{\partial A(t,y)}{\partial t} - y\right)\frac{\partial A(t,y)}{\partial t}|_{t\leq j-3}\right)$$

or, equivalently, that

$$\frac{\partial^2 A(t,y)}{\partial t^2}|_{t\leq j-3} = \left(\frac{\partial A(t,y)}{\partial t} + \left(\frac{\partial A(t,y)}{\partial t} - y\right)\frac{\partial A(t,y)}{\partial t}\right)|_{t\leq j-3}. \tag{54}$$

If we take the coefficient of t^s where $0 \leq s \leq t^{j-3}$ on both sides of (54), then we must show that

$$\begin{aligned}
\frac{A_{s+2}(y)}{s!} &= \frac{A_{s+1}(y)}{s!} + \sum_{k=1}^{s} \frac{A_{k+1}(y)}{k!}\frac{A_{s-k+1}(y)}{(s-k)!} \\
&= \frac{A_{s+1}(y)}{s!} + \sum_{k=3}^{s+2} \frac{A_{k-1}(y)}{(k-2)!}\frac{A_{s+2-(k-1)}(y)}{(s+2-k)!}.
\end{aligned}$$

Thus if we multiply both sides by $s!$, we see that we must show that for $0 \leq s \leq j-3$,

$$A_{s+2}(y) = A_{s+1}(y) + \sum_{k=3}^{s+2} \binom{s+2}{k-2} A_{k-1}(y)A_{s+2-(k-1)}(y). \tag{55}$$

However this follows from our analysis of Cases 1, 2, and 3 above for the recursion of $A_{s+2}(y)$. That is, since $s+2 \leq j-1$, Case 2 does not apply so that we only get the contributions from Cases 1 and 3 which is exactly (55).

Thus we have shown that $A(y,t)$ satisfies the partial differential equation where

$$\frac{\partial^2 A(t,y)}{\partial t^2} = (1-y-y^{\mathrm{des}(\tau)})\frac{\partial A(t,y)}{\partial t} + \left(\frac{\partial A(t,y)}{\partial t}\right)^2 \tag{56}$$

with intial conditions that $A(y,0) = 0$, $A(y,t)|_t = y$, and $A(y,t)|_{\frac{t^2}{2!}} = y$. It is then easy to check that the solution to this PDE is

$$A(y,t) = ln\left(\frac{1}{1 - \int_0^t e^{(1-y)s + y^{\mathrm{des}(\tau)}\frac{s^{j-1}}{(j-1)!}} ds}\right). \tag{57}$$

Thus

$$\begin{aligned} A(y,t) &= \sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)} \\ &= ln\left(\frac{1}{1 - \int_0^t e^{(1-y)s + y^{\mathrm{des}(\tau)}\frac{s^{j-1}}{(j-1)!}} ds}\right). \end{aligned} \tag{58}$$

But then we know by Theorem 2, that

$$\begin{aligned} NCM_\tau(t,x,y) &= e^{x\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)}} \\ &= e^{xln\left(\frac{1}{1 - \int_0^t e^{(1-y)s + y^{\mathrm{des}(\tau)}\frac{s^{j-1}}{(j-1)!}} ds}\right)} \\ &= \left(\frac{1}{1 - \int_0^t e^{(1-y)s + y^{\mathrm{des}(\tau)}\frac{s^{j-1}}{(j-1)!}} ds}\right)^x \end{aligned}$$

which is what we wanted to prove. □

We end this section by showing how one can compute $NCM_\tau(t,x,y)$ where $\tau \in S_m$ is of the form $\tau = 1\ 2\ \ldots\ (j-1)\ \gamma\ j$ where γ is a permutation of the elements $j+1,\ldots,m$ where $m \geq j+1$. We let $p = m - j$ so that $\mathrm{red}(\gamma) \in S_p$. We shall assume that $j \geq 3$ since we have already dealt with permutations that start with 1 and end with 2.

Using our previous theorems as a guide, we shall assume that $NCM_\tau(t,x,y)$ is of the form

$$NCM_\tau(t,x,y) = e^{x\sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{cdes}(C)}} = \frac{1}{(U_\tau(t,y))^x}$$

where

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}\frac{t^n}{n!}. \tag{59}$$

We have been unable to find a closed form for $U_\tau(t,y)$. However, we can show that the coefficients of $U_{n,\tau}(y)$ satisfy a simple recursion. That is, we shall prove the following.

Theorem 13. *Suppose that $\tau = 1\ 2\ \ldots j-1\ \gamma\ j$ where γ is a permutation of $j+1,\ldots,j+p$ and $j \geq 3$. Then*

$$NCM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$$

where

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}(y)\frac{t^n}{n!} \tag{60}$$

and

$$U_{n+j,\tau}(y) = (1-y)U_{n+j-1,\tau}(y) - y^{\mathrm{des}(\tau)}\binom{n}{p}U_{n-p+1,\tau}(y). \tag{61}$$

Proof. Taking the natural logarithm of both sides (59) and using (36), we see

$$-\,ln(U_\tau(t,y)) = \sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in\mathcal{L}_n^{ncm}(\tau)} y^{\mathrm{des}(\sigma)+1}\sum_{n\geq 1}\sum_{\sigma\in S^1_{n,\tau}} y^{\mathrm{des}(\sigma)+1}. \tag{62}$$

Before proceeding, we need to establish some notation. Fix τ of the form $1\ 2\ \ldots j-1\gamma j$ where $j\geq 3$. For any $\sigma\in S_n^1$, we let τ-imch(σ) be the indicator function that the initial segment of size m in σ is a τ-match. Thus τ-imch$(\sigma)=1$ if $\mathrm{red}(\sigma_1\ldots\sigma_m)=\tau$ and we let τ-imch$(\sigma)=0$ otherwise. For $i=1,\ldots,j-1$, we let $\tau^{(i)}=\mathrm{red}(i\ i+1\ \ldots j-1\ \gamma\ j)$. Our first goal is to compute

$$A(t,y) = \sum_{n\geq 1} A_n(y)\frac{t^n}{n!} \tag{63}$$

where

$$A_n(y) = \sum_{\sigma\in S^1_{n,\tau}} y^{1+\mathrm{des}(\sigma)}.$$

For $i=2,\ldots,k-1$, we shall also need the following functions

$$B_i(t,y) = 1+\sum_{n\geq 1} B_{i,n}(y)\frac{t^n}{n!} \tag{64}$$

where

$$B_{i,n}(y) = \sum_{\substack{\sigma\in S_n^1\\ \tau\text{-mch}(\sigma)=0\\ \tau^{(2)}\text{imch}(\sigma)=0\\ \tau^{(3)}\text{imch}(\sigma)=0\\ \vdots\\ \tau^{(i)}\text{imch}(\sigma)=0}} y^{1+\mathrm{des}(\sigma)}.$$

Thus $B_{i,n}(y)$ is the sum of $y^{1+(des)(\sigma)}$ over all permutation σ in S_n^1 such that σ has no τ-matches and σ does not start with a $\tau^{(j)}$-match for $j=2,\ldots,i$.

First we develop recursions for $A_n(y)$ for $n\geq 2$. Let $\mathcal{E}_{n,k,\tau}$ denote the set of all $\sigma=\sigma_1\ldots\sigma_n\in S^1_{n,\tau}$ such that $\sigma_k=2$. We then consider two cases for $\sigma\in S^1_{n,\tau}$ depending on which $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma\in\mathcal{E}_{n,2,\tau}$.
Thus $\sigma=1\ 2\ \sigma_3\ldots\sigma_n$. To ensure that σ has no τ-matches, we must ensure that there are no τ matches in $2\ \sigma_3\ldots\sigma_n$ and that σ does not start with a τ-match which is equivalent to ensuring that $2\ \sigma_3\ldots\sigma_n$ does not start with $\tau^{(2)}$-match. Thus in this case, the permtuations of $\mathcal{E}_{n,2,\tau}$ contribute $B_{2,n-1}(y)$ to $A_n(y)$.

Case 2 $\sigma\in\mathcal{E}_{n,k,\tau}$ where $3\leq k\leq n$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k\ldots\sigma_n$ or in $\sigma_1\ldots\sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose that elements that will constitute $\sigma_2\ldots\sigma_{k-1}$

and $A_{k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$. Once we have picked $\sigma_2 \ldots \sigma_{k-1}$, there are $A_{n-k+1}(1)$ ways to order the remaining elements so that there are no τ-matches in $\sigma_k \ldots \sigma_n$. Having picked σ, we have that

$$\mathrm{des}(\sigma) + 1 = \mathrm{des}(\sigma_1 \ldots \sigma_{k-1}) + 1 + \mathrm{des}(\sigma_k \ldots \sigma_n) + 1$$

since $\sigma_{k-1} > 2$. Hence in this case, the permutations in $\mathcal{E}_{n,k,\tau}$ where will contribute $\binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y)$ elements to $A_n(y)$.

It follows that for $n \geq 2$,

$$A_n(y) = B_{2,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y). \tag{65}$$

We can develop similar recursions for $B_{2,n}(y)$ for $n \geq 2$. However we have to consider the cases $j = 3$ and $j > 3$ separately.

First consider, the case where $j = 3$. Note in this case $\tau^{(2)} = \mathrm{red}(2\ \gamma\ 3) = 1\ \alpha\ 2$ where α is a permutation of $3, \ldots, p+2$ such that $\mathrm{red}(\alpha) = \mathrm{red}(\gamma)$. We then consider three cases for $\sigma \in S^1_{n,\tau}$ depending on which $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in E_{n,2,\tau}$.
Thus $\sigma = 1\ 2\ \sigma_3 \ldots \sigma_n$. To ensure that σ has no τ-matches, we must ensure that there are no τ matches in $2\ \sigma_3 \ldots \sigma_n$ and that σ does not start with a τ-match which is equivalent to ensuring that $2\ \sigma_3 \ldots \sigma_n$ does not start with $\tau^{(2)}$-match. It might seem that to ensure that σ does not start with a $\tau^{(2)}$-match that we must ensure that $2\ \sigma_3 \ldots \sigma_n$ does start with $\tau^{(3)}$-match. However, in this case $\tau^{(3)} = \mathrm{red}(\gamma\ 3)$ does not start with 1 so that is automatically true that $2\ \sigma_3 \ldots \sigma_n$ does start with $\tau^{(3)}$-match. Thus the permutations in $\mathcal{E}_{n,2,\tau}$ contribute $B_{2,n-1}(y)$ to $B_{2,n}(y)$.

Case 2. $\sigma \in \mathcal{E}_{n,p+2,\tau}$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_{p+1} \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_p$. Now we have $\binom{n-2}{p}$ ways to choose that elements that will constitute $\sigma_2 \ldots \sigma_{p+1}$. We can order these elements in any way that we want except that we cannot have $\mathrm{red}(\sigma_2 \ldots \sigma_{p+1}) = \mathrm{red}(\gamma)$ since otherwise σ would start with at $\tau^{(2)}$ match. Note that $B_{2,p+1}(y) = \sum_{\beta \in S^1_{p+1}} y^{\mathrm{des}(\beta)+1}$ since no permutation of length $p+1$ can contain a τ-match or start with $\tau^{(2)}$-match. Since

$$\mathrm{des}(1\ \sigma_2 \ldots \sigma_{p+1}) + 1 + \mathrm{des}(2\ \sigma_{p+2} \ldots \sigma_n) + 1 = \mathrm{des}(\sigma)$$

and $\mathrm{des}(1\ \gamma) + 1 = \mathrm{des}(\tau)$, the permutations in $\mathcal{E}_{n,p+2,\tau}$ will contribute $\binom{n-2}{p}(B_{2,p+1}(y) - y^{\mathrm{des}(\tau)}) A_{n-p-1}(y)$ to $B_{2,n}(y)$.

Case 3. $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$ and $k \notin \{2, p+2\}$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose that elements that will constitute $\sigma_2 \ldots \sigma_{k-1}$ and $B_{2,k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$ and $\sigma_1 \ldots \sigma_{k-1}$ does not start with a $\tau^{(2)}$ match and $A_{n-k+1}(1)$ ways to order $\sigma_k \ldots \sigma_n$ that it contains no τ-match. It follows that the permutations in $\mathcal{E}_{n,k,\tau}$ will contribute $\binom{n-2}{k-2} B_{2,k-1}(y) A_{n-k+1}(y)$ to

$B_{2,n}(y)$.

Thus if $n \geq p+2$, we have the recursion

$$B_{2,n}(y) = B_{2,n-1}(y) + \left(\sum_{k=3}^{n} \binom{n-2}{p-2} B_{2,k-1}(y) A_{n-k+1}(y)\right) - \binom{n-2}{p} y^{\mathrm{des}(\tau)} A_{n-p-1}(y). \quad (66)$$

For $2 \leq n \leq p+1$, Case 2 does not apply so that we have the recursion

$$B_{2,n}(y) = B_{2,n-1}(y) + \left(\sum_{k=3}^{n} \binom{n-2}{p-2} B_{2,k-1}(y) A_{n-k+1}(y)\right). \quad (67)$$

Before considering the case where $j > 3$, we shall show how we can derive a recursion (61) for the $U_{n,\tau}(y)$s in this case. We have shown that for all $n \geq 2$,

$$\begin{aligned} A_n(y) &= B_{2,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y) A_{n-k+1}(y) \text{ and} \\ B_{2,n}(y) &= B_{2,n-1}(y) + \left(\sum_{k=3}^{n} \binom{n-2}{p-2} B_{2,k-1}(y) A_{n-k+1}(y)\right) - \\ &\quad \chi(n \geq p+2) y^{\mathrm{des}(\tau)} \binom{n-2}{p} A_{n-p-1}(y) \end{aligned}$$

where for any statement A, we let $\chi(A)$ equal 1 if A is true and equal 0 if A is false. Thus we have that for all $n \geq 2$,

$$\begin{aligned} \frac{A_n(y)}{(n-2)!} &= \frac{B_{2,n-1}(y)}{(n-2)!} + \sum_{k=3}^{n} \frac{A_{k-1}(y)}{(k-2)!} \frac{A_{n-k+1}(y)}{(n-k)!} \text{ and} \\ \frac{B_{2,n}(y)}{(n-2)!} &= \frac{B_{2,n-1}(y)}{(n-2)!} + \left(\sum_{k=3}^{n} \frac{B_{2,k-1}(y)}{(k-2)!} \frac{A_{n-k+1}(y)}{(n-k)!}\right) - \chi(n \geq p+2) \frac{y^{\mathrm{des}(\tau)}}{p!} \frac{A_{n-p-1}(y)}{(n-p)!}. \end{aligned}$$

Multiplying by t^{n-2} and summing, we obtain the following differential equations when we think of $A = A(t,y)$ and $B_2 = B_2(t,y)$ as just functions of t:

$$\begin{aligned} A'' &= B_2' + (A' - y)A' \text{ and} \\ B_2'' &= B_2' + (B_2' - y)A' - \frac{y^{\mathrm{des}(\tau)} t^p}{p!} A'. \end{aligned}$$

Now if $U = U(t,y) = U_\tau(t,y)$, then $A = -ln(U)$. Thus

$$A' = \frac{-U'}{U} \text{ and} \quad (68)$$

$$A'' = \frac{-U''}{U} + \left(\frac{U'}{U}\right)^2. \quad (69)$$

Making these substititions in our first differential equation and solving for B_2', we see that

$$B_2' = -\frac{U'' + yU'}{U}. \quad (70)$$

Thus

$$B_2'' = -\frac{U''' + yU''}{U} + \frac{(U'' + yU')U'}{U^2}. \tag{71}$$

Substituting these expressions into our second differential equation and simplifying, we obtain the following differential equation for U,

$$U''' = (1-y)U'' - \frac{y^{\mathrm{des}(\tau)}t^p}{p!}U'. \tag{72}$$

Taking the coefficient of $\frac{t^n}{n!}$ on both sides of (72), we set that

$$U_{n+3,\tau}(y) = (1-y)U_{n+2}(y) - \binom{n}{p} y^{\mathrm{des}(\tau)} U_{n-p+1}(y). \tag{73}$$

in the case where $\tau = 12\gamma 3$ and γ is permutation of $4, \ldots, 3+p$.

Now consider the recursion for $B_{2,n}(y)$ where $j > 3$. We then consider two cases for $\sigma \in S^1_{n,\tau}$ depending on which set $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.
Thus $\sigma = 1\ 2\ \sigma_3 \ldots \sigma_n$. To ensure that σ has no τ-matches, we must ensure that there are no τ matches in $2\ \sigma_3 \ldots \sigma_n$ and that σ does not start with a τ-match which is equivalent to ensuring that $2\ \sigma_3 \ldots \sigma_n$ does not start with $\tau^{(2)}$-match. However in this case, we must also ensure that σ does not start with at $\tau^{(2)}$ which means that $2\ \sigma_3 \ldots \sigma_n$ must not start with $\tau^{(3)}$-match. Thus in this case, the $\sigma \in \mathcal{E}_{n,2,\tau}$ contribute $B_{3,n-1}(y)$ to $B_{2,n}(y)$.

Case 2 $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose that elements that will constitute $\sigma_2 \ldots \sigma_{k-1}$ and $B_{2,k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$ and $\sigma_1 \ldots \sigma_{k-1}$ does not start with a $\tau^{(2)}$ match and there are $A_{n-k+1}(1)$ ways to order $\sigma_k \ldots \sigma_n$ so that there is no τ-match. It follows that the permutations in $\mathcal{E}_{n,k}$ will contribute $\binom{n-2}{k-2} B_{2,k-1}(y) A_{n-k+1}(y)$ elements to $B_{2,n}(y)$.

It follows that if $j \geq 3$, then for $n \geq 2$,

$$B_{2,n}(y) = B_{3,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{2,k-1}(y) A_{n-k+1}(y). \tag{74}$$

One can repeat this type of argument to show that in general, for $2 \leq i \leq j-2$

$$B_{i,n}(y) = B_{i+1,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{i,k-1}(y) A_{n-k+1}(y). \tag{75}$$

The recursion for $B_{j-1,n}(y)$ is similar to the recursion for $B_{2,n}(y)$ when $j=3$. That is, $\tau^{(j-1)} = \mathrm{red}(j-1\ \gamma\ j) = 1\ \alpha\ 2$, where α is a permutation of $3, \ldots, p+2$ and $\mathrm{red}(\gamma) = \mathrm{red}(\alpha)$. Then we have to consider three cases depending on which set $\mathcal{E}_{n,k,\tau}$ contains σ.

Case 1. $\sigma \in \mathcal{E}_{n,2,\tau}$.

Thus $\sigma = 1\ 2\ \sigma_3 \ldots \sigma_n$. To ensure that σ has no τ-matches and does not start with $\tau^{(i)}$-match for $i = 2, \ldots, j-1$, we clearly have to ensure that $2\ \sigma_3 \ldots \sigma_n$ has no τ-matches and does not start with $\tau^{(i)}$-match for $i = 2, \ldots, j-1$. However, we do not have to worry about $2\ \sigma_3 \ldots \sigma_n$ starting with $\tau^{(j)} = \text{red}(\sigma\ j)$ since $\tau^{(j)}$ does not start with its least element. Thus in this case, the permutations in $\mathcal{E}_{n,2,\tau}$ contribute $B_{j-1,n-1}(y)$ to $B_{j-1,n}(y)$.

Case 2. $\sigma \in \mathcal{E}_{n,p+2,\tau}$ In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_{p+1} \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_p$. Now we have $\binom{n-2}{p}$ ways to choose that elements that will constitute $\sigma_2 \ldots \sigma_{p+1}$. We can order these elements in any way that we want except that we cannot have $\text{red}(\sigma_2 \ldots \sigma_{p+1}) = \text{red}(\gamma)$ since otherwise σ would start with at $\tau^{(j-1)}$ match. Note that $B_{j-1,p+1}(y) = \sum_{\beta \in S^1_{p+1}} y^{\text{des}(\beta)+1}$ since no permutation of length $p+1$ can contain a τ-match or start with $\tau^{(i)}$-match for $i = 2, \ldots j-1$. Thus since

$$\text{des}(1\ \sigma_2 \ldots \sigma_{p+1}) + 1 + \text{des}(2\ \sigma_{p+2} \ldots \sigma_n) + 1 = \text{des}(\sigma)$$

and $\text{des}(1\ \gamma) + 1 = \text{des}(\tau)$, the permutations in $\mathcal{E}_{n,p+2,\tau}$ will contribute $\binom{n-2}{p}(B_{j-1,p+1}(y) - y^{\text{des}(\tau)})A_{n-p+1}(y)$ to $B_{j-1,n}(y)$.

Case 3. $\sigma \in \mathcal{E}_{n,k,\tau}$ where $3 \leq k \leq n$ and $k \notin \{2, p+2\}$.
In this case, it is easy to see that the only possible τ-matches must occur in $\sigma_k \ldots \sigma_n$ or in $\sigma_1 \ldots \sigma_{k-1}$. Thus we have $\binom{n-2}{k-2}$ ways to choose that elements that will constitute $\sigma_2 \ldots \sigma_{k-1}$ and $B_{j-1,k-1}(1)$ ways to order them so that there are no τ-matches in $\sigma_1 \ldots \sigma_{k-1}$ and $\sigma_1 \ldots \sigma_{k-1}$ does not start with a $\tau^{(i)}$-match for $i = 2, \ldots, j-1$ and there are $A_{n-k+1}(1)$ ways to order $\sigma_k \ldots \sigma_n$ so that there is no τ-match. Thus the permutations in $\mathcal{E}_{n,k,\tau}$ will contribute $\binom{n-2}{k-2}B_{j-1,k-1}(y)A_{n-k+1}(y)$ to $B_{j-1,n}(y)$.

It follows that for $n \geq 2$,

$$\begin{aligned} B_{j-1,n}(y) &= B_{j-1,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{j-1,k-1}(y) A_{n-k+1}(y) - \\ & \qquad \chi(n \geq p+2) \binom{n-2}{p} y^{\text{des}(\tau)} A_{n-p-1}(y). \end{aligned} \tag{76}$$

Thus for all $n \geq 2$, we have proved that in general

$$A_n(y) = B_{2,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} A_{k-1}(y)A_{n-k+1}(y)$$
$$B_{2,n}(y) = B_{3,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{2,k-1}(y)A_{n-k+1}(y)$$
$$B_{3,n}(y) = B_{4,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{3,k-1}(y)A_{n-k+1}(y)$$
$$\vdots$$
$$B_{j-2,n}(y) = B_{j-1,n-1}(y) + \sum_{k=3}^{n} \binom{n-2}{k-2} B_{j-2,k-1}(y)A_{n-k+1}(y)$$
$$B_{j-1,n}(y) = B_{j-1,n-1}(y) + \left(\sum_{k=3}^{n} \binom{n-2}{k-2} B_{j-1,k-1}(y)A_{n-k+1}(y)\right) - \chi(n \geq p+2)\binom{n-2}{p} y^{\mathrm{des}(\tau)} A_{n-p-1}(y)$$

As in the case for $j = 3$, if we multiply everything by $\frac{t^n}{n!}$ and then sum over n we get the following system of differential equations where we think of $A(t, y)$ and $B_i(t, y)$ for $i = 2, \ldots, j-1$ as functions of t.

$$\begin{aligned}
&(D_1) \quad A'' = B_2' + A'^2 - yA' \\
&(D_2) \quad B_2'' = B_3' + B_2'A' - yA' \\
&(D_3) \quad B_3'' = B_4' + B_3'A' - yA' \\
&\vdots \\
&(D_{j-2})\ B_{j-2}'' = B_{j-1}' + B_{j-2}'A' - yA' \\
&(D_{j-1})\ B_{j-1}'' = B_{j-1}' + B_{j-1}'A' - yA' - \frac{t^p}{(p)!} y^{\mathrm{des}(\tau)} A'
\end{aligned}$$

As in the case $j = 3$, we let $A(t, y) = -\log(U(t, y))$ so that $A' = \frac{-U'}{U}$ and $A'' = \frac{-U''}{U} + \frac{U'^2}{U^2}$. Thus under this substitution, the first differential equations becomes

$$\frac{-U''}{U} + \frac{U'^2}{U^2} = B_2' + \frac{U'^2}{U^2} + y\frac{U'}{U}$$

so that

$$B_2' = \frac{-U'' - yU'}{U} \tag{77}$$

In fact, we have the following lemma.

Lemma 14. *For* $2 \leq i \leq j-1$,

$$B_i' = \frac{-U^{(i)} - y\sum_{k=1}^{i-1} U^{(k)}}{U}. \tag{78}$$

Proof. We proceed by induction on i. We have already shown that (78) in the case where $i = 2$. Now suppose that

$$B_i' = \frac{-U^{(i)} - y\sum_{k=1}^{i-1} U^{(k)}}{U}. \tag{79}$$

Then we must show that

$$B_{i+1}' = \frac{-U^{(i+1)} - y\sum_{k=1}^{i} U^{(k)}}{U}. \tag{80}$$

Taking the derivative of both sides of (79) with respect to t, we see that

$$B_i'' = \frac{-U^{(i+1)} - y\sum_{k=2}^{i} U^{(k)}}{U} + (\frac{U^{(i)} + y\sum_{k}^{i-1} U^{(k)}}{U})(\frac{U'}{U}).$$

Pluggin our expression for B_i'' and B_i' into the differential equation (D_i), we see that

$$\begin{aligned}&\frac{-U^{(i+1)} - y\sum_{k=2}^{i} U^{(k)}}{U} + (\frac{U^{(i)} + y\sum_{k}^{i-1} U^{(k)}}{U})(\frac{U'}{U})\\ &= B_{i+1}' + (\frac{-U^{(i)} - y\sum_{k=1}^{i-1} U^{(k)}}{U})(\frac{-U'}{U}) - y(\frac{-U'}{U}).\end{aligned}$$

Solving for B_{i+1}' we see that

$$B_{i+1}' = \frac{-U^{(i+1)} - y\sum_{k=1}^{i} U^{(k)}}{U}.$$

□

By the Lemma, we know that

$$B_{j-1}' = \frac{-U^{(j-1)} - y\sum_{k=1}^{j-2} U^{(k)}}{U},$$

and, hence,

$$B_{j-1}'' = \frac{-U^{(j)} - y\sum_{k=2}^{j-1} U^{(k)}}{U} + (\frac{U^{(j-1)} + y\sum_{k}^{j-2} U^{(k)}}{U})(\frac{U'}{U}).$$

Thus plugging these expressions into the differential equation (D_{j-1}), we obtain that

$$\begin{aligned}&\frac{-U^{(j)} - y\sum_{k=2}^{j-1} U^{(k)}}{U} + (\frac{U^{(j-1)} + y\sum_{k}^{j-2} U^{(k)}}{U})(\frac{U'}{U})\\ &= \frac{-U^{(j-1)} - y\sum_{k=1}^{j-2} U^{(k)}}{U} +\\ &(\frac{-U^{(j-1)} - y\sum_{k=1}^{j-2} U^{(k)}}{U})(\frac{-U'}{U}) - y(\frac{-U'}{U}) - \frac{t^p}{p!}y^{\mathrm{des}(\tau)}(\frac{-U'}{U}).\end{aligned}$$

Simplifying this expression yields that

$$U^{(j)} = (1-y)U^{(j-1)} - \frac{t^p}{p!}y^{\mathrm{des}(\tau)}U'. \tag{81}$$

Then taking the coefficient of $\frac{t^n}{n!}$ on both side of (81) gives that

$$U_{n+j} = (1-y)U_{n+j-1} + y^{\mathrm{des}(\tau)}\binom{n}{p}U_{n-p+1}$$

which is what we wanted to prove. □

We end this section with an example of the use of Theorem 13. Let $\tau = 1243$ and

$$A_{n,\tau}(t,y) = \sum_{n\geq 1} A_{n,\tau}(y)\frac{t^n}{n!} = \sum_{n\geq 1}\frac{t^n}{n!}\sum_{\sigma\in S^1_{n,\tau}} y^{\mathrm{des}(\sigma)+1} = \sum_{n\geq 1}\frac{t^n}{n!}\sum_{C\in\mathcal{L}^{ncm}_n(\tau)}.$$

Thus it is easy to check that $A_{1,\tau}(y) = y$, $A_{2,\tau}(y) = y$, $A_{3,\tau}(y) = y+y^2$, and $A_{4,\tau}(y) = y+3y^2+y^3$. Now

$$U_\tau(t,y) = \sum_{n\geq 0} U_{n,\tau}(y) = e^{-A_\tau(t,y)}$$

so that one can use Mathematica to compute that $U_{0,\tau}(y) = 1$, $U_{1,\tau}(y) = -y$,
$U_{2,\tau}(y) = -y + y^2y$, $U_{3,\tau}(y) = -y + 2y^2 - y^3$, and $U_{4,\tau}(y) = -y + 4y^2 - 3y^3 + y^4$.

By Theorem 13, we know that we have the recursion that

$$U_{n+3,\tau}(y) = (1-y)U_{n+2,\tau}(y) - yU_{n,\tau}(y).$$

Thus we can use this recursion to compute that
$U_{5,\tau}(y) = -y + 6y^2 - 8y^3 + 4y^4 - y^5$,
$U_{6,\tau}(y) = -y + 8y^2 - 16y^3 + 13y^4 - 5y^5 + y^6$,
$U_{7,\tau}(y) = -y + 10y^2 - 28y^3 + 32y^4 - 19y^5 + 6y^6 - y^7$, and
$U_{8,\tau}(y) = -y + 12y^2 - 44y^3 + 68y^4 - 55y^5 + 26y^6 - 7y^7 + y^8$.

But then we know that $NCM_\tau(t,x,y) = \frac{1}{(U_\tau(t,y))^x}$. Thus one can use Mathematica to show that

$$NCM_\tau(t,x,y) = \sum_{n\geq 0} S^{ncm}_{n,\tau}(x,y)\frac{t^n}{n!},$$

where $S^{ncm}_{0,\tau}(x,y) = 1$, $S^{ncm}_{1,\tau}(x,y) = xy$, $S^{ncm}_{2,\tau}(x,y) = xy + x^2y^2$,

$S^{ncm}_{3,\tau}(x,y) = xy + xy^2 + 3x^2y^2 + x^3y^3$,

$S^{ncm}_{4,\tau}(x,y) = xy + 3xy^2 + 7x^2y^2 + xy^3 + 4x^2y^3 + 6x^3y^3 + x^4y^4$,

$S^{ncm}_{5,\tau}(x,y) = xy+9xy^2+15x^2y^2+8xy^3+25x^2y^3+25x^3y^3+xy^4+5x^2y^4+10x^3y^4+10x^4y^4+x^5y^5$,

$S^{ncm}_{6,\tau}(x,y) = xy + 23xy^2 + 31x^2y^2 + 45xy^3 + 119x^2y^3 + 90x^3y^3 + 20xy^4 + 73x^2y^4 + 105x^3y^4 + 65x^4y^4 + xy^5 + 6x^2y^5 + 15x^3y^5 + 20x^4y^5 + 15x^5y^5 + x^6y^6$,

$S^{ncm}_{7,\tau}(x,y) = xy + 53xy^2 + 63x^2y^2 + 217xy^3 + 490x^2y^3 + 301x^3y^3 + 192xy^4 + 623x^2y^4 + 749x^3y^4 + 350x^4y^4 + 47xy^5 + 196x^2y^5 + 343x^3y^5 + 315x^4y^5 + 140x^5y^5 + xy^6 + 7x^2y^6 + 21x^3y^6 + 35x^4y^6 + 35x^5y^6 + 21x^6y^6 + x^7y^7$, and

$S^{ncm}_{8,\tau}(x,y) = xy + 115xy^2 + 127x^2y^2 + 916xy^3 + 1838x^2y^3 + 966x^3y^3 + 1500xy^4 + 4333x^2y^4 + 4466x^3y^4 + 1701x^4y^4 + 765xy^5 + 2810x^2y^5 + 4214x^3y^5 + 3164x^4y^5 + 1050x^5y^5 + 105xy^6 + 495x^2y^6 + 1008x^3y^6 + 1148x^4y^6 + 770x^5y^6 + 266x^6y^6 + xy^7 + 8x^2y^7 + 28x^3y^7 + 56x^4y^7 + 70x^5y^7 + 56x^6y^7 + 28x^7y^7 + x^8y^8$.

4 Conclusions

As mentioned in the introduction, we know of two other ways to compute $NCM_\tau(t,x,y)$ and $NCM_\Upsilon(t,y)$ for various τ's and Υ's.

Our second approach again uses the function $U_\tau(t,x,y)$ as defined in the previous section where

$$NCM_\tau(t,x,y) = \sum_{n\geq 0} ncmS_{n,\tau}(x,y)\frac{t^n}{n!} = \frac{1}{(U_\tau(t,y))^x}.$$

It follows that

$$U_\tau(t,y) = \frac{1}{NCM_\tau(t,1,y)} = \frac{1}{\sum_{n\geq 0} ncmS_{n,\tau}(x,y)\frac{t^n}{n!}}. \tag{82}$$

Remmel and his coathors [3, 12, 15, 16, 17, 18, 21, 26] developed a method called the homomorphism method to show that many generating functions involving permutation statistics can be applied to simple symmetric function identities such as

$$H(t) = 1/E(-t) \tag{83}$$

where

$$H(t) = \sum_{n\geq 0} h_n t^n = \prod_{i\geq 1} \frac{1}{1-x_i t}$$

is the generating function of the homogeneous symmetric functions h_n in infinitely many variables $x_1, x_2, \ldots$ and

$$E(t) = \sum_{n\geq 0} e_n t^n = \prod_{i\geq 1} 1 + x_i t$$

is the generating function of the elementary symmetric functions e_n in infinitely many variables $x_1, x_2, \ldots$. Now if we define a homomorphism θ and the ring of symmetric function so that

$$\theta(e_n) = \frac{(-1)^n}{n!} ncmS_{n,\tau}(1,y),$$

then

$$\theta(E(-t)) = \frac{1}{\sum_{n\geq 0} ncmS_{n,\tau}(1,y)\frac{t^n}{n!}}.$$

Thus $\theta(H(t))$ should equal $U_\tau(t,y)$. One can then use the combinatorial methods associated with the homomorphism method to develop recursions for the coefficient of $U_\tau(t,y)$ much like we did in Theorem 13. For example, we can show that

$$U_{n,1324}(y) = (1-y)U_{n-1,1324}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} C_{k-1} U_{n-2k+1,1324}(y)$$

where C_k is k-th Catalan number and

$$U_{n,1423}(y) = (1-y)U_{n-1,1423}(y) + \sum_{k=2}^{\lfloor n/2 \rfloor} (-y)^{k-1} \binom{n-k-1}{k-1} U_{n-2k+1,1423}(y).$$

The second author has developed a third way to approach the problem of computing $NCM_\Upsilon(t)$ which is completely different from the other two approaches. That method involves defining a certain bijection between the set of derangements and certain fillings of brick tabloids. That bijection allows one to compute generating functions the number derangements that have no cycle Υ-matches by applying an appropriate ring homomorphism defined on the ring of symmetric functions Λ in infinitely many variables $x_1, x_2, \ldots$ to certain simple symmetric function identities as described above. One can then multiply the resulting generating function by e^t to obtain generating for $ncmS_n(\tau)$. This approach is generally much more complicated than the first two approaches. However, it allows us to compute $NCM_{\Upsilon}(t)$ for a number of sets of permutations Υ which seem beyond the either the techniques employed in this paper or the second approach described above. For example, one can show that

$$NCM_{\Upsilon}(t) = \frac{e^t}{1 - \sum_{n \geq 3} \frac{2(-t)^n}{n!}}$$

where Υ is the set of permutations that contain 1234 and all permutations $\sigma = \sigma_1\sigma_2\sigma_3\sigma_4\sigma_5$ such such that $\sigma_1 < \sigma_2 > \sigma_3 < \sigma_4 > \sigma_5$. This approach will be described in a forth coming paper.

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